(72) <u>Thm.</u> (Riesz Representation Theorem)
Let X be an LCH space, and
$I: C_c(X) \to \mathbb{C}$ be a positive
linear functional.
Then there ex. a unique Radon
measure μ on X s.t.

(*) $I(f) = \int f \, d\mu$ for all $f \in C_c(X)$

<u>Rem.</u> If μ is a Radon meas., then
(*) defines a pos. lin. functional
on X:

$$|I(f)| \le \underbrace{\int |f| \, d\mu}_{<\infty} \le \overbrace{\|f\|_u}^{<\infty} \cdot \underbrace{\mu(\operatorname{supp}(K))}_{<\infty}$$

I linear.
I positive, if $f \in C_c(K)$, $f \ge 0$,
then $\int f \, d\mu \ge 0$.

<u>Proof of Riesz Rep. Thm.:</u>
1) <u>Uniqueness:</u> If μ exists, then
$\mu(U)$ uniquely determined by I
for all open sets.
Let $U \subseteq X$ be arb. i let $K \subseteq U$
be compact. Then by Urysohn
there ex. $f \in C_c(X)$ with
$K \prec f \prec U$ i.s. $\chi_K \le f \le \chi_U$
which implies

$$\mu(K) \le \underset{\overset{\|}{I(f)}}{\int f \, d\mu} \le \mu(U).$$

(73) If we take the sup of over K, then inner reg. for open sets implies

$$\mu(U) = \sup\{\mu(K) : K \subseteq U \text{ comp.}\}$$

(1) $$= \sup\{I(f) : f \prec U\}$$

So $\mu(U)$ uniquely determined by I. By outer regularity, μ uniquely determined by $\mu(U)$ for all $U \subseteq X$ open.

Existence: Idea: Define a set function on open sets by (1); use this to construct outer measure μ^*; μ^* will give μ.

If $U \subseteq X$ open we define

$$\rho(U) := \sup\{I(f) : f \prec U\} \in [0,\infty]$$

$$\rho(\emptyset) = 0.$$

and for each $E \subseteq X$

$$\mu^*(E) := \inf\left\{\sum_{n=1}^{\infty} \rho(U_n) : U_n \subseteq X \text{ open}, n \in \mathbb{N}, X \subseteq \bigcup_{n \in \mathbb{N}} U_n\right\} \in [0,\infty]$$

μ^* is an outer measure on X:

i) $\mu^*(\emptyset) = 0$

ii) $\mu^*(A) \leq \mu^*(B)$ for $A \subseteq B \subseteq X$.

iii) $\mu^*\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$ whenever $A_n \subseteq X$, $n \in \mathbb{N}$.

(easy exercise!)

(74) By Caratheodory's thm. the outer measure μ^* induces a measure μ on the σ-algebra $\mathcal{A}$ of μ^*-measurable sets A characterized by

$$\mu^*(T) \geq \mu^*(T \cap A) + \mu^*(T \cap A^c) \quad \text{for all } T \subseteq X.$$

WTS each Borel set is μ^*-meas., outer regularity on Borel sets, inner reg. on open sets, finiteness of μ on comp. sets, formula $(*)$.

1) Outer regularity

Claim $\mu^*(U) = \rho(U)$ for each $U \subseteq X$ open

$\leq$: obvious (U cover of U)

$\geq$: WTS $\rho(U) = \sup\{I(f) : f \prec U\}$
$\leq \inf\{\sum_{n=1}^{\infty} \rho(U_n) : \{U_n\}$ open cover of $U\}$

Let $f \prec U$ and an open cover $\{U_n\}$ of U be arb.

Enough to show: $I(f) \leq \sum_{n=1}^{\infty} \rho(U_n)$

$K := \operatorname{supp}(f) \subseteq U$.

So $\{U_n\}$ is cover of compact set K, and so has a finite subcover, say $K \subseteq U_1 \cup \ldots \cup U_k$.

