(84) **Thm.** Let X be a LCH space.

Define
$$\mathcal{M}(X) = \{\mu : \mu \text{ complex Radon measure on } X\},$$
$\|\mu\| := |\mu|(X)$ for $\mu \in \mathcal{M}(X)$.

Then $\mathcal{M}(X)$ is a vector space and $\|\cdot\|$ a norm on $\mathcal{M}(X)$.

Proof: If $\alpha, \beta \in \mathbb{C}$, $\mu, \nu \in \mathcal{M}(X)$ we define $\alpha\mu + \beta\nu$ by
$$(\alpha\mu + \beta\nu)(A) := \alpha\mu(A) + \beta\nu(A), \text{ for each Borel set } A \subseteq X.$$
Then $\alpha\mu + \beta\nu$ is a complex Borel measure on X. It is a Radon measure (easily follows from $|\alpha\mu + \beta\nu| \leq |\alpha||\mu| + |\beta||\nu|$).

$\|\cdot\|$ norm on $\mathcal{M}(X)$:

$\|\mu\| = |\mu|(X) \geq 0$. If $\|\mu\| = 0$, then $|\mu|(X) = 0$; so $|\mu| = 0$ and $\mu = 0$.

$\|\alpha\mu\| = |\alpha\mu|(X) = |\alpha||\mu|(X) = |\alpha|\,\|\mu\|$.

$|\mu + \nu| \leq |\mu| + |\nu|$

So $\|\mu + \nu\|(X) = |\mu + \nu|(X) \leq |\mu|(X) + |\nu|(X) = \|\mu\| + \|\nu\|$. $\square$

Thm. (Riesz Representation Thm.)

Let X be a LCH space.

For $\mu \in \mathcal{M}(X)$ define
$$I_\mu(f) = \int f \, d\mu \quad \text{for } f \in C_0(X).$$
Then $I_\mu \in C_0(X)^*$, i.e., I_μ is a

(85) bdd. linear functional on $C_0(X)$
and the map $\mu \in \mathcal{M}(X) \longmapsto I_\mu \in C_0(X)^*$
defines an isometric isomorphism between
normed spaces.

Proof: Recall: if $\mu \in \mathcal{M}(X)$ and
$\mu = \mu_r^+ - \mu_r^- + i\mu_i^+ - i\mu_i^-$, then we define
$\int f\,d\mu = \int f\,d\mu_r^+ - \int f\,d\mu_r^- + i\int f\,d\mu_i^+ - i\int f\,d\mu_i^-$
for $f \in C_0(X)$ HW 3, Prob. 1
(all integrals finite, because meas.
are finite and f bdd.)
Alternatively, $\mu \ll |\mu|$ and so
$\mu = g\,|\mu|$, where $g = \frac{d\mu}{d|\mu|}$ Radon-Nikodym deriv.

We showed (proof of Prop. on $|\nu|$)
that if $d\mu = f\,d\nu$, ν positive,
then $d|\mu| = |f|\,d\nu$.
Take $\nu = |\mu|$, $f = g$, then
$d|\mu| = |g|\,d|\mu|$; by uniqueness of
Radon-Nikodym deriv.

$|g| = 1$ $|\mu|$-a.e.
Then (HW 3, Prob. 1c)
$\int f\,d\mu = \int f g\,d|\mu|$, $f \in C_0(X)$.
I_μ defines linear function on $C_0(X)$.
It is bdd.:
$|I_\mu(f)| = \left|\int f\,d\mu\right| = \left|\int f g\,d|\mu|\right|$
$\leq \int |fg|\,d|\mu| = \underbrace{\|f\|_u \cdot |\mu|(X)}_{|g|=1\ |\mu|\text{-a.e.}}$, $|\mu|(X) = \|\mu\|$.

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In particular,
$\|I_\mu\| \le \|\mu\|$.

For the other ineq.: $\|\mu\| \le \|I_\mu\|$

consider $g = \frac{d\mu}{d|\mu|}$. w.l.o.g. $|g| \le 1$.

$|g| = 1$ $|\mu|$-a.e.; so $g \in L^1(|\mu|)$.

