Here $\nu^+ \perp \nu^-$.

(13) By assumption ν and so $|\nu| = \nu^+, \nu^-$ are σ-finite. So we can apply Case II to the pairs ν^+, μ and ν^-, μ.

This gives Lebesgue decomp.

$\nu^+ = \lambda^+ + \rho^+$

$\nu^- = \lambda^- + \rho^-$, where $\lambda^+, \lambda^- \perp \mu$,

$d\rho^+ = f^+ d\mu$, $d\rho^- = f^- d\mu$,

$f^+, f^- : X \to [0, \infty)$, meas.

Since $\nu^+ \perp \nu^-$, we have $\lambda^+ \perp \lambda^-$ and $\rho^+ \perp \rho^-$ ($\to f^+ \cdot f^- = 0$ μ-a.e.)

Hence

$\lambda = \lambda^+ - \lambda^-$ and $\rho = \rho^+ - \rho^- = (f^+ - f^-) d\mu$

are well-def. signed measures and

$\lambda \perp \mu$ (exercise!), $\rho \ll \mu$, $\nu = \lambda + \rho$,

λ, ρ σ-finite.

Proof of the existence of Lebesgue decomp. is complete.

<u>Uniqueness</u> (Outline)

<u>Case I</u>: ν finite signed meas.,

μ finite signed meas.

Suppose we have two Lebesgue decomp. of ν w.r.t. μ:

$\nu = \lambda_1 + \rho_1 = \lambda_2 + \rho_2$, where

$\lambda_1, \lambda_2 \perp \mu$, $d\rho_1 = f_1 d\mu$, $d\rho_2 = f_2 d\mu$.

Then $\lambda_1, \lambda_2, \rho_1, \rho_2$ are also finite, (exercise!)

and

$\lambda = \lambda_1 - \lambda_2 = \rho_2 - \rho_1$.

So $\lambda = \lambda_1 - \lambda_2 \perp \mu$, but $d\lambda = d(\rho_2 - \rho_1) = (f_2 - f_1) d\mu$

and $\lambda \ll \mu$. Hence $\lambda = 0$ which implies

(14) $\lambda_1 = \lambda_2$, $g_2 = g_1$, and $f_2 = f_1$ μ-a.e.

<u>Case II</u> (general case):

ν σ-finite signed meas.,

μ σ-finite pos. meas.

Partition X into countably many meas. sets on which ν and μ are finite and apply considerations of Case I on each set of partition. $\square$

<u>Complex measures</u>

$(X, \mathcal{A})$ meas. space. A <u>complex measure</u> ν on $(X, \mathcal{A})$ is a function

$\nu : \mathcal{A} \to \mathbb{C}$ s.t.

(i) $\nu(\emptyset) = 0$.

(ii) ν is countably additive, i.e., if $A_n \in \mathcal{A}$, $n \in \mathbb{N}$, are pairwise disj., then

$$\nu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n).$$

(Convergence of series is part of the statement).

If ν is a complex measure, then

$\nu = \nu_r + i\nu_i$, where ν_r (real part of ν) and ν_i (imaginary part of ν) are signed measures given by

$$\nu_r(A) := \operatorname{Re} \nu(A) \qquad \text{for } A \in \mathcal{A}.$$
$$\nu_i(A) := \operatorname{Im} \nu(A)$$

Since ν only takes finite values (i.e., values in $\mathbb{C}$), ν_r and ν_i take values in $\mathbb{R}$ and are hence finite measures.

(15) $\nu_r = \nu_r^+ - \nu_r^-$ Jordan decomp.
$\nu_r^+(X), \nu_r^-(X) < \infty$; so
$|\nu_r|(X) = \nu_r^+(X) + \nu_r^-(X) < \infty$.
Similarly, $|\nu_i|(X) < \infty$.

Radon-Nikodym Thm. for complex meas.
$(X, \mathcal{A})$ meas. space, μ σ-finite pos. meas., ν complex meas. on $(X, \mathcal{A})$.
Then $\nu = \lambda + \rho$, where
$\lambda \perp \mu$ (i.e., $\lambda_r, \lambda_i \perp \mu$) and
$\rho \ll \mu$ (i.e., $\rho_r, \rho_i \ll \mu$).
Moreover, there ex. $f \in L^1(\mu)$ s.t.
$d\rho = f\, d\mu$.
The meas. λ, ρ are unique, and f is unique μ-a.e.