Let $h_1, \ldots, h_k \in C_c(X)$ be a partition of unity on K subordinate

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to open cover $U_1, \dots, U_k$ of K, i.e.,
$0 \le h_i \le 1$, $0 \le h_1 + \dots + h_k \le 1$,
$h_1 + \dots + h_k \equiv 1$ on K, $\operatorname{supp}(h_i) \subseteq U_i$.
Then
$$f = \underbrace{f h_1}_{g_1} + \dots + \underbrace{f h_k}_{g_k} = g_1 + \dots + g_k,$$
where $g_i \prec U_i$.
So
$$I(f) = I(g_1) + \dots + I(g_k)$$
$$\le \rho(U_1) + \dots + \rho(U_k)$$
$$\le \sum_{n=1}^{\infty} \rho(U_n) \text{ as desired.}$$

Claim 1 follows.

Claim 2

$\mu^*(E) = \inf\{\mu^*(U) : U \supseteq E \text{ open}\}$
for each $E \subseteq X$
($\longrightarrow$ outer regularity of μ, once we know that Borel sets are meas.)

$\le$ obvious

$\ge$ if $\{U_n\}$ is an arb. open cover of E, then $U = \bigcup_{n\in\mathbb{N}} U_n \supseteq E$ is open and
$$\sum_{n=1}^{\infty} \rho(U_n) \underset{\text{Claim 1}}{=} \sum_{n=1}^{\infty} \mu^*(U_n) \underset{\substack{\text{countable}\\ \text{subadditivity}}}{\ge} \mu^*(U).$$

2) Measurability of Borel sets

Enough to show that open sets are μ^*-measurable ($\longrightarrow$ Borel sets are meas.)

(76) <u>Claim 3</u> Let $U \subseteq X$ be open, $T \subseteq X$ be arb. Then

(2) $\mu^*(T) \geq \mu^*(T \cap U) + \mu^*(T \cap U^c)$.

<u>1. Case</u>: $\mu^*(T) = +\infty$. Then (2) obvious.

<u>2. Case</u>: $\mu^*(T) < \infty$.

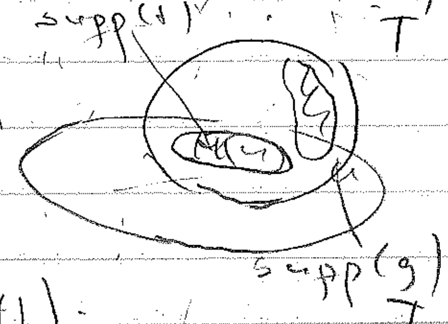
<u>Subcase a</u>: T open.

Then $T \cap U$ open and $\beta(T \cap U) = \mu^*(T \cap U) \leq \mu^*(T) < \infty$.

So for each $\varepsilon > 0$, we can find $f \prec T \cap U$ s.t.

(3) $\mu^*(T \cap U) - \varepsilon \leq I(f) \leq \mu^*(T \cap U)$

$T \setminus \operatorname{supp}(f)$ open; so we can find $g \prec T \setminus \operatorname{supp}(f)$ with



(4) $\mu^*(T \setminus \operatorname{supp}(f)) - \varepsilon \leq I(g) \leq \mu^*(T \setminus \operatorname{supp}(f))$.

Then $f + g \prec T$ and

$\mu^*(T) \geq I(f+g) = I(f) + I(g) \geq \mu^*(T \cap U) - \varepsilon + \mu^*(T \setminus \operatorname{supp}(f)) - \varepsilon$

$\geq \mu^*(T \cap U) + \mu^*(T \cap U^c) - 2\varepsilon$

(n.b. $T \cap U^c = T \setminus U \subseteq T \setminus \operatorname{supp}(f)$)

Since $\varepsilon > 0$ arb., (2) follows.

<u>Subcase b</u>: $T \subseteq X$ with $\mu^*(T) < \infty$ arb.

