(74) By Fatou's lemma we have for $n > N$,

$$\int |f - f_n|^p = \int \liminf_{k\to\infty} |f_{n_k} - f_n|^p$$

$$\leq \liminf_{K\to\infty} \underbrace{\int |f_{n_k} - f_n|^p}_{\leq \varepsilon^p \text{ for } k \text{ large}} \leq \varepsilon^p < \infty$$

So $f - f_n \in \mathcal{L}^p$ and so $f = f_n + (f - f_n) \in \mathcal{L}^p$

and $\|f - f_n\|_p \leq \varepsilon$ for $n > N$.

Hence $\|f - f_n\|_p \longrightarrow 0$ as $n \longrightarrow \infty$. $\square$

<u>Thm.</u> Let $1 \leq p < \infty$. Then simple functions $s \in L^p$ are dense in $L^p$, i.e., for each $f \in L^p$ and each $\varepsilon > 0$ there ex. a simple function

$$s = \sum_{k=1}^{n} \alpha_k \chi_{A_k} \quad \text{with } \alpha_k \neq 0,\ \alpha_k \in \mathbb{C},\ A_k \in \mathcal{A},\ \mu(A_k) < \infty \text{ for } k = 1, \ldots, n$$

s.t. $\|f - s\|_p < \varepsilon$.

<u>Proof:</u> It is enough to prove this for non-neg. functions $f \in L^p$. (why?)
If $f \in L^p$ is non-neg., there ex. a sequence of simple functions $s_n \geq 0$ with $s_n \nearrow f$.

Then $f(x) - s_n(x) \longrightarrow 0$ for all $x \in X$,

$|f - s_n|^p \leq 2^p \max(f, s_n) \leq 2^p f^p =: g$

$\int g = \int 2^p f^p = 2^p \int f^p = 2^p \|f\|_p^p < \infty.$

(75) So by LDC
$$\int |f-s_n|^p \longrightarrow \int 0 = 0$$

equiv. $\|f-s_n\|_p \longrightarrow 0$.

So for each $\varepsilon>0$ there ex. $n \in \mathbb{N}$ s.t. for $s = s_n$ we have
$$\|f-s\|_p < \varepsilon.$$

If we write
$$s = \sum_{k=1} \alpha_k \chi_{A_k}$$
with $\alpha_k > 0$ and pairwise disj. $A_k$,

then
$$\int s^p = \int \sum_{k=1}^{n} \alpha_k^p \chi_{A_k}$$
$$= \sum_{k=1}^{n} \alpha_k^p \mu(A_k) \leq \int f^p < \infty,$$

and so
$$\mu(A_k) < \infty \quad \text{for } k = 1, \dots, n. \quad \square$$

<u>Def.</u>: Let $X$ be a locally compact Hausdorff space.

A <u>Borel measure</u> $\mu$ on $X$ is called a <u>Radon measure</u> if

i) $\mu(K) < \infty$ for all $K \subseteq X$ compact,

ii) $\mu$ is <u>outer regular</u>, i.e., for each Borel set $M \subseteq X$ we have
$$\mu(M) = \inf\{\mu(U) : U \supseteq M \text{ open}\}$$

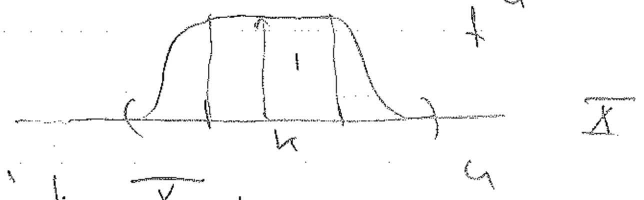
iii) $\mu$ is <u>inner regular for open sets</u>, i.e.,

(76) for all $U \subseteq X$ open we have
$$\mu(U) = \sup\{\mu(K) : K \subseteq U \text{ compact}\}$$

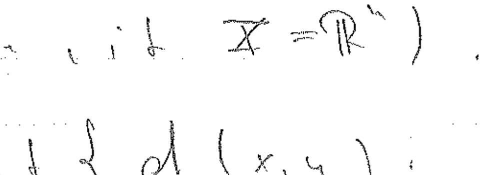
<u>Note</u>: Lebesgue measure on $\mathbb{R}^n$ (restricted to Borel sets) is a Radon measure.

