(31) Then $B \subseteq 5\hat{B}$ by beginning of proof.
Hence $\hat{\mathcal{B}} \subseteq \mathcal{B}$ disjointed family and
$$\bigcup_{B \in \mathcal{B}} B \subseteq \bigcup_{\hat{B} \in \hat{\mathcal{B}}} 5\hat{B}$$ as desired. $\square$

The Hardy-Littlewood maximal functions

We consider $\mathbb{R}^n$ equipped with Lebesgue measure λ_n.
We write $L^p = L^p(\lambda_n)$ or $L^p(\mathbb{R}^n) := L^p(\lambda_n)$.
$f \in L^1_{loc}$ (f is locally integrable)
if $f: \mathbb{R}^n \to \mathbb{C}$ is (Lebesgue) measurable
and $\int_B |f| < \infty$ for all balls $B \subseteq \mathbb{R}^n$

equiv. $\int_K |f| < \infty$ for all $K \subseteq \mathbb{R}^n$ compact.

If $f \in L^1_{loc}$, we define

$$(Hf)(x) := \sup_{r>0} \fint_{B(x,r)} |f| \, d\lambda_n \quad , x \in \mathbb{R}^n$$

"centered Hardy-Littlewood max. function"

$$(Mf)(x) := \sup_{B \ni x} \fint_B |f| \, d\lambda_n \quad , x \in \mathbb{R}^n$$

"Supremum taken over all open balls B containing x"

"uncentered Hardy-Littlewood max function"

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Lem. i) if $f \in L^1_{loc}$, then
$Hf : \mathbb{R}^n \longrightarrow [0,\infty]$ and $Mf : \mathbb{R}^n \longrightarrow [0,\infty]$
are measurable and
$Hf \leq Mf$.
ii) The maximal operators M and H are sublinear, i.e.,
$M(f+g) \leq Mf + Mg$ for $f, g \in L^1_{loc}$,
$M(\alpha f) = |\alpha| \, Mf$ for $f \in L^1_{loc}$, $\alpha \in \mathbb{C}$.
Same for H.

Proof: i) $Hf \leq Mf$ follows from definitions. (sup for $Mf(x)$ taken over larger collections of balls).
Measurability of Hf proved in Folland, p. 96.
Measurability of Mf:
it suffices to show that $\{Mf > \alpha\}$ is open for each $\alpha \in \mathbb{R}$.
If $x \in \{Mf > \alpha\}$, then there ex. an open ball B with $x \in B$ and

$$⨍_B |f| \, d\lambda_n > \alpha.$$

Then $Mf(y) > \alpha$ for all $y \in B$ and so $B \subseteq \{Mf > \alpha\}$.

Rem. i) We'll see that for $f \in L^1_{loc}$,
$|f| \leq Hf \leq Mf$ a.e.,
but Mf is "typically" not much larger than $|f|$.

(33) In particular, we'll see that if $f \in L^p$, $1 < p \leq \infty$, then
$\|Mf\|_p \leq C(n,p) \|f\|_p$

L^p-boundedness of max. function.
$1 < p \leq \infty$.

2) If $f \in L^p$, $1 \leq p < \infty$, then
$\lambda_n\{|f| > \alpha\} \leq \frac{C}{\alpha^p}$ for $\alpha > 0$,

where C is a constant ind. of α
"weak-type L^p-estimate"

Indeed:
$\alpha^p \cdot \lambda_n\{|f| > \alpha\} \leq \int_{\{|f|>\alpha\}} |f|^p$

$\leq \int |f|^p = \|f\|_p^p =: C$.

In general, $Mf \notin L^1$ if $f \in L^1$, but Mf still satisfies a "weak-type L^1-estimate".

Thm. (Weak-type L^1-estimate for Hardy-Littlewood max. function)
There ex. a constant $C = C(n) = 5^n$ s.t.
$\lambda_n\{Mf > \alpha\} \leq \frac{C}{\alpha} \|f\|_1$, $\alpha > 0$,
for each $f \in L^1(\mathbb{R}^n)$.

Proof: Let $f \in L^1$ and $\alpha > 0$ be arb.

(34) If $x \in A := \{Mf > \alpha\}$, then we can find an open ball $B_x \subseteq \mathbb{R}^n$ with $x \in B_x$ and

$$(Mf)(x) \geq \fint_{B_x} |f| \, d\lambda_n > \alpha .$$

Let $\mathcal{B} := \{B_x : x \in A\}$ be the collection of these balls.
Note that if $B = B(a,r) \in \mathcal{B}$, then

$\alpha < \fint_B |f|$ equiv.

