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Math 245A Real Analysis
UCLA, Fall 2015

Measure theory =
math. theory that puts concept of volume and integrals on a sound basis.

$\mathbb{R}^n$	X
volume of sets (area in $\mathbb{R}^2$, length in $\mathbb{R}^1$)	measure μ
system of sets that can be assigned a volume	σ-algebra $\mathcal{A}$ "measurable" subsets of X.

<u>Def.</u> (σ-algebras)
X set, $\mathcal{A}$ family of subsets of X.
Then $\mathcal{A}$ is called a <u>σ-algebra</u> if the following axioms are true:

i) $\emptyset, X \in \mathcal{A}$,

ii) if $A \in \mathcal{A}$, then $A^c = X \setminus A \in \mathcal{A}$,
"complement of A (in X)"

iii) if $A_n \in \mathcal{A}$, $n \in \mathbb{N}$, then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$.
($\mathcal{A}$ is closed under taking countable unions)

<u>Note:</u> 1) If $\mathcal{A}$ is a σ-algebra, then $\mathcal{A}$ is closed under countable intersections as well: if $A_n \in \mathcal{A}$ for $n \in \mathbb{N}$, then $\bigcap_{n \in \mathbb{N}} A_n \in \mathcal{A}$.

② Proof: $A_n^c \in \mathcal{A}$ by ii). So $\bigcup_{n\in\mathbb{N}} A_n^c \in \mathcal{A}$ by i). So

$$\bigcap_{n\in\mathbb{N}} A_n = \Big(\bigcap_{n\in\mathbb{N}} A_n\Big)^{cc} \underset{\text{de Morgan}}{=} \Big(\bigcup_{n\in\mathbb{N}} A_n^c\Big)^c \in \mathcal{A}. \square$$

2) If $A, B \in \mathcal{A}$, then $A \setminus B = \{a \in A, a \notin B\} = A \cap B^c \in \mathcal{A}$.

So a σ-algebra is closed under all countable set-theoretic operations.

Ex. 1) $\mathcal{P}(X)$ "power set of X" = set of all subsets of X is a σ-algebra on X.

2) X set (such as $X = \mathbb{R}$)

$\mathcal{A} = \{A \subseteq X : A$ countable or $X \setminus A$ countable$\}$

Then $\mathcal{A}$ is a σ-alg. on X (exercise!)

3) If $\mathcal{A}_i$, $i \in I$, is a family of σ-alg. on X, then

$$\mathcal{A} := \bigcap_{i\in I} \mathcal{A}_i = \{A \subseteq X : A \in \mathcal{A}_i \text{ for all } i \in I\}$$

is also a σ-alg. on X.

Proof: Straightforward; for ex. prop. (iii). If $A_n \in \mathcal{A}$, for $n \in \mathbb{N}$. Then $A_n \in \mathcal{A}_i$ for all $i \in I, n \in \mathbb{N}$; so $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}_i$ for all $i \in I$; so $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$. $\square$

③ 4) If $\mathcal{S} \subseteq \mathcal{P}(X)$ is any family of subsets of X, then there exists a smallest σ-algebra on X that contains $\mathcal{S}$, denoted $\sigma(\mathcal{S})$ "the σ-algebra generated by $\mathcal{S}$";

more precisely:

1) $\sigma(\mathcal{S})$ is a σ-alg. with $\mathcal{S} \subseteq \sigma(\mathcal{S})$
2) if $\mathcal{A}$ is any σ-alg. with $\mathcal{S} \subseteq \mathcal{A}$, then $\sigma(\mathcal{S}) \subseteq \mathcal{A}$.
3) $\sigma(\mathcal{S})$ is uniquely determined based on 1) + 2).

<u>Proof:</u>

Define
$$\sigma(\mathcal{S}) = \bigcap_{\substack{\mathcal{A}\ \sigma\text{-alg.}\\ \text{on } X}} \mathcal{A} \quad \text{with } \mathcal{S} \subseteq \mathcal{A}$$

Note: $\mathcal{S} \subseteq \mathcal{P}(X)$; so intersection is over non-empty system of σ-alg.

This is a σ-alg. with $\mathcal{S} \subseteq \sigma(\mathcal{S})$.