Since $C_c(X)$ is dense in $L^1(|\mu|)$

there ex. $g_n \in C_c(X)$ s.t.

$$g_n \longrightarrow \bar{g} \quad \text{in } L^1(|\mu|).$$

Passing to a subsequence, we may assume

$$g_n \longrightarrow \bar{g} \quad |\mu|\text{-a.e. on } X.$$

W.l.o.g. $|g_n| \le 1$ (otherwise compose g_n with projection $P: \mathbb{C} \to \mathbb{C}$

$$= \begin{cases} z, & |z| \le 1, \\ \frac{z}{|z|} & |z| \ge 1. \end{cases}$$

Hence

$$\|\mu\| = |\mu|(X) = \Big|\int \overbrace{g\bar{g}}^{=1}\, d|\mu|\Big|$$

$$\underset{LDC}{=} \lim_{n\to\infty} \Big|\int g\, g_n\, d|\mu|\Big|$$

$$= \lim_{n\to\infty} \Big|\int g_n\, d\mu\Big| = \lim_{n\to\infty} \underbrace{|I_\mu(g_n)|}_{\le \|I_\mu\|,\ \|g_n\|_u \le 1}$$

$$\le \|I_\mu\|.$$

It is clear that $\mu \in M(X) \longmapsto I_\mu \in C_0(X)^*$ is linear; so this gives an linear isometric embedding of $M(X)$ <u>into</u> $C_0(X)^*$.

The embedding is actually <u>onto</u>, i.e., surjective.

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To see this let $I \in C_0(X)^*$ be arb. Let $I_r := \operatorname{Re} I$, $I_i := \operatorname{Im} I$, i.e.,

$$I_r(f) = \operatorname{Re} I(f), \quad I_i(f) = \operatorname{Im} I(f) \quad \text{for } f \in C_0(X).$$

This defines bdd. linear functionals on $C_0(X, \mathbb{R})$.

Let $I_r = I_r^+ - I_r^-$, $I_i = I_i^+ - I_i^-$ decomp. into positive bdd. lin. functionals on $C_0(X, \mathbb{R}) \supseteq C_c(X, \mathbb{R})$.

By Riesz Repr. Thm. for positive linear functionals, there ex. pos. Radon $\mu_r^+, \mu_r^-, \mu_i^+, \mu_i^-$ s.t.

$I_r^+ = I_{\mu_r^+}$, etc.

(exercise!)

Note $\|\mu_r^+\| \leq \|I_r^+\| < \infty$ etc.

So all these measures are finite.

Let $\mu = \mu_r^+ - \mu_r^- + i\mu_i^+ - i\mu_i^- \in \mathcal{M}(X)$

Then

$I(f) = I_\mu(f)$ for all real-valued function $f \in C_c(X)$,

hence for all complex-valued $f \in C_c(X)$; hence for all $f \in C_0(X)$ (because $C_c(X)$ dense in $C_0(X)$). $\square$

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Cor. Let X be a compact Hausdorff space.

Then
$C(X)^*$ is isometrically isomorphic to $\mathcal{M}(X) = \{\mu$ complex Borel meas. on $X\}$.

Proof: Follows from Riesz Rep. Thm.:
Have $C_0(X) = C(X)$.
Every complex Borel meas. on a compact metric space is regular.
(exercise; regularity of $|\mu|$ proved similar as in 245B, HW1, Prob. 3). □

Weak topologies

Let X be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

A seminorm on X is a map
$p: X \longrightarrow [0, \infty)$ s.t.

i) $p(\alpha x) = |\alpha|\, p(x)$ for each $\alpha \in \mathbb{F}$, $x \in X$,

ii) $p(x+y) \leq p(x) + p(y)$.

Ex. X Banach space, $f \in X^*$.
Then $p(x) := |f(x)| = |\langle x, f \rangle|$ is seminorm on X.

Prop.: Let X be a vector space, and $\{p_\alpha\}_{\alpha \in A}$ be a family of seminorms on X.

(19) We call a set $U \subseteq X$ open if for all $x \in U$ there ex. $\varepsilon > 0$ and a finite set $F \subseteq A$ s.t.

$$N_{F,\varepsilon}(x) := \{y \in X : p_\alpha(y-x) < \varepsilon \text{ for all } \alpha \in F\} \subseteq U.$$

Then the open sets form a topology $\mathcal{T}$ on X (the topology generated by the family of semi-norms).

Proof: $\emptyset, X \in \mathcal{T}$ clear.

unions of open sets open: clear

Let $U, V \in \mathcal{T}$.

WTS $U \cap V \in \mathcal{T}$.

Let $x \in U \cap V$ be arb.

Since U, V are open, there ex. $\varepsilon_1, \varepsilon_2 > 0$, $F_1, F_2 \subseteq A$ finite s.t.

$N_{F_1,\varepsilon_1}(x) \subseteq U$,

$N_{F_2,\varepsilon_2}(x) \subseteq V$.

Let $\varepsilon := \min(\varepsilon_1, \varepsilon_2) > 0$, $F := F_1 \cup F_2$.

