(43) **Thm.** Let ν be a complex measure on $\mathbb{R}^n$ and $d\nu = dg + f\,d\lambda_n$ be its Lebesgue decomposition w.r.t. Lebesgue measure λ_n (so $g \perp \lambda_n$, $f \in L^1(\lambda_n)$). Suppose for each $x \in \mathbb{R}^n$ we have a family $\{E_r(x)\}_{r>0}$ of Borel sets shrinking nicely to x.
Then
$$\lim_{r \to 0^+} \frac{\nu(E_r(x))}{\lambda_n(E_r(x))} = f(x) \quad \text{for } \lambda_n\text{-a.e. } x \in \mathbb{R}^n.$$

So in particular,
$$\lim_{r \to 0^+} \frac{\nu(B(x,r))}{\lambda_n(B(x,r))} = f(x) \quad \text{for } \lambda_n\text{-a.e. } x \in \mathbb{R}^n.$$

Proof: Note:
$$\frac{\nu(E_r(x))}{\lambda_n(E_r(x))} = \frac{g(E_r(x))}{\lambda_n(E_r(x))} + \fint_{E_r(x)} f\,d\lambda_n$$

Now
$$\lim_{r \to 0^+} \fint_{E_r(x)} f\,d\lambda = f(x) \quad \text{for each Lebesgue point } x \in \mathbb{R}^n,$$
i.e. for λ_n-a.e. $x \in \mathbb{R}^n$.
(Improved Lebesgue Diff. Thm.)

$$\left| \frac{g(E_r(x))}{\lambda_n(E_r(x))} \right| \le \frac{|g|(E_r(x))}{\lambda_n(E_r(x))}$$
$$\le \frac{|g|(B(x,r))}{\lambda_n(B(x,r))} \cdot \frac{\lambda_n(B(x,r))}{\lambda_n(E_r(x))}$$

(4)

$$\le \frac{1}{\alpha(x)} \frac{|g|(B(x,r))}{\lambda_n(B(x,r))} \overset{?}{\longrightarrow} 0 \quad \lambda_n\text{-a.e.}$$

(from def. of nicely shrinking sets)

Here $|g| \perp \lambda_n$ ($g \perp \lambda_n$ equiv.
$g_1, g_2 \perp \lambda_n \to |g| \le |g_1| + |g_2| \perp \lambda_n$)

So it suffices to show:

Claim if μ is a finite pos. measure on $\mathbb{R}^n$ with $\mu \perp \lambda_n$, then

(*) $$\lim_{r \to 0^+} \frac{\mu(B(x,r))}{\lambda_n(B(x,r))} = 0 \quad \text{for } \lambda_n\text{-a.e. } x \in \mathbb{R}^n.$$

Since $\mu \perp \lambda_n$, there ex. a Borel set $A \subseteq \mathbb{R}^n$ s.t. $\mu(A) = 0$ and $\lambda_n(A^c) = 0$.

Let $k \in \mathbb{N}$ be arb. and consider

$$F_k := \left\{ x \in \mathbb{R}^n : \limsup_{r \to 0^+} \frac{\mu(B(x,r))}{\lambda_n(B(x,r))} > \frac{1}{k} \right\}$$

Since μ is regular (245A, HW1, Prob. 1), for each $\varepsilon > 0$ there ex. an open set $U_\varepsilon \supseteq A$ s.t. $\mu(U_\varepsilon \setminus A) < \varepsilon$ equiv.

$$= \mu(U_\varepsilon) = \underbrace{\mu(U_\varepsilon \setminus A)}_{<\varepsilon} + \underbrace{\mu(A)}_{=0} < \varepsilon.$$

For each $x \in F_k$ there ex. a ball $B_x \subseteq U_\varepsilon$ centered at x s.t.