Proof: Apply Radon-Nikodym Thm. to the finite signed meas. ν_r and ν_i. $\square$

Let ν be a complex meas. on $(X, \mathcal{A})$.
Then we define the total variation $|\nu|$ of ν as the set function $|\nu| : \mathcal{A} \to [0, \infty]$ given by

$$|\nu|(A) := \sup \left\{ \sum_{n=1}^{\infty} |\nu(A_n)| : A_n \in \mathcal{A} \text{ pairw. disj.}, \bigcup_{n \in \mathbb{N}} A_n = A \right\}$$

Note:
For a signed meas. ν this agrees with the total variation $|\nu| = \nu^+ + \nu^-$ (HW 2 Prob. 1)
defined by using Jordan decomp. $\nu = \nu^+ - \nu^-$

(16) <u>Prop.</u> If ν is a complex measure on $(X, \mathcal{A})$, then $|\nu|$ is a finite pos. meas. on $(X, \mathcal{A})$.

<u>Proof:</u> We can find a finite pos. meas. μ on $(X, \mathcal{A})$ s.t. $\nu \ll \mu$ (take $\mu = |\nu_r| + |\nu_i|$ for example).
Then by Radon-Nikodym there ex. $f \in L^1(\mu)$ s.t. $d\nu = f\, d\mu$.
<u>Claim</u>
$$|\nu|(A) = \int_A |f|\, d\mu \quad \text{for } A \in \mathcal{A},$$
(i.e. $d|\nu| = |f|\, d\mu$ and so $|\nu|$ is a finite pos. meas.).

"$\leq$": Let $A \in \mathcal{A}$ be arb. and $A = \bigcup_{n\in\mathbb{N}} A_n$ be a meas. partition of A.
Then
$$\sum_{n=1}^{\infty} |\nu(A_n)| = \sum_{n=1}^{\infty} \left| \int_{A_n} f\, d\mu \right| \leq \sum_{n=1}^{\infty} \int_{A_n} |f|\, d\mu$$
$$\overset{MCT}{=} \int_A |f|\, d\mu.$$
So $|\nu|(A) \leq \int_A |f|\, d\mu$.

"$\geq$": Let $A \in \mathcal{A}$ and $\varepsilon > 0$ be arb.
Since simple functions are dense in $L^1(\mu)$ there ex.
$$s = \sum_{i=1}^{k} c_k \chi_{B_k}, \quad c_1, \dots, c_k \in \mathbb{C}, \; B_1, \dots, B_k \in \mathcal{A} \text{ disj.}$$
s.t. $\int |f-s|\, d\mu < \varepsilon$

(17) Let $B = B_1 \cup \dots \cup B_k$,
$A_0 = A \cap B^c$, $A_i = A \cap B_i$ for $i = 1, \dots, k$.

Then
$A_0 \cup \dots \cup A_k$ finite meas. partition of A.

$$\sum_{i=0}^{k} |\nu(A_i)| = \sum_{i=0}^{k} \left| \int_{A_i} (f-s)\,d\mu + \int_{A_i} s\,d\mu \right|$$

$$\geq \sum_{i=0}^{k} \int_{A_i} |s|\,d\mu - \underbrace{\int |f-s|\,d\mu}_{<\varepsilon}$$

s const. on A_i

$$\geq \sum_{i=0}^{k} \int_{A_i} |f|\,d\mu - 2\varepsilon = \int_A |f|\,d\mu - 2\varepsilon.$$

Since $\varepsilon > 0$ arb., we conclude
$$|\nu|(A) \geq \int_A |f|\,d\mu. \quad \square$$

Review of basic functional analysis over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

X, Y normed vector spaces, $T : X \to Y$ linear map.
Then T is called <u>bounded</u> if there ex. $C \geq 0$ s.t.
$$\|T(x)\| \leq C \|x\| \quad \text{for all } x \in X.$$

<u>Thm.</u> X, Y normed vector spaces, $T : X \to Y$ linear. TFAE:

i) T is continuous,
ii) T is continuous at $0 \in X$,
iii) T is bounded.

