(iii) Note: $|J_T|$ continuous on U; so if f meas., then $(f\circ T)\cdot J_T$ meas.
This takes care of measurability issues.

For the proof of (i) we need a

<u>Lemma</u> Suppose $Q \subseteq U$ is a cube.
Then
$$\lambda_n(T(Q)) \leq \int_Q |J_T| \, d\lambda_n .$$

We postpone the proof of the lemma, and first finish the proof of the theorem assuming the lemma is true.

<u>Claim</u>
$$\int_V f \, d\lambda_n \leq \int (f\circ T)\, |J_T| \, d\lambda_n$$
for all measurable $f \geq 0$ on V.

This is true if $\quad (\chi_{T(E)}\circ T = \chi_E)$

1) $f = \chi_{T(Q)}$ $\quad Q \subseteq U$ cube by lemma

2) $f = \chi_{T(W)}$ where $W \subseteq U$ is any open set.

Indeed, if $W \subseteq U$ is open, then one can find a countable collection Q_i, $i \in \mathbb{N}$, of cubes $Q_i \subseteq W$ with pairwise disjoint interiors s.t. $W = \bigcup_{i\in I} Q_i$
(this is related to the "Whitney cube de-

(112) composition; we'll discuss this later).

Note that $\lambda_n(\partial Q) = 0$ for any cube $Q \subseteq \mathbb{R}^n$; so the cubes and their images are essentially disjoint, i.e.,

$$\sum_i \chi_{Q_i} = \chi_W \quad \text{a.e. and}$$

$$\sum_i \chi_{T(Q_i)} = \chi_{T(W)} \quad \text{a.e.}$$

Hence

$$\lambda_n(T(W)) = \int \chi_{T(W)} = \int \sum_i \chi_{T(Q_i)}$$

$$\overset{MCT}{=} \sum_i \int \chi_{T(Q_i)} = \sum_i \lambda(T(Q_i))$$

$$\overset{\text{Claim 1)}}{\leq} \sum_i \int_{Q_i} |J_T| \overset{MCT}{=} \int \left(\sum \chi_{Q_i}\right) |J_T|$$

$$= \int \chi_W |J_T| = \int_W |J_T|.$$

3) $f = \chi_{T(B)}$, where $B \subseteq U$ meas. set with $\overline{B} \subset\subset U$.

By outer regularity of Lebesgue measure we can find open sets with $\overline{W_k} \subset\subset U$ and $W_k \supseteq B$ with $\lambda_n(B) \leq \lambda_n(W_k) \leq \lambda_n(B) + \frac{1}{k}$.

Let $\Omega_k = W_1 \cap \dots \cap W_k$; then $\Omega_k \searrow \Omega = \bigcap_{k=1}^{\infty} \Omega_k \supseteq B$, and

$$\lambda(\Omega_k) \longrightarrow \lambda(\Omega) = \lambda(B).$$

So $\lambda(\Omega \setminus B) = 0$, i.e., $\chi_{\Omega_k} \longrightarrow \chi_\Omega = \chi_B$ a.e.

(113) Since T preserves sets of measure zero, we also have

$$\chi_{\Omega_n} \circ T^{-1} = \chi_{T(\Omega_n)} \longrightarrow \chi_B \circ T^{-1} = \chi_{T(B)} \quad \text{a.e.}$$

Note $0 \le \chi_{T(\Omega_k)} \le \underbrace{\chi_{T(\Omega_1)}}_{\text{bdd.}} \in L^1(\lambda_n)$,

and $0 \le \chi_{\Omega_k} \cdot |J_T| \le \underbrace{\chi_{\Omega_1} \cdot |J_T|}_{\text{bdd. on } \Omega_1,\ \in L^1(\lambda)}$.

So by LDC and Claim 2):

$$\lambda(T(B)) = \int \chi_{T(B)} \overset{LDC}{=} \lim_{k\to\infty} \int \chi_{T(\Omega_k)} \overset{\text{Claim 2)}}{\le} \lim_{k\to\infty} \int \chi_{\Omega_k} \cdot |J_T| \overset{LDC}{=} \int \chi_B |J_T|$$

$$= \int_B |J_T| .$$

4) $f = \chi_{T(M)}$, where $M \subseteq U$ is an arb. meas. set

Take a compact exhaustion of U, i.e., $K_n \subseteq U$ compact with $K_n \nearrow U$.
Then $B_n = K_n \cap U \subset\subset U$ meas. and $B_n \nearrow M$.
Hence $T(B_n) \nearrow T(M)$ and

$$\chi_{B_n} \cdot |J_T| \longrightarrow \chi_M \cdot |J_T| .$$

By MCT:

$$\lambda(T(M)) = \lim_{k\to\infty} \lambda(T(B_n)) \le \lim_{k\to\infty} \int_{B_n} |J_T|$$

