Homework 3 (due: Fr, Jan. 26)

Problem 1:

a) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $\mathbb{R}^n$. Show that then these norms are *equivalent*, i.e., there exists a constant $C \geq 1$ such that
$$\frac{1}{C}\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1$$
for each $x \in \mathbb{R}^n$.

b) Let X be a real normed vector space and $T\colon \mathbb{R}^n \to X$ be a linear map. Show that T is bounded.

c) Let X be a real normed vector space, and $U \subseteq X$ be a finite-dimensional subspace of X. Show that U is closed in X.

Problem 2: A *bump function* on $\mathbb{R}^n$ is a C^∞-smooth function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ with compact support such $\varphi \geq 0$ and $\varphi \not\equiv 0$. The purpose of this problem is to show the existence of certain bump functions.

a) Define $f(x) = \exp(-1/x^2)$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$. Show that f is C^∞-smooth on $\mathbb{R}$. Hint: Show that $f^{(n)}$ exists by induction on n. For this it helps to find a general type of expression that represents $f^{(n)}$.

b) Show that if f is as in (a) and $a, b \in \mathbb{R}$ with $a < b$, then $x \in \mathbb{R} \mapsto g(x) = f(x-a)f(b-x)$ defines a bump function on $\mathbb{R}$ with support in the interval $[a, b]$.

c) Show that if $a, b \in \mathbb{R}$ with $a < b$, then there exists a C^∞-smooth function h on $\mathbb{R}$ with $0 \leq h \leq 1$ such that $h(x) = 1$ for $x \leq a$ and $h(x) = 0$ for $x \geq b$.

d) Let $0 < r < R$. Show that there exists a C^∞-smooth function φ on $\mathbb{R}^n$ with $0 \leq \varphi \leq 1$ such that $\varphi(x) = 1$ for $x \in \mathbb{R}^n$ with $|x| \leq r$ and $\varphi(x) = 0$ for $x \in \mathbb{R}^n$ with $|x| \geq R$.

Problem 3: Let V be a (possibly infinite-dimensional) vector space over the field $\mathbb{F}$.

a) Use Zorn's lemma to show that V contains a maximal linearly independent subset I (i.e., any finite subset of I is linearly independent and there exists no linearly independent subset of V that strictly contains I).

b) Show that a set I as in (a) is a basis of V, i.e., every element in V can be uniquely written as a linear combination of elements in I.

Problem 4:

a) Let X be a normed vector space and $U \neq X$ be a closed subspace. Show that then there exists a non-zero functional $f \in X^*$ such that $f|U = 0$. Hint: Pick $x \in X \setminus U$, consider the span of U and x, and apply the Hahn-Banach Theorem.

b) Let X be a normed vector space. Suppose there exist vectors x_n for $n \in \mathbb{N}$ whose span is dense in X. Show that then X is separable, i.e., there exists a countable dense subset in X.

c) Let X be a normed vector space and X^* be its dual space. Show that if X^* is separable, then X is also separable.

   Hint: Let $\{f_n : n \in \mathbb{N}\}$ be a countable dense subset in X^*. Then we can find $x_n \in X$ with $\|x_n\| \leq 1$ such that $|f_n(x_n)| \geq \frac{1}{2}\|f_n\|$.

   Remark: Part (c) is Problem 6 from the Analysis Qual, Fall 2014 (the hint was not given).