(18) Proof: i) $\to$ ii) obvious.

ii) $\to$ iii): Note that $T(0)=0$. So if T is cont. at 0, then there ex. $\delta > 0$ s.t.

$$\|x\| \le \delta \longrightarrow \|T(x)\| \le 1 \quad \text{for } x \in X.$$

If $u \in X$, $u \ne 0$, is arb., define $x = \frac{\delta}{\|u\|} u$. Then $\|x\| = \delta$; so

$$1 \ge \|T(x)\| = \left\|\frac{\delta}{\|u\|} T(u)\right\| \quad \text{equiv.}$$

$$\|T(u)\| \le \frac{1}{\delta} \|u\|.$$

So T is bdd.

iii) $\to$ i) If T is bdd. then there ex. $C \ge 0$ s.t.

$$\|T(x)\| \le C\|x\| \quad \text{for all } x \in X.$$

Then

$$\|T(x) - T(x_0)\| = \|T(x-x_0)\| \le C\|x-x_0\|.$$

So T is Lipschitz and hence continuous. $\square$

If $T: X \to Y$ is a bdd. linear map ("bounded operator") we define

$$\|T\| := \sup\left\{\frac{\|T(x)\|}{\|x\|} : x \in X \setminus \{0\}\right\}$$

$$= \inf\{C \ge 0 : \|T(x)\| \le C\|x\| \text{ for all } x \in X\}$$

$$= \sup\{\|T(x)\| : x \in X, \|x\| \le 1\}$$

$\|T\|$ is the <u>operator norm</u> of T.

(19) $L(X,Y) = \{T : X \to Y$ bdd. lin. operator$\}$

$L(X,Y)$ equipped with operator norm is a normed vector space over $\mathbb{F}$

$(\alpha S + \beta T)(x) = \alpha S(x) + \beta T(x)$, $x \in \mathbb{R}$

$\|T\| = 0$ iff $T = 0$

$\|\alpha T\| = |\alpha| \cdot \|T\| \qquad \alpha \in \mathbb{F}$

$\|S + T\| \leq \|S\| + \|T\|$

The operator norm is submultiplicative: if $T \in L(X,Y)$, $S \in L(Y,Z)$, then $S \circ T \in L(X,Z)$, and

$\|S \circ T\| \leq \|S\| \cdot \|T\|$

<u>Prop.</u>: Let X and Y be normed spaces over $\mathbb{F}$. If Y is complete, then $L(X,Y)$ is complete (equiv. if Y is a Banach space, then $L(X,Y)$ is a Banach space).

<u>Proof</u> (Outline): Let $\{T_n\}$ be a Cauchy seq. in $L(X,Y)$. Then for each $x \in X$, $\{T_n(x)\}$ is a Cauchy seq. in Y (given that Y is complete).
Define
$$T(x) := \lim_{n \to \infty} T_n(x).$$

Then one can show $T \in L(X,Y)$ and $T_n \to T$. $\square$

(20) Bounded linear functionals and dual spaces

X normed vector spaces over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.
A bounded linear map $f: X \to \mathbb{F}$ is called a <u>bounded linear functional</u>.
So f is a bdd. lin. funct. if $f \in L(X, \mathbb{F})$.
The space $L(X, \mathbb{F})$ of all bdd. lin. functionals is called the <u>dual space</u> of X, denoted X^*.
So

$$X^* = \{ f: X \to \mathbb{F} : f \text{ bdd. lin. funct.} \}$$

$$\|f\| = \sup \{ |f(x)| : \|x\| \le 1 \}$$

norm on X^* (operator norm).
If X is a Banach space, then X^* is also a Banach space (by Prop.; note that $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ with Eucl. norm is complete).

<u>Thm.</u> (Hahn-Banach Thm; simple version)
Let X be a normed vector space, $M \subseteq X$ a (linear) subspace, and $f: M \to \mathbb{F}$ a bdd. linear functional. Then there ex. a bdd. linear functional $F: X \to \mathbb{F}$ s.t. $F|M = f$ and $\|F\| = \|f\|$.

<u>Proof:</u> Folland, p. 157 - 158

(21) <u>Cor.</u> For each $x \in X$ we have

$$\|x\| = \sup\{|f(x)| : f \in X^*, \|f\| \le 1\}$$

and this sup. is attained as a max.