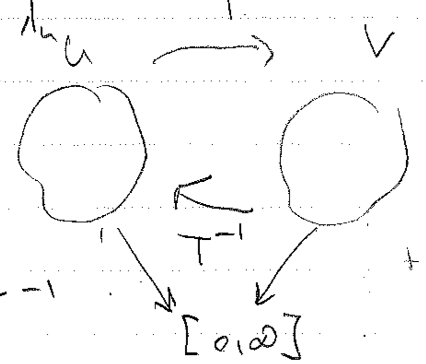
$$= \int_M |J_T| .$$

(114) 5) $f \geq 0$ meas.
Use 4) and the usual monotone class machine.
This finishes the proof of the Claim

The Claim implies the transformation formula (1):

1) Let $f : V \longrightarrow [0,\infty]$ be meas.
By Claim applied to f:

$$\int_V f \, d\lambda_n \leq \int_U (f \circ T) \, |J_T| \, d\lambda_n$$



We can also apply the Claim to the meas. function $g = (f \circ T) \cdot |J_T| \geq 0$ and T^{-1}

Hence

$$\int_U g \, d\lambda_n \leq \int_V (g \circ T^{-1}) \cdot |J_{T^{-1}}| \, d\lambda_n$$

$$= \int_V f(y) \cdot \underbrace{|J_T(T^{-1}(y))| \cdot |J_{T^{-1}}(y)|}_{J_{T \circ T^{-1}}(y) = 1} \, d\lambda_n(y) \quad \text{Chainrule}$$

$$= \int_V f \, d\lambda_n$$

Formula (1) follows.

2) If $f \in L^1(\lambda_n)$, we split f first into real and imaginary part, then into positive and negative part and apply (1) for non-neg. functions. Form. (1) also follows in this case. □

(115) <u>Proof of Lemma</u>

Basic idea: Decompose Q into small subcubes $Q_1, \dots, Q_N$ on which T behaves almost like an affine map

$$\left(T|Q_i \approx x \in Q_i \longmapsto \underbrace{f(x_i) + DT(x_i)(x - x_i)}_{A_i(x)} \right)$$

x_i center of cubes

Then

$$\lambda_n(T(Q)) = \sum_{i=1}^{N} \lambda(T(Q_i))$$

$$\approx \sum_{i=1}^{N} \lambda_n(A_i(Q_i)) = \sum_{i=1}^{N} \underbrace{|\det(DT(x_i))|}_{J_T(x_i)} \lambda_n(Q_i)$$

$$\approx \sum_{i=1}^{N} \int_{Q_i} |J_T| \, d\lambda_n = \int_Q |J_T| \, d\lambda_n .$$

To make this precise we use careful estimates.

If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $A = (a_{ij})$ $n\times n$-matrix define

$$\underbrace{\|x\|_\infty := \max_{1 \le i \le n} |x_i|}_{\ell^\infty\text{-norm}}, \quad \underbrace{\|A\| = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|}_{\text{matrix norm}}$$

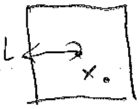
$\|x\|_1 = \sum_{i=1}^{n} |x_i|$ ℓ^1-norm

Then $\|Ax\|_\infty \le \|A\| \|x\|_\infty$ and $\|AB\| \le \|A\| \cdot \|B\|$

$\|x \cdot y\| \le \|x\|_1 \cdot \|y\|_\infty$ for $x \in \mathbb{R}^n$, A, B $n\times n$-matrices.

(exercise!)

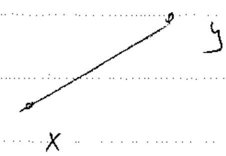
Note: $\{ x \in \mathbb{R}^n : \|x - x_0\| \le l \}$ is cube with center x_0 and sidelength $2l$.



Suppose $S = (g_1, \dots, g_n)$ is C^1-smooth map on $U \supseteq Q$.