If $\mathcal{A}$ is another σ-alg. with $\mathcal{S} \subseteq \mathcal{A}$, then $\sigma(\mathcal{S}) \subseteq \mathcal{A}$ by def. of $\sigma(\mathcal{S})$.

Finally, if $\mathcal{A}$ is another σ-alg. with prop. 1+2, then $\sigma(\mathcal{S}) \subseteq \mathcal{A}$ and $\mathcal{A} \subseteq \sigma(\mathcal{S})$. So $\mathcal{A} = \sigma(\mathcal{S})$.

5) The <u>Borel σ-algebra</u> on a topological space.

Let $(X, \mathcal{O})$ be a topological space, i.e., X set and $\mathcal{O}$ family of subsets of X ("open sets") satisfying

(4) the usual axioms.

Then the <u>Borel σ-algebra</u> on X, denoted $\mathcal{B}_X$ is the smallest σ-alg. on X containing all open sets, i.e., $\mathcal{B}_X = \sigma(\mathcal{O})$.

The elements in $\mathcal{B}_X$ are called <u>Borel sets</u> in X.

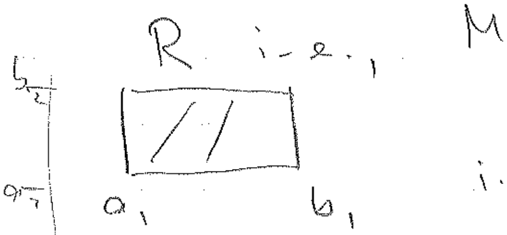
So every open set is Borel; every closed set = compl. of open set is Borel.

Countable set-theoretic operations of Borel sets give Borel s.

For ex.: $\mathbb{Q}$ = set of rat. numbers is Borel set in $\mathbb{R}$. (Why?)

<u>Rem.</u>: Often it is desirable to generate a given σ-alg. by a "small" family of sets.

<u>Ex.</u> Let $\mathcal{R} = \{R \subseteq \mathbb{R}^n : R$ rectangle in $\mathbb{R}^n$, i.e., $M = [a_1, b_1] \times \cdots \times [a_n, b_n] \}$



Then $\mathcal{B}_{\mathbb{R}^n} = \sigma(\mathcal{R})$, i.e., $\mathcal{R}$ generated $\mathcal{B}_{\mathbb{R}^n}$.

<u>Proof</u> (Outline):

$\mathcal{R} \subseteq \mathcal{B}_{\mathbb{R}^n}$ (why?); so $\sigma(\mathcal{R}) \subseteq \mathcal{B}_{\mathbb{R}^n}$.

every open set in $\mathbb{R}^n$ is a countable union of rectangles (exercise!); so $\mathcal{O} \subseteq \sigma(\mathcal{R})$; hence $\mathcal{B}_{\mathbb{R}^n} = \sigma(\mathcal{O}) \subseteq \sigma(\mathcal{R})$.

(5) We conclude $\sigma(\mathcal{R}) = \mathcal{B}_{\mathbb{R}^n}$

Def.: (measures)

X set equipped with σ-alg. $\mathcal{A}$.
A <u>(positive) measure</u> μ on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ s.t.

i) $\mu(\emptyset) = 0$,

ii) if A_n, $n \in \mathbb{N}$, are pointwise disjoint set in $\mathcal{A}$ (i.e., $A_n \cap A_k = \emptyset$ for $n \neq k$), then

$$\mu\Big(\bigcup_{n \in \mathbb{N}} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n) \in [0, \infty].$$
$$= \lim_{N \to \infty} \sum_{n=1}^{N} \mu(A_n).$$

(countable additivity)

The pair $(X, \mathcal{A})$ is called a <u>measurable space</u>, and the triple $(X, \mathcal{A}, \mu)$ a <u>measure space</u>.

