Homework 4 (due: Fr, Feb. 2)

Problem 1: If $\leq$ is a partial order on some set X we write $x < y$ for $x, y \in X$ if $x \leq y$ and $x \neq y$. If $a \in X$ we use the notation $X_a = \{x \in X : x < a\}$.

a) A partially ordered set $(C, \leq)$ is called *well-ordered* if it is totally ordered and if every non-empty subset $M \subseteq C$ has a *least element* $m_0 \in M$ such that $m_0 \leq m$ for all $m \in M$. A subset $I \subseteq C$ of the well-ordered set $(C, \leq)$ is called an *initial segment* if for all $x, y \in C$ we have that $y \in I$ and $x \leq y$ imply $x \in I$.

   Show that if $I \subseteq C$ is an initial segment of the well-ordered set $(C, \leq)$ with $I \neq C$, then there exists a unique element $a \in C$ such that $I = C_a$.

b) Show that if I is an initial segment of a well-ordered set $(C, \leq)$, then $I_a = C_a$ for all $a \in I$.

c) Suppose $\mathcal{C}$ is a family of well-ordered subsets of a partially ordered set $(X, \leq)$ such that for all $C, C' \in \mathcal{C}$ we have that C is an initial segment of C' or C' is an initial segment of C. Show that then

$$V = \bigcup_{C \in \mathcal{C}} C$$

is also a well-ordered subset of X. Moreover, each $C \in \mathcal{C}$ is an initial segment of V.

Problem 2: The purpose of this problem is to give a proof of Zorn's lemma. We use the notation of Problem 1. Let $(X, \leq)$ be a partially ordered set such that every chain $C \subseteq X$ has an upper bound. We choose an upper bound $s(C) \in X$ for each chain $C \subseteq X$ (including the empty chain $C = \emptyset$). If possible, we choose $s(C)$ so that not only $c \leq s(C)$ for all $c \in C$, but even $c < s(C)$ for all $c \in C$ (when can we do this?).

a) We call a chain $C \subseteq X$ *admissible* if C is well-ordered and if $x = s(C_x)$ for all $x \in C$. Show that if $C, C' \subseteq X$ are two admissible chains in X, then C is an initial segment of C' or C' is an initial segment of C. Hint: Consider the union U of all sets S that are initial segments of both C and C'. Show that U itself is an initial segment of both C and C', and that $U = C$ or $U = C'$.

b) Show that the union V of all admissible chains $C \subseteq X$ is itself an admissible chain.

c) Show that for the chain V in (b) we have $s(V) \in V$ and that $s(V)$ is a maximal element in $(X, \leq)$.

Problem 3: Let (X, d) be a metric space. As usual, we denote by $B(a, r) = \{x \in X : d(x, a) < r\}$ the open ball in X of radius $r > 0$ centered at $a \in X$. If $B = B(a, r)$ is an open ball and $\lambda > 0$, we use the notation $\lambda B = B(a, \lambda r)$.

a) Let $B = B(a, r)$ and $B' = B(b, s)$ be balls in X and assume that $B \cap B' \neq \emptyset$ and $s \geq r/2$. Show that then $B \subseteq 5B'$.

b) Let $\mathcal{B}$ be a family of balls in X with uniformly bounded radius. Show that then there exists a subfamily $\mathcal{A} \subseteq \mathcal{B}$ such that the balls in $\mathcal{A}$ are pairwise disjoint and

$$\bigcup_{B \in \mathcal{B}} B \subseteq \bigcup_{B \in \mathcal{A}} 5B.$$

Hint: Consider the sets $\mathcal{F} \subseteq \mathcal{B}$ with the following property: the balls in $\mathcal{F}$ are pairwise disjoint, and if a ball $B = B(a, r) \in \mathcal{B}$ meets a ball in $\mathcal{F}$, then B meets a ball $B' = B(b, s) \in \mathcal{F}$ with $s \geq r/2$. Use Zorn's lemma.

Problem 4: Let X be a real normed vector space.

a) Show that if $V \subseteq X$ is a finite-dimensional subspace of X, then for each $x \in X$ there exists $v_0 \in V$ such that

$$\|x - v_0\| = \inf\{\|x - v\| : v \in V\}.$$

b) Let $V \subseteq X$ is a finite-dimensional subspace of X with $V \neq V$. Show that then there exists $x_0 \in X$ with

$$\|x_0\| = 1 = \inf\{\|x_0 - v\| : v \in V\}.$$

c) Show that X is finite-dimensional if and only the closed unit ball $B = \{x \in X : \|x\| \leq 1\}$ in X is compact. Hint: For one of the directions construct a sequence in B that does not have a convergent subsequence.