Then

$$N_{F,\varepsilon}(x) \subseteq N_{F_1,\varepsilon_1}(x) \cap N_{F_2,\varepsilon_2}(x) \subseteq U \cap V.$$

So $U \cap V$ open. $\square$

Note: Each set $N_{F,\varepsilon}(x)$ is open: Let $z \in N_{F,\varepsilon}$ be arb.

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Define $\delta := \min\{\varepsilon - p_\alpha(z-x) : \alpha \in F\} > 0$.
Then for $y \in N_{F,\delta}(z)$, we have

$$p_\alpha(y-x) \le p_\alpha(y-z) + p_\alpha(z-x)$$
$$< p_\alpha(y-z) + \delta \le \varepsilon \quad \text{for } \alpha \in F.$$

So $N_{F,\delta}(z) \subseteq N_{F,\varepsilon}(x)$.

Def.: Let X be a Banach space. Then the week topology on X is the topology induced by the family $\{p_f\}_{f \in X^*}$ of seminorms

$$p_f(x) := |\underbrace{\langle x, f \rangle}_{f(x)}|, \quad x \in X.$$

The week-$*$ topology is the topology on X^* induced by the family of seminorms $\{q_x\}_{x \in X}$

$$q_x(f) := |\langle x, f \rangle|, \quad f \in X^*.$$

Lem. X Banach space.
Every open set in the week topology is also open in the norm topology.
Similarly, every open subset of X^* in the weak* topology is also open in the norm topology.

Proof: Let $U \subseteq X$ be weekly open, and $x \in U$ be arb. Then there ex. $\varepsilon > 0$ and $f_1, \dots, f_n \in X^*$ s.t.

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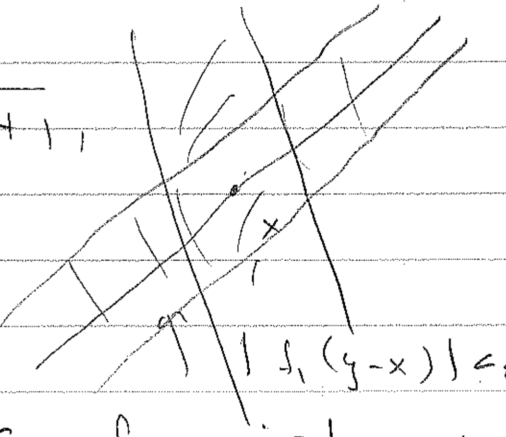
$N := \{ y \in \mathbb{X} : |f_1(y-x)| < \varepsilon, \ldots, |f_n(y-x)| < \varepsilon \}$
$\subseteq G.$



Let $\delta := \dfrac{\varepsilon}{\max\{\|f_1\|, \ldots, \|f_n\|\} + 1}$,

and $B := B(x, \delta)$
$= \{ y \in \mathbb{X} : \|y - x\| < \delta \}.$

If $y \in B$, then
$|f_i(y-x)| \leq \|f_i\| \cdot \delta \leq \varepsilon$ for $i = 1, \ldots, n$.

Then $x \in B \subseteq N \subseteq G$, and so G is open in norm topology.

Proof for weak*-top similar. □

So the weak topology on $\mathbb{X}$ is coarser or weaker (i.e., has fewer open sets) than the norm-topology.

Rem. Weak topology on a Banach space $\mathbb{X}$ and weak-$*$ topology on $\mathbb{X}^*$ are Hausdorff, but are not metrizable for infinite-dimensional Banach spaces.

This requires some adjustments.
In a metric space $\mathbb{X}$ a set $A \subseteq \mathbb{X}$ is closed iff for every convergent sequence $\{x_n\}$ in A with $x_n \to x \in \mathbb{X}$, we have $x \in A$ (A contains all of its limit pts.)

In arb. topological spaces this is not true. One has to use <u>nets</u> in X.

A <u>directed set</u> A is a set with a relation $\leq$ s.t.

i) $\alpha \leq \alpha$ for all $\alpha \in A$

ii) $\alpha \leq \beta$ and $\beta \leq \gamma$ implies $\alpha \leq \gamma$ for all $\alpha, \beta, \gamma \in A$

(so $\leq$ is a partial order on A)

iii) for all $\alpha, \beta \in A$ there exist $\gamma \in A$ s.t. $\alpha \leq \gamma$ and $\beta \leq \gamma$.

<u>Ex.</u> 1) $\mathbb{N}$ with usual order relation

2) All finite subsets $F \subseteq [0,1]$ with inclusion $\subseteq$:
if $F_1, F_2 \subseteq [0,1]$ finite, then $F_1, F_2 \subseteq F := F_1 \cup F_2$ finite.

