Homework 7 (due: Fr, Feb. 19)

Problem 1: Consider the Poisson kernel P_t on $\mathbb{R}^n$ for $t > 0$ defined as

$$P_t(x) = c_n \frac{t}{(t^2 + |x|^2)^{(n+1)/2}}, \quad x \in \mathbb{R}^n,$$

where $c_n = \Gamma((n+1)/2)/\pi^{(n+1)/2}$.

For $f \in L^1(\mathbb{R}^n)$ and $t > 0$ we define

$$(P_t f)(x) := (P_t * f)(x) = \int P_t(x-y) f(y)\, d\lambda_n(y), \quad x \in \mathbb{R}^n.$$

a) Show that for each $t > 0$ the function $P_t f$ is continuous.

b) Show that if we define

$$(\mathcal{M}f)(x) := \sup_{t>0} |(P_t f)(x)|, \quad x \in \mathbb{R}^n,$$

then $\mathcal{M}f$ is measurable. Hint: Consider superlevel sets.

c) Show that there exists a constant $C_0 > 0$ such that

$$(\mathcal{M}f)(x) \le C_0 (Mf)(x),$$

whenever $f \in L^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Here Mf is the Hardy-Littlewood maximal function of f.

Problem 2: For $x \in \mathbb{R}^n$ and $A \subseteq \mathbb{R}^n$ we use the notation

$$\operatorname{dist}(x, A) := \inf\{|x-y| : y \in A\}.$$

a) Show that if $A \subseteq \mathbb{R}^n$ is measurable and $x \in A$ is a Lebesgue density point, then

$$\lim_{y \to x} \frac{\operatorname{dist}(y, A)}{|x-y|} = 0.$$

b) Suppose $f\colon \mathbb{R}^n \to \mathbb{C}$ is Lipschitz and $A = \{x \in \mathbb{R}^n : f(x) = 0\}$. Show that the derivative $Df(x)$ exists and $Df(x) = 0$ for a.e. $x \in A$.

Problem 3: Let $F \in NBV$ and $T_F \in NBV$ be the associated total variation function. We consider the unique complex Borel measures μ and ν on $\mathbb{R}$ such that $F_\mu = F$ and $F_\nu = T_F$. Note that ν is a positive finite measure, because T_F is bounded and increasing.

a) Show that $|\mu(I)| \le \nu(I)$ for each h-interval $I \subseteq \mathbb{R}$.

b) Show that $\nu(I) \le |\mu|(I)$ for each h-interval $I \subseteq \mathbb{R}$.

c) Show that similar inequalities as in (a) and (b) hold for all open intervals $I \subseteq \mathbb{R}$.

d) Show that $|\mu| = \nu$ (or, equivalently $F_{|\mu|} = T_F$).

Problem 4: We consider the space $L^1(\mathbb{R}^n)$ equipped with the multiplicative operation given by convolution $*$. So for $f, g \in L^1(\mathbb{R}^n)$ we set

$$(f * g)(x) = \int f(x - y) g(y) \, d\lambda_n(y).$$

Then $(f * g)(x)$ is defined for a.e. $x \in \mathbb{R}^n$ and $f * g \in L^1(\mathbb{R}^n)$ (HW 2, Prob. 3).

Show that for the multiplication $*$ there exists no unit in $L^1(\mathbb{R}^n)$, i.e., there exists no $e \in L^1(\mathbb{R}^n)$ such that $f * e = f$ in $L^1(\mathbb{R}^n)$ for each $f \in L^1(\mathbb{R}^n)$ (recall that $f = g$ in $L^1(\mathbb{R}^n)$ means $f = g$ a.e. on $\mathbb{R}^n$).