<u>Urysohn's Lemma (locally compact version)</u>
Let $X$ be a locally compact Hausdorff space, $K \subseteq X$ compact, $U \subseteq X$ open with $K \subseteq U$.
Then there ex. a continuous function $f : X \to [0,1]$ s.t. $f|K \equiv 1$, $f|\underbrace{X \setminus U}_{U^c} \equiv 0$.

<u>Proof</u>: See Folland, p. 131, 4.32.




This is easy to prove if $X$ is a metric space (for example, if $X = \mathbb{R}^n$).
Let
$$\delta := \operatorname{dist}(K, U^c) = \inf\{d(x,y) : x \in K, y \in U^c\}$$
So $\delta > 0$, because $K$ compact, $U^c$ closed and $K \cap U^c = \emptyset$.
Define
$$f(x) = \left(1 - \frac{\operatorname{dist}(x,K)}{\delta}\right)_+ .$$




Then $0 \le f \le 1$, $f$ is continuous, $f|K \equiv 1$, $f|U^c \equiv 0$. □

$$C_c(X) = \{f : X \to \mathbb{C} : \operatorname{supp}(f) \subseteq X \text{ compact}\}$$

(77) Rem. Actually, the function $f$ in the lemma can be chosen to have compact ~~support, i.e.,~~ such that $\operatorname{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}}$ compact:

By a covering argument we can choose an open set $V$ with $\overline{V}$ compact s.t. $K \subseteq V \subseteq \overline{V} \subseteq U$.

Then there ex. $f : X \to [0,1]$ cont. s.t. $f|K \equiv 1$, $f|V^c \equiv 0$.

Then $\operatorname{supp}(f) \subseteq \overline{V}$ compact, $f|U^c \equiv 0$.

Thm. Let $1 \leq p < \infty$, $X$ a locally compact Hausdorff space, and $\mu$ be a Radon measure on $X$.

Then the continuous functions with compact support are dense in $L^p$, i.e., for each $f \in L^p(\mu)$ and each $\varepsilon > 0$ there ex. $g \in C_c(X)$ s.t.

$$\| f - g \|_p \leq \varepsilon.$$

Proof: Since simple functions are dense in $L^p$, it is enough to show this for a function

$$f = \chi_A, \text{ where } A \subseteq X \text{ Borel and } \mu(A) < \infty.$$

By outer regularity of $\mu$ and inner reg. of $\mu$ on opensets, we can find an open set $U \supseteq A$ with

$$\mu(A) \le \mu(U) \le \mu(A) + \left(\frac{\varepsilon}{2}\right)^p$$

and a compact set $K \subseteq U$ with

$$\mu(U) - \left(\frac{\varepsilon}{2}\right)^p \le \mu(K) \le \mu(U).$$

By Urysohn's Lemma there ex. $g \in C_c(X)$ s.t. $0 \le f \le 1$, $g|K \equiv 1$, $g|U^c \equiv 0$.

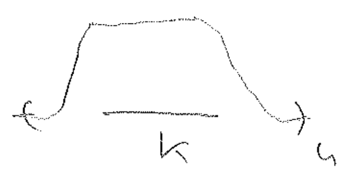
Then

$$\|\chi_A - \chi_U\|_p = \|\chi_{U\setminus A}\|_p$$

$$= \left(\int \chi_{U\setminus A}^p\right)^{1/p} = \mu(U\setminus A)^{1/p} \le \frac{\varepsilon}{2};$$

moreover

$$\chi_K \le g \le \chi_U, \text{ so}$$




$$|\chi_U - g| \le \chi_{U\setminus K} \quad \text{and}$$

$$\|\chi_U - g\|_p \le \mu(U\setminus K)^{1/p} \le \frac{\varepsilon}{2}.$$

So by Minkowski's ineq.