3) The family $\mathcal{U}_x$ of all neighborhoods of a point x in a top space X with reverse inclusion $\supseteq$:
$U, V \in \mathcal{U}_x$, then $U \cap V \in \mathcal{U}_x$ and $U, V \supseteq U \cap V$.

A <u>net</u> in a space X is a family $\{x_\alpha\}_{\alpha \in A}$ of points $x_\alpha \in X$ indexed by a directed set A.

If X is a topological space and $x \in X$, then we say that the net $\{x_\alpha\}_{\alpha \in I}$

converges to x, written $x_\alpha \longrightarrow x$, if for every neighborhood U of x there ex. $\alpha_0 \in I$ s.t.

$x_\alpha \in U$ for all $\alpha \in A$ with $\alpha_0 \le \alpha$.

Ex. f Riemann integrable on $[0,1]$.
A set of all finite subsets F of the form $F = \{x_0 = 0 < x_1 < \dots < x_n = 1\}$ with inclusion is directed set.

$$L_F = \sum_{k=1}^{n} \inf_{x \in [x_{k-1}, x_k]} f(x) \, (x_k - x_{k-1})$$

(lower sum for partition given by F)
$\{L_F\}_{F \in A}$ is a convergent net and

$$L_F \longrightarrow I = \int_0^1 f(x)\,dx. \quad \text{(exercise!)}$$

Lem. Let M be a subset of a top. space X. Then M is closed in X if and only if the following condition is true: if $\{x_\alpha\}_{\alpha \in A}$ is a net in M (i.e., $x_\alpha \in M$ for $\alpha \in A$) with $x_\alpha \longrightarrow x \in X$, then $x \in M$.

Proof: "$\Longrightarrow$" Suppose M is closed.
Let $\{x_\alpha\}_{\alpha \in A}$ be a net in M with $x_\alpha \longrightarrow x \in X$. If U is a neighborhood of x, then there ex. $\alpha_0 \in A$ s.t. $x_\alpha \in U$ for $\alpha_0 \le \alpha$.

(94) In particular, $x_{\alpha_0} \in U \cap M$.
So $U \cap M \neq \emptyset$ for each neighborhood U of x, i.e., $x \in \overline{M} = M$, because M is closed.

"$\Leftarrow$": Suppose condition with nets is true.

<u>WTS</u> M is closed equiv. $\overline{M} = M$.

equiv. if $x \in \overline{M}$, then $x \in M$.

Let $x \in \overline{M}$ be arb. Then for each neighbourhood U of x, $U \cap M \neq \emptyset$; so there ex. $x_U \in U \cap M$.

We consider the net $\{x_U\}_{\mathcal{U}}$ indexed by the set $\mathcal{U}$ of neighbourhoods of x directed by inverse inclusion $\supseteq$.

<u>Claim</u> For the net $\{x_U\}_{U \in \mathcal{U}}$ we have

$x_U \longrightarrow x$.

<u>Proof:</u> Let $V \in \mathcal{U}$ be arb.

Then $x_U \in U \subseteq V$ for all $U \in \mathcal{U}$ with $\underbrace{V \supseteq U}_{\alpha_0 \leq \alpha}$.

So $\{x_U\}_{U \in \mathcal{U}}$ is a net in M with limit x; by our condition $x \in M$, i.e., $\overline{M} = M$ as desired. $\square$

By a very similar argument:

A map $f : X \to Y$ between top. spaces X and Y is continuous iff the following condition is true: if $\{x_\alpha\}_{\alpha \in A}$ is a convergent net in X with $x_\alpha \to x \in X$, then $\{f(x_\alpha)\}_{\alpha \in A}$ converges in Y and

(95) $f(x_\alpha) \longrightarrow f(x)$.

A topology on a space is in general not uniquely determined by knowing the convergent sequences in the space; knowledge of convergent nets determines topology uniquely:

$\mathcal{O}_1, \mathcal{O}_2$ topologies on $\mathbb{X}$.

$\mathcal{O}_1 = \mathcal{O}_2$ iff every convergent net in $(\mathbb{X}, \mathcal{O}_1)$ also converges in $(\mathbb{X}, \mathcal{O}_2)$ with the same limit.

Ex. Let $\mathbb{X}$ be a Banach space.
If a net $\{x_\alpha\}_{\alpha \in A}$ converges w.r.t. the weak topology on $\mathbb{X}$ to x one often writes

$$x_\alpha \xrightarrow{w} x \quad \text{or} \quad x_\alpha \rightharpoonup x.$$

Claim $x_\alpha \xrightarrow{w} x$ iff $f(x_\alpha) \longrightarrow f(x)$ for all $f \in \mathbb{X}^*$.