<u>Rem.</u> 1) <u>Countable additivity</u> implies finite additivity:
if μ is a measure, $A_1, \dots, A_n \in \mathcal{A}$ pointwise disj., then
$$\mu\Big(\bigcup_{k=1}^{n} A_k\Big) = \sum_{k=1}^{n} \mu(A_k)$$

<u>Proof</u>: Define $\emptyset = A_{n+1} = A_{n+2} = \dots$, and use $\mu(\emptyset) = 0$ and count. additivity. $\square$

(6) **Ex.** X set, $\mathcal{A} = \mathcal{P}(X)$, $a \in X$

$\mu = \delta_a$ "Dirac measure at a"

$$\delta_a(M) = \begin{cases} 0 & \text{if } a \notin M \\ 1 & a \in M \end{cases}.$$

Ex. (Counting measure)

X set, $\mathcal{A} = \mathcal{P}(X)$

$$\mu(M) = \begin{cases} +\infty & \text{if } M \text{ infinite} \\ \#M = \text{no. of elements in } M & \text{if } M \text{ finite} \end{cases}$$

Thm. $(X, \mathcal{A}, \mu)$ measure space

i) if $A, B \in \mathcal{A}$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$ (monotonicity)

ii) if $A_n \in \mathcal{A}$ for $n \in \mathbb{N}$, then

$$\mu\Big(\bigcup_{n \in \mathbb{N}} A_n\Big) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

(subadditivity)

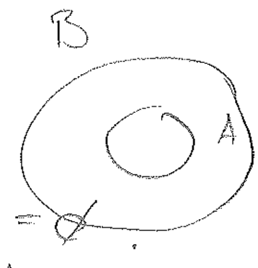
iii) if $A_n \in \mathcal{A}$ for $n \in \mathbb{N}$ and $A_n \nearrow$ (i.e., $A_n \subseteq A_{n+1}$ for $n \in \mathbb{N}$), then

$$\mu\Big(\bigcup_{n \in \mathbb{N}} A_n\Big) = \lim_{n \to \infty} \mu(A_n)$$

(continuity from below)

iv) if $A_n \in \mathcal{A}$ for $n \in \mathbb{N}$, $A_n \searrow$ (i.e., $A_n \supseteq A_{n+1}$ for $n \in \mathbb{N}$), and $\mu(A_1) < \infty$, then

$$\mu\Big(\bigcap_{n \in \mathbb{N}} A_n\Big) = \lim_{n \to \infty} \mu(A_n).$$



Proof: i) $B = A \cup (B \setminus A)$; $A \cap (B \setminus A) = \emptyset$.

So $\mu(B) = \mu(A) + \underbrace{\mu(B \setminus A)}_{\geq 0} \geq \mu(A)$.

(7) ii) $B_n := A_n \setminus \left(\bigcup_{k=1}^{n-1} A_k\right) \subseteq A_n$.
$(= \emptyset$ for $n=1)$

Then $\bigcup_{n\in\mathbb{N}} B_n = \bigcup_{n\in\mathbb{N}} A_n$, and B_n, $n\in\mathbb{N}$, pairwise disjoint.

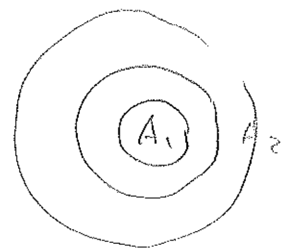
So

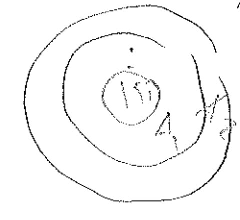
$$\mu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \mu\Big(\bigcup_{n\in\mathbb{N}} B_n\Big) = \sum_{n=1}^{\infty} \mu(B_n) \underset{\text{monotonicity}}{\leq} \sum_{n=1}^{\infty} \mu(A_n).$$

iii) if $A_n \nearrow$, then $B_n = A_n \setminus \left(\bigcup_{k=1}^{n-1} A_k\right) = A_n \setminus A_{n-1}$. $A_0 = \emptyset$

So

$$\mu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \mu\Big(\bigcup_{n\in\mathbb{N}} B_n\Big) = \lim_{N\to\infty} \underbrace{\sum_{n=1}^{N} \mu(A_n \setminus A_{n-1})}_{\mu(A_N)} = \lim_{n\to\infty} \mu(A_n).$$





iv) $B_n := A_1 \setminus A_n \nearrow$

$\mu(A_1) = \mu(A_n) + \mu(B_n)$

$$\bigcup_{n\in\mathbb{N}} B_n = \bigcup_{n\in\mathbb{N}} (A_1 \cap A_n^c) = A_1 \cap \Big(\bigcup_{n\in\mathbb{N}} A_n^c\Big) \underset{\text{de Morgan}}{=} A_1 \cap \Big(\bigcap_{n\in\mathbb{N}} A_n\Big)^c = A_1 \setminus \bigcap_{n\in\mathbb{N}} A_n.$$

