Homework 2 (due: Fr, Jan. 19)

Problem 1: Let $(X, \mathcal{A})$ be a measurable space. We denote by $\mathcal{M}$ the set of all finite signed measures on $(X, \mathcal{A})$.

a) If $a, b \in \mathbb{R}$ and $\mu, \nu \in \mathcal{M}$, we define
$$(a\mu + b\nu)(A) = a\mu(A) + b\nu(A)$$
for $A \in \mathcal{A}$. Show that $a\mu + b\nu \in \mathcal{M}$ and that $\mathcal{M}$ is a vector space over $\mathbb{R}$ with this linear structure.

b) For $\mu \in \mathcal{M}$ define $\|\mu\| = |\mu|(X)$. Show that $\mu \in \mathcal{M} \mapsto \|\mu\|$ defines a norm on $\mathcal{M}$.

c) Show that the vector space $\mathcal{M}$ equipped with the norm defined in (b) is a Banach space.

Problem 2: Let ν be a complex measure on a measurable space $(X, \mathcal{A})$.

a) Let $\nu_r = \nu_r^+ - \nu_r^-$ and $\nu_i = \nu_i^+ - \nu_i^-$ be the Jordan decompositions of the real part ν_r and the imaginary part ν_i of ν. Show that if $|\nu|$ denotes the total variation of ν, then $\nu_r^+, \nu_r^-, \nu_i^+, \nu_i^- \leq |\nu|$.

b) We say that a measurable function f on $(X, \mathcal{A})$ is *ν-integrable* if it is integrable with respect to $|\nu|$. So if we denote the space of these functions f by $L^1(\nu)$, then $L^1(\nu) = L^1(|\nu|)$. Show that if $f \in L^1(\nu)$, then f is integrable with respect to each of the measures $\nu_r^+, \nu_r^-, \nu_i^+, \nu_i^-$ and so
$$\int f\, d\nu := \int f\, d\nu_r^+ - \int f\, d\nu_r^- + i \int f\, d\nu_i^+ - i \int f\, d\nu_i^-$$
is well-defined.

c) Suppose μ is a σ-finite positive measure on $(X, \mathcal{A})$ such that $\nu << \mu$ and let $g = d\nu/d\mu$ be the Radon-Nikodym derivative of ν with respect to μ. Show that if $f \in L^1(\nu)$, then $fg \in L^1(\mu)$ and
$$\int f\, d\nu = \int fg\, d\mu.$$

Problem 3:

a) Let $f \colon \mathbb{R}^n \to \mathbb{C}$ and $g \colon \mathbb{R}^n \to \mathbb{C}$ be Borel measurable functions on $\mathbb{R}^n$. Show that the function $F \colon \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ defined as
$$F(x, y) = f(x - y)g(y) \text{ for } x, y \in \mathbb{R}^n$$
is also Borel measurable.

b) Let $f\colon \mathbb{R}^n \to \mathbb{C}$ and $g\colon \mathbb{R}^n \to \mathbb{C}$ be (Lebesgue) integrable functions. Show that then the *convolution* of f and g given by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)\, d\lambda_n(y)$$

is well-defined for almost every $x \in \mathbb{R}^n$ and that

$$\|f * g\|_1 \leq \|f\|_1 \cdot \|g\|_1.$$

c) Show that if $f\colon \mathbb{R}^n \to \mathbb{C}$ and $g\colon \mathbb{R}^n \to \mathbb{C}$ are integrable functions, then

$$(f * g)(x) = (g * f)(x) \text{ for almost every } x \in \mathbb{R}^n.$$

Problem 4: Let X be a topological space, and $f\colon X \to X$ be a continuous map. A Borel measure μ on X is called f-*invariant* if $f_*\mu = \mu$, or equivalently, if $\mu(f^{-1}(B)) = \mu(B)$ for all Borel sets $B \subseteq X$.

Consider two f-invariant Borel probability measures ν and μ on X and let $\nu = \nu_s + \nu_a$ be the Lebesgue decomposition of ν with respect to μ, where $\nu_s \perp \mu$ and $\nu_a << \mu$. Show that then ν_s and ν_a are also f-invariant.