$$\|f - g\|_p = \|\chi_A - g\|_p \le \|\chi_A - \chi_U\|_p + \|\chi_U - g\|_p$$

$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \square$$

<u>Note.</u> $C_c(X) \subseteq L^p(\mu)$

(if $f \in C_c(X)$, then there ex. $K \subseteq X$ comp. and $C > 0$ s.t. $|f| \le C \cdot \chi_K$. Hence

$$\int |f|^p \le C^p \int \chi_K = C^p \mu(K) < \infty.$$

(79) By previous thm. $C_c(X)$ is a dense subspace of $L^p(\mu)$, and so

$$\overline{C_c(X)} = L^p(\mu).$$

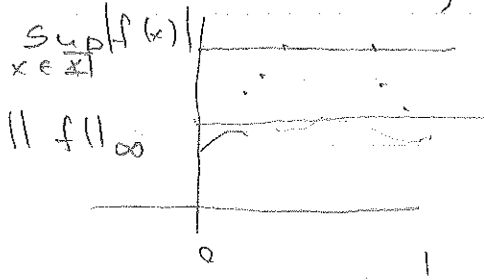
## $L^\infty$ and the essential supremum

$(X, \mathcal{A}, \mu)$ measure space, $f : X \to \mathbb{C}$ measurable

$$\|f\|_\infty = \underbrace{\operatorname{ess\,sup}_{x \in X} |f(x)|}_{\text{"essential supremum"}}$$

$$= \inf \{ \lambda : \mu(\{x \in X : |f(x)| > \lambda\}) = 0 \}$$

$$\mathcal{L}^\infty := \{ f : X \to \mathbb{C} : \|f\|_\infty < \infty \}$$




$\|\cdot\|_\infty$ seminorm on $\mathcal{L}^p$.

$f \sim g$ iff $f = g$ a.e. iff $f - g = 0$ a.e. iff $\|f - g\|_\infty = 0$.

$$L^\infty := \{ [f] : f \in \mathcal{L}^\infty \},$$

$[f] + [g] = [f + g]$

$\alpha [f] = [\alpha f]$

$\|[f]\|_\infty = \|f\|_\infty$.

**Thm.** i) $\|\cdot\|_\infty$ norm on $L^\infty$.

ii) With this norm $L^\infty$ is a Banach space

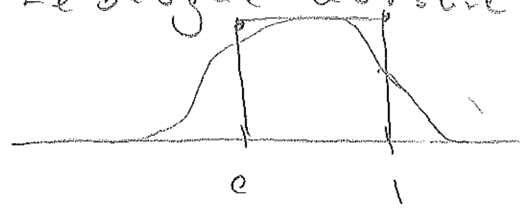
iii) Simple functions are dense in $L^\infty$

(80) <u>Note</u>: If $X$ locally comp. Hausdorff space and $\mu$ Radon measure on $X$, then
$$C_c(X) \subseteq L^\infty(\mu), \text{ but in general } C_c(X) \text{ is not dense in } L^\infty(\mu).$$

<u>Example</u>: $f = \chi_{[0,1]} \in L^\infty(\lambda)$, but

Lebesgue measure

$\|f - g\|_\infty \geq \frac{1}{2}$ for all $g \in C_c(\mathbb{R})$.



## <u>Lebesgue and Riemann integral</u>

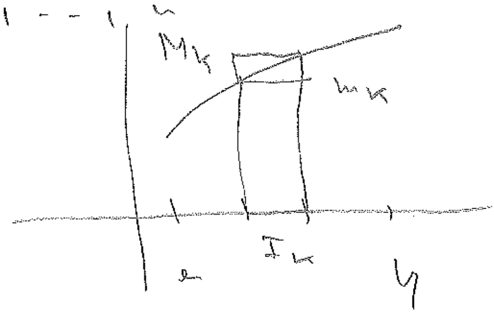
$f : [a,b] \longrightarrow \mathbb{R}$ bdd. function.

$P = \{t_0, t_1, \ldots, t_n\}$ partition of $[a,b]$, i.e., $a = t_0 < t_1 < \ldots < t_n = b$.