By iii)

$$\mu(A_1) = \mu\Big(\bigcap_{n\in\mathbb{N}} A_n\Big) + \mu\Big(\bigcup_{n\in\mathbb{N}} B_n\Big) = \ldots + \lim_{n\to\infty} \mu(B_n)$$

(8) $= \cdots + \lim_{n\to\infty}[\mu(A_1) - \mu(A_n)]$; so

$$\mu(A_1) = \mu\Big(\bigcap_{n\in\mathbb{N}} A_n\Big) + \mu(A_1) - \lim_{n\to\infty}\mu(A_n).$$

Since $\mu(A_1) < \infty$, $\mu\Big(\bigcap_{n\in\mathbb{N}} A_n\Big) = \lim_{n\to\infty}\mu(A_n)$. $\square$

$(X, \mathcal{A}, \mu)$ measure space

$P(x)$ statement about point $x \in X$ (that may be true or false depending on x).

We say that $P(x)$ is true <u>for μ-almost every x</u> (for μ-a.e. x) if there ex. a set $E \in \mathcal{A}$ with $\mu(E) = 0$ ("E is exceptional set") s.t. $P(x)$ is true for each $x \in X \setminus E$.

"statement is true for all points except for pts. in a set of μ-measure 0".

One would like to say: $P(x)$ hold for μ-a.e. x if $\mu(\{x \in X : P(x) \text{ false}\}) = 0$,

but $\underbrace{\{x \in X : P(x) \text{ false}\}}_{F} \notin \mathcal{A}$ in general.

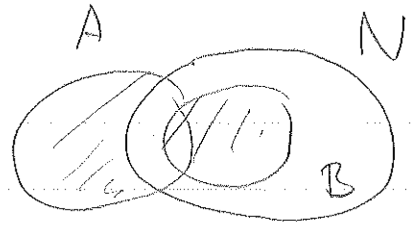
If $P(x)$ true for μ-a.e. x, then ex. $E \in \mathcal{A}$ with $\mu(E) = 0$ s.t. $F \subseteq E$.

(9) A measure space $(X, \mathcal{A}, \mu)$ is called <u>complete</u> (often one calls μ complete with X and $\mathcal{A}$ understood) if subsets of μ-null sets are μ-null sets, i.e., if $A, B \subseteq X$, $A \in \mathcal{A}$, $\mu(A) = 0$ ("A μ-null set"), and $B \subseteq A$, then $B \in \mathcal{A}$ (and so $0 \le \mu(B) \le \mu(A) = 0$, and $\mu(B) = 0$).

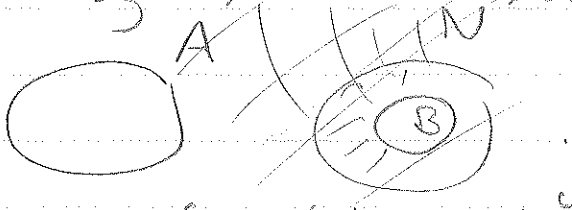
<u>Thm</u> $(X, \mathcal{A}, \mu)$ measure space, $\mathcal{N} = \{N \in \mathcal{A} : \mu(N) = 0\}$ "family of μ-null sets"
$\bar{\mathcal{A}} = \{A \cup B : A \in \mathcal{A}, \text{ ex. } N \in \mathcal{N} \text{ s.t. } B \subseteq N\}$.
Then $\bar{\mathcal{A}}$ is a σ-alg. on X, with $\mathcal{A} \subseteq \bar{\mathcal{A}}$; moreover, μ can uniquely be extended to a measure $\bar{\mu}$ on $\bar{\mathcal{A}}$ s.t. $\bar{\mu}$ is complete ("So every measure space can be completed")

<u>Proof:</u> $\mathcal{N}$ is closed under countable unions; so $\bar{\mathcal{A}}$ is closed under countable unions; obviously $\bar{\mathcal{A}} \supseteq \mathcal{A}$, and so, $\phi, X \in \bar{\mathcal{A}}$.
Let $M \in \bar{\mathcal{A}}$; then $M = A \cup B$, where $A \in \mathcal{A}$, ex. $N \in \mathcal{N}$ s.t. $B \subseteq N$.