(1) $$\lambda_n(B) = c_n r^n < \frac{1}{\alpha} \int_B |f| \leq \frac{1}{\alpha} \|f\|_1 ;$$

so

$r \leq$ Const. ind. of B.

Hence the balls in $\mathcal{B}$ have unif. bdd. radius.
By the 5B-covering lemma there ex. a disjointed family $\tilde{\mathcal{B}} \subseteq \mathcal{B}$ s.t.

$$A \subseteq \bigcup_{B \in \mathcal{B}} B \subseteq \bigcup_{\hat{B} \in \tilde{\mathcal{B}}} 5\hat{B} .$$

Note that $\tilde{\mathcal{B}}$ is countable, say $\tilde{\mathcal{B}} = \{B_n : n \in \mathbb{N}\}$

Then

$$\begin{aligned}
\lambda_n(A) &\leq \lambda_n\Big(\bigcup_{\hat{B} \in \tilde{\mathcal{B}}} 5\hat{B}\Big) \\
&\underset{\text{count. sub. add.}}{\leq} \sum_{n=1}^{\infty} \lambda_n(5B_n) \\
&= 5^n \sum_{n=1}^{\infty} \lambda_n(B_n) \\
&\underset{(1)}{\leq} 5^n \sum_{n=1}^{\infty} \frac{1}{\alpha} \int_{B_n} |f|
\end{aligned}$$

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$$= \frac{5^n}{\alpha} \int_{\bigcup_{n\in\mathbb{N}} B_n} |f| \le \frac{5^n}{\alpha} \|f\|_1$$

B_n's are disjoint

as desired. $\square$

Thm. Let $n \in \mathbb{N}$ and $1 < p \le \infty$. Then there ex. a constant

$$C = C(p,n) = 2 \cdot \left(\frac{p \cdot 5^n}{p-1}\right)^{1/p}$$

s.t.

$$\|Mf\|_p \le C \|f\|_p \text{ for all } f \in L^p(\mathbb{R}^n).$$

"L^p-boundedness of Hardy-Littlewood max. function"

Proof: The L^∞-case is trivial: if $f \in L^\infty$, then

$$\fint_B |f| \le \fint_B \|f\|_\infty = \|f\|_\infty \text{ and}$$

so $0 \le Mf \le \|f\|_\infty$, i.e.,

$\|Mf\|_\infty \le \|f\|_\infty$.

So M satisfies a weak-type L^1-estimate and is bdd. in L^∞.

The L^p-boundedness is derived from an "interpolation technique" (see Folland, Sect. 6.5, in particular Marcinkiewicz's Interpolation Thm.)

Basic idea: Split the given function $f \in L^p$, $1 < p < \infty$, into "small part" g and "large part" h and use diff. estimates for g and h.

(36)

Let $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$ arb.

Then

$$\|Mf\|_p^p = \int |Mf|^p \, d\lambda_n$$

$$= p \int_0^\infty \alpha^{p-1} \lambda_n\{|Mf| > \alpha\} \, d\alpha$$

(245a, HW 8, Prob. 3)

To show that $\|Mf\|_p < \infty$, we need good estimates for $\lambda(\alpha) := \lambda_n\{|Mf| > \alpha\}$.

Fix $\alpha > 0$.

Define

$$g = f \cdot \chi_{\{|f| \leq \frac{\alpha}{2}\}} \quad \text{"small part"}$$

$$h = f \cdot \chi_{\{|f| > \frac{\alpha}{2}\}} \quad \text{"large part"}$$

Then $f = g + h$, $\|g\|_\infty \leq \frac{\alpha}{2}$ and

$$\|h\|_1 = \int_{\{|f| > \frac{\alpha}{2}\}} |f| \leq \left(\frac{2}{\alpha}\right)^{p-1} \int_{\{|f| > \frac{\alpha}{2}\}} |f|^p < \infty.$$

So

$$Mf \leq Mg + Mh \quad \text{(subadditivity)}$$

$$\leq \frac{\alpha}{2} + Mh \quad (L^\infty\text{-boundedness})$$

This implies that

$$\lambda(\alpha) = \{Mf > \alpha\}$$

$$\leq \{Mh > \tfrac{\alpha}{2}\}$$

$$\leq \frac{5^n}{\alpha/2} \|h\|_1 \quad \text{(weak-type } L^1\text{-estimate)}$$

$$= \frac{2 \cdot 5^n}{\alpha} \int_{\{|f| > \frac{\alpha}{2}\}} |f|$$

(37)