<u>Proof:</u> "$\ge$": if $f \in X^*$, $\|f\| \le 1$, then
$|f(x)| \le \|f\| \cdot \|x\| \le \|x\|$.
"$\le$": Let $M = \{\lambda x : \lambda \in \mathbb{F}\}$ and define
$g : M \to \mathbb{F}$ by $g(\lambda x) := \lambda \|x\|$, $\lambda \in \mathbb{F}$.
wlog $x \neq 0$.
Then $|g(\lambda x)| = |\lambda| \cdot \|x\| = \|\lambda x\|$
equiv. $|g(u)| = \|u\|$ for $u \in M$,
and so $\|g\| = 1$. By Hahn-Banach
there ex. $f \in X^*$ with $f|M = g$ and $\|f\| = 1$.
Then $|f(x)| = |g(1 \cdot x)| = \|x\|$. $\square$

<u>Note:</u> $\|f\| = \sup\{|f(x)| : x \in X, \|x\| \le 1\}$
by def. of $\|f\|$. In general, sup. is <u>not</u>
attained as max.

<u>The dual of L^p</u>

$(X, \mathcal{A}, \mu)$ measure space, $1 \le p \le \infty$.
$X = L^p(\mu)$
$= \{[f] : f : X \to \mathbb{C}$ meas. and
$\|f\|_p := \left(\int |f|^p d\mu\right)^{1/p} < \infty\}$.
$f \sim g$ iff $f = g$ μ-a.e. $\quad \|f\|_\infty := \operatorname{ess\,sup}_{x \in X} |f(x)|$.

(22) Let q be the conjugate exp. of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$ equiv. $q = \frac{p}{p-1}$ ($= \infty$ if $p = 1$).

If $g \in L^q(\mu)$, we define a lin. funct. by

$\Phi_g : L^p(\mu) \longrightarrow \mathbb{C}$ by setting

$\Phi_g(f) = \int f g \, d\mu$ (well-def. because only dep. on $[f]$ and $[g]$).

Φ_g is bdd. lin. function, because by Hölder's ineq.

$|\Phi_g(f)| = \left| \int f g \, d\mu \right| \leq \|f\|_p \cdot \|g\|_q$,

and so

$\|\Phi_g\| \leq \|g\|_q$.

<u>Thm.</u> (L^p-L^q duality)

Let $1 \leq p < \infty$ and $q \in (1, \infty]$ be the conjugate exp., $(X, \mathcal{A}, \mu)$ a σ-finite measure space.

Then for each bdd. lin. funct. $\Phi : L^p(\mu) \longrightarrow \mathbb{C}$ there ex. $g \in L^q(\mu)$ s.t.

$\Phi = \Phi_g$.

Moreover, g is unique μ-a.e. (i.e., if $\Phi = \Phi_g = \Phi_{\tilde{g}}$, then $g = \tilde{g}$ μ-a.e.)

and $\|\Phi\| = \|g\|_q$.

(23) Rem. 1) If we define a map
$T: L^q(\mu) \to L^p(\mu)^*$ by
$T(g) := \Phi_g$ for $g \in L^q(\mu)$,
then
i) T is well-def. (Φ_g only depends on $[g]$)
ii) T is linear,
iii) T is surjective by Thm.
iv) $\|T(g)\| = \|\Phi_g\| = \|g\|_q$; so
T is norm-preserving and so
injective ($\|T(g)\| = 0 \to \|g\|_q = 0 \to [g] = 0$).

So T is an isomorphism between
$L^q(\mu)$ and $L^p(\mu)^*$; under this isomorph.:
$L^p(\mu)^* = L^q(\mu)$ for $1 \le p < \infty$.

2) In general: $L^\infty(\mu)^* \neq L^1(\mu)$
(examples in discussion session).

<u>Proof of Thm.</u> (Outline) $1 \le p < \infty$, $1 < q \le \infty$.

<u>Uniqueness</u>: Suppose $\Phi_g = \Phi = \Phi_{\tilde g}$ for
$g, \tilde g \in L^q = L^q(\mu)$.
Then
$$\int f g \, d\mu = \int f \tilde g \, d\mu \quad \text{for all } f \in L^p$$
equiv.
$$\int f (g - \tilde g) \, d\mu = 0 \quad \text{for all } f \in L^p$$
$$\to \int \chi_E (g - \tilde g) \, d\mu = 0 \quad \text{whenever } E \in \mathcal{A}, \ \mu(E) < \infty$$
(exercise!)
$\to g - \tilde g = 0$ μ-a.e. equiv. $g = \tilde g$ μ-a.e.