Proof: "$\Rightarrow$" Suppose $x_\alpha \xrightarrow{w} x$ and let $f \in \mathbb{X}^*$ and $\varepsilon > 0$ be arb.
Then there ex. $\alpha_0 \in A$ s.t.

$$x_\alpha \in N_{\{f\},\varepsilon}(x) = \{ y \in \mathbb{X} : |f(y-x)| < \varepsilon \} \quad \text{for } \alpha_0 \leq \alpha.$$

equiv.

$$|f(x_\alpha) - f(x)| < \varepsilon \quad \text{for } \alpha_0 \leq \alpha.$$

Hence $f(x_\alpha) \longrightarrow f(x)$.

(96) "$\Leftarrow$". Suppose $f(x_\alpha) \to f(x)$ for all $f \in X^*$. Consider an arb. neighborhood in the weak top. of the form
$$N_{F,\varepsilon}(x) = \{ y : |f(y-x)| < \varepsilon \text{ for } f \in F \}.$$
Here $F = \{f_1, \dots, f_n\} \subseteq X^*$ finite.
Then $f_i(x_\alpha) \to f_i(x)$ for $i = 1, \dots, n$.
In particular, there ex. $\alpha_i \in A$ s.t.

(*) $\quad |f_i(x_\alpha) - f_i(x)| < \varepsilon$ for $\alpha_i \le \alpha$.

Since A is a directed set there ex. $\beta \in A$ s.t. $\alpha_1 \le \beta, \dots, \alpha_n \le \beta$ (exercise!)
Then if $\beta \le \alpha$ we have $\alpha_i \le \alpha$ and so by (*):
$$|f_i(x_\beta) - f_i(x)| < \varepsilon \quad \text{for } i = 1, \dots, n$$
$$= |f_i(x_\beta - x)|$$
equiv.
$$x_\beta \in N_{F,\varepsilon}(x) \quad \text{for } \beta \le \alpha.$$
Hence
$$x_\beta \to x.$$

Ex. Similarly, Let $\{f_\alpha\}_{\alpha \in A}$ be a net in the dual X^* of a Banach space X. Then $f_\alpha \xrightarrow{w^*} f \in X^*$ iff
$$f_\alpha(x) \to f(x) \quad \text{for all } x \in X.$$

Ex. X_i, $i \in I$, top. spaces,
$X = \prod_{i \in I} X_i$ prod. space equipped

(97) with prod. topology, and $p_j : X \longrightarrow X_j$ for $j \in I$ projection map: if $x = (x_i)_{i \in I}$, then $p_j(x) = x_j$.

One can show that for a net $\{x_\alpha\}_{\alpha \in A}$ in X we have

$$x_\alpha \longrightarrow x \in X \text{ iff } p_i(x_\alpha) \longrightarrow p_i(x) \text{ for all } i \in I.$$

Ex. $(X, \mathcal{A}, \mu)$ σ-finite m.s. space $1 \le p < \infty$, $X = L^p(\mu)$ is Banach space $\{f_n\}$ seq. in X

When do we have convergence $f_n \xrightarrow{w} f$?

Note: $X^* = L^q(\mu)$ where q conjugate exp p

Dual pairing: $\langle f, g \rangle = \int f g \, d\mu$ for $f \in L^p(\mu)$, $g \in L^q(\mu)$.

So

$$f_n \xrightarrow{w} f \text{ iff } \int f_n g \, d\mu \longrightarrow \int f g \, d\mu \text{ for all } g \in L^q(\mu)$$

Ex. X LCH space,

$Z = \mathcal{M}(X)$ set of all complex Radon measures on X.

If $Y = C_0(X)$, then $Z = Y^*$ equiv.

$\mathcal{M}(X) = C_0(X)^*$

Dual pairing: $\langle f, \mu \rangle = \int f \, d\mu$ for $f \in C_0(X)$, $\mu \in \mathcal{M}(X)$.

Let $\{\mu_n\}$ be a seq. in $\mathcal{M}(X)$

Then $\mu_n \xrightarrow{w^*} \mu$ iff

$$\int f\,d\mu_n \longrightarrow \int f\,d\mu \quad \text{for all } f \in C_0(X).$$

Instead of weak-* convergence, one often speaks of <u>vague convergence</u> of measures in this context.

<u>Ex.</u> $X = [0,1]$, $\mu_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{k/n}$.
Dirac measure

Then

$\mu_n \xrightarrow{w^*} \lambda_1 | [0,1]$
Lebesgue meas. on $[0,1]$.

equiv.

$$\int f\,d\mu_n = \underbrace{\sum_{k=1}^{n} f\left(\tfrac{k}{n}\right)\cdot\tfrac{1}{n}}_{\text{Riemann sum}} \longrightarrow \int_0^1 f(t)\,dt$$

for all $f \in C[0,1]$.