So

$$\|Mf\|_p^p = p \int_0^\infty \alpha^{p-1} \lambda(\alpha)\, d\alpha$$

$$\leq 2p\, 5^n \int_0^\infty \alpha^{p-2} \left(\int \chi_{\{|f| > \frac{\alpha}{2}\}} |f|\, d\lambda_n \right) d\alpha$$

$$= 2p\, 5^n \int_0^\infty \int_{\mathbb{R}^n} \alpha^{p-2} |f|\, \chi_{\{(\alpha,x) : |f|(x) > \frac{\alpha}{2}\}}\, d\lambda_n\, d\alpha$$

Fubini

$$= 2p\, 5^n \int_{\mathbb{R}^n} |f(x)| \int_0^\infty \alpha^{p-2} \chi_{\{(\alpha,x) : |f|(x) > \frac{\alpha}{2}\}}\, d\alpha\, d\lambda_n(x)$$

$$= 2p\, 5^n \int_{\mathbb{R}^n} |f(x)| \left(\int_0^{2|f(x)|} \alpha^{p-2} d\alpha \right) d\lambda_n(x)$$

$$= \frac{2p\, 5^n}{p-1} \int_{\mathbb{R}^n} |f(x)| \cdot 2^{p-1} |f(x)|^{p-1}\, d\lambda_n(x)$$

$$= \frac{p\, 2^p \cdot 5^n}{p-1} \int |f|^p = \frac{p \cdot 2^p \cdot 5^n}{p-1} \|f\|_p^p.$$

So $\|Mf\|_p \leq C(p,n) \|f\|_p$ with

$$C(p,n) = 2 \left(\frac{p \cdot 5^n}{p-1} \right)^{1/p} \quad \square$$

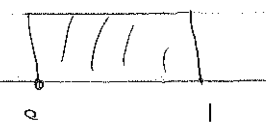
<u>Lebesgue points</u>

Let $f : \mathbb{R}^n \to \mathbb{C}$ be measurable.
Then $x \in \mathbb{R}^n$ is called a <u>Lebesgue point of f</u> if

$$\lim_{r \to 0^+} \fint_{B(x,r)} |f(y) - f(x)|\, d\lambda_n(y) = 0.$$

Note: if f is continuous, then <u>every</u> point is a Lebesgue point of f. (exercise!)

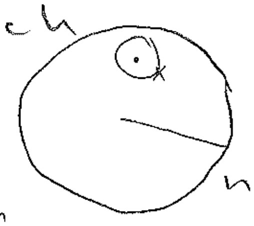
<u>Ex.</u> $f = \chi_{[0,1]}$. The the Lebesgue points of f are the points in $\mathbb{R}\setminus\{0,1\}$.



<u>Thm.</u> (Lebesgue Differentiation Theorem)
Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then almost every point (w.r.t. λ_n) is a Lebesgue point of f, i.e.,

$$\lim_{r\to 0^+} \fint |f(y)-f(x)|\,d\lambda_n(y) = 0 \quad \text{for a.e. } x\in\mathbb{R}^n.$$

<u>Proof:</u> Wlog $f \in L^1(\mathbb{R}^n)$
(note: the non-Lebesgue points of $f \in L^1_{loc}(\mathbb{R}^n)$ and $f\chi_{B(0,n)} \in L^1(\mathbb{R}^n)$ are the same in $B(0,n)$ for each $n\in\mathbb{N}$).



For $r>0$ define

$$T_r f(x) := \fint_{B(x,r)} |f - f(x)|, \quad x\in\mathbb{R}^n,$$

and

$$Tf(x) := \limsup_{r\to 0^+} T_r f(x)$$

<u>WTS</u> $Tf(x) = 0$ for a.e. $x\in\mathbb{R}^n$.

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Let $k \in \mathbb{N}$ be arb. Since $C_c(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$, there ex. $g \in C_c(\mathbb{R}^n)$ s.t. $\|f-g\|_1 < 1/k$.

Since g is continuous, $Tg \equiv 0$.

Let $h = f-g$. Then $f = h+g$,

$T_r f \leq T_r g + T_r h$, and so

(1) $\quad Tf \leq Th \quad$ (since $T_r g \to 0$)

Now

$$Th(x) = \fint_{B(x,r)} |h - h(x)|$$

$$\leq \fint_{B(x,r)} |h| + |h(x)|$$

$$\leq (Mh)(x) + h(x)$$

maximal function; so

(2) $\quad Th \leq Mh + |h|$. Let $\alpha > 0$ be arb.