(24) <u>Existence</u>: $\Phi \in (L^p)^*$ arb.

<u>Case I</u>: μ is a finite meas, i.e., $\mu(X) < \infty$.

For $A \in \mathcal{A}$ we have

$$\int \chi_A^p \, d\mu = \int \chi_A \, d\mu = \mu(A) < \infty,$$

so $\chi_A \in L^p$.

Define

$$\nu(A) := \Phi(\chi_A) \in \mathbb{C}.$$

Then ν is a complex measure on $(X, \mathcal{A})$:

i) $\nu(\emptyset) = \Phi(\chi_\emptyset) = \Phi(0) = 0$

ii) ν is countably additive:

Let $A_n \in \mathcal{A}$ for $n \in \mathbb{N}$ be pairwise disj. and $A = \bigcup_{n \in \mathbb{N}} A_n$.

Define $B_n = A_1 \cup \dots \cup A_n$.

Then $\chi_{B_n} \xrightarrow{p} \chi_A$ in L^p; indeed:

$$\|\chi_{B_n} - \chi_A\|_p^p = \int |\chi_{B_n} - \chi_A|^p \, d\mu$$

$$= \int |\chi_{A \setminus B_n}|^p \, d\mu = \int \chi_{A \setminus B_n} \, d\mu = \mu(A \setminus B_n)$$

$\longrightarrow 0$ as $n \to \infty$, because $A \setminus B_n \searrow \emptyset$.

By continuity of Φ we have

$$\nu(A) = \Phi(\chi_A) = \lim_{n \to \infty} \Phi(\chi_{B_n})$$

$$= \lim_{n \to \infty} \Phi(\chi_{A_1} + \dots + \chi_{A_n})$$

$$= \lim_{n \to \infty} \Phi(\chi_{A_1}) + \dots + \Phi(\chi_{A_n}) = \lim_{n \to \infty} \nu(A_1) + \dots + \nu(A_n)$$

(25) $= \sum_{n=1}^{\infty} \nu(A_n)$.

We have $\nu \ll \mu$; indeed, if $A \in \mathcal{A}$ and $\mu(A) = 0$, then $\chi_A = 0$ μ-a.e., i.e., $[\chi_A] = [0]$ and so

$$\nu(A) = \Phi(\chi_A) = \Phi([\chi_A]) = \Phi([0]) = 0.$$

So by Radon-Nikodym there ex. $g \in L^1$ s.t.

$$\Phi(\chi_A) = \nu(A) = \int_A g \, d\mu$$

(1) $$= \int \chi_A g \, d\mu \quad \text{for all } A \in \mathcal{A}.$$

By linearity this extends to all simple functions, i.e.,

(2) $\Phi(s) = \int s g \, d\mu$ for all simple function s on X.

Let $h \in L^\infty \subseteq L^p$. Since simple functions are dense in L^∞, there ex. a seq. s_n of simple functions s.t.

$$s_n \to h \text{ in } L^\infty \text{ equiv. } \|s_n - h\|_\infty \to 0$$

Hence

$$\|s_n - h\|_p^p = \int |s_n - h|^p \, d\mu = \int \|s_n - h\|_\infty^p \, d\mu$$
$$= \mu(X) \|s_n - h\|_\infty^p \to 0.$$

So $s_n \to h$ in L^p and so

$$\Phi(h) = \lim_{n\to\infty} \Phi(s_n) = \lim_{n\to\infty} \int s_n g \, d\mu$$
$$= \lim_{n\to\infty} \int h g \, d\mu$$

(26) Note:

$$\limsup_{n\to\infty} \left| \int s_n g \, d\mu - \int hg \, d\mu \right| \leq$$

$$\limsup_{n\to\infty} \int |s_n - h| \, |g| \, d\mu \leq$$

$$\limsup_{n\to\infty} \|s_n - h\|_\infty \int |g| \, d\mu = 0.$$

So

(3) $\Phi(h) = \int hg \, d\mu$ for all $h \in L^\infty$.