(1) $$\frac{\mu(B_x)}{\lambda_n(B_x)} > \frac{1}{k}.$$

(45)

By the 5B-covering lemma, we can find a disjoint subfamily $\{B_n : n \in \mathbb{N}\}$ of $\{B_x\}_{x \in F_k}$ s.t.

$$F_k \subseteq V_\varepsilon := \bigcup_{x \in F_k} B_x \subseteq \bigcup_{n \in \mathbb{N}} 5B_n .$$

Then
$$\lambda_n(V_\varepsilon) \leq \sum_{n \in \mathbb{N}} \lambda_n(5B_n)$$
$$\leq 5^n \sum_{n \in \mathbb{N}} \lambda_n(B_n) \underset{(1)}{\leq} 5^n K \sum_{n \in \mathbb{N}} \mu(B_n)$$
$$\underset{\substack{B_n, n \in \mathbb{N} \\ \text{disj.}}}{\leq} 5^n K \mu\Big(\bigcup_{n \in \mathbb{N}} B_n\Big) \underset{B_n \subseteq U_\varepsilon}{\leq} 5^n K \mu(U_\varepsilon)$$
$$\leq 5^n K \cdot \varepsilon .$$

For fixed K this is true for each $\varepsilon > 0$; so
$$\lambda_n(F_k) = 0 \qquad \Big(F_k \subseteq \bigcap_{m \in \mathbb{N}} V_{1/m} \Big)$$

Hence
$$\lambda_n\Big(A^c \cup \bigcup_{k \in \mathbb{N}} F_k\Big) \leq \underbrace{\lambda_n(A^c)}_{=0} + \sum_{k=1}^{\infty} \underbrace{\lambda_n(F_k)}_{=0}$$
$= 0$, and for $x \notin A^c \cup \bigcup_{k \in \mathbb{N}} F_k$, $(*)$ is true. $\square$

Complex measures on $\mathbb{R}$

Want to describe all complex measures on $\mathbb{R}$.

Idea: μ complex m.s. on $\mathbb{R}^1$

(46) $\longrightarrow F_\mu(x) := \mu(-\infty, x]$

F_μ is a special type of function, namely a function of bounded varition.

complex measures on $\mathbb{R}$
$\longleftrightarrow$ functions of bdd. variation with some normalizations.

Def. (Functions of bounded varition or BV-functions)
We say that a function $F: \mathbb{R} \longrightarrow \mathbb{C}$ has bounded variation (or is a BV-function) if there ex. a constant s.t.

$$(*) \quad \sum_{k=1}^{n} |F(x_k) - F(x_{k-1})| \leq M$$

whenever $n \in \mathbb{N}$ and $x_0 < x_1 < \dots < x_n$.
The set of all BV functions on $\mathbb{R}$ is denoted by $BV(\mathbb{R})$ or by BV.
A function $F: [a,b] \longrightarrow \mathbb{C}$ is of bdd. variation if an ineq. as in (1) holds for $a \leq x_0 < \dots < x_n \leq b$.
$BV[a,b]$ = set of all BV-functions on $[a,b]$
If $F \in BV$ and $x \in \mathbb{R}$, we define
$T_F(x) := \sup \{ \sum_{k=1}^{n} |F(x_k) - F(x_{k-1})| :$

(47)

$n \in \mathbb{N}, \ x_0 < \dots < x_n \le x \}$
"total variation of F up to x".

Thm. (Properties of BV-functions)

(i) $F \in BV$ iff $\operatorname{Re} F, \operatorname{Im} F \in BV$

(ii) A real-valued function F on $\mathbb{R}$ is in BV iff F is the difference of two bdd. increasing functions (G increasing equiv. $x \le y \to G(x) \le G(y)$)

(iii) If $F \in BV$, then

$$F(x+) := \lim_{y \to x^+} F(y), \text{ and}$$

$$F(x-) := \lim_{y \to x^-} F(y),$$

exist for all $x \in \mathbb{R}$.
Moreover, the limits

$$F(+\infty) := \lim_{x \to +\infty} F(x)$$

$$F(-\infty) := \lim_{x \to -\infty} F(x) \quad \text{exist.}$$

(iv) If $F \in BV$, then F has at most countably many discontinuities.