<u>Thm.</u> (Banach-Alaoglu)
Let X be a Banach space and $B = \{f \in X^* : \|f\| \le 1\}$ be the closed unit ball in X^*.
Then B is compact w.r.t. the weak-* topology.

<u>Cor.</u> Let X be a comp. metric space and $\{\mu_n\}$ be a seq. of Borel probability meas. on X (i.e., each μ_n is a pos. Borel meas. with $\mu_n(X) = 1$).

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Then there ex. a subseq. $\{\mu_{n_k}\}$ and a Borel prob. meas. μ on $\mathbb{X}$ s.t.

$$\mu_{n_k} \xrightarrow{w^*} \mu.$$

Proof: (Outline): $\mu_n \in \mathcal{M}(\mathbb{X})$ and $\|\mu_n\| = |\mu_n(\mathbb{X})| = 1$.

So by Banach - Alaoglu there ex a subseq. $\{\mu_{n_k}\}$ with

$$\mu_{n_k} \xrightarrow{w^*} \mu, \text{ where } \mu \in \mathcal{M}(\mathbb{X}).$$

Here μ must be positive

($\int f \, d\mu_{n_k} \longrightarrow \int f \, d\mu$ for all $f \in C(\mathbb{X})$,

this implies $\int f \, d\mu \geq 0$ for all $f \geq 0$, because μ_{n_k} pos. and so μ positive (exercise!))

μ prob. meas.:

$$\mu(\mathbb{X}) = \int 1 \cdot d\mu = \lim_{k \to \infty} \int 1 \, d\mu_{n_k}$$

$$= 1, \text{ because } \mu_{n_k}(\mathbb{X}) = 1. \quad \square$$

Proof of Banach - Alaoglu Thm.

$\mathbb{X}$ Banach space,

$\mathcal{B} = \{ f \in \mathbb{X}^* : \|f\| \leq 1 \}$

unit ball in $\mathbb{X}^*$.

For $x \in \mathbb{X}$ let $D_x := \{ z \in \mathbb{C} : |z| \leq \|x\| \}$

and $P = \prod_{x \in \mathbb{X}} D_x$ product space.

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Then P equipped with prod. top.
is compact (Tychonoff's Thm.)
We define a map
$I : B \to P$ by setting
$I(f) = (f(x))_{x \in \bar{X}} \in \prod_{x \in \bar{X}} D_x$
for $f \in B$.
(Note: $|f(x)| \leq \|f\| \cdot \|x\| \leq \|x\|$
and so $f(x) \in D_x$ for $x \in \bar{X}$, $f \in B$).

Claim 1

$I : B \to I(B) \subseteq P$ is
a homeo. onto its image if B is
equipped with the weak-* top.
and $I(B)$ carries induced topology.

Proof: $I : B \to I(B)$ surjective.
Also injective: if $f, g \in B$ and
$I(f) = \{f(x)\}_{x \in \bar{X}} = I(g) = \{g(x)\}_{x \in \bar{X}}$,
then $f(x) = g(x)$ for all $x \in \bar{X}$;
so $f = g$.
To show that $I : B \to I(B)$ is
a homeo., we have to show continuity
in both directions or equivalently:
for every net $\{f_\alpha\}$ in B we
have
$f_\alpha \xrightarrow{w^*} f \in B$ iff $I(f_\alpha) \to I(f)$ in P.

Indeed:
$f_\alpha \xrightarrow{w^*} f$ iff $f_\alpha(x) \to f(x)$ for $x \in \bar{X}$
iff $p_x(I(f_\alpha)) \to p_x(I(f))$ for $x \in \bar{X}$

(101) where $p_x : P \to D_x$ projection onto factor D_x

iff $I(f_\alpha) \to I(f)$ in P.

Claim 2 $I(B)$ is closed in P
($\Rightarrow I(B)$ compact $\Rightarrow B$ compact)

An arb. net in $I(B)$ has the form
$\{I(f_\alpha)\}_{\alpha \in A}$, where $\{f_\alpha\}_{\alpha \in A}$ is a net in B.

Suppose $I(f_\alpha) \to \underset{\in P}{z} = \{z_x \in D_x\}_{x \in X}$.

WTS Ex. $f \in B$ s.t. $z = I(f)$.

Define
$f(x) := z_x$ for $x \in X$.

Then f is a linear functional on X.

Let $a, b \in \mathbb{F}$, $x, y \in X$ be arb.

