Homework 1 (due: Fr, Jan. 8)

Problem 1:

a) If $\{a_n\}$ is a sequence in $\mathbb{C}$, then the statement that it has a limit $a \in \mathbb{C}$ can be expressed with quantifiers as follows:

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq N : |a_n - a| < \epsilon.$$

Express the statement that $\{a_n\}$ is a Cauchy sequence by quantifiers, where (in contrast to the previous expression) the quantifiers only range over countable sets.

b) Let $(X, \mathcal{A})$ be a measurable space and $f_n \colon X \to \mathbb{C}$ be measurable functions for $n \in \mathbb{N}$. Consider the set

$$E = \{x \in X : \{f_n(x)\} \text{ is a convergent sequence}\}.$$

By using ideas from (a) express E in terms of the sets

$$A_{n,m,k} = \{x \in X : |f_n(x) - f_m(x)| < 1/k\},$$

where only countably many set operations are involved.

c) Show that the set E in (b) is measurable, i.e., $E \in \mathcal{A}$.

Problem 2: Let f be a measurable function on $\mathbb{R}^n$. Its *Riesz potential* at $x \in \mathbb{R}^n$ is defined as

$$I(x) = \int \frac{f(y)}{|x-y|^{n-1}} \, d\lambda_n(y)$$

if the integral exists. Here λ_n denotes Lebesgue measure on $\mathbb{R}^n$.

Show that if f has compact support and $f \in L^p(\lambda_n)$ with $n < p < \infty$, then $I(x)$ exists for each $x \in \mathbb{R}^n$. Hint: Use Hölder's inequality and polar coordinates with a suitable origin.

Problem 3: If $f \colon \mathbb{R} \to \mathbb{C}$ is a measurable function on $\mathbb{R}$ and $\delta \in \mathbb{R}$, we denote by $T_\delta f \colon \mathbb{R} \to \mathbb{C}$ the function defined as

$$T_\delta f(x) = f(x + \delta) \text{ for } x \in \mathbb{R}.$$

Show that if $f \in L^1$ (with Lebesgue measure as the underlying measure), then $T_\delta f \in L^1$ and

$$\|T_\delta f - f\|_{L^1} \to 0 \text{ as } \delta \to 0.$$

Hint: Show this first for functions $f \in C_c(\mathbb{R})$, i.e., for continuous functions with compact support.

Problem 4: Let A be a positive definite $(n \times n)$-matrix. Show that

$$\int_{\mathbb{R}^n} \exp(-x^t Ax)\, d\lambda_n(x) = \frac{\pi^{n/2}}{\det(A)^{1/2}}.$$

Here x^t denotes the transpose of the column vector $x \in \mathbb{R}^n$.

Hint: Use the transformation formula and Fubini.