$I_k = [t_{k-1}, t_k]$ intervals of partition $k = 1, \ldots, n$

$m_k = \inf\{f(x) : x \in I_k\}$

$M_k = \sup\{f(x) : x \in I_k\}$




$L(f,P) = \sum_{k=1}^{n} m_k (t_k - t_{k-1})$ "lower sum"

$U(f,P) = \sum_{k=1}^{n} M_k (t_k - t_{k-1})$ "upper sum"

$\text{mesh}(P) := \max_{k=1,\ldots,n} (t_k - t_{k-1})$

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Fact: if $P, Q$ partitions of $[a,b]$, then
$L(f,P) \leq U(f,Q)$.

$$\begin{aligned} I(f) &:= \sup\{L(f,P) : P \text{ partition of } [a,b]\} \\ &\leq \inf\{U(f,P) : Q \text{ partition of } [a,b]\} \\ &= U(f). \end{aligned}$$

<u>Def.</u>: A bdd. function $f : [a,b] \to \mathbb{R}$ is called <u>Riemann integrable</u> if $I(f) = U(f)$.
In this case, we define
$$\int_a^b f(x)\,dx := I(f) = U(f).$$

<u>Fact</u>: If $f$ is Riemann integrable, and $P_n, n \in \mathbb{N}$, sequence of partitions of $[a,b]$ with
$P_n \subseteq P_{n+1}$ for $n \in \mathbb{N}$ and "mesh $(P_n) \to 0$", partitions get finer and finer"
then
$$I(f,P_n) \to \int_a^b f(x)\,dx \quad \text{and}$$
$$U(f,P_n) \to \int_a^b f(x)\,dx.$$

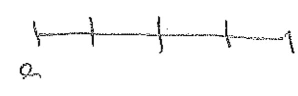
<u>Thm.</u> Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable and define
$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in [a,b] \\ 0 & \text{elsewhere} \end{cases}$$

(82) Then $\hat{f}$ is Lebesgue measurable and
$$\int \hat{f}\, d\lambda = \int_a^b f(x)\,dx.$$

(So essentially, each Riemann integrable function is Lebesgue measurable and the Riemann integral agrees with Lebesgue integral).

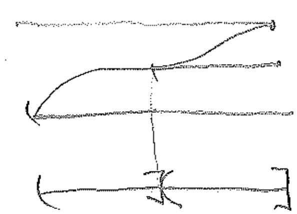
<u>Proof</u>: Pick seq. $P_n$, $n \in \mathbb{N}$, of partitions s.t. $P_n \subseteq P_{n+1}$ f.a. $n \in \mathbb{N}$ and
$$\text{mesh}(P_n) \longrightarrow 0.$$
$$\left( \text{e.g. } P_n = \{ t_k^n : t_k^n = a + (b-a)\cdot\frac{k}{2^n},\ k = 0,1,\dots,2^n \} \right)$$

For a partition $P$ we define



$P_2$

$$g_P = \sum_k m_k \chi_{(t_{k-1},t_k]}$$
$$G_P = \sum M_k \chi_{(t_{k-1},t_k]}$$



$g_Q$

$g_P$

If $Q \supseteq P$ is another partition, then
$$-M \le g_P \le g_Q \le f \le G_Q \le G_P \le M \text{ on } (a,b],$$
where $M = \sup_{x\in[a,b]} |f(x)|$

Define $g_n := g_{P_n}$, $G_n := G_{P_n}$.

Then $g_n \nearrow$, $G_n \searrow$

(83) $g_n, G_n$ simple functions (hence measurable)

Let $g = \lim_{n\to\infty} g_n$, $G = \lim_{n\to\infty} G_n$. measurable.

Then $g \leq f \leq G$ on $[a,b]$.

