Homework 2 (due: Fr, Jan. 15)

Problem 1: Let $(X, \mathcal{A})$ be a measurable space.

a) Let μ be a signed measure on $(X, \mathcal{A})$ and $\mu = \mu^+ - \mu^-$ be its Jordan decomposition. Show that if λ and ν are positive measures on $(X, \mathcal{A})$ with $\mu = \lambda - \nu$, then $\lambda \geq \mu^+$ and $\nu \geq \mu^-$, i.e., $\lambda(A) \geq \mu^+(A)$ and $\nu(A) \geq \mu^-(A)$ for all $A \in \mathcal{A}$.

b) Let μ be a signed measure on $(X, \mathcal{A})$ and $|\mu|$ be its total variation. Show that

$$|\mu|(A) = \sup \left\{ \sum_{n \in \mathbb{N}} |\mu(B_n)| : B_n \in \mathcal{A} \text{ pairwise disjoint for } n \in \mathbb{N} \text{ and } \bigcup_{n \in \mathbb{N}} B_n = A \right\}$$

for each $A \in \mathcal{A}$.

c) Let μ and ν be signed measure on $(X, \mathcal{A})$ that both omit $+\infty$ or $-\infty$. Show that then $\mu + \nu$ is a signed measure on $(X, \mathcal{A})$ with $|\mu + \nu| \leq |\mu| + |\nu|$.

Problem 2: Let $(X, \mathcal{A})$ be a measurable space. We denote by $\mathcal{M}$ the set of all finite signed measures on $(X, \mathcal{A})$.

a) If $a, b \in \mathbb{R}$ and $\mu, \nu \in \mathcal{M}$, we define

$$(a\mu + b\nu)(A) = a\mu(A) + b\nu(A)$$

for $A \in \mathcal{A}$. Show that $a\mu + b\nu \in \mathcal{M}$ and that $\mathcal{M}$ is a vector space over $\mathbb{R}$ with this linear structure.

b) For $\mu \in \mathcal{M}$ define $\|\mu\| = |\mu|(X)$. Show that $\mu \in \mathcal{M} \mapsto \|\mu\|$ defines a norm on $\mathcal{M}$.

c) Show that the vector space $\mathcal{M}$ equipped with the norm defined in (b) is a Banach space.

Problem 3:

a) Let $f \colon \mathbb{R}^n \to \mathbb{C}$ and $g \colon \mathbb{R}^n \to \mathbb{C}$ be Borel measurable functions on $\mathbb{R}^n$. Show that the function $F \colon \mathbb{R}^{2n} \to \mathbb{C}$ defined as

$$F(x, y) = f(x - y)g(y) \text{ for } x, y \in \mathbb{R}^n$$

is also Borel measurable.

b) Let $f\colon \mathbb{R}^n \to \mathbb{C}$ and $g\colon \mathbb{R}^n \to \mathbb{C}$ be (Lebesgue) integrable functions. Show that then the *convolution* of f and g given by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)\, d\lambda_n(y)$$

is well-defined for almost every $x \in \mathbb{R}^n$ and that

$$\|f * g\|_1 \le \|f\|_1 \cdot \|g\|_1.$$

c) Show that if $f\colon \mathbb{R}^n \to \mathbb{C}$ and $g\colon \mathbb{R}^n \to \mathbb{C}$ are integrable functions, then

$$(f * g)(x) = (g * f)(x) \text{ for almost every } x \in \mathbb{R}^n.$$

Problem 4: (Analysis Qual, Spring 2012)

a) Suppose $f\colon [0,1) \to \mathbb{C}$ is integrable with respect to Lebesgue measure on $[0,1)$. For $n \in \mathbb{N}$ define

$$f_n(x) = n \int_{(k-1)/n}^{k/n} f(t)\, dt \quad \text{if } x \in [(k-1)/n, k/n) \text{ for } k = 1, \dots n.$$

Show that $f_n \to f$ in L^1.

b) Let S be the set of all complex-valued integrable functions f on $\mathbb{R}^3$ with $f \in L^2$ and $\int f\, d\lambda_3 = 0$. Show that S is dense in L^2.