Homework 4 (due: Fr, Jan. 29)

Problem 1: Let $(X, \mathcal{A}, \mu)$ be a measure space. Show that simple functions are dense in $L^\infty(\mu)$, i.e., if $f\colon X \to \mathbb{C}$ is an essentially bounded measurable function, then there exits a sequence $\{s_n\}$ of simple functions on X such that

$$\|s_n - f\|_\infty = \operatorname{ess\,sup}_{x\in X} |s_n(x) - f(x)| \to 0.$$

Problem 2: Establish the following fact that was used in the proof that $(L^1)^* = L^\infty$. If $(X, \mathcal{A}, \mu)$ is a measure space and $g\colon X \to \mathbb{C}$ an integrable function such that

$$\left|\int_A g\, d\mu\right| \le C\mu(A)$$

for all $A \in \mathcal{A}$, then $\|g\|_\infty \le C$.

Problem 3: The purpose of this problem is to complete the proof of L^p-L^q duality for σ-finite measures based on the case of finite measures.

Let $1 \le p < \infty$ and q be the conjugate exponent of p. Suppose μ is a σ-finite measure on a measurable space $(X, \mathcal{A})$ and $\Phi\colon L^p(\mu) \to \mathbb{C}$ a bounded linear functional. Then we can find measurable sets E_n with $E_n \nearrow X$ and $\mu(E_n) < \infty$ for $n \in \mathbb{N}$.

a) Let $n \in \mathbb{N}$. Use the finite measure $\mu|E_n$ to show that there exists a function $g_n \in L^q(\mu)$ with $g_n|E_n^c = 0$ and $\|g_n\|_q \le \|\Phi\|$ such that

$$\Phi(f) = \int f g_n\, d\mu$$

for all $f \in L^p(\mu)$ with $f = 0$ μ-a.e. on E_n^c.

b) Show that if $n \le k$, then we have $g_n = g_k$ μ-a.e. on E_n for the functions constructed in (b).

c) Show that $g = \lim_{n\to\infty} g_n$ exists μ-a.e. on X, $g \in L^q(\mu)$, $\|g\|_q = \|\Phi\|$, and that

$$\Phi(f) = \int f g\, d\mu$$

for all $f \in L^p(\mu)$.

Problem 4:

a) Let X be a normed vector space and $M \ne X$ be a closed subspace. Show that then there exists a non-zero functional $f \in X^*$ such that $f|M = 0$. Hint: Pick $x \in X \setminus M$, consider the span of M and x, and apply the Hahn-Banach Theorem.

b) Let X be a normed vector space. Suppose there exist vectors x_n for $n \in \mathbb{N}$ whose span is dense in X. Show that then X is separable, i.e., there exists a countable dense subset in X.

c) Let X be a normed vector space and X^* be its dual space. Show that if X^* is separable, then X is also separable.

Hint: Let $\{f_n : n \in \mathbb{N}\}$ be a countable dense subset in X^*. Then we can find $x_n \in X$ with $\|x_n\| \leq 1$ such that $|f_n(x_n)| \geq \frac{1}{2}\|f_n\|$.

Remark: Part (c) is Problem 6 from the Analysis Qual, Fall 2014 (the hint was not given).