Math 245B Real Analysis
UCLA, Winter 2016

$(X, \mathcal{A})$ measurable space. A <u>signed measure</u> μ on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to \overline{\mathbb{R}} = [-\infty, \infty]$ s.t.

(i) $\mu(\emptyset) = 0$

(ii) μ takes at most one of the values $\pm\infty$, i.e., either $\mu : \mathcal{A} \to [-\infty, \infty)$ or $\mu : \mathcal{A} \to (-\infty, +\infty]$

(iii) μ is countably additive, i.e., if $A_n \in \mathcal{A}, n \in \mathbb{N}$, are pairwise disjoint, then

$$\mu\Big(\bigcup_{n \in \mathbb{N}} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n).$$

Part of this statement is the convergence of the series on the right (as an improper limit).

<u>Ex.</u> 1) Every measure is a signed measure. For emphasis we often say that a measure is a <u>positive measure</u>.

2) Let ν be a (positive) measure on $(X, \mathcal{A})$, $f : X \to \mathbb{R}$ measurable s.t.

$$\int f^+ \, d\nu < \infty \quad \text{or} \quad \int f^- \, d\nu < \infty.$$

Define

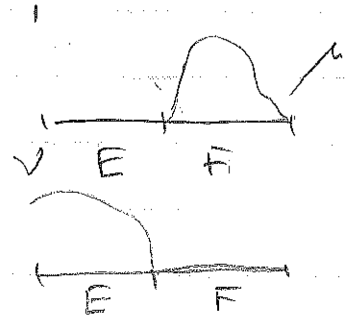
$$\mu(A) := \int_A f \, d\nu = \underbrace{\int_A f^+ \, d\nu}_{\mu^+(A)} - \underbrace{\int_A f^- \, d\nu}_{\mu^-(A)}$$

for $A \in \mathcal{A}$.

Then μ is a signed measure on $(X, \mathcal{A})$ (follows from MCT).

(2) Note: $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are pos. measures that "live" on disj. sets; namely, if $P = \{f > 0\}$, $N = \{f \le 0\}$, then $P \cup N = X$, $P \cap N = \emptyset$, and $\mu^+(N) = 0$, $\mu^-(P) = 0$.

<u>Def.</u> Let $(X, \mathcal{A})$ be a measurable space, and μ and ν be (positive) measures on $(X, \mathcal{A})$. We say that μ and ν are <u>mutually singular</u>, written $\mu \perp \nu$ if there ex. $E, F \in \mathcal{A}$ s.t. $E \cap F = \emptyset$, $E \cup F = X$, $\mu(E) = 0$, $\nu(F) = 0$.



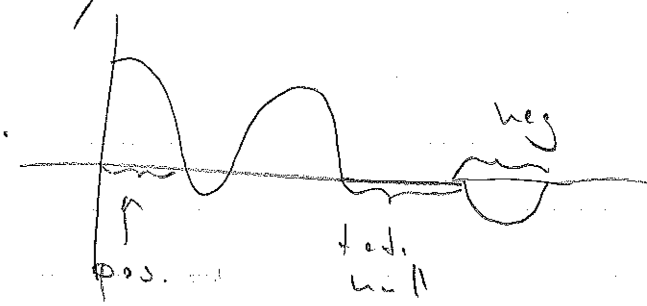
<u>Thm.</u> (Jordan Decomposition Theorem)
Let μ be a signed measure on $(X, \mathcal{A})$. Then there ex. unique pos. measures μ^+, μ^- on $(X, \mathcal{A})$ s.t.
$\mu = \mu^+ - \mu^-$ and $\mu^+ \perp \mu^-$.

($\mu = \mu^+ - \mu^-$ means that $\mu(A) = \mu^+(A) - \mu^-(A)$ for all $A \in \mathcal{A}$, where not both terms on the RHS are $+\infty$).

For the proof we need another definition: $(X, \mathcal{A})$ measure space, μ signed measure on $(X, \mathcal{A})$. A set $P \in \mathcal{A}$ is called <u>positive</u> if $\mu(A) \ge 0$ for each $A \in \mathcal{A}$ with $A \subseteq P$.

