Homework 3 (due: Fr, Jan. 22)

Problem 1: Let ν be a complex measure on a measurable space $(X, \mathcal{A})$.

a) Let $\nu_r = \nu_r^+ - \nu_r^-$ and $\nu_i = \nu_i^+ - \nu_i^-$ be the Jordan decompositions of the real part ν_r and the imaginary part ν_i of ν. Show that if $|\nu|$ denotes the total variation of ν, then $\nu_r^+, \nu_r^-, \nu_i^+, \nu_i^- \leq |\nu|$.

b) We say that a measurable function f on $(X, \mathcal{A})$ is *ν-integrable* if it is integrable with respect to $|\nu|$. So if we denote the space of these functions f by $L^1(\nu)$, then $L^1(\nu) = L^1(|\nu|)$. Show that if $f \in L^1(\nu)$, then f is integrable with respect to each of the measures $\nu_r^+, \nu_r^-, \nu_i^+, \nu_i^-$ and so

$$\int f\, d\nu := \int f\, d\nu_r^+ - \int f\, d\nu_r^- + i \int f\, d\nu_i^+ - i \int f\, d\nu_i^-$$

is well-defined.

c) Suppose μ is a σ-finite positive measure on $(X, \mathcal{A})$ such that $\nu << \mu$ and let $g = d\nu/d\mu$ be the Radon-Nikodym derivative of ν with respect to μ. Show that if $f \in L^1(\nu)$, then $fg \in L^1(\mu)$ and

$$\int f\, d\nu = \int fg\, d\mu.$$

Problem 2: A *bump function* on $\mathbb{R}^n$ is a C^∞-smooth function φ on $\mathbb{R}^n$ with compact support such $\varphi \geq 0$ and $\varphi \not\equiv 0$. The purpose of this problem is to show the existence of certain bump functions.

a) Define $f(x) = \exp(-1/x^2)$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$. Show that f is C^∞-smooth on $\mathbb{R}$. Hint: Show that $f^{(n)}$ exists by induction on n. For this it helps to find a general type of expression that represents $f^{(n)}$.

b) Show that if f is as in (a) and $a, b \in \mathbb{R}$ with $a < b$, then $x \in \mathbb{R} \mapsto g(x) = f(x-a)f(b-x)$ defines a bump function on $\mathbb{R}$ with support in the interval $[a, b]$.

c) Show that if $a, b \in \mathbb{R}$ with $a < b$, then there exists a C^∞-smooth function h on $\mathbb{R}$ with $0 \leq h \leq 1$ such that $h(x) = 1$ for $x \leq a$ and $h(x) = 0$ for $x \geq b$.

d) Let $0 < r < R$. Show that there exists a C^∞-smooth function φ on $\mathbb{R}^n$ with $0 \leq \varphi \leq 1$ such that $\varphi(x) = 1$ for $x \in \mathbb{R}^n$ with $|x| \leq r$ and $\varphi(x) = 0$ for $x \in \mathbb{R}^n$ with $|x| \geq R$.

Problem 3: Let f and g be a functions on $\mathbb{R}^n$. Suppose f is locally integrable and g is C^∞-smooth with compact support. Show that then the convolution $h(x) = (f * g)(x)$ is defined for each $x \in \mathbb{R}^n$. Moreover, h is C^∞-smooth and if $\mathscr{D}$ denotes any partial derivative of any order, then $f * \mathscr{D}g$ is everywhere defined and

$$\mathscr{D}h = f * \mathscr{D}g.$$

Problem 4:

a) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $\mathbb{R}^n$. Show that then these norm are *equivalent*, i.e., there exists a constant $C \geq 1$ such that

$$\frac{1}{C}\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1$$

for each $x \in \mathbb{R}^n$.

b) Let X be a normed real vector space and $T\colon \mathbb{R}^n \to X$ be a linear map. Show that T is bounded.