We use Claim 2:

if $V \supseteq T$ open, then by Case 1 + Case 2a:

$\mu^*(V) \geq \mu^*(V \cap U) + \mu^*(V \cap U^c)$

$\geq \mu^*(T \cap U) + \mu^*(T \cap U^c)$

(77) If we take the infimum over all V here, then (2) follows.

Summary: We now know that all open sets and so all Borel sets are μ^*-meas. (since they form a σ-alg.)

Define
$$\mu := \mu^* \mid \mathcal{B}_X \longleftarrow \text{Borel } \sigma\text{-alg. on } X.$$

By Carathéodory this is a meas.
By Claim 2 it is outer regular and by Claim 1

(5) $\mu(U) = \sup\{I(f) : f \prec U\}$ for $U \subseteq X$ open.

3) Finiteness of μ on compact sets

Claim 4 If $K \subseteq X$ compact, then

(6) $\mu(K) = \inf\{I(f) : K \prec f\}$

($\rightarrow \mu(K) < \infty$, because ex. f with $K \prec f \prec X$ by Urysohn and so $\mu(K) \leq I(f) < \infty$).

"$\mu \leq$" Let f with $K \prec f$ and $\varepsilon \in (0,1)$ be arb.

Define $U_\varepsilon = \{f > 1-\varepsilon\}$ open, $K \subseteq U_\varepsilon$. If $g \prec U_\varepsilon$, then

$$g \leq \underbrace{\frac{1}{1-\varepsilon}\, f}_{\geq 1 \text{ on } U_\varepsilon \supseteq K} \quad \text{equiv.}$$

$\frac{1}{1-\varepsilon} f - g \geq 0$ and so $\frac{1}{1-\varepsilon} I(f) - I(g) \geq 0$

equiv. $\frac{1}{1-\varepsilon} I(f) \geq I(g)$

(78) By positivity of I.
Taking supremum over all g, we conclude

$$\mu(K) \leq \mu(U_\varepsilon) = \sup\{I(g) : g \prec U_\varepsilon\} \leq \frac{1}{1-\varepsilon} I(f).$$

Letting $\varepsilon \to 0$, we see
$\mu(K) \leq I(f)$ and "$\leq$" in (6) follows.

"$\geq$" Let $U \supseteq K$ open be arb.
Then by Urysohn there ex. f with $K \prec f \prec U$. Then

$$\mu(K) \leq I(f) \leq \mu(U).$$

If we take infimum over all $U \supseteq K$ open, then "=" in (6) follows from outer regularity of μ.
Claim 4 follows.

4) <u>Inner regularity of μ on open sets.</u>
Let $U \subseteq X$ be open, and $\alpha \geq 0$ with $\alpha < \mu(U)$ be arb.
<u>WTS</u> Ex. $K \subseteq U$ compact with

$$\alpha \leq \mu(K) \leq \mu(U)$$

$(\rightarrow \mu(U) = \sup\{\mu(K) : K \subseteq U \text{ comp.}\})$
By (5), there ex. $f \prec U$ with

$$\alpha < I(f) \leq \mu(U).$$

Let $K = \operatorname{supp}(f)$. Then $K \subseteq U$ compact.

(79) If $K \prec g$, then $f \leq g$ and so
$I(f) \leq I(g)$ (by positivity of I)
If we take infimum over all such g,
then by Claim 4
$\alpha < I(f) \leq \inf\{I(g) : K \prec g\}$
$= \mu(K)$ as desired.

5) $\underline{\mu \text{ represents } I}$, i.e.,

(*) $I(f) = \int f \, d\mu$ for all $f \in C_c(X)$.

Reductions: By linearity it is enough to show (*) for real-valued functions, and enough to show

(7) $I(f) \leq \int f \, d\mu$ for all real-valued $f \in C_c(X)$.

$(\Rightarrow I(-f) = -I(f) \leq \int -f \, d\mu = -\int f \, d\mu$
equiv. $I(f) \geq \int f \, d\mu$ (and so (*) follows).

