Math 245A, Real Analysis

Fall 2017

Final

Name:

There are six problems (with 10 pts each). You can choose five problems to work on. They will be counted towards the maximal score of 50 points. After you have completed the exam, indicate here which problem should NOT be counted:

Problem 1: Let $(X, \mathcal{A})$ be a measurable space and $f_n \colon X \to \mathbb{R}$ for $n \in \mathbb{N}$ be measurable functions. Consider the set E of all points $x \in X$ for which the limit $\lim\limits_{n\to\infty} f_n(x)$ exists. Show directly from the definitions (not using $\limsup_{n\to\infty} f_n$ or $\liminf_{n\to\infty} f_n$) that $E \in \mathcal{A}$. Hint: Use quantifiers and translate to set-theoretic language. (10 pts)

Problem 2: Let $f_n \in L^2([0,1])$ for $n \in \mathbb{N}$. Suppose that $f_n(x) \to f(x)$ for all $x \in [0,1]$, where $f\colon [0,1] \to \mathbb{C}$ is some function, and that there exists a constant $C \geq 0$ such that $\|f_n\|_{L^2} \leq C$ for all $n \in \mathbb{N}$.

(a) Show that then $f \in L^2([0,1])$. (3 pts)

(b) Show that if $g \in L^2([0,1])$ is arbitrary, then

$$\int_0^1 f_n(x)g(x)\,dx \to \int_0^1 f(x)g(x)\,dx$$

as $n \to \infty$. Hint: Use Egorov and that the integral of an L^1-function over a set of small Lebesgue measure is small. (7 pts)

Problem 3: Let $f\colon \mathbb{R}^n \to [0,\infty)$ be a measurable function on $\mathbb{R}^n$. Its *Riesz potential* at $x \in \mathbb{R}^n$ is defined as

$$I(x) = \int \frac{f(y)}{|x-y|^{n-1}}\, d\lambda_n(y).$$

Show that if f has compact support and $f \in L^p(\mathbb{R}^n)$ with $n < p < \infty$, then $I(x) < +\infty$ for each $x \in \mathbb{R}^n$. Hint: Introduce suitable polar coordinates. (10 pts)

Problem 4: Let $A \subseteq [0,1]$ be a Borel set. We are interested in the set $S = \{x+y : x, y \in A\}$ of all sums of elements in A.

(a) Show that the function $(x,y) \mapsto \chi_A(x-y)\chi_A(y)$ is Borel measurable on $\mathbb{R}^2$. (2 pts)

(b) Consider the function

$$f(x) = \int \chi_A(x-y)\chi_A(y)\, dy.$$

Show that f is defined for each $x \in \mathbb{R}$ and is a continuous function on $\mathbb{R}$. (5 pts)

(c) Show that if $\lambda_1(A) > 0$, then S contains a non-empty open interval. (3 pts)

Problem 5: Let $f\colon \mathbb{R} \to \mathbb{C}$ be a measurable function on $\mathbb{R}$. For $\delta \in \mathbb{R}$ we denote by $T_\delta f\colon \mathbb{R} \to \mathbb{C}$ the function defined as

$$T_\delta f(x) = f(x + \delta) \text{ for } x \in \mathbb{R}.$$

Show that if $f \in L^1(\mathbb{R})$, then $T_\delta f \in L^1(\mathbb{R})$ and

$$\|T_\delta f - f\|_{L^1} \to 0 \text{ as } \delta \to 0.$$

Hint: First consider functions in a suitable subclass of $L^1(\mathbb{R})$. (10 pts)

Problem 6: Let μ be a Borel probability measure on $\mathbb{R}$ (so $\mu(\mathbb{R}) = 1$) with $\displaystyle\int |x|\, d\mu(x) < \infty$. Consider the function $\varphi\colon \mathbb{R} \to \mathbb{R}$ defined as

$$\varphi(u) = \int \cos(ux)\, d\mu(x)$$

for $u \in \mathbb{R}$. Then φ is C^1-smooth and

$$\varphi'(u) = -\int x \sin(ux)\, d\mu(x).$$

(you can use these facts without further justification). Show that $\varphi''(0)$ exists if and only if $\displaystyle\int x^2\, d\mu(x) < \infty$.

Hint: For one of the directions consider $\dfrac{\varphi(u) + \varphi(-u) - 2\varphi(0)}{u^2}$. (10 pts)