Homework 6 (due: Fr, Feb. 12)

Problem 1:

a) Let μ be a complex measure on a measurable space $(X, \mathcal{A})$. Suppose A_n for $n \in \mathbb{N} \cup \{\infty\}$ are sets in $\mathcal{A}$ with $A_n \nearrow A_\infty$. Show that then
$$\mu(A_n) \to \mu(A_\infty) \quad \text{as } n \to \infty.$$

b) Let μ and ν be complex Borel measures on $\mathbb{R}$ and suppose
$$\mu((-\infty, x]) = \nu((-\infty, x]) \quad \text{for each } x \in \mathbb{R}.$$
Show that then $\mu = \nu$.

Problem 2: In the following $F\colon \mathbb{R} \to \mathbb{R}$ is a right-continuous and increasing function. Note that then then limits $F(+\infty) := \lim_{x\to+\infty} F(x) \in (-\infty, +\infty]$, $F(-\infty) := \lim_{x\to-\infty} F(x) \in [-\infty, +\infty)$, and $F(x+) := \lim_{y\to x^+} F(y)$ for $x \in \mathbb{R}$ exist.

Recall that an *h-interval* $I \subseteq \mathbb{R}$ is an interval of the form
$$I = (a, b] := \{x \in \mathbb{R} : a < x \leq b\}$$
with $a, b \in [-\infty, +\infty]$ and $a \leq b$. If $I = (a, b]$ is an h-interval, we define
$$F(I) := F(b) - F(a+) \in [0, \infty].$$

a) Let $I = (a, b]$ be an h-interval and $\{a_n\}$ and $\{b_n\}$ be sequences in $\overline{\mathbb{R}}$ with $a_n \searrow a$ and $b_n \searrow b$. Show that if we define $I'_n = (a_n, b]$ and $I''_n = (a, b_n]$ for $n \in \mathbb{N}$, then
$$F(I) = \lim_{n\to\infty} F(I'_n) = \lim_{n\to\infty} F(I''_n).$$

b) Show that if $\mathcal{M}$ and $\mathcal{N}$ are finite families of h-intervals such that $\mathcal{M}$ is disjointed and
$$\bigcup_{I\in\mathcal{M}} I \subseteq \bigcup_{J\in\mathcal{N}} J,$$
then

(1)
$$\sum_{I\in\mathcal{M}} F(I) \leq \sum_{J\in\mathcal{N}} F(J).$$

Hint: Use a "common refinement".

c) Show that the inequality (1) remains true under the same assumptions if $\mathcal{M}$ and $\mathcal{N}$ are countable familes.

Hint: Shrink the intervals in $\mathcal{M}$ and enlarge the intervals in $\mathcal{N}$ slightly and use a covering argument to reduce to (b).

d) Show that there exists a unique positive Borel measure on μ on $\mathbb{R}$ such that $\mu(I) = F(I)$ for each h-interval $I \subseteq \mathbb{R}$.

Problem 3: Let $f \in L^1_{loc}(\mathbb{R}^n)$ and Mf be the maximal function of f. For fixed $R > 0$ and $x \in \mathbb{R}^n$ and we define

$$I(x) = \int_{B(x,R)} \frac{f(y)}{|x-y|^{n-1}} \, d\lambda_n(y).$$

Show that then

$$|I(x)| \leq CR(Mf)(x)$$

for each $x \in \mathbb{R}^n$, where $C = C(n) > 0$ is a constant only depending on n.

Problem 4: Let $f\colon [0,1) \to \mathbb{C}$ be an integrable function. For $n \in \mathbb{N}$ and $x \in [0,1)$ we define

$$f_n(x) = 2^n \int_{I_x} f(y) \, dy,$$

where I_x is the unique interval of the form $I_x = [(k-1)/2^n, k/2^n)$ with $k \in \{1, \ldots, 2^n\}$ that contains x. Show that then $f_n \to f$ pointwise almost everywhere on $[0,1)$.