(3) $N \in \mathcal{A}$ is called <u>negative</u> if $\mu(A) \le 0$ for each $A \in \mathcal{A}$ with $A \subseteq N$.
$E \in \mathcal{A}$ is called a total null set if $\mu(A) = 0$ for each $A \in \mathcal{A}$ with $A \subseteq E$.

$\mu(A) = \int_A f \, d\nu$.



<u>Facts</u> μ signed measure on $(X, \mathcal{A})$

1. Let $A_n \in \mathcal{A}$, $n \in \mathbb{N}$, $A \in \mathcal{A}$.
If $A_n \nearrow A$, then
$$\lim_{n\to\infty} \mu(A_n) = \mu(A).$$
If $A_n \searrow A$ and $\mu(A_1) \in \mathbb{R}$, then
$$\lim_{n\to\infty} \mu(A_n) = \mu(A).$$
(proof similar as for pos. meas; exercise!)

2. Meas. subsets of pos. sets are positive.
Countable unions of pos. sets are positive (exercise!).

<u>Lem.</u> μ signed meas. ~~$\neq +\infty$~~ on $(X, \mathcal{A})$, $A \in \mathcal{A}$, $\mu(A) \neq -\infty$. Then A contains a pos. set P with $\mu(P) \ge \mu(A)$.

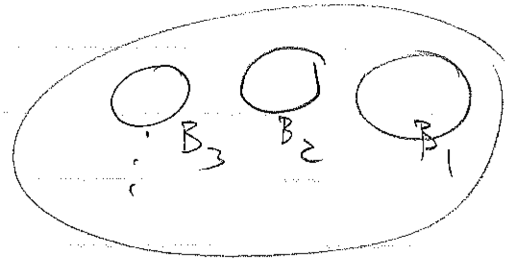
Proof: <u>Claim</u> For each $\varepsilon > 0$ there ex. $A_\varepsilon \in \mathcal{A}$ with $\mu(A_\varepsilon) \ge \mu(A)$ s.t.
$$B \in \mathcal{A},\ B \subseteq A_\varepsilon \Longrightarrow \mu(B) \ge -\varepsilon.$$

Otherwise, we can inductively find pairwise disj. meas. sets $B_k \subseteq A$ for $k \in \mathbb{N}$ s.t.

A

(4) $\mu(B_k) \le -\varepsilon$ for $k \in \mathbb{N}$.

Put $B = \bigcup_{k \in \mathbb{N}} B_k$.



Then $\mu(B) = \sum_{k=1}^{\infty} \underbrace{\mu(B_k)}_{\le -\varepsilon} = -\infty$,

and

$\mu(A) = \underbrace{\mu(B)}_{= -\infty} + \underbrace{\mu(B \setminus A)}_{< +\infty} = -\infty$, contradiction.

Now we use the claim for $\varepsilon = 1/n$, $n \in \mathbb{N}$, to find meas. sets

$A \supseteq A_1 \supseteq A_2 \supseteq \cdots$ s.t.

$\mu(A_n) \nearrow$ and

$B \subseteq A_n$ meas. $\Longrightarrow \mu(B) \ge -1/n$.

Let $P = \bigcap_{n \in \mathbb{N}} A_n \in \mathcal{A}$. Then $\mu(A_1) \neq \pm\infty$

and so

$\mu(P) = \lim_{n \to \infty} \mu(A_n) \ge \mu(A)$.

Moreover, P is positive; otherwise, ex.

$B \subseteq P \subseteq A_n$ meas. s.t.

$\mu(B) < 0$ and so $\mu(B) < -1/n$ for n large. Contradiction!

<u>Thm. (Hahn Decomposition Thm.)</u>

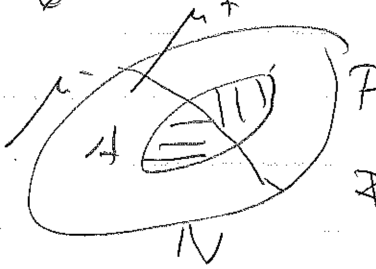
Let μ be a signed measure on $(X, \mathcal{A})$. Then there ex. a pos. set P and a neg. set N s.t. $P \cap N = \emptyset$ and $P \cup N = X$. If P', N' is another such pair, then $\underbrace{P' \Delta P}_{(P' \setminus P) \cup (P \setminus P')}$ and $N \Delta N'$ are total null sets.

