Homework 9 (due: Fr, March 4)

Problem 1: Let X be a locally compact Hausdorff space.

a) Show that if μ is a finite positive Radon measure, then it is regular in the following sense: if $E \subseteq X$ is a Borel set and $\epsilon > 0$, then there exists a compact set $K \subseteq U$, and an open set $U \subseteq X$ with $K \subseteq E \subseteq U$ such that $\mu(U \setminus K) < \epsilon$.

Hint: First choose U and then use inner regularity of μ on open sets to find an auxiliary compact set $K' \subseteq U$. Set $K = K' \cap (X \setminus V)$, where V is a suitable open set with $U \setminus E \subseteq V$.

b) Suppose μ is a finite positive Borel measure on X that has the regularity property in (a). Show that then μ is a Radon measure.

c) Let μ be a complex Borel measure on X and $\mu = \mu_r + i\mu_i$ its decomposition into real and imaginary parts. Let $\mu_r = \mu_r^+ - \mu_r^-$ and $\mu_i = \mu_i^+ - \mu_i^-$ be the Jordan decompositions of μ_r and μ_i, respectively. Show that then

$$\mu_r^+, \mu_r^-, \mu_i^+, \mu_i^- \le |\mu| \le \mu_r^+ + \mu_r^- + \mu_i^+ + \mu_i^-.$$

d) Recall that a complex Borel measure μ on X is a Radon measure if the positive measures $\mu_r^+, \mu_r^-, \mu_i^+, \mu_i^-$ as in (b) are Radon measures. Show that μ is a Radon measure if and only if $|\mu|$ is a Radon measure.

Problem 2: Let $1 \le p < \infty$.

a) For $y \in \mathbb{R}^n$ and $f \in L^p(\mathbb{R}^n)$ we define $\tau_y f \colon \mathbb{R}^n \to \mathbb{C}$ as

$$\tau_y f(x) = f(x+y), \quad x \in \mathbb{R}^n.$$

Show that $\tau_y f \in L^p(\mathbb{R}^n)$ and that the operator τ_y has the following continuity property:

$$\lim_{z \to y} \|\tau_z f - \tau_y f\|_p = 0.$$

Hint: Approximate f by a function in $C_c(\mathbb{R}^n)$.

b) Let φ be a mollifier on $\mathbb{R}^n$ (see HW 8, Prob. 1). For $t > 0$ we define

$$\varphi_t(x) = \frac{1}{t^n}\varphi(x/t), \quad x \in \mathbb{R}^n.$$

Show that if $f \in L^p(\mathbb{R}^n)$, then $f * \varphi_t \in L^p(\mathbb{R}^n)$ and

$$f * \varphi_t \to f \quad \text{in } L^p(\mathbb{R}^n) \text{ as } t \to 0^+.$$

c) Show that the space $C_c^\infty(\mathbb{R}^n)$ of C^∞-smooth functions on $\mathbb{R}^n$ with compact support is dense in $L^p(\mathbb{R}^n)$.

Problem 3:

a) Show that if three open intervals in $\mathbb{R}$ have non-empty intersection, then one of the intervals is contained in the union of the other two.

b) Let $\mathcal{U}$ be a finite cover of a set $M \subseteq \mathbb{R}$ by open intervals. Show that then there exist two subfamilies $\mathcal{U}_1$ and $\mathcal{U}_2$ of $\mathcal{U}$ that are each disjointed and whose union $\mathcal{U}_1 \cup \mathcal{U}_2$ is a cover of M.

c) Let $M \subseteq \mathbb{R}$ be a measurable set with $\lambda_1(M) < \infty$ and $\mathcal{U}$ be a cover of M by open intervals. Show that then there exists a disjointed finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that

$$\lambda_1\left(\left(\bigcup_{I \in \mathcal{V}} I\right) \cap M\right) \geq \frac{1}{3}\lambda_1(M).$$

d) Prove *Vitali's covering theorem*: Let $M \subseteq \mathbb{R}$ be a measurable set with $\lambda_1(M) < \infty$ and $\mathcal{U}$ be a cover of M by open intervals such that each point $x \in M$ is contained in arbitrarily small intervals that belong to $\mathcal{U}$. Show that then there exists a disjointed subfamily $\mathcal{U}' \subseteq \mathcal{U}$ such that

$$\lambda_1\left(M \setminus \left(\bigcup_{I \in \mathcal{U}'} I\right)\right) = 0.$$

Problem 4: The following problem was part of the Basic Exam in Spring 2015. Solve it by using measure-theoretic methods (this was not intended!):

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Suppose that for every $x \in \mathbb{R}$,

$$\lim_{n\to\infty} n[f(x+1/n) - f(x)] = 0.$$

Prove that f is differentiable.