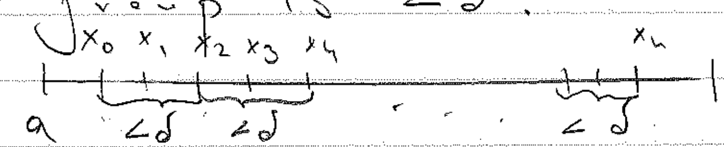
(60) Rem. $F \in AC(\mathbb{R}) \not\Rightarrow F \in BV(\mathbb{R})$

Ex. $F(x) = \sin x$

Then $F'(x) = \cos x$ and $|F'| \le 1$,
so F is Lipschitz and so $F \in AC(\mathbb{R})$,
but $F \notin BV(\mathbb{R})$.

Lem. $[a,b] \subseteq \mathbb{R}$ finite interval
Then $AC[a,b] \subseteq BV[a,b]$
(every AC-function for $[a,b]$ is a BV-function)

Proof. Let $F \in AC[a,b]$ and pick $\delta > 0$ for $\varepsilon = 1$ as in def. of abs. cont. for F.
To show $F \in BV[a,b]$ let
$a \le x_0 < \dots < x_n \le b$ be arb.
By adding pts. we may wlog assume that $x_k - x_{k-1} < \delta$ for all k.
We split the points $x_0, \dots, x_n$ into max. groups s.t. total length of intervals in each group is $< \delta$.
If N is the total no. of groups, then



$$N \le N_0 := \frac{2(b-a)}{\delta} + 1$$

(Note: if L_i is the "total" length of intervals in the i-th group, then $L_i + L_{i+1} \ge \delta$, and so

$$\delta(N-1) \le (l_1+l_2) + (l_2+l_3) + \dots + (l_{N-1}+l_N)$$
$$\le 2(l_1 + \dots + l_N) \le 2(b-a).)$$

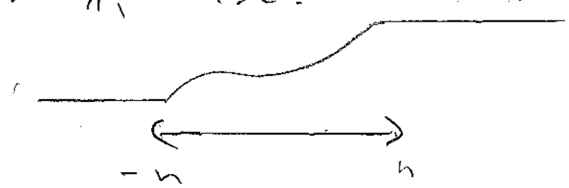
Hence

$$\sum_k |F(x_k) - F(x_{k-1})|$$
$$= \sum_{i=1}^{N} \underbrace{\sum_{\substack{x_k \text{ in} \\ i\text{-th group}}} |F(x_k) - F(x_{k-1})|}_{\le 1 \text{ by def. of } \delta}$$
$$\le N,.$$

So $F \in BV[a\,b]$. □

Thm. Let $F: \mathbb{R} \to \mathbb{R}$ be increasing. Then $F'(x)$ exist for λ_1-a.e. $x \in \mathbb{R}$.

Proof: Wlog F is bdd.



We define
$$G(x) := F(x+) \text{ for } x \in \mathbb{R}.$$
Then G is increasing and right-continuous. Moreover, the set where $H(x) := G(x) - F(x) \neq 0$ is countable (exercise!)

There ex. a (finite) Borel measure μ s.t.
$$\mu(a,b] = G(b) - G(a)$$
for each h-interval $(a,b] \subseteq \mathbb{R}$.
(HW 6, Prob. 2)

In particular,

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$$G(x+h)-G(x)=\begin{cases}\mu(x,x+h], & h>0,\\ -\mu(x-h,x], & h<0.\end{cases}$$

The sets $(x, x+r]$, $(x-r, x]$ shrink nicely to x as $r \to 0$.

It follows that if $d\mu = f\,d\lambda_1 + d\nu$, $\nu \perp \lambda_1$, then

$$\lim_{h\to 0^+}\frac{G(x+h)-G(x)}{h}=\lim_{h\to 0^+}\frac{\mu(x,x+h]}{\lambda_1(x,x+h]}$$

$= f(x)$ for a.e. $x \in \mathbb{R}$.

Similarly,

$$\lim_{h\to 0^+}\frac{G(x+h)-G(x)}{h}=\lim_{h\to 0^+}\frac{\mu(x-h,x]}{\lambda_1(x-h,x]}$$

$= f(x)$ for a.e. $\mathbb{R}$.

Hence $G'(x)$ exists and $G'(x) = f(x)$ for λ_1-a.e. x.

<u>Claim</u> $H'(x)$ exists and $H'(x) = 0$ for λ_1-a.e. x ($\Rightarrow F'(x) = G'(x) - H'(x)$ exists and $F'(x) = f(x)$ for λ_1-a.e. $x \in \mathbb{R}$).

Let $E = \{x_n : n \in \mathbb{N}\}$ be an enumeration of the pts x where $H(x) \neq 0$, δ_n a point mass at x_n, and

$$\varrho := \sum_n H(x_n)\,\delta_n$$

(note $H(x_n) := G(x_n) - F(x_n)$
$= F(x_n+) - F(x_n) > 0$).