To prove (7) let $f \in C_c(X)$, f real-valued be arb.

Let $M = \|f\|_u$, and $\epsilon \in (0,1)$ be arb. Then $f(X) \subseteq [-M, M]$.

Pick pts.

$y_0 < -M \leq y_1 < \dots < y_n = M$

s.t. $y_i - y_{i-1} < \epsilon$ for $i = 1, \dots, n$.

Let $K = \operatorname{supp}(f)$ and

$E_i := \{y_{i-1} < f \leq y_i\} \cap K$ Borel

$E_1, \dots, E_n$ disjoint with

$E_1 \cup \dots \cup E_n = K$

By outer regularity we can find open sets $U_i \supseteq E_i$ s.t.

(80)

$\mu(G_i) \le \mu(E_i) + \frac{\varepsilon}{n}$. Wlog $G_i \subseteq \{f < y_i + \varepsilon\} \cap E_i$.

Then $G_1, \dots, G_n$ open cover of K.

Let $h_1, \dots, h_n$ be a corr. partition of unity on K subordinate to this cover, i.e. so

$h_i \prec G_i$ for $i = 1, \dots, n$,

$K \prec h_1 + \dots + h_n$.

Then by Claim 4

$$\mu(K) \le I(h_1 + \dots + h_n) = I(h_1) + \dots + I(h_n).$$

Moreover

$f = fh_1 + \dots + fh_n$.

So

$$I(f) = \sum_{i=1}^{n} I(fh_i) \qquad \text{(linearity of } I\text{)}$$

$$\le \sum_{i=1}^{n} I((y_i + \varepsilon) h_i) \qquad \left(\begin{array}{l}\text{positivity of } I;\\ fh_i \le (y_i+\varepsilon) h_i,\\ \text{because } h_i \prec G_i\\ \text{and } f < y_i + \varepsilon \text{ on } G_i\end{array}\right)$$

$$= \sum_{i=1}^{n} (y_i + \varepsilon) I(h_i)$$

$$= \sum_{i=1}^{n} (\underbrace{M + y_i + \varepsilon}_{\ge 0}) I(h_i) - M \sum_{i=1}^{n} I(h_i)$$

$$\le \sum_{i=1}^{n} (\underbrace{M + y_i + \varepsilon}_{\ge 0}) \left(\mu(E_i) + \frac{\varepsilon}{n}\right) - M \mu(K)$$

$\left(h_i \prec G_i \text{ so } I(h_i) \le \mu(G_i) \le \mu(E_i) + \frac{\varepsilon}{n}\right)$

$$\le (2M+1)\frac{\varepsilon}{n} + \sum (y_i + \varepsilon) \mu(E_i)$$

$\left(E_1 \cup \dots \cup E_n = K \text{ disj. union}\right)$

$$\le \sum_{i=1}^{n} \int_{E_i} f \, d\mu + (2M+1)\varepsilon + \mu(K) 2\varepsilon$$

$\left(f \ge y_{i-1} \ge y_i - \varepsilon \text{ on } E_i\right)$

$$= \int f \, d\mu + O(\varepsilon)$$

Letting $\varepsilon \to 0$, we conclude (7). $\square$

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Lem. (Jordan decomp. for linear functionals)
Let X be a LCH space, and
$I : C_0(X, \mathbb{R}) = \{ f$ real-valued functions in $C_0(X) \}$
$\to \mathbb{R}$ be a bdd. linear functional.
Then there ex. bdd. linear functionals
$I^+, I^- : C_0(X, \mathbb{R}) \to \mathbb{R}$ that are positive
and satisfy $I = I^+ - I^-$.