(*) Then $\|S(y) - S(x)\|_\infty \le \overbrace{\sup_{u \in Q} \|DS(u)\|}^{M_Q} \cdot \|y - x\|_\infty$ for $x, y \in Q$.

(116) Consider the component functions g_k:



$$
\begin{aligned}
|g_k(y) - g_k(x)| &= \left| \int_0^1 \frac{d}{dt}\, g_k(x+t(y-x))\, dt \right| \\
&= \left| \int_0^1 \nabla g_k(x+t(y-x)) \cdot (y-x)\, dt \right| \qquad \text{dot-product} \\
&\le \int_0^1 \underbrace{\|\nabla g_k(x+t(y-x))\|_1}_{\le \sup_{u\in Q}\|DS(u)\|} \cdot \|y-x\|_\infty \, dt \\
&\le \sup_{u\in Q} \|DS(u)\| \cdot \|y-x\|_\infty
\end{aligned}
$$

Ineq. (*) follows.

Consequence: If Q is a cube of side length l and center x, then $S(Q)$ is contained in a cube with center $S(x)$ and sidelength $\sup_{u\in Q}\|DS(u)\| \cdot l$; so

$$\lambda_n(S(Q)) \le M_Q^n \cdot \lambda_n(Q).$$

Let $\varepsilon > 0$ be arb. Then we can decompose Q into subcubes $Q_1, \dots, Q_N$ with centers $x_1, \dots, x_N$ resp and sidelength $\delta > 0$ s.t.

i) $|J_T(x) - J_T(y)| < \varepsilon$ for $x, y \in Q_i$, $i = 1, \dots, N$ arb.

ii) $\|\underbrace{DT(x)^{-1}}_{L_x} \underbrace{DT(y)}_{L_y}\| \le 1+\varepsilon$ for $x, y \in Q_i$

(117) $\|L_x^{-1} \circ L_y\| = \|L_x^{-1} \circ (L_y - L_x) + I_n\|$

$\leq 1 + \underbrace{\|L_x^{-1}\|}_{\text{unif. bdd. on } Q} \cdot \underbrace{\|L_y - L_x\|}_{\text{small if } \delta \text{ small}} \leq 1+\varepsilon.$

On Q_i let $L_i := DT(x_i)$ and
$S_i := L_i^{-1} \circ T$.
Then $\|DS_i(u)\| = \|DT(x_i)^{-1} DT(u)\|$
$\leq 1+\varepsilon$ for $u \in Q_i$
Hence $S_i(Q)$ contained in cube $\tilde{Q}_i$ with
side length $(1+\varepsilon)\delta$
$T(Q_i) = L_i(S_i(Q_i)) \subseteq L_i(\tilde{Q}_i)$. $\lambda_n(Q_i)$
So
$\lambda_n(T(Q_i)) \leq \lambda_n(L_i(\tilde{Q}_i)) = |\det L_i| \cdot (1+\varepsilon)^n \overbrace{\delta^n}^{\lambda_n(Q_i)}$
$(1+\varepsilon)^n J_T(x_i) \cdot \lambda(Q_i)$.

Hence

$$\lambda_n(T(Q)) = \sum_{i=1}^{N} \lambda_n(T(Q_i))$$

$$\leq (1+\varepsilon)^n \sum_{i=1}^{N} |J_T(x_i)|\, \lambda_n(Q_i)$$

$$= (1+\varepsilon)^n \sum_{i=1}^{n} \int_{Q_i} |J_T(x_i)|\, d\lambda_n$$

$$= (1+\varepsilon)^n \sum_{i=1}^{n} \int_{Q_i} \underbrace{|J_T(x_i) - J_T(x)|}_{\leq \varepsilon} + |J_T(x)|\, d\lambda_n$$

$$= (1+\varepsilon)^n \int_Q |J_T|\, d\lambda_n + \varepsilon \lambda_n(Q).$$

Letting $\varepsilon \to 0$ the lemma follows. $\square$

(118) <u>Thm.</u> (Whitney cube decomposition of open sets in $\mathbb{R}^n$)

Let $\Omega \subseteq \mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$ be non-empty and open. Then there exists a countable collection Q_k, $k \in \mathbb{N}$, of cubes ("Whitney cubes") s.t.

i) $\Omega = \bigcup_{k \in \mathbb{N}} Q_k$

ii) the cubes do not "overlap", i.e.,
$$\mathring{Q}_k \cap \mathring{Q}_l = \emptyset \quad \text{for } k \neq l$$

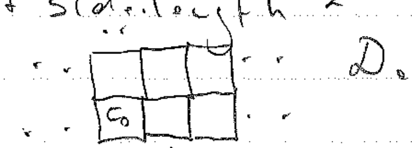
iii) $$\operatorname{diam}(Q_k) \leq \operatorname{dist}(Q_k, \partial\Omega) \leq 4 \operatorname{diam}(Q_k) \quad \text{for } k \in \mathbb{N}$$

the diameter of each cube is comparable to its distance to the boundary.

