Homework 5 (due: Fr, Feb. 5)

Problem 1:

a) Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, $1 < p < \infty$, and q be the conjugate exponent of p. Show that if $f\colon (X, \mathcal{A}) \to [0, \infty]$ is a measurable function, then

$$\|f\|_p = \sup\left\{ \int fg\, d\mu : g\colon X \to [0, \infty] \text{ measurable, } \|g\|_q \le 1 \right\} \in [0, \infty].$$

Hint: This does not directly follow from L^p-L^q-duality, because we allow $\|f\|_p = \infty$ here and pair f only with *non-negative* functions g.

b) Prove *Minkowski's inequality for integrals*: Let $1 \le p < \infty$. Suppose that $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces and

$$h\colon (X \times Y, \mathcal{A} \otimes \mathcal{B}) \to [0, \infty]$$

is measurable. Then

$$\left(\int \left(\int h(x, y)\, d\nu(y) \right)^p d\mu(x) \right)^{1/p} \le \int \left(\int h(x, y)^p\, d\mu(x) \right)^{1/p} d\nu(y).$$

c) Show how derive the usual Minkowski inequality $\|f + g\|_p \le \|f\|_p + \|g\|_p$ from the integral version in (b).

Problem 2: Show that the L^1-boundedness of the Hardy-Littlewood maximal function fails: If $f \in L^1(\mathbb{R}^n)$ is an arbitrary non-zero integrable function on $\mathbb{R}^n$, then $Mf \notin L^1(\mathbb{R}^n)$. Hint: Find a lower bound for Mf that implies $\int Mf = +\infty$.

Problem 3: Let (X, d) be a metric space. A Borel measure μ on X is called *doubling* if there exists a constant $C \ge 0$ such that

$$\mu(2B) \le C\mu(B)$$

for all (open) balls B in X. To rule out trivial cases we also assume that there exists a ball B_0 in X such that $0 < \mu(B_0) < \infty$.

a) Show that if μ is a doubling measure μ on X, then $0 < \mu(B) < \infty$ for *all* balls B in X.

b) Suppose μ is a doubling measure on X and $f\colon X \to \mathbb{C}$ is Borel measurable. We define the *maximal function* $Mf\colon X \to [0, \infty]$ of f as

$$(Mf)(x) = \sup_{x \in B} \frac{1}{\mu(B)} \int_B |f|\, d\mu \quad \text{for } x \in X,$$

where the supremum is taken over all balls B in X that contain x.

Show that Mf is measurable and that if $f \in L^1(\mu)$, then

$$\mu\{Mf > \alpha\} \le C_0 \frac{\|f\|_1}{\alpha} \quad \text{for all } \alpha > 0 \ ,$$

where $C_0 \ge 0$ is a constant independent of f.

Problem 4: (Analysis Qual 2010) Let T be a linear transformation on the space $C_c(\mathbb{R}^n)$ of continuous functions on $\mathbb{R}^n$ with compact support. Suppose that $\|Tf\|_\infty \le \|f\|_\infty$ for all $f \in C_c(\mathbb{R}^n)$ and

$$|\{|Tf| > \lambda\}| \le \frac{\|f\|_1}{\lambda}$$

for all $\lambda > 0$ and $f \in C_c(\mathbb{R}^n)$, where $|A|$ denotes Lebesgue measure of a measurable set $A \subseteq \mathbb{R}^n$. Show that then there exists a constant $C \ge 0$ such that

$$\int |Tf|^2 \le C \int |f|^2 \quad \text{for all } f \in C_c(\mathbb{R}^n).$$

Here integration is with respect to Lebesgue measure.