Then

$$p_{ax+by}(I(f_\alpha)) = f_\alpha(ax+by)$$

$$\to p_{ax+by}(z) = z_{ax+by} = f(\alpha x+\beta y)$$

On the other hand, by linearity of f_α

$$f_\alpha(ax+by) = a f_\alpha(x) + b f_\alpha(y)$$

$$\to a f(x) + b f(y) \quad \left(f_\alpha(x) \to z_x = f(x),\ f_\beta(y) \to z_y = f(y) \right.$$

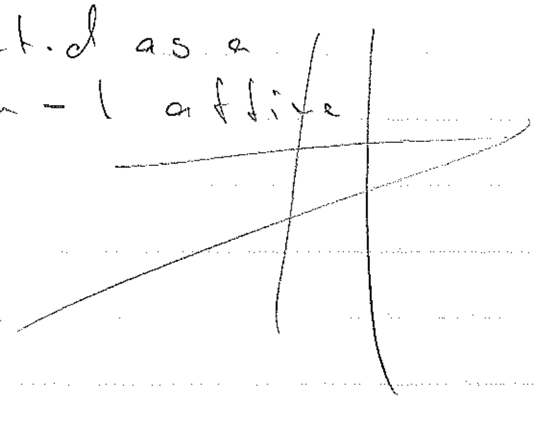
and comp rules for limits of nets)

By uniqueness of net limits in $\mathbb{C}$,

$f(ax+by) = a f(x) + b f(y)$ as desired.
Since $z \in P$, we have $z_x = f(x) \in D_x$ equiv. $|f(x)| \le \|x\|$ for $x \in \overline{X}$;
so $\|f\| \le 1$.
Hence $f \in B$ and $z = I(f)$ as desired. □

<u>Thm.</u> (Baire's Category Thm.)
Let $\overline{X}$ be a complete metric space.
i) if G_n, $n \in \mathbb{N}$, are dense open sets in $\overline{X}$, then $\bigcap G_n$ is dense.
ii) if $A = \bigcup_{n\in\mathbb{N}} A_n$ is a countable union of closed sets with empty interior, then A has empty interior; in particular $A \neq \overline{X}$.
(such sets A are often called sets of the <u>first category</u> or <u>meagre</u> sets.

<u>Ex.</u> $\mathbb{R}^n$ cannot be represented as a countable union of codim-1 affine hyperplanes



<u>Proof of Baire's Cat. Thm.</u>
i) Let $D = \bigcap_{n\in\mathbb{N}} G_n$.

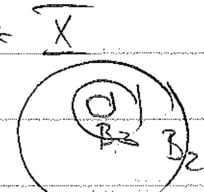
<u>WTS</u> $\overline{D} = \overline{X}$ equiv. if $x \in \overline{X}$ and $\varepsilon > 0$ arb., then $\underbrace{B(x,\varepsilon)}_{= B_0} \cap D \neq \emptyset$.

(103) We'll inductively construct closed balls
$B_n := \bar{B}(x_n, r_n)$ s.t. $r_n \le \frac{1}{2^n}$
$B_{n+1} \subseteq B_n \cap G_1 \cap \dots \cap G_{n+1} \subseteq B_0$

Induction beginning:
G_1 dense; so $G_1 \cap \mathring{B}_0 \neq \emptyset$.
Pick $x_1 \in G_1 \cap \mathring{B}_0$ and $r_1 \le \frac{1}{2}$ so small
s.t. $B_1 := \bar{B}(x_1, r_1) \subseteq G_1 \cap \mathring{B}_0$.

Induction step:
If $B_n = \bar{B}(x_n, r_n)$ has been chosen,
then $G_{n+1} \cap \mathring{B}_n \neq \emptyset$, because G_{n+1} is dense. Pick $x_{n+1} \in G_{n+1} \cap \mathring{B}_n$ and
$r_{n+1} \le \frac{1}{2^{n+1}}$ so small s.t. $B_{n+1} \subseteq G_{n+1} \cap \mathring{B}_n$.
Then by induction hyp.:
$B_{n+1} \subseteq B_n \cap G_{n+1} \subseteq B_n \cap G_1 \cap \dots \cap G_{n+1}$.

Now
$d(x_n, x_{n+1}) \le r_n \le \frac{1}{2^n}$, because $x_{n+1} \in B_n$.
Hence $\{x_n\}$ is a Cauchy seq. (exercise!)
Since X is complete, there ex. $x \in X$
s.t. $x_n \longrightarrow x$.