Proof: (i) Follows from
$$|\operatorname{Re} F|, |\operatorname{Im} F| \le |F| \le |\operatorname{Re} F| + |\operatorname{Im} F|.$$

(ii) $\Leftarrow$ Suppose $F = G - H$, where G, H bdd. and

Then if $x_0 < \cdots < x_n$,

$$\sum_{k=1}^{n} |F(x_k) - F(x_{k-1})|$$

$$\le \sum_{k=1}^{n} |G(x_k) - G(x_{k-1})| + \sum_{k=1}^{n} |H(x_k) - H(x_{k-1})|$$

triangle ineq.

$$= \sum_{k=1}^{n} (G(x_k) - G(x_{k-1})) + \sum_{k=1}^{n} (H(x_k) - H(x_{k-1}))$$

$$= G(x_n) - G(x_0) + H(x_n) - H(x_0)$$

telescoping sum

$$\le 2M_G + 2M_H := M,$$

where $|G| \le M_G$, $|H| \le M_H$ (G, H bdd.).

Suppose F is real-valued and of bdd. variation. Consider total variation T_F of F. If $x \le y$, then

(*) $$T_F(x) + |F(y) - F(x)| \le T_F(y)$$

(exercise!)

equiv. $$T_F(x) \pm (F(y) - F(x)) \le T_F(y)$$

equiv. $$T_F(x) - F(x) \le T_F(y) - F(y)$$

and $$T_F(x) + F(x) \le T_F(y) + F(y).$$

So $T_F - F$ and $T_F + F$ are both increasing.

Obviously, T_F is bdd. since $F \in BV$.

(*) implies that F is also bdd.

So $G = \frac{1}{2}(T_F + F)$ and (exercise!)

$H = \frac{1}{2}(T_F - F)$ are

bdd. and increasing; moreover

$$G - H = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F) = F.$$

(iii) Statements true for bdd. and increasing functions (exercise!)
Hence true for all BV-functions by (i) and (ii).

(iv) A bdd. increasing functions F has only countably many discontinuities!

$$\{x : F \text{ discont. at } x\}$$
$$= \{x \in \mathbb{R} : F(x+) - F(x-) > 0\}$$
$$= \underbrace{\bigcup_{k \in \mathbb{N}} \underbrace{\{x \in F(x+) - F(x-) > \tfrac{1}{k}\}}_{\text{finite if } F \text{ bdd. increasing}}}_{\text{countable.}}$$

So again statement true for (i) + (ii). $\square$

A function $F : \mathbb{R} \to \mathbb{C}$ is called a <u>normalized BV-function</u> if

i) F has bounded variation

ii) F is right-continuous, i.e.,
$F(x+) \overset{!}{=} F(x)$ for all $x \in \mathbb{R}$
(note that $F(x+)$ exists by prev. thm.)

iii) $F(-\infty) = 0$
($F(-\infty)$ exists by prev. thm.)

Let NBV the family of all normalized BV-functions on $\mathbb{R}$.

(50) Thm. (Complex measures on $\mathbb{R}$)
$F \in NBV$ if and only if there ex. a complex Borel measure μ on $\mathbb{R}$ s.t. $F = F_\mu$, where $F_\mu(x) = \mu(-\infty, x]$ for $x \in \mathbb{R}$.
Moreover, for given $F \in NBV$ the complex Borel measure μ with $F_\mu = F$ is unique.

In other words, $\mu \longmapsto F_\mu$ gives a bijection

$$\varphi : \{\mu : \mu \text{ complex Borel meas. on } \mathbb{R}\} =: \mathcal{M} \longleftrightarrow NBV.$$

Proof: The proof consists of three parts:

I. $\mu \in \mathcal{M} \to F_\mu \in NBV$.
II. $F \in NBV \to \exists \mu \in \mathcal{M} : F = F_\mu$
III. $\mu, \nu \in \mathcal{M} \wedge F_\mu = F_\nu \to \mu = \nu$.

I. Let $\mu \in \mathcal{M}$ be arb. Then $|\mu| \in \mathcal{M}$ is a finite pos. Borel meas. on $\mathbb{R}$.
Let $F = F_\mu$. Then $F \in NBV$.
Indeed:
i) F has bdd. variation.
if $x_0 < \dots < x_n$, then

$$F(x_k) - F(x_{k-1}) = \mu(-\infty, x_k] - \mu(-\infty, x_{k-1}]$$
$$= \mu(x_{k-1}, x_k].$$

51

So
$$\sum_{k=1}^{n} |F(x_k) - F(x_{k-1})| \le \sum_{k=1}^{n} |\mu(x_{k-1}, x_k]|$$
$$\le \sum_{k=1}^{n} |\mu|(x_{k-1}, x_k] = |\mu|(x_0, x_n]$$
$$\le |\mu|(\mathbb{R}) < \infty.$$

ii) $F(x+) = F(x)$ for each $x \in \mathbb{R}$ and $F(-\infty) = 0$.