(10)



Wlog $A \cap N = \emptyset$. Otherwise, replace A by $A \setminus N$ and B by $B \cup \underbrace{(A \cap N)}_{\subseteq N}$



Then $M^c = (A \cup B)^c = A^c \cap B^c$
$= \underbrace{(A^c \cap N^c)}_{\in \mathcal{A}} \cup \underbrace{(N \setminus B)}_{\subseteq N} \in \bar{\mathcal{A}}$.

So $\bar{\mathcal{A}}$ is σ-alg.

Define $\bar{\mu}(A \cup B) = \mu(A)$ if $A \in \mathcal{A}$, $B \subseteq N$, $N \in \mathcal{N}$.

This is well-defined: $A_1 \cup B_1 = A_2 \cup B_2$, $A_1, A_2 \in \mathcal{A}$, $B_1 \subseteq N_1$, $B_2 \subseteq N_2$, where $N_1, N_2 \in \mathcal{N}$.

Then $A_1 \subseteq A_2 \cup B_2 \subseteq A_2 \cup N_2$, and so

$\mu(A_1) \leq \mu(A_2 \cup N_2) \leq \mu(A_2) + \underbrace{\mu(N_2)}_{=0}$
$= \mu(A_2)$;

similarly, $\mu(A_2) \leq \mu(A_1)$; so $\mu(A_1) = \mu(A_2)$.

It is easy to see that $\bar{\mu}$ is a measure, is complete, extends μ, and is uniquely determined with these properties. (exercise!) □

(11) Construction of non-trivial measures

premeasure $\longrightarrow$ outer measure $\longrightarrow$ measure.

Def. X set, $\mu^*: \mathcal{P}(X) \longrightarrow [0,\infty]$
is called an outer measure on X if
(i) $\mu^*(\emptyset)=0$
(ii) $\mu^*(A) \le \mu^*(B)$ whenever $A \subseteq B \subseteq X$
(monotonicity)
(iii) $\mu^*(\bigcup_{n\in\mathbb{N}} A_n) \le \sum_{n=1}^{\infty} \mu^*(A_n)$,
whenever $A_n \subseteq X, n\in\mathbb{N}$. (countable additivity).

Carathéodory's Thm. (informal version)
If μ^* is an outer measure on X, then there
ex. a natural σ-alg. $\mathcal{A}$ (depending on μ^*)
s.t. $\mu := \mu^*|\mathcal{A}$ is a measure.

We define $\mathcal{A}$ to be the set of all
$A \subseteq X$ s.t.

(*) $\mu^*(T) = \mu^*(T\cap A) + \mu^*(T\cap A^c)$
for all $T \subseteq X$ ("test sets")

Note: Inequality $\le$ in in is always
true by (finite) subadditivity;
so (*) is equiv. to

(**) $\mu^*(T) \ge \mu^*(T\cap A) + \mu^*(T\cap A^c)$
for all $T \subseteq X$.

(12) <u>Carathéodory's Thm.</u> (precise version)

Let μ^* be an outer measure on X, and $\mathcal{A}$ be the family of all sets $A \subseteq X$ satisfying $(*)$.

Then $\mathcal{A}$ is a σ-alg. on X and $\mu = \mu^* | \mathcal{A}$ is a complete measure.

For the proof we need:

<u>Def.</u> (Algebra) A family $\mathcal{A}$ of subsets of X is called an <u>algebra</u> on X if $\emptyset, X \in \mathcal{A}$, $\mathcal{A}$ is closed under taking complements, and closed under taking <u>finite</u> unions.

<u>Lem.</u> Let $\mathcal{A}$ be an algebra on X, and suppose that $A_n \in \mathcal{A}$ for $n \in \mathbb{N}$ and $A_n \nearrow$ implies $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$. Then $\mathcal{A}$ is a σ-algebra.

<u>Proof.</u> Let $A_n \in \mathcal{A}$, $n \in \mathbb{N}$.
Define $B_n = A_1 \cup \dots \cup A_n \in \mathcal{A}$, $n \in \mathbb{N}$.
Then $B_n \nearrow$ and
$$\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} B_n \in \mathcal{A}$$ by hypotheses.
So $\mathcal{A}$ is closed under countable unions and so a σ-alg.

