Exercises

Problem 1:

a) Let $\mathcal{O}$ be the family of all sets $U \subseteq \mathbb{R}$ such that $U = \emptyset$ or $\mathbb{R} \setminus U$ is a countable set. Show that $\mathcal{O}$ is a topology on $\mathbb{R}$.

b) Find a simple necessary and sufficient condition for a sequence $\{x_n\}$ in $\mathbb{R}$ to converge to a point $x \in \mathbb{R}$ in the topological space $(\mathbb{R}, \mathcal{O})$.

c) Show that in $(\mathbb{R}, \mathcal{O})$ not every set is closed, but every set $M \subseteq \mathbb{R}$ has the following property: if $\{x_n\}$ is a sequence in M and $x_n \to x$, then $x \in M$.

Problem 2:

a) Show that in a Hausdorff space X every convergent net has a unique limit; in other words, if $\{x_\alpha\}_{\alpha \in A}$ is a net in X with $x_\alpha \to x$ and $x_\alpha \to y$, then $x = y$.

b) Show that for nets in $\mathbb{C}$ similar computational rules hold as for sequences: if $\{z_\alpha\}_{\alpha \in A}$ and $\{w_\alpha\}_{\alpha \in A}$ are nets in $\mathbb{C}$ (indexed by the same directed set A!) and $z_\alpha \to z$ and $w_\alpha \to w$, then $z_\alpha + w_\alpha \to z + w$ and $z_\alpha w_\alpha \to zw$.

Problem 3: A topology on a space is uniquely determined by its convergent nets and their limits: Let X be a set with two topologies $\mathcal{O}_1$ and $\mathcal{O}_2$. Suppose that with respect to these topologies convergent nest are the same with identical limits; more precisely, assume that for every net $\{x_\alpha\}_{\alpha \in A}$ in X and every point $x \in X$ we have $x_\alpha \to x$ in $(X, \mathcal{O}_1)$ if and only if $x_\alpha \to x$ in $(X, \mathcal{O}_2)$. Show that then $\mathcal{O}_1 = \mathcal{O}_2$.

Problem 4: Let X_i, $i \in I$, be a family of topological spaces indexed by a set I. Let $X = \prod_{i \in I} X_i$ be the product of these spaces equipped with the product topology and $p_j \colon X \to X_j$ for $j \in I$ be the natural projection map that sends each $x = (x_i)_{i \in I} \in X$ to its j-th component x_j.

Show that a net $\{x_\alpha\}_{\alpha \in A}$ in X converges if and only if the net $\{p_j(x_\alpha)\}_{\alpha \in A}$ in X_j converges for each $j \in I$.

Problem 5: Let X be a Banach space equipped with its weak topology.

a) Show that X is a Hausdorff space.

b) Prove that vector addition is continuous in the weak topology: Consider the map $A \colon X \times X \to X$ given by $A((x, y)) = x + y$ for $(x, y) \in X \times X$. Show that A is continuous. Here $X \times X$ carries the product topology induced by the weak topology on each factor.

Problem 6:

a) Let X be a Banach space and suppose that $\{x_n\}$ is a sequence in X that converges in the weak topology to the limit $x \in X$. Show that then

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|.$$

Hint: Hahn-Banach.

b) Find an example of a space X and a sequence $\{x_n\}$ as in (a), where $\|x_n\| = 1$ for all $n \in \mathbb{N}$, but $x = 0$. Hint: Consider suitable L^p-spaces X.