<u>Note:</u> $\partial\Omega \neq \emptyset$; otherwise, $\partial\Omega = \emptyset$ and $\Omega = \Omega \cup \partial\Omega = \overline{\Omega}$ is open and closed. This is a contradiction, because $\Omega \neq \emptyset, \mathbb{R}^n$ and $\mathbb{R}^n$ is connected.

<u>Proof (Outline):</u> $C_0 := [0,1]^n$ unit cube, $\operatorname{diam} C_0 = \sqrt{n}$.

$\mathcal{D}_k := \{ 2^{-k}(u + C_0) : u \in \mathbb{Z}^n \}$ dyadic cubes of sidelength 2^{-k}, $k \in \mathbb{Z}$



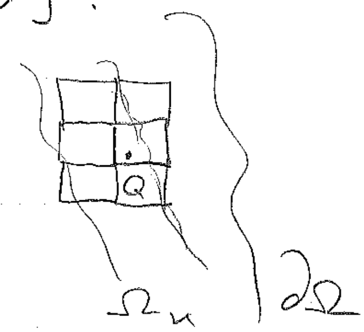
If $Q \in \mathcal{D}_k$, then $\operatorname{diam} Q = 2^{-k} \cdot \sqrt{n}$

Layers $\Omega_k = \{ x \in \Omega : c \cdot 2^{-k} < \operatorname{dist}(x, \partial\Omega) \leq c\, 2^{-k+1} \}$

$k \in \mathbb{Z}$ $\qquad c = 2\sqrt{n}$ $\qquad \Omega = \bigcup_{k \in \mathbb{Z}} \Omega_k$

(11) $\mathcal{F}_0 = \bigcup_{k \in \mathbb{Z}} \{ Q \in \mathcal{D}_k : Q \cap \Omega_k \neq \emptyset \}$.

initial choice of cubes



Then if $Q \in \mathcal{F}_0$:

$$\operatorname{dist}(\bar{Q}, \partial\Omega) \geq c \cdot 2^{-k} - \operatorname{diam}(Q)$$
$$= \sqrt{n}\, 2^{-k} = \operatorname{diam}(Q)$$
$$\operatorname{dist}(Q, \partial\Omega) \leq 2 \cdot c\, 2^{-k}$$
$$= 4 \cdot \operatorname{diam}(Q).$$

So (iii) true.

Moreover, $\bigcup_{Q \in \mathcal{F}_0} Q = \Omega$.

(ii) not necessarily true!

Let $\mathcal{F} \subseteq \mathcal{F}_0$ be the family of maximal cubes Q in $\mathcal{F}_0$, i.e., cubes Q not contained in any other cube in $\mathcal{F}_0$.

Then (i) - (iii) true:

(iii) clear,

(ii) if $Q, Q' \in \mathcal{F}_0$, $Q \neq Q'$, then neither $Q \subseteq Q'$, nor $Q' \subseteq Q$; so $\mathring{Q} \cap \mathring{Q}' = \emptyset$, because Q, Q' are dyadic cubes

(i) Obviously, $\underbrace{\dot{\bigcup}_{Q \in \mathcal{F}} Q}_{\hat{\Omega}} \subseteq \bigcup_{Q \in \mathcal{F}_0} Q = \Omega$.

To see the converse ineq. let $x \in \Omega$ be arb. Then $x \in Q_1$ for some $Q_1 \in \mathcal{F}_0$. If Q_1 maximal, then $x \in \hat{\Omega}$. If not, then there ex. $Q_2 \in \mathcal{F}_0$ with $Q_1 \subsetneq Q_2$. If Q_2 is maximal, then $x \in \hat{\Omega}$. If not ex. $Q_3 \in \mathcal{F}_0$ s.t. $Q_1 \subsetneq Q_2 \subsetneq Q_3$, etc. This process must end giving $x \in \hat{\Omega}$. $\square$