By LDC (n.b. that all functions are uniformly bdd. on $[a,b]$), we have

$$\int g = \lim_{n\to\infty} \int g_n = \lim_{n\to\infty} L(f,P_n) = I$$
$$= \lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} \int G_n$$
$$= \int G.$$

So $G-g \geq 0$ and $\int(G-g) = \int G - \int g = \underbrace{I-I}_{=0}$;

hence $g = G$ a.e., and so $\tilde{f} = g = G$ a.e. on $[a,b]$. This implies that $\tilde{f}$ is Lebesgue measurable, and $\int \tilde{f} = \int g = \int G = I = \int_a^b f(x)\,dx$. $\square$

Ex. $f(x) = \begin{cases} 0 & \text{elsewhere.} \\ \frac{1}{x^\alpha} & \text{for } x \geq 1 \end{cases}$ $\quad \alpha > 0$

$f$ is measurable (exercise!)

For which $\alpha > 0$ is $f$ integrable?

Solution: $f|[1,n]$ cont. hence Riemann integrable; so $f_n := \chi_{[1,n]} f$ Lebesgue integrable, $f_n \nearrow f$. So by monotone conv. thm.

(84) $\int f = \lim_{n\to\infty} \int f_n = \lim_{n\to\infty} \int_1^n \frac{dx}{x^\alpha} = \begin{cases} \infty & \alpha \le 1, \\ \frac{1}{\alpha-1} & \alpha > 1. \end{cases}$

$$\int_1^n \frac{dx}{x^\alpha} = \begin{cases} [\log x]_1^n = \log n & \longrightarrow \infty \\ \frac{1}{1-\alpha}\left[x^{1-\alpha}\right]_1^n = \frac{1}{1-\alpha}\left(n^{1-\alpha}-1\right) \longrightarrow & \infty \quad \alpha < 1 \\ & \frac{1}{\alpha-1} \quad \alpha > 1 \end{cases}$$

So $f$ integrable iff $\alpha > 1$.

## Modes of convergence

$(X, \mathcal{A}, \mu)$ measure space, $f : X \to \mathbb{C}$, $f_n : X \to \mathbb{C}$, $n \in \mathbb{N}$, measurable functions.

We consider several types of convergence $f_n \to f$

i) $f_n \to f$ (pointwise) a.e. iff there ex. $N \in \mathcal{A}$, $\mu(N) = 0$ s.t. $f_n(x) \to f(x)$ for all $x \in X \setminus N$.

ii) $f_n \to f$ in $L^p$ $(1 \le p < \infty)$ iff $\| f_n - f \|_p \to 0$ equiv. $\int |f_n - f|^p \to 0$

iii) $f_n \to f$ in measure iff

$$\lim_{n\to\infty} \mu(\underbrace{\{ x \in X : |f_n(x) - f(x)| \ge \varepsilon \}}) = 0 \quad \text{for all } \varepsilon > 0.$$

Prop. If $f_n \to f$ in $L^p$ $(1 \le p < \infty)$, then $f_n \to f$ in measure.

(85) Proof: Let $\varepsilon > 0$ and $E_{n,\varepsilon} = \{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}$

Then
$$\int |f_n - f|^p \geq \int \varepsilon^p \cdot \chi_{E_{n,\varepsilon}} = \varepsilon^p \mu(E_{n,\varepsilon});$$

so $\mu(E_{n,\varepsilon}) \leq \frac{1}{\varepsilon^p} \int |f_n - f|^p \longrightarrow 0$ as $n \to \infty$,

and $\mu(E_{n,\varepsilon}) \longrightarrow 0$; hence $f_n \longrightarrow f$ in measure. □

Prop. If $f_n \longrightarrow f$ in measure, then there ex. a subsequence $\{f_{n_k}\}$ s.t. $f_{n_k} \longrightarrow f$ ptwise a.e.

Proof: We can find a seq. $n_1 < n_2 < \dots$ s.t.
$$\mu(\underbrace{\{x : |f_{n_k}(x) - f(x)| \geq \tfrac{1}{2^k}\}}_{E_k}) < \frac{1}{2^k}.$$

$$F_n := \bigcup_{k \geq n} E_k, \quad F = \bigcap_{n \in \mathbb{N}} F_n \qquad F_n \searrow F$$

$$\mu(F_n) \leq \sum_{k \geq n}^{\infty} \mu(E_k) \leq \sum_{k \geq n}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n-1}} \longrightarrow 0 \text{ as } n \to \infty.$$

So $\mu(F) = \lim_{n \to \infty} \underbrace{\mu(F_n)}_{\leq \frac{1}{2^{n-1}}} = 0$.