Then ϱ is a finite Borel meas. on $\mathbb{R}$ with $\varrho \perp \lambda_1$. ($\lambda_1(E) = 0$, $\varrho(\mathbb{R}\setminus E) = 0$).

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Note: $\varsigma(A) = \sum_{x_n \in A} H(x_n)$.

Then
$$\left|\frac{H(x+h)-H(x)}{h}\right| \le \frac{H(x+h)+H(x)}{|h|}$$
$$\le 2\,\frac{\varsigma[x-|h|,\ x+|h|]}{2|h|} = 2\,\frac{\varsigma[x-|h|,\ x+|h|]}{\lambda_1[x-|h|,\ x+|h|]}$$

$\longrightarrow 0$ as $h \longrightarrow 0$ for λ_1-a.e. x

by Lebesgue differentiation,
($\varsigma \perp \lambda_1$).

So $H'(x)$ ex. and $H'(x) = 0$ for λ_1-a.e. $x \in \mathbb{R}$. $\square$

Cor. Let $F \in NBV$. Then $F'(x)$ exists for λ_1-a.e. $x \in \mathbb{R}$.
Moreover, if μ is the unique complex Borel meas. with $F = F_\mu$ and $d\mu = f\,d\lambda_1 + \nu$ the Lebesgue decomp. of μ,
$$F'(x) = f(x) \quad \text{for } \lambda_1\text{-a.e. } x \in \mathbb{R}.$$
In particular, $F' \in L^1(\lambda_1)$.
Finally, if $F \in AC \cap NBV$, then
$$F(x) = \int_{-\infty}^{x} F'(t)\,dt = \int_{[-\infty, x]} F'\,d\lambda_1$$
for each $x \in \mathbb{R}$.

Proof: The first part follows from prev. thm. and its proof if $F \in NBV$ is increasing. In the general case, we

(64) split F into real and img. part, and repr. these as differences of increasing functions in NBV, etc.
If $F \in AC \cap NBV$, then $\mu << \lambda_1$ and so $d\mu = f\, d\lambda_1 = F'\, d\lambda_1$.
Hence for each $x \in \mathbb{R}$,

$$F(x) = \mu(-\infty, x] = \int_{(-\infty,x]} F'\, d\lambda_1 = \int_{-\infty}^{x} F'(t)\, dt. \quad \square$$

<u>Thm.</u> (Fundamental Thm. of Calculus for Lebesgue Integrals)
Let $[a,b] \subseteq \mathbb{R}$ be a finite interval.
<u>TFAE:</u>

i) $F \in AC[a,b]$

ii) F is diff. a.e. on $[a,b]$, $F' \in L^1[a,b]$, and
$$F(x) - F(a) = \int_a^x F'(t)\, dt \quad \text{for each } x \in [a,b]$$

iii) there ex. $f \in L^1[a,b]$ s.t.
$$F(x) - F(a) = \int_a^x f(t)\, dt \quad \text{for each } x \in [a,b].$$

<u>Proof.</u> i) $\rightarrow$ ii) By subtracting a const., wlog $F(a) = 0$. Since $F \in AC[a,b]$, $F \in BV[a,b]$. We extend F to $\mathbb{R}$ by setting
$$F(x) := F(a) = 0 \quad \text{for } x \le a$$
$$F(x) := F(b) \quad \text{for } x \ge b.$$

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Then $F \in NBV \cap AC$. By Cor. $F'(x)$ ex. a.e. on $\mathbb{R}$, and $F' \in L^1(\mathbb{R})$.
Obviously, $F' \equiv 0$ on $(-\infty, a) \cup (b, \infty)$.
Moreover, for $x \in [a,b]$

$$F(x) = \int_{-\infty}^{x} F'(t)\,dt = \int_{a}^{x} F'(t)\,dt + F(a).$$

ii) $\Rightarrow$ iii) trivial: take $f = F'$.
iii) $\Rightarrow$ ii) We extend f by 0 outside $[a,b]$. Then $f \in L^1(\mathbb{R})$, and so

$d\mu = f\,d\lambda_1$ is a complex Borel meas. on $\mathbb{R}$ with $\mu \ll \lambda_1$.
Then for each $x \in \mathbb{R}$,

$$F_\mu(x) = \mu(-\infty, x] = \int_{(-\infty, x]} f\,d\lambda_1$$

$$= \begin{cases} 0 = F(x) & x < a \\ \int_a^x f(t)\,dt = F(x) & x \in [a,b] \\ \int_a^b f(t)\,dt = F(x) & x \geq b \end{cases}$$

$= F(x)$.
Since $\mu \ll \lambda_1$, we have $F \in AC(\mathbb{R})$, and so $F|[a,b] \in AC[a,b]$. □

Locally compact Hausdorff spaces

X top. space.
X is <u>locally compact</u> if every point $x \in X$ has a compact neighborhood N (i.e., N compact and $x \in \mathring{N}$)

(66) $\mathbb{X}$ is <u>Hausdorff</u> if for all $x, y \in \mathbb{X}$, $x \neq y$, there ex. neighborhoods U of x and V of y s.t. $U \cap V = \emptyset$.