<u>Note.</u> In the lemma we can also require $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$ whenever $A_n, n \in \mathbb{N}$, pairwise disjoint. (T.k. $B_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k$ & $\mathcal{A}$ in the proof)

(13) <u>Proof of Carathéodory's Thm.</u>

1. $\mathcal{A}$ consisting of all satisfying (#) or equiv. (##) is a σ-alg. on X.

We'll show that $\mathcal{A}$ is an algebra that is closed under monotone limits (Lemma $\Longrightarrow$ $\mathcal{A}$ σ-alg.)

(#) symmetric in A, A^c, so $A \in \mathcal{A}$ iff $A^c \in \mathcal{A}$. Hence $\mathcal{A}$ is closed under taking complements.

$\emptyset \in \mathcal{A}$, because for all $T \subseteq X$

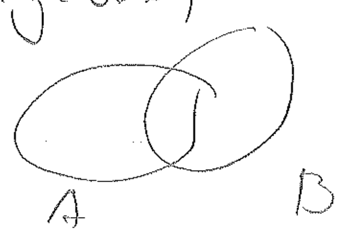
$$\mu^*(T \cap \emptyset) + \mu^*(T \cap \emptyset^c) = \underbrace{\mu^*(\emptyset)}_{0} + \mu^*(T) = \mu^*(T).$$

$X = \emptyset^c \in \mathcal{A}$.

Let $A, B \in \mathcal{A}$, $T \subseteq X$ arb.

<u>WTS</u> $A \cup B \in \mathcal{A}$ ($\Longrightarrow$ $\mathcal{A}$ algebra)

Note $A \cup B = (A \cap B^c) \cup (A \cap B) \cup (A^c \cap B)$.



So

$$\mu^*(T) \overset{A \in \mathcal{A}}{=} \mu^*(T \cap A) + \mu^*(T \cap A^c)$$

$$\overset{B \in \mathcal{A}}{=} \underbrace{\mu^*(T \cap (A \cap B)) + \mu^*(T \cap (A \cap B^c)) + \mu^*(T \cap A^c \cap B)}_{\geq \mu^*(T \cap (A \cup B))} + \mu^*(T \cap \underbrace{A^c \cap B^c}_{(A \cup B)^c})$$

$$\geq \mu^*(T \cap (A \cup B)) + \mu^*(T \cap (A \cup B)^c)$$

i.e., $A \cup B$ satisfies (##), and so $A \cup B \in \mathcal{A}$.

(14) Note that on $\mathcal{A}$, μ^* is finitely additive, if $A, B \in \mathcal{A}$, and $A \cap B = \emptyset$

$$\mu^*(A \cup B) = \mu^*(\underbrace{(A \cup B) \cap B}_{B}) + \mu^*(\underbrace{(A \cup B) \cap B^c}_{A})$$

$$= \mu^*(A) + \mu^*(B).$$

Let $A_n \in \mathcal{A}$, $n \in \mathbb{N}$, arbitrary and pairwise disjoint. $B_n = A_1 \cup \ldots \cup A_n \in \mathcal{A}$

$B = \bigcup_{n \in \mathbb{N}} A_n$.

<u>WTS</u> $B \in \mathcal{A}$ (Lem. + Rem. $\to$ $\mathcal{A}$ σ-alg.)

$$\mu^*(T \cap B_n) = \mu^*(\overbrace{T \cap B_n \cap A_{n+1}}^{A_{n+1}}) \quad A_{n+1} \in \mathcal{A}$$

$$+ \mu^*(T \cap \underbrace{B_n \cap A_n^c}_{B_{n-1}})$$

$$= \mu^*(T \cap A_n) + \mu^*(T \cap B_{n-1})$$

So inductively:

$$\mu^*(T \cap B_n) = \sum_{k=1}^{n} \mu^*(T \cap A_k).$$

$$T \cap B \supseteq$$

Hence $B_n \in \mathcal{A}$

$$\mu^*(T) = \mu^*(T \cap B_n) + \mu^*(T \cap B_n^c)$$

$$\geq \sum_{k=1}^{n} \mu^*(T \cap A_k) + \mu^*(T \cap B^c)$$

Letting $n \to \infty$, and using subadditivity:

$$\mu^*(T) \geq \sum_{k=1}^{\infty} \mu^*(T \cap A_k) + \mu^*(T \cap B^c)$$

$$\geq \mu^*(T \cap B) + \mu^*(T \cap B^c) \geq \mu^*(T)$$