Claim $f_{n_k}(x) \longrightarrow f(x)$ for all $x \in X \setminus F$
($\Rightarrow f_{n_k} \longrightarrow f$ ptw. a.e.)

Let $x \in X \setminus F$ be arb. Then there ex. $n \in \mathbb{N}$ s.t. $x \notin F_n$; so

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$x \notin F_n = \bigcup_{k \geq n} E_k$.

So $|f_{n_k}(x) - f(x)| \leq \frac{1}{2^k}$ for $k \geq n$,
and
$f_{n_k}(x) \longrightarrow f(x)$ as $k \longrightarrow \infty$. $\square$

<u>Cor.</u>: If $f_n \longrightarrow f$ in $L^p$ $(1 \leq p < \infty)$, then there ex. a subsequence $\{f_{n_k}\}$ s.t. $f_{n_k} \longrightarrow f$ ptw. a.e.

<u>Proof</u>: $f_n \longrightarrow f$ in $L^p$ implies $f_n \longrightarrow f$ in measure. Statement follows from previous prop. $\square$

<u>Thm.</u> (Egorov's Thm.) Suppose $\mu(X) < \infty$ and $f_n \longrightarrow f$ ptw. a.e. Then for each $\varepsilon > 0$ there ex. $E \subseteq X$ measurable s.t. $\mu(E) < \varepsilon$ and $f_n \longrightarrow f$ uniformly on $X \setminus E$.

<u>Proof</u>: Midterm, Prob. 3. $\square$

<u>Cor.</u> Suppose $\mu(X) < \infty$ and $f_n \longrightarrow f$ ptw. a.e. Then $f_n \longrightarrow f$ in measure.

<u>Proof</u>: Let $\varepsilon > 0$ be arb. and
$E_{n,\varepsilon} := \{x : |f_n(x) - f(x)| \geq \varepsilon\}$.

(87) <u>WTS</u> $\mu(E_{n,\varepsilon}) \longrightarrow 0$ as $n \longrightarrow \infty$.

Let $\delta > 0$ be arb. Then by Egorov's thm ex. $E$ with $\mu(E) < \delta$ s.t. $f_n \longrightarrow f$ uniformly on $X \setminus E$.

Then $|f_n - f| < \varepsilon$ on $X \setminus E$ for large $n$, say for $n \geq N$.

Hence $E_{n,\varepsilon} \subseteq E$ for $n \geq N$, and

$\mu(E_{n,\varepsilon}) \leq \mu(E) < \delta$.

Since $\delta > 0$ was arb., $\mu(E_{n,\varepsilon}) \longrightarrow 0$ as $n \to \infty$. □

<u>Summary</u>

$$
\begin{tikzcd}
f_n \longrightarrow f_n \text{ in } L^p \arrow[d, shift right, "\text{subseq.}"'] \arrow[dr, shift left] & \\
f_n \longrightarrow f \text{ ptw. a.e.} \arrow[u, shift right, "/"'] \arrow[r, shift left, "\mu(X)<\infty"] & f_n \longrightarrow f \text{ in measure} \arrow[l, shift left, "\text{subseq.}"] \arrow[ul, shift left, "/"]
\end{tikzcd}
$$

<u>Thm.</u> (Lusin's Thm.)
Let $\mu$ be a Radon measure on a loc. comp. Hausdorff space, $f: X \to \mathbb{C}$ measurable, and $A \subseteq X$ a Borel set with $\mu(A) < \infty$.
Then for each $\varepsilon > 0$ there ex. $E \subseteq A$ Borel s.t. $\mu(E) < \varepsilon$ and $f|A \setminus E$ is continuous.

<u>Proof:</u> Cont. functions are dense in $L^p(\mu)$