Note. If $\mu = \mu_r + i\mu_i$, μ_r, μ_i real and im. part of μ, then
$$F_\mu = F_{\mu_r} + i F_{\mu_i}, \text{ where } \mu_r, \mu_i \in \mathcal{M}.$$
Moreover, if $\mu \in \mathcal{M}$ signed measure and $\mu = \mu_+ - \mu_-$ Jordan decomp., then
$$F_\mu = F_{\mu_+} - F_{\mu_-} \text{ and } \mu_+, \mu_- \in \mathcal{M}$$
pos. meas.

So wlog μ finite pos. Borel meas.
Let $x \in \mathbb{R}$ be arb. and $x_n \searrow x$.
Then $(-\infty, x_n] \searrow (-\infty, x]$ and so (F_μ is finite!),
$$F(x_n) = \mu(-\infty, x_n] \longrightarrow \mu(-\infty, x] = F(x)$$
Hence
$F(x+) = F(x)$ ($F(x+)$ exists, because $F \in BV$).
The proof that $F(-\infty) = 0$ is similar.
This completes Part I.

(52) II. Let $F \in NBV$. We have to find a meas. $\mu \in \mathcal{M}$ s.t. $F = F_\mu$.
Wlog F is real-valued (otherwise split F into real and imaginary parts)

Lemma If $F \in NBV$, then $T_F \in NBV$.

Proof: T_F is bdd. increasing; so $T_F \in BV$.
Let $\varepsilon > 0$ and $b \in \mathbb{R}$ be arb.
We can find $x_0 < x_1 < \dots x_n \leq b$ s.t.

$$(1) \quad T_F(b) - \varepsilon \leq \Sigma_n = \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \leq T_F(b)$$

Then

$$(2) \quad T_F(x_k) - \varepsilon \leq \Sigma_k := \sum_{i=1}^{k} |F(x_i) - F(x_{i-1})| \leq T_F(x_k)$$

for all $k \leq n$.
(if $\Sigma_k < T_F(x_k) - \varepsilon$, then we could replace Σ_k by a similar sum Σ_k' s.t. $\Sigma_k' > \Sigma_k + \varepsilon$.
If we replace the Σ_k terms in Σ_n by Σ_k', we get a sum $\Sigma_n' > \Sigma_n + \varepsilon \geq T_F(b)$ which is impossible).

(53) To show $F(-\infty) = 0$, we pick $b = 0$ in (1).
Then for $k = 0$ we have
Σ_0 = empty sum $= 0$, and so

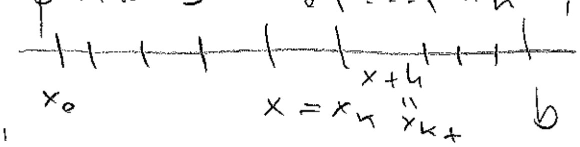
$T_F(x_0) - \varepsilon \leq \Sigma_0 = 0$ equiv.

$0 \leq T_F(-\infty) \leq T_F(x_0) \leq \varepsilon$.

Since $\varepsilon > 0$ arb., we conclude

$$T_F(-\infty) = \lim_{x \to -\infty} T_F(x) = 0.$$

To show $T_F(x+) = T_F(x)$ at $x \in \mathbb{R}$, we pick $b = x+1$, say.
Wlog x is one of the points $x_0, \ldots, x_n$, say $x = x_k$.



Since $F(x+) = F(x)$, we can find small $h \in (0,1)$ s.t.
$|F(x+h) - F(x)| < \varepsilon$.
Wlog $x_{k+1} = x+h$.
Then by (2)

$$T_F(x) - \varepsilon \leq \Sigma_k \leq T_F(x)$$

(3) $\longrightarrow \quad |T_F(x) - \Sigma_x| \leq \varepsilon$

Similarly, by (2)

(4) $|T_F(x+h) - \Sigma_{k+1}| \leq \varepsilon$

and Σ_k and Σ_{k+1} differ by one term, namely $|F(x+h) - F(x)| < \varepsilon$

(5) So $|\Sigma_{k+1} - \Sigma_k| \leq \varepsilon$

(54) By (3) - (5)

$$3\varepsilon \geq T_F(x+h) - T_F(x)$$
$$\geq T_F(x+) - T_F(x).$$

Since $\varepsilon > 0$ was arb, $T_F(x+) = T_F(x)$.
This proves the lemma, namely that $T_F \in NBV$.