LCH = loc. comp. Hausdorff

Facts about LCH spaces $\mathbb{X}$:

1) If $K \subseteq U \subseteq \mathbb{X}$, K compact, and U open, then there ex. an open set V with $\bar{V}$ compact s.t.

$$K \subseteq V \subseteq \bar{V} \subseteq U$$

(245A, HW6, Prob.1)

2) <u>Urysohn's lemma</u> (loc. comp. version)

if $K \subseteq U \subseteq \mathbb{X}$, K comp., U open, then there ex. a cont. function $f: \mathbb{C} \to \mathbb{R}$ s.t.

$$\left.\begin{array}{l} f|K \equiv 1, \\ 0 \leq f \leq 1 \\ \underbrace{\text{supp}(f) \subseteq U \text{ compact}}_{K \prec f \prec U} \end{array}\right\} \text{notation: } f \prec U$$

"f dominated by U"

$(\to 0 \leq f \leq \chi_U)$

3) <u>Tietze's extension theorem</u>:

if $K \subseteq \mathbb{X}$ compact, and $f: K \to \mathbb{C}$ continuous, then there ex. $F: \mathbb{X} \to \mathbb{C}$ cont. s.t. supp(F) comp. and $F|K = f$. (Folland, 4.16, 4.34)

4) <u>Partitions of unity</u>

Let $K \subseteq \mathbb{X}$ be compact, and $\mathcal{U} = \{U_i : i = 1, \ldots, n\}$ be an open cover

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of K. Then there ex. cont. functions $h_1, \dots, h_n : X \to \mathbb{R}$ s.t.

$h_i \prec G_i$ (i.e., $0 \le h_i \le 1$, $\mathrm{supp}(h_i) \subseteq G_i$ compact) for $i = 1, \dots, n$.

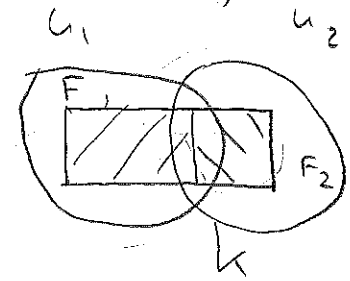
$0 \le h_1 + \dots + h_n \le 1$,

$h_1 + \dots + h_n | K \equiv 1$.

The functions $h_1, \dots, h_n$ form a "partition of unity" on K subordinate to the cover $\mathcal{U}$.

<u>Proof</u> (Outline; details: Folland 4.41)

By a covering argument we can find compact sets $F_1, \dots, F_n$ s.t. $F_i \subseteq G_i$ for $i = 1, \dots, n$ and $K \subseteq F_1 \cup \dots \cup F_n$.



Pick g_i s.t. $F_i \prec g_i \prec G_i$ (Urysohn).
Then

$U := \{ g_1 + \dots + g_n > 0 \} \supseteq K$ open

By Urysohn we can find f with $K \prec f \prec U$ $\quad g_{n+1}|K = 0$

If $g_{n+1} := 1 - f$, then $g_{n+1} | U^c \equiv 1 > 0$

$g := \underbrace{g_1 + \dots + g_n}_{>0 \text{ on } U} + \underbrace{g_{n+1}}_{>0 \text{ on } U^c} > 0$ on X.

Define $h_i = \dfrac{g_i}{g}$ for $i = 1, \dots, n$. $\square$

(68) $\bar{X}$ LCH space

$C_c(\bar{X}) := \{ f : \bar{X} \to \mathbb{C}$ cont., $\operatorname{supp}(f)$ compact $\}$

vector space over $\mathbb{R}$.

$\|f\|_u := \sup_{x \in \bar{X}} |f(x)|$

(often denoted $\|f\|_\infty$)

"uniform norm" on $C_c(\bar{X})$

$\|\cdot\|_u$ is indeed a norm on $C_c(\bar{X})$.

defines topology on $\bar{X}$:

$f_n \to f$ iff $\|f_n - f\|_u \to 0$

iff $f_n \to f$ uniformly on $\bar{X}$.

$C_c(\bar{X})$ is not a Banach space in general!

Def.: $f : \bar{X} \to \mathbb{C}$ cont.

We say that f vanishes <u>at infinity</u> if for all $\varepsilon > 0$ there ex. a compact set $K \subseteq \bar{X}$ s.t. $|f(x)| \leq \varepsilon$ for all $x \in \bar{X} \setminus K$.