(5) <u>Proof:</u> Wlog μ doesn't take $+\infty$.
Let
$$s = \sup \{\mu(A) : A \in \mathcal{A}\} \in [0, +\infty].$$
By the lemma above there ex. a seq. of pos. sets P_n s.t.
$$\mu(P_n) \longrightarrow s \quad \text{as} \quad n \longrightarrow \infty.$$
Then $P := \bigcup_{n \in \mathbb{N}} P_n$ is positive and
$$\mu(P) \geq \limsup_{n \to \infty} \mu(P_n) = s.$$
So $\mu(P) = s < +\infty$.
The set $N := X \setminus P \in \mathcal{A}$ is negative; otherwise, there ex. $E \subseteq N$ with $\mu(E) > 0$. Then
$$\mu(P \cup E) = \underbrace{\mu(P)}_{\in [0, \infty)} + \underbrace{\mu(E)}_{> 0} > s, \text{ contradiction.}$$
This shows existence of the Hahn decomp.
If P', N' is another Hahn decomp., then $P' \setminus P \subseteq N \cap P'$ is pos. and neg., hence total null.
Similarly, $P \setminus P'$ is total null, so $P' \triangle P$ is tot. null.
In the same way, $N' \triangle N$ is tot. null. $\square$

<u>Proof of the Jordan Decomp. Thm.:</u>
Let $X = P \cup N$ be the Hahn decomp. for μ. Define
$$\mu^+(A) := \mu(A \cap P)$$
$$\mu^-(A) := -\mu(A \cap N) \quad \text{for } A \in \mathcal{A}.$$
Then μ^+, μ^- are pos. meas. on $(X, \mathcal{A})$ s.t.

(6) $\mu = \mu^+ - \mu^-$. Since $N, P \in \mathcal{A}$, $P \cup N = X$, $P \cap N = \emptyset$, $\mu^+(N) = \mu(\underbrace{N \cap P}_{\emptyset}) = 0$, and $\mu^-(P) = -\mu(\underbrace{P \cap N}_{\emptyset}) = 0$, we have $\mu^+ \perp \mu^-$.

Existence of Jordan decomp. follows.



Uniqueness: Suppose $\mu = \nu^+ - \nu^-$, where ν^+, ν^- are pos. meas. on $(X, \mathcal{A})$ and $\nu^+ \perp \nu^-$.

Then there ex. $E, F \in \mathcal{A}$ s.t. $E \cup F = X$, $E \cap F = \emptyset$, $\nu^+(F) = 0 = \nu^-(E)$.

Then for $A \in \mathcal{A}$,

$$\mu(A) = \mu(A \cap E) + \mu(A \cap F) = \nu^+(A \cap E) - \nu^-(A \cap F).$$

This implies that E is positive and F is neg. (w.r.t. μ).

So $X = E \cup F$ is a Hahn decomp. of X.

By uniqueness of Hahn decomp. $E \Delta P$ is total null set w.r.t. μ.

Hence for $A \in \mathcal{A}$,

$$\nu^+(A) = \nu^+(A \cap E) + \overbrace{\nu^+(A \cap F)}^{0}$$
$$= \mu(A \cap E) = \mu(A \cap P) \quad (E \Delta P \text{ total null})$$
$$= \mu^+(A \cap P) = \mu^+(A)$$

So $\nu^+ = \mu^+$. Similarly, $F \Delta N$ total null for μ implies $\nu^- = \mu^-$. $\square$

The unique measures μ^+, μ^- in the Jordan decomp. $\mu = \mu^+ - \mu^-$ are called the <u>positive</u> and <u>negative variations</u> of μ, resp.

(7) The *total variation* $|\mu|$ of μ is defined as the pos. meas. on $(X, \mathcal{A})$ given by
$$|\mu|(A) = \mu^+(A) + \mu^-(A) \quad \text{for } A \in \mathcal{A},$$
i.e., $|\mu| = \mu^+ + \mu^-$.
A signed meas. is *finite* or *σ-finite* if $|\mu|$ is finite or σ-finite, resp.