Since $F = \frac{1}{2}\underbrace{(T_F + F)}_{\substack{\in NBV \\ \text{increasing}}} - \frac{1}{2}\underbrace{(T_F - F)}_{\substack{\in NBV \\ \text{increasing}}}$,

we may wlog assume that $F \in NBV$ is real-valued and increasing in order to construct a measure μ s.t. $F_\mu = F$.
If F is such a function, and $I = (a, b] \subseteq \mathbb{R}$ an h-interval, we define

(1) $F(I) = F(b) - F(a)$

(here $a = -\infty$, or $b = +\infty$ allowed).
We use the function in (1) to define a premeasure μ on the algebra $\mathcal{A}$ generated by the h-intervals:
$A \in \mathcal{A}$ if $A = I_1 \cup \cdots \cup I_n$ is a finite disjoint union of h-intervals.
We set $\mu(A) = \sum_{k=1}^{n} F(I_k)$.

(55) To show that μ is well-def. and a premeasure, we need the

Basic Lemma:

if $\mathcal{M}, \mathcal{N}$ are countable families of h-intervals, $\mathcal{M}$ is disjointed, and $\bigcup_{I \in \mathcal{M}} I \subseteq \bigcup_{J \in \mathcal{N}} J$, then

$$\sum_{I \in \mathcal{M}} F(I) \leq \sum_{J \in \mathcal{N}} F(J).$$

(HW 6, Prob. 2).

The Carathéodory Extension Thm. implies that there ex. a pos. meas. μ on $\sigma(\mathcal{A})$ = Borel σ-alg. on $\mathbb{R}$ that extends the premeasure.

Then

$$\mu(I) = F(I) = F(b) - F(a)$$
for each h-interval $(a, b]$.

In particular,

$$\mu(-\infty, x] = F((-\infty, x]) = F(x) - \overbrace{F(-\infty)}^{0}$$
$$= F(x) \quad \text{for } x \in \mathbb{R}.$$

(μ can be regarded as a complex measure, because it is finite:

$$\mu(\mathbb{R}) = F((-\infty, \infty]) = F(+\infty) - \underbrace{F(-\infty)}_{0}$$
$$= F(+\infty) < \infty,$$

because $F \in NBV$ and so F bdd.

This completes Part II.

(56) Part III. Let μ, ν be complex measures with $F_\mu = F_\nu$ equiv. $\mu(-\infty, x] = \nu(-\infty, x]$ for all $x \in \mathbb{R}$.
WTS $\mu = \nu$.
This follows from the Dynkin π-λ-Thm. (HW 6, Prob. 1 b)). $\square$

General problem:
How are the properties of $\mu \in \mathcal{M}$ related to properties of $F_\mu \in NBV$?
F_μ continuous iff μ has no atoms.
We'll consider one important case of this problem: if $\mu \ll \lambda$, what can we say about F_μ?

Def. (Absolutely continuous functions or AC-functions)
We say that a function $F: \mathbb{R} \to \mathbb{C}$ is absolutely continuous (or is an AC-function) if for all $\varepsilon > 0$ there ex. $\delta > 0$ with the following property: if $(a_1, b_1), \ldots, (a_n, b_n) \subseteq \mathbb{R}$ are disjoint interval, then we have the implication

$$\sum_{i=1}^{n} (b_i - a_i) < \delta \longrightarrow \sum_{i=1}^{n} |F(b_i) - F(a_i)| < \varepsilon,$$

57

We denote by AC or $AC(\mathbb{R})$ the set of all AC-functions on $\mathbb{R}$.
AC-functions on an interval $[a,b]$ are defined similarly (in this case we require $[a_i,b_i] \subseteq [a,b]$ in the def.)
$AC[a,b]$ = set of all AC-functions $[a,b]$.