<u>Claim</u> $g \in L^q$.

<u>Subcase 1:</u> $1 < p < \infty$. Then $1 < q < \infty$.

There ex. a meas. function α on X s.t. $|\alpha| = 1$ and $g = \alpha |g|$ (exercise!)

Let $E_n = \{|g| \leq n\} \in \mathcal{A}$ and

$$f_n := \bar{\alpha} |g|^{q-1} \cdot \chi_{E_n} \in L^\infty \subseteq L^p$$

So by (3): $\bar{\alpha}\alpha = 1$

$$\int_{E_n} |g|^q \, d\mu = \left| \int f_n g \, d\mu \right| = |\Phi(f_n)|$$

$$\leq \|\Phi\| \, \|f_n\|_p$$

$$= \|\Phi\| \left(\int_{E_n} |g|^{(q-1)\cdot p} \, d\mu \right)^{1/p} \qquad p = \frac{q}{q-1}, \quad p\cdot(q-1) = q$$

$$= \|\Phi\| \cdot \underbrace{\left(\int_{E_n} |g|^q \, d\mu \right)}_{<\infty}^{1/p}$$

$$\Rightarrow \left(\int_{E_n} |g|^q \right)^{1 - 1/p} = \left(\int_{E_n} |g|^q \right)^{1/q} \leq \|\Phi\|$$

(27) So by MCT ($\chi_{E_n} \nearrow 1 = \chi_X$)

$$\|g\|_q = \lim_{n\to\infty} \left(\int_{E_n} |g|^q\right)^{1/q} \le \|\Phi\| < \infty.$$

So indeed, $g \in L^q$ and $\|g\|_q \le \|\Phi\|$.

In particular,

$$f \in L^p \longmapsto \Phi_g(f) := \int f g \, d\mu$$

defines a bdd. linear functional on L^p.

By (2)

$$\Phi(s) = \Phi_g(s) \quad \text{for all simple functions } s,$$

and simple functions are dense in L^p, we have $\Phi = \Phi_g$;

moreover

$$\|g\|_q \le \|\Phi\| \le \|g\|_q; \text{ so } \|\Phi\| = \|g\|.$$

<u>Subcase 2:</u> $p = 1$, $q = \infty$.

Based on (3) one shows $\|g\|_\infty \le \|\Phi\|$ and completes the proof as in Subcase 2.

The existence proof in Case 1 is complete.

<u>Case 2:</u> (general case) μ is σ-finite.

One uses the usual idea:

There ex. meas. sets $E_n \nearrow X$ s.t. $\mu(E_n) < \infty$ for $n \in \mathbb{N}$.

One considers the restriction

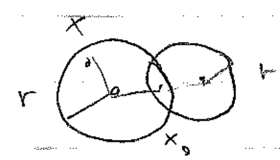
$$\Phi_n = \Phi \mid L^p(\mu|E_n) = \{ f \in L^p(\mu) : f = 0 \text{ on } E_n^c \ \mu\text{-a.e.}\}$$

and uses Case 1 (details in homework!). $\square$

(28) Notation: $B = B(a,r) = \{x \in X : d(a,x) < r\}$
open ball centered at $a \in X$ with radius $r > 0$
in a metric space (X,d).
$\lambda > 0$, $\lambda B := B(a, \lambda r)$.

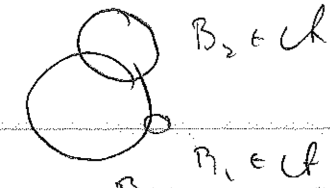
Thm. (Basic 5B - covering lemma)
Let (X,d) be a metric space and
$\mathcal{B}$ be a collection of balls in X
with uniformly bounded radius.
Then there ex. a collection $\tilde{\mathcal{B}} \subseteq \mathcal{B}$
of disjoint ball ("a disjointed family")
s.t.
$$\bigcup_{B \in \mathcal{B}} B \subseteq \bigcup_{B' \in \tilde{\mathcal{B}}} 5B'.$$

Proof: If $B = B(a,r)$ and $B' = B(a',r')$
are balls in X with $B \cap B' \neq \emptyset$ and
$r' \geq \frac{1}{2} r$, then $B \subseteq 5B'$.
Indeed, ex. $x_0 \in B \cap B'$.



