Homework 5 (due: Fr, Feb. 9)

Problem 1: Establish the following fact used in the proof that $(L^1)^* = L^\infty$. If $(X, \mathcal{A}, \mu)$ is a measure space, $g\colon X \to \mathbb{C}$ an integrable function, and $C \geq 0$ a constant such that

$$\left| \int_A g \, d\mu \right| \leq C\mu(A)$$

for all $A \in \mathcal{A}$, then $\|g\|_\infty \leq C$.

Problem 2: Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and $f\colon X \to \mathbb{C}$ be a measurable function.

a) Show that

$$M = \{(x,t) \in X \times [0, \infty) : |f(x)| > t\}$$

is a measurable subset of $X \times [0, \infty)$.

b) We use the notation $\{|f| > t\} := \{x \in X : |f(x)| > t\}$. Show that then

$$\int |f|^p \, d\mu = p \int_0^\infty t^{p-1} \mu(\{|f| > t\}) \, dt$$

for each $p > 0$. Here we allow the possibility that these expressions have an infinite value.

c) Show the identity in (b) without the assumption that μ is σ-finite.

Problem 3: a) Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and Mf the uncentered maximal function of f. Show that if $B \subseteq \mathbb{R}^n$ an open ball and $t \geq 1$, then

$$\int_{tB} |f| \, d\lambda_n \leq t^n \int_B Mf \, d\lambda_n.$$

b) Let $1 \leq p < \infty$, $t \geq 1$, $\{B_k\}_{k \in \mathbb{N}}$ be a family of open balls in $\mathbb{R}^n$, and $a_k \geq 0$ for $k \in \mathbb{N}$. Show that then there exists a constant $C = C(p, n, t) \geq 1$ only depending on p, n, t such that

$$\left\| \sum_{k \in \mathbb{N}} a_k \chi_{tB_k} \right\|_p \leq C \left\| \sum_{k \in \mathbb{N}} a_k \chi_{B_k} \right\|_p.$$

Hint: Use L^p-L^q-duality and the maximal function.

c) Show that an inequality as in (b) is not true in general for $p = \infty$.

Problem 4: We know that a C^1-diffeomorphism on $\mathbb{R}^n$ preserves sets of measure zero. The purpose of this problem is to extend this theorem to a larger class of homeomorphisms on $\mathbb{R}^2$.

In the following, we assume that $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism with an *upper gradient* ρ in L^2_{loc}. By definition this means that ρ is a non-negative Borel function with $\rho \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ such that

$$|f(x) - f(y)| \leq \int_\gamma \rho\, ds := \int_0^1 \rho(\gamma(t))|\gamma'(t)|\, dt,$$

whenever $x, y \in \mathbb{R}^2$ and $\gamma\colon [0,1] \to \mathbb{R}^2$ is a C^1-smooth path with $\gamma(0) = x$ and $\gamma(1) = y$.

a) Show that if $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a C^1-smooth homeomorphism, then g has an upper gradient ρ in L^2_{loc}.

b) Let $B = B(a, R) \subseteq \mathbb{R}^2$ be an open ball and

$$\operatorname{osc}(f, B) := \sup\{|f(x) - f(y)| : x, y \in B\}.$$

Show that then

$$\operatorname{osc}(f, B) \leq \frac{C}{R} \int_{2B} \rho\, d\lambda_2,$$

where $C > 0$ is a constant independent of B and f. Hint: Integrate over circles and use that f is a homeomorphism.

c) Show that if $N \subseteq \mathbb{R}^2$ is a set of measure zero, then $f(N)$ is also a set of measure zero. Hint: We may assume that N is bounded. We can cover N by a collection of balls $B'_k = B(a_k, r_k)$ with $\sum r_k^2 < \epsilon$. Use (b) and in combination with suitable estimates to find a cover of $f(N)$ showing that $f(N)$ has small measure.