Homework 8 (due: Fr, Feb. 26)

Problem 1: A function φ on $\mathbb{R}^n$ is called a *mollifier* if it is a non-negative function in $C_c^\infty(\mathbb{R}^n)$ (the space of all C^∞-smooth functions on $\mathbb{R}^n$ with compact support), and $\int \varphi \, d\lambda_n = 1$.

a) Show that mollifiers on $\mathbb{R}^n$ exist for each $n \in \mathbb{N}$. Hint: HW3, Prob. 2.

b) Fix a mollifier φ on $\mathbb{R}^n$. If $t > 0$ we define

$$\varphi_t(x) = \frac{1}{t^n}\varphi(x/t), \quad x \in \mathbb{R}^n.$$

If $f \in L^1_{loc}(\mathbb{R}^n)$, we consider the *t-mollification* of f defined as

$$(f * \varphi_t)(x) = \int f(x-y)\varphi_t(y)\, d\lambda_n(y), \quad x \in \mathbb{R}^n.$$

Show that $f * \varphi_t$ is C^∞-smooth and that $(f * \varphi_t)(x) \to f(x)$ as $t \to 0$ for almost every $x \in \mathbb{R}^n$.

c) Let $g, h \in L^1_{loc}(\mathbb{R}^n)$ and suppose that

$$\int fg \, d\lambda_n = \int fh \, d\lambda_n$$

for all $f \in C_c^\infty(\mathbb{R}^n)$. Show that then $g = h$ almost everywhere on $\mathbb{R}^n$.

Problem 2: Let f be a continuous function on $[-1, 1]$.

a) Show that if f is absolutely continuous on $[-1, 0]$ and on $[0, 1]$, then f is absolutely continuous on $[-1, 1]$.

b) Show (by finding a suitable counterexample) that if f is absolutely continuous on each closed subinterval of $[-1, 1] \setminus \{0\}$, then f may fail to be absolutely continuous.

c) Let $E \subseteq [-1, 1]$ be a closed set with $\lambda_1(E) = 0$. Show that if f is absolutely continuous on each closed subinterval of $[-1, 1] \setminus E$, then $f'(x)$ exists for almost every $x \in [-1, 1]$.

d) Assume in addition to the conditions in (c) that E is countable and that the almost everywhere defined derivative f' of f is integrable on $[-1, 1]$. Show that then f is absolutely continuous on $[-1, 1]$.

e) Show that the conclusion in (d) fails if we do not assume that E is countable.

Problem 3: Let X be a locally compact Hausdorff space. We set $\widehat{X} = X \cup \{\infty\}$, where ∞ is an additional point not in X. We call a set $U \subseteq \widehat{X}$ open in $\widehat{X}$ if $U \subseteq X$ and U is open in X or if $\infty \in U$ and $X \setminus U$ is a compact set in X.

a) Show that the open sets in $\widehat{X}$ form a topology on $\widehat{X}$.

b) Show that $\widehat{X}$ equipped with this topology is a compact Hausdorff space.

c) Show that the subspace topology of X in $\widehat{X}$ agrees with the original topology of X. Hint: Recall that a set $V \subseteq X$ is open in the subspace topology if there exists an open set $U \subseteq \widehat{X}$ such that $V = X \cap U$.

d) Consider the family $\mathcal{F}$ of all continuous functions F on $\widehat{X}$ with $F(\infty) = 0$. Show that if $F \in \mathcal{F}$, then $F|X \in C_0(X)$ and $F \mapsto F|X$ defines a bijection between $\mathcal{F}$ and $C_0(X)$.

Problem 4: (Analysis Qual, Spring 2002) Let $T\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the linear operator defined as

$$(Tf)(x) = f(x+1), \quad x \in \mathbb{R},$$

for $f \in L^2(\mathbb{R})$. Show that T has no (non-zero) eigenvectors, i.e., there exists no $f \neq 0$ in $L^2(\mathbb{R})$ such that $Tf = \lambda f$ for some $\lambda \in \mathbb{C}$.