So, if $x \in B$, then
$d(a', x') \leq d(a', x_0) + d(x_0, a) + d(a, x)$
$< r' + 2r \leq 5r'$; hence $x \in 5B'$
and $B \subseteq 5B'$.
We now consider the collection
$M = \{ \mathcal{A} \subseteq \mathcal{B} :$ $\mathcal{A}$ disjointed family and
if $B \in \mathcal{B}$ meets a ball in $\mathcal{A}$,
then it meets one in $\mathcal{A}$ whose radius
is at least half the radius of B $\}$

(29) M is partially ordered by inclusion: $\mathcal{A}_1 \subseteq \mathcal{A}_2$.



Note that:

i) $M \neq \emptyset$. Indeed we can find a ball $B \in \mathcal{B}$ s.t.

radius of $B \geq \frac{1}{2}$ sup of all radii of balls in $\mathcal{B}$ $(<\infty)$.

Then $\mathcal{A} = \{B\} \in M$.

ii) every chain C in M (i.e., every totally ordered subset of M) has an upper bound $\tilde{\mathcal{A}}$;

take
$$\tilde{\mathcal{A}} := \bigcup \{\mathcal{A} : \mathcal{A} \in C\}$$

Indeed, then

1) $\tilde{\mathcal{A}}$ consists of disjoint balls: if $B_1, B_2 \in \tilde{\mathcal{A}}$, $B_1 \neq B_2$, there ex. $\mathcal{A}_1, \mathcal{A}_2 \in C$ s.t. $B_1 \in \mathcal{A}_1$, $B_2 \in \mathcal{A}_2$. Since C is a chain $\mathcal{A}_1 \subseteq \mathcal{A}_2$ or $\mathcal{A}_2 \subseteq \mathcal{A}_1$, wlog $\mathcal{A}_1 \subseteq \mathcal{A}_2$. Then $B_1, B_2 \in \mathcal{A}_2$ and so $B_1 \cap B_2 = \emptyset$.

2) if $B \in \mathcal{B}$ and B meets a ball $\hat{B} \in \tilde{\mathcal{A}}$, then $\hat{B} \in \mathcal{A}$, where $\mathcal{A} \in C$. Then B meets a ball $B' \in \mathcal{A} \subseteq \tilde{\mathcal{A}}$ with

radius $B' \geq \frac{1}{2}$ radius B.

So $\tilde{\mathcal{A}} \in C$ and is an upper bound for C ($\mathcal{A} \subseteq \tilde{\mathcal{A}}$ for all $\mathcal{A} \in C$).

30

By Zorn's lemma M has a maximal element $\tilde{\mathcal{B}} \in C$

Then $\tilde{\mathcal{B}}$:

i) is a disjointed family

ii) every ball $B \in \mathcal{B}$ meets a ball in $\tilde{\mathcal{B}}$.

otherwise, we can find $B' \in \mathcal{B}$ s.t. $B' \cap \hat{B} = \emptyset$ for all $\hat{B} \in \tilde{\mathcal{B}}$ and radius $B' \geq \frac{1}{2}$ sup of radii of balls disj. from all balls in $\tilde{\mathcal{B}}$.

Then $\mathcal{B}' := \tilde{\mathcal{B}} \cup \{B'\}$ disj. family, $\tilde{\mathcal{B}} \subsetneq \mathcal{B}'$.

If B meets a ball in $\mathcal{B}'$, then

1) B meets a ball in $\tilde{\mathcal{B}}$ and hence one in $\tilde{\mathcal{B}} \in \mathcal{B}'$ with radius $\geq \frac{1}{2}$ radius of B.

2) B doesn't meet a ball $\tilde{\mathcal{B}}$, but meets B'. Then radius $B' \geq \frac{1}{2}$ radius B by definition of B'.

So $\mathcal{B}' \in C$ which contradicts maximality of $\tilde{\mathcal{B}}$.

So if $B \in \mathcal{B}$ is arb., then by (ii) B meets a ball in $\tilde{\mathcal{B}}$; since $\tilde{\mathcal{B}} \in C$, B meets a ball $\hat{B} \in \tilde{\mathcal{B}}$ s.t. radius $\hat{B} \geq \frac{1}{2}$ radius B.