Homework 6 (due: Fr, Feb. 16)

Problem 1:

a) Let μ be a complex measure on a measurable space $(X, \mathcal{A})$. Suppose A_n for $n \in \mathbb{N} \cup \{\infty\}$ are sets in $\mathcal{A}$ with $A_n \nearrow A_\infty$. Show that then
$$\mu(A_n) \to \mu(A_\infty) \quad \text{as } n \to \infty.$$

b) Let μ and ν be complex Borel measures on $\mathbb{R}$ and suppose
$$\mu((-\infty, x]) = \nu((-\infty, x]) \quad \text{for each } x \in \mathbb{R}.$$
Show that then $\mu = \nu$.

Problem 2: In the following $F\colon \mathbb{R} \to \mathbb{R}$ is a right-continuous and increasing function. Note that then the limits $F(+\infty) := \lim_{x \to +\infty} F(x) \in (-\infty, +\infty]$ and $F(-\infty) := \lim_{x \to -\infty} F(x) \in [-\infty, +\infty)$ exist.

Recall that an *h-interval* $I \subseteq \mathbb{R}$ is an interval of the form
$$I = (a, b] := \{x \in \mathbb{R} : a < x \leq b\}$$
with $a, b \in [-\infty, +\infty]$ and $a \leq b$. If $I = (a, b]$ is an h-interval, we define
$$F(I) := F(b) - F(a) \in [0, \infty].$$

a) Let $I = (a, b] \subseteq \mathbb{R}$ be an h-interval and $\{a_n\}$ and $\{b_n\}$ be sequences in $\overline{\mathbb{R}}$ with $a_n \searrow a$ and $b_n \searrow b$. Show that if we define $I'_n = (a_n, b]$ and $I''_n = (a, b_n]$ for $n \in \mathbb{N}$, then
$$F(I) = \lim_{n \to \infty} F(I'_n) = \lim_{n \to \infty} F(I''_n).$$

b) Show that if $\mathcal{M}$ and $\mathcal{N}$ are finite families of h-intervals such that $\mathcal{M}$ is disjointed and
$$\bigcup_{I \in \mathcal{M}} I \subseteq \bigcup_{J \in \mathcal{N}} J,$$
then
$$\sum_{I \in \mathcal{M}} F(I) \leq \sum_{J \in \mathcal{N}} F(J). \tag{1}$$
Hint: Use a "common refinement".

c) Show that the inequality (1) remains true under the same assumptions if $\mathcal{M}$ and $\mathcal{N}$ are countable familes.

Hint: Shrink the intervals in $\mathcal{M}$ and enlarge the intervals in $\mathcal{N}$ slightly and use a covering argument to reduce to (b).

Problem 3: Show that if F is as in Problem 2, then there exists a unique positive Borel measure on μ on $\mathbb{R}$ such that $\mu(I) = F(I)$ for each h-interval $I \subseteq \mathbb{R}$.

Problem 4: Let $F \in NBV$ and $T_F \in NBV$ be the associated total variation function. We consider the unique complex Borel measures μ and ν on $\mathbb{R}$ such that $F_\mu = F$ and $F_\nu = T_F$. Note that ν is a positive finite measure, because T_F is bounded and increasing.

a) Show that $|\mu(I)| \leq \nu(I)$ for each h-interval $I \subseteq \mathbb{R}$.
b) Show that $\nu(I) \leq |\mu|(I)$ for each h-interval $I \subseteq \mathbb{R}$.
c) Show that similar inequalities as in (a) and (b) hold for all open intervals $I \subseteq \mathbb{R}$.
d) Show that $|\mu| = \nu$ (or, equivalently $F_{|\mu|} = T_F$).