$C_0(\bar{X}) := \{ f : \bar{X} \to \mathbb{C}$ cont., f vanishes at infinity $\}$.

<u>Thm.</u> Let $\bar{X}$ be a LCH space. Then $C_0(\bar{X})$ with uniform norm is a Banach space. Moreover, $C_c(\bar{X}) \subseteq C_0(\bar{X})$ is dense in $C_0(\bar{X})$.

(69) Proof (outline): If $f \in C_0(X)$, then f bdd. ; so $\|f\|_u < \infty$, and $\|\cdot\|_u$ well-def. norm on vector space $C_0(X)$.

With this norm, $C_0(X)$ is complete: $\{f_n\}$ Cauchy seq. in $C_0(X)$. Then ex. cont. function $f: X \to \mathbb{C}$ s.t. $\|f_n - f\|_u \to 0$ as $n \to \infty$.

Then $f \in C_0(X)$: if $\varepsilon > 0$ pick $n \in \mathbb{N}$ s.t. $\|f_n - f\|_u < \varepsilon/2$. Since $f_n \in C_0(X)$, there ex. $K \subseteq X$ comp. s.t.

$$|f_n| \leq \frac{\varepsilon}{2} \quad \text{on } X \setminus K.$$

Then

$$|f| \leq |f_n| + |f - f_n| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{on } X \setminus K.$$

$C_c(X) \subseteq C_0(X)$ closer.

Let $f \in C_0(X)$ and $\varepsilon > 0$ be arb.

Pick $K \subseteq X$ comp. s.t.

$$|f| \leq \varepsilon \quad \text{on } X \setminus K.$$

There ex. $g \in C_c(X)$ with $K \prec g \prec X$ (Urysohn).

Then $h = fg \in C_c(X)$,

$$|f - h| = \underbrace{(1-g)}_{\in [0,1]}|f| = \begin{cases} = 0 & \text{on } K, \\ \leq |f| \leq \varepsilon & \text{on } X \setminus K. \end{cases}$$

So $\|f - h\|_u \leq \varepsilon$. □

(70) Rem.: If X is an LCH space, one can define the Alexandroff or one-point compactification as follows:

We add a point ∞ to X:

$\hat{X} = X \cup \{\infty\}$, and define a topology on $\hat{X}$:

$U \subseteq \hat{X}$ open iff

$U \subseteq X$ and U open in X, or

$\infty \in U$ and $\hat{X} \setminus U$ compact set in X

($U = X \setminus K \cup \{\infty\}$, K compact are the open neighborhoods of ∞).

This defines a topology on $\hat{X}$ s.t. $X \subseteq \hat{X}$ subspace, and $\hat{X}$ compact Hausdorff space.
(exercise!)

If $f \in C_0(X)$, then we can define a unique extension $\hat{f}$ to $\hat{X}$ by setting $\hat{f}(\infty) = 0$.

Then

$$f \in C_0(X) \underset{\text{bijection}}{\longleftrightarrow} \hat{f} \in C(\hat{X}),\ \hat{f}(\infty) = 0.$$

Basic problem

X LHC space.

What is the dual Z^* of the Banach space $Z = C_0(X)$?

X compact Hausdorff space,

$Z = C(X)$. What is Z^*? Answer.

$C(X)^* = \mathcal{M}(X)$ space of complex Borel measures on X

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If $\mu \in \mathcal{M}(\underline{Y})$, then
$L_\mu : C(\underline{Y}) \longrightarrow \mathbb{C}$,
$L_\mu(f) = \int f \, d\mu$ defines a bdd. linear functional on $C(\underline{Y})$,
$\|L_\mu\| = \|\mu\| := |\mu|(\underline{Y})$.
(Riesz representation thm.)
If $\underline{X}$ LCH space, then
$C_0(\underline{X})^* = \mathcal{M}(\underline{X})$
= space of complex Radon measures on $\underline{X}$.

Recall from 245A:
A pos. meas. μ on a LCH space $\underline{X}$ is called a <u>Radon measure</u> if
i) it is a Borel measure,
ii) $\mu(K) < \infty$ whenever $K \subseteq \underline{X}$ comp.
iii) μ is <u>outer regular</u>
$\mu(E) = \inf \{ \mu(U) : U \supseteq E \text{ open} \}$
for each Borel set $E \subseteq \underline{X}$,
μ is <u>inner regular on open sets</u>, i.e.,
$\mu(U) = \sup \{ \mu(K) : K \subseteq U \text{ compact} \}$
for each open set $U \subseteq \underline{X}$.

A linear functional $I : C_c(\underline{X}) \longrightarrow \mathbb{C}$ is called <u>positive</u> if $I(f) \geq 0$ whenever $f \in C_c(\underline{X})$ and $f \geq 0$.