Note $x_k \in B_n$ for $k \ge n$, and
so $x \in B_n \subseteq G_n$ for all $n \in \mathbb{N}$.
Hence $x \in \bigcap_{n \in \mathbb{N}} G_n = D$. Moreover,
$x \in B_1 \subseteq B_0$, and so $B_0 \cap D \supseteq \{x\} \neq \emptyset$ as desired.

ii) Our assumptions imply that
$G_n = X \setminus A_n$ is open and dense;
so $\bigcap_{n \in \mathbb{N}} G_n = \bigcap_{n \in \mathbb{N}} A_n^c = \left(\bigcup_{n \in \mathbb{N}} A_n\right)^c = A^c$
dense which means that A has empty interior. □

Thm. (Banach-Steinhaus Thm. or Uniform Boundedness Principle)
Let X be a Banach space, Y be a normed vector space, and $\{T_i\}_{i\in I}$ be a family of bdd. operators $T_i : X \to Y$.
Then if $\sup_{i\in I} \|T_i(x)\| < \infty$ for each $x \in X$, we have
$$\sup_{i\in I} \|T_i\| < \infty.$$
(If the family is pointwise bdd., then it is uniformly bdd.)

Proof: For $n \in \mathbb{N}$ define
$$A_n = \{x \in X : \sup_{i\in I} \|T_i(x)\| \le n\}$$
$$= \bigcap_{i\in I} \{x \in X : \|T_i(x)\| \le n\}$$
$$= \bigcap_{i\in I} T_i^{-1}(\bar{B}_Y(0,n)) \quad \text{closed}$$
(in norm topology).
Moreover, $\bigcup_{n\in\mathbb{N}} A_n = X$.
By Baire's Category Thm. one of the sets A_n must have non-empty interior (otherwise, $\bigcup_{n\in\mathbb{N}} A_n \neq X$).
Say $\mathring{A}_k \neq \emptyset$.
Then there ex. $x_0 \in X$, $r > 0$, s.t. $\bar{B}(x_0, r) \subseteq A_k$.

(105) If $x \in \bar{B}(0,r)$ is arb., then $x_0 + x \in \bar{B}(x_0,r)$, and so

$$\|T_i(x)\| = \|T_i(x_0+x) - T_i(x_0)\|$$
$$\leq \|T_i(x_0+x)\| + \|T_i(x_0)\|$$
$$\leq k + k = 2k \qquad \begin{pmatrix} x_0, x_0+x \\ \in \bar{B}(x_0,r) \end{pmatrix}$$

for $i \in I$. So

$$\|x\| \leq r \longrightarrow \|T_i(x)\| \leq 2k$$
for $x \in X$, $i \in I$.

By homogenity,

$$\|T_i(x)\| \leq \frac{2k}{r}\|x\| \quad \text{for } x \in X, i \in I,$$

i.e., $\|T_i\| \leq \frac{2k}{r}$ for $i \in I$. □

<u>Cor.</u> X Banach space, Y normed space, $T_n : X \to Y$ bdd. lin. op. for $n \in \mathbb{N}$. If $\{T_n(x)\}$ converges in Y for each $x \in X$, then

$$\sup_{n \in \mathbb{N}} \|T_n\| < \infty.$$

(Pointwise convergence implies uniform boundedness.)

<u>Proof:</u> Since $\{T_n(x)\}$ converges, this sequence is bdd. for each $x \in X$. So the claim follows from Banach-Steinhaus.

<u>Cor.</u> Let X be a Banach space and X^* be its dual

Then every weakly convergent sequence $\{x_n\}$ in $\mathbb{X}$ is bdd. in norm, i.e., $\sup_{n \in \mathbb{N}} \|x_n\| < \infty$. Similarly, for every sequence $\{f_n\}$ in $\mathbb{X}^*$ that converges w.r.t. weak-$*$ topology we have
$$\sup_{n \in \mathbb{N}} \|f_n\| < \infty.$$

Proof: Each $x \in \mathbb{X}$ can be considered as a bdd. lin. operator
$T_x : \mathbb{X}^* \longrightarrow \mathbb{F}$ by setting
$$T_x(f) := f(x) = \langle x, f \rangle$$
for $f \in \mathbb{X}^*$.

Then $\|T_x\| = \sup \{ |f(x)| : \|f\| \leq 1\}$
$= \|x\|$ as follows from Hahn-Banach.

If $x_n \xrightarrow{w} x \in \mathbb{X}$, then
$f(x_n) \xrightarrow{w} f(x)$ for all $f \in \mathbb{X}^*$
equiv. $T_{x_n}(f) \longrightarrow T_x(f)$ for all $f \in \mathbb{X}^*$.

So by Banach-Steinhaus
$$\sup_{n \in \mathbb{N}} \|T_{x_n}\| = \sup_{n \in \mathbb{N}} \|x_n\| < \infty. \quad \square$$