Prop. Let μ be a complex measure and $F = F_\mu$.
Then $\mu \ll \lambda_1$ if and only if F is absolutely continuous.

Proof: "$\Rightarrow$" If $\mu \ll \lambda_1$, then by Radon-Nikodym there ex. $f \in L^1(\mu)$ s.t. $d\mu = f\, d\lambda_1$. Then (this was proved earlier)
$d|\mu| = |f|\, d\lambda_1$. In particular, $|\mu| \ll \lambda_1$.
Hence (245A, HW2, Prob. 4) for each $\varepsilon > 0$ there ex. $\delta > 0$ s.t.
$$\lambda_1(A) < \delta \to |\mu|(A) < \varepsilon$$
for each Borel set $A \subseteq \mathbb{R}$.

Now suppose $(a_1,b_1), \dots, (a_n,b_n) \subseteq \mathbb{R}$ are disjoint intervals.
If $\sum_{i=1}^{n} (b_i - a_i) < \delta$, then for $A = \bigcup_{i=1}^{n} (a_i,b_i)$

(58) we have

$\lambda_1(A) < \delta$ and so

$$\sum_{i=1}^{n} |F(b_i) - F(a_i)| = \sum_{i=1}^{n} |\mu(a_i, b_i]|$$

$$\leq \sum_{i=1}^{n} |\mu|(a_i, b_i] \underset{|\mu|\{b_i\}=0}{=} \sum_{i=1}^{n} |\mu|(a_i, b_i)$$

$$= |\mu|(A) < \varepsilon.$$

So $F \in AC$.

"$\Leftarrow$" Suppose $F = F_\mu \in AC$.

WTS $\mu \ll \lambda_1$, equiv. if $E \subseteq \mathbb{R}$ Borel set with $\lambda_1(E) = 0$, then $\mu(E) = 0$.

Let $\varepsilon > 0$ be arb. and $\delta > 0$ as in the def. of abs. cont. for F.

Since $|\mu|$ and λ_1 are regular, we can find open sets $V_n \supseteq E$ for $n \in \mathbb{N}$ s.t.

$|\mu|(V_n) \longrightarrow |\mu|(E)$

and an open set $U \supseteq E$ s.t.

$\lambda_1(U) < \delta$.

Define $U_n := V_n \cap U \subseteq U$

Then U_n is open, $V_n \supseteq U_n \supseteq E$,

$|\mu|(U_n) \longrightarrow |\mu|(E)$

and $\lambda_1(U_n) < \delta$ for $n \in \mathbb{N}$.

Then

$$|\mu(U_n) - \mu(E)| = |\mu(U_n \setminus E)|$$

59

$$\leq |\mu|(G_n \setminus E) = |\mu|(G_n) - |\mu|(E) \to 0.$$

So

(1) $$\mu(G_n) \to \mu(E).$$

Since G_n is open,

$$G_n = \bigcup_{i \in \mathbb{N}} (a_i, b_i), \text{ where } (a_i, b_i) \text{ are disj. intervals.}$$

Moreover,

$$\sum_{i=1}^{\infty} (b_i - a_i) = \lambda_1(G_n) < \delta.$$

So by abs. cont. of F,

$$\varepsilon > \sum_{i=1}^{N} |F(b_i) - F(a_i)|$$

$$= \sum_{i=1}^{N} |\mu(a_i, b_i]|$$

$$= \sum_{i=1}^{N} |\mu(a_i, b_i)| \qquad \left(\begin{array}{l} F \text{ cont.;} \\ \text{so } \mu \text{ has} \\ \text{no atoms} \end{array}\right)$$

$$\geq \Big|\mu\Big(\underbrace{\bigcup_{i=1}^{N} (a_i, b_i)}_{G_n^N}\Big)\Big|$$

$G_n^N \nearrow G_n$ as $N \to \infty$, so $\mu(G_n^N) \to \mu(G_n)$

and $$|\mu(G_n)| = \lim_{N \to \infty} |\mu(G_n^N)| \leq \varepsilon.$$

Hence (1) implies that $|\mu(E)| \leq \varepsilon$.
Since $\varepsilon > 0$ was arb. $\mu(E) = 0$. $\square$