Proof: For $f \geq 0$ in $C_0(X, \mathbb{R})$ we
define
$$I^+(f) := \sup \{ I(g) : g \in C_0(X, \mathbb{R}), \ 0 \leq g \leq f \}.$$
Note: if $0 \leq g \leq f$, then $\|g\|_u \leq \|f\|_u$,
and so
$$|I(g)| \leq \|I\| \cdot \|g\|_u \leq \|I\| \cdot \|f\|_u.$$
So $0 \leq I^+(f) \leq \|I\| \cdot \|f\|_u < \infty$.
Then:
i) $I^+(\alpha f) = \alpha I^+(f)$, $\alpha \geq 0$, $f \in C_0(X, \mathbb{R})$, $f \geq 0$.

ii) $I^+(f_1 + f_2) = I^+(f_1) + I^+(f_2)$, $f_1, f_2 \in C_0(X, \mathbb{R})$, $f_1, f_2 \geq 0$.

(i): easy exercise!
(ii): $\geq$ if $0 \leq g_1 \leq f_1$, $0 \leq g_2 \leq f_2$,
then $0 \leq g_1 + g_2 \leq f_1 + f_2$; so
$$I^+(f_1 + f_2) \geq I(g_1) + I(g_2)$$
Taking sup over all g_1, g_2, ineq.
$\geq$ follows!

(82) "$\leq$": If $0 \leq g \leq f_1 + f_2$, let $g_1 = \min(g, f_1)$. Then $g_1 \in C_0(X, \mathbb{R})$, $0 \leq g_1 \leq f_1$.

Define $g_2 := g - g_1$. Then $g_2 \in C_0(X, \mathbb{R})$, $0 \leq g_2 \leq f_2$.

Then

$$I(g) = I(g_1) + I(g_2) \leq I^+(f_1) + I^+(f_2).$$

Taking sup over all g, we get "$\leq$".

If $f \in C_0(X, \mathbb{R})$ is arb., we define

$$I^+(f) = I^+(f^+) - I^+(f^-)$$

where $f^+ = \max(f, 0)$ positive part, and $f^- = -\min(f, 0)$ negative part of f.

Then:

i) $I^+ : C_0(X, \mathbb{R}) \to \mathbb{R}$ is linear. (follows from (i) + (ii); exercise!)

ii) I^+ is positive: if $f \geq 0$, then $I^+(f) = I^+(f^+) \geq 0$.

iii) I^+ is bounded:

$$|I(f)| \leq I^+(f^+) + I^+(f^-)$$
$$\leq \|I\| \, \|f\|_u + \|I\| \, \|f\|_u$$
$$\leq 2\|I\| \, \|f\|_u$$

Define $I^- := I - I^+$. This is a bdd. lin. functional on $C_0(X, \mathbb{R})$. It is positive: if $f \geq 0$, then

$$I^-(f) = I(f) - I^+(f) \geq 0$$

by def. of I^+. $\square$

(83)

Def. Let X be a LCH space.
A signed measure μ on X is called Radon if μ^+, μ^- are Radon.
A complex measure μ is called Radon if its real part μ_r and its imaginary part are Radon.

Rem: 1) A finite pos. Radon meas. has the following stronger regularity property.
For each Borel set $E \subseteq X$ and each $\varepsilon > 0$, there ex. a compact set K and an open set U with $K \subseteq E \subseteq U$ s.t. $\mu(U \setminus K) < \varepsilon$
($\longrightarrow$ implies outer regularity + inner regularity on all Borel sets).
2) Let μ be a complex Borel meas. on a LCH space,
$\mu = \mu_r + i\mu_i$, $\mu_r = \mu_r^+ - \mu_r^-$, $\mu_i = \mu_i^+ - \mu_i^-$.

Then μ is Radon iff
$\mu_r^+, \mu_r^-, \mu_i^+, \mu_i^-$ are Radon (by def.)
iff $|\mu|$ is Radon.
(follows from
$\mu_r^+, \mu_r^-, \mu_i^+, \mu_i^- \leq |\mu| \leq \mu_r^+ + \mu_r^- + \mu_i^+ + \mu_i^-$)