Then by (1) and (2), $\{Tf > \alpha\} \subseteq$

$$\{Th > \alpha\} \subseteq \underbrace{\{Mh > \tfrac{\alpha}{2}\} \cup \{|h| > \tfrac{\alpha}{2}\}}_{A(k,\alpha)}$$

By weak-type L^1-estimate for Mh and h:

(40) $\lambda_n(A(k,\alpha))$

$$\leq \frac{5^n}{\alpha/2}\|h\|_1 + \frac{1}{\alpha/2}\|h\|_1$$

$$< \frac{1}{k}\ \frac{2(5^n+1)}{\alpha} \longrightarrow 0 \text{ as } k \to \infty$$

So $\lambda_n\left(\bigcap_k A(k,\alpha)\right) = 0$;

since

$\{Tf > \alpha\} \subseteq A(k,\alpha)$ for all $k \in \mathbb{N}$,

we conclude

$\{Tf > \alpha\} \subseteq \bigcap_k A(k,\alpha)$ and so

$\{Tf > \alpha\}$ is Lebesgue measurable and

$\lambda_n\{Tf > \alpha\} = 0$.

Since this is true for all $\alpha = \frac{1}{L}$, $L \in \mathbb{N}$,

we have $Tf = 0$ a.e. as desired. $\blacksquare$

Cor. If $f \in L^1_{loc}(\mathbb{R}^n)$, then

$$\lim_{r \to 0^+} ⨍_{B(x,r)} f = f(x) \text{ for a.e. } x \in \mathbb{R}^n.$$

Proof: Note that if x is a Lebesgue point of f, then

$$\limsup_{r\to 0^+}\left| ⨍_{B(x,r)} f - f(x)\right| \leq \limsup_{r \to 0^+} ⨍_{B(x,r)} |f - f(x)|$$

$$= 0. \quad \square$$

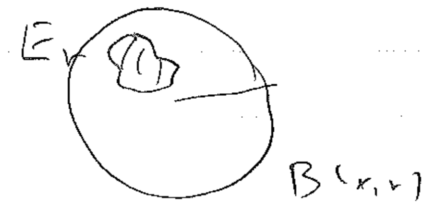
A family $\{E_r\}_{r>0}$ of measurable sets $E_r \subseteq \mathbb{R}^n$ <u>shrinks nicely</u> to $x \in \mathbb{R}^n$ if

i) $E_r \subseteq B(x,r)$ for $r>0$

ii) there ex. $\alpha > 0$ s.t.

$$\lambda_n(E_r) \geq \alpha\, \lambda_n(B(x,r)) \quad \text{for } r>0.$$

<u>Thm.</u> (Improved version of Lebesgue Diff. Thm.)



Let $f \in L^1_{loc}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ a Lebesgue point of f, and $\{E_r\}_{r>0}$ a family of measurable sets that shrinks nicely to x.

Then

$$\lim_{r\to 0^+} ⨍_{E_r} |f(y)-f(x)|\, d\lambda_n(y) = 0$$

$$\left(\lim_{r\to 0^+} ⨍_{E_r} f(y)\, d\lambda_n(y) = f(x) \right)$$

<u>Proof:</u>

$$⨍_{E_r} |f(y)-f(x)|\, d\lambda_n(y)$$

$$= \frac{1}{\lambda_n(E_r)} \int_{E_r} |f-f(x)| \leq \frac{1}{\lambda_n(E_r)} \int_{B(x,r)} |f-f(x)|$$

$$\leq \underbrace{\frac{\lambda_n(B(x,r))}{\lambda_n(E_r)}}_{\leq \frac{1}{\alpha}} ⨍_{B(x,r)} |f-f(x)| \longrightarrow 0 \quad \text{as } r \to 0^+.$$

$\square$

(42) Let $E \subseteq \mathbb{R}^n$ be measurable.
The (metric) density of E at $x \in \mathbb{R}^n$ is defined as

$$D_E(x) := \lim_{r \to 0^+} \frac{\lambda_n(E \cap B(x,r))}{\lambda_n(B(x,r))}$$

if the limit exists.
In this case, $0 \leq D_E(x) \leq 1$.
A point with $D_E(x) = 1$ is called a Lebesgue density point of E.

Thm Let $E \subseteq \mathbb{R}^n$ be meas.
Then $D_E(x)$ exists for a.e. $x \in \mathbb{R}^n$.
Actually,

$$D_E(x) = \begin{cases} 0 & \text{for a.e. } x \in \mathbb{R}^n \setminus E \\ 1 & \text{a.e. } x \in E. \end{cases}$$

In particular, a.e. $x \in E$ is a Lebesgue density point of E.

Proof: Let $f = \chi_E \in L^1_{loc}$. Then for each Lebesgue point x of f (i.e., for a.e. x) we have

$$\frac{\lambda_n(E \cap B(x,r))}{\lambda_n(B(x,r))} = \fint_{B(x,r)} \chi_E$$

$$\longrightarrow \chi_E(x) \in \{0,1\} \text{ as } r \to 0^+. \quad \square$$