Ex. $(X, \mathcal{A})$ meas. space, ν pos. meas. on $(X, \mathcal{A})$. Let $f : X \to \mathbb{R}$ be *ν-integrable in the extended sense*, i.e., f is meas. and $\int f^+ d\nu < \infty$ or $\int f^- d\nu < \infty$.
Let μ be the signed measure defined as
$$\mu(A) = \int_A f \, d\nu \quad \text{for } A \in \mathcal{A}.$$
"For abbreviation we write
$$d\mu = f \, d\nu."$$
Define
$$P := \{f \geq 0\} \text{ and } N := \{f < 0\}.$$
Then $X = P \cup N$ is a Hahn decomp. for μ.
$P' := \{f > 0\}$, $N' = \{f \leq 0\}$ is also a Hahn decomp.
Note: $\left.\begin{array}{l} P \triangle P' = \{f = 0\} \\ N \triangle N' = \{f = 0\} \end{array}\right\}$ is total null for μ
$\mu^+, \mu^-, |\mu|$ are given by
$$d\mu^+ = f^+ d\nu,$$
$$d\mu^- = f^- d\nu,$$
$$d|\mu| = |f| \, d\nu.$$

(8) $(X, \mathcal{A})$ meas. space, μ pos. meas., and ν signed meas. on $(X, \mathcal{A})$.
We say that ν is absolutely continuous w.r.t. μ, written $\nu \ll \mu$, if we have the implication
$$\mu(A) = 0 \longrightarrow \nu(A) = 0 \quad \text{for each } A \in \mathcal{A}.$$

Remarks: 1) Let ν be finite. Then $\nu \ll \mu$ iff for all $\varepsilon > 0$ there ex. $\delta > 0$ s.t.
$$\mu(A) < \delta \longrightarrow |\nu(A)| < \varepsilon \quad \text{for each } A \in \mathcal{A}$$
(exercise!)

2) $\nu \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$ iff $|\nu| \ll \mu$

(exercise!)

3) Two signed measures σ, τ are mutually singular, written $\sigma \perp \tau$ if there ex. meas. sets E, F s.t. $E \cup F = X$, $E \cap F = \emptyset$, F is total null for σ, E is total null for τ.

If μ pos. meas., ν signed meas., $\nu \ll \mu$, $\nu \perp \mu$, then $\nu = 0$. Proof: Since $\nu \perp \mu$, there ex. meas. sets E, F s.t. $E \cup F = X$, $E \cap F = \emptyset$, and F (total) null for μ, E total null for ν. If $A \in \mathcal{A}$, then
$$\nu(A) = \underset{0}{\nu(\underbrace{A \cap E}_{\subseteq E \ \nu\text{-tot. null}})} + \overset{0}{\nu(\underbrace{A \cap F}_{\subseteq F \text{ (tot.) null for } \mu,\ \nu \ll \mu})} = 0.$$

(9) **Thm.** (Radon-Nikodym)
Let μ be a σ-finite pos. meas., and ν be a σ-finite signed meas. on $(X, \mathcal{B})$. Then there ex. unique σ-finite signed meas. λ, ρ on $(X, \mathcal{A})$ s.t.

$$\nu = \lambda + \rho \text{ , where}$$
$$\lambda \perp \mu \text{ and } \rho \ll \mu.$$

Moreover, there ex. an extended μ-meas. function $f: X \to \mathbb{R}$ s.t.

$d\rho = f\, d\mu$. and any two such functions f are equal μ-a.e.

$\nu = \lambda + \rho$ is the Lebesgue decomp. of ν w.r.t. to μ

Note: if $\nu \ll \mu$, then

$\lambda = 0$, $\rho = \nu$, and so

$d\rho = \underline{\underline{d\nu = f\, d\mu}}$ for some f,

i.e.,

$$\nu(A) = \int_A f\, d\mu \quad \text{for } A \in \mathcal{A}.$$

We need a lemma for the proof.

Lem. Let ν and μ be finite pos. measures on $(X, \mathcal{B})$. Then either $\nu \perp \mu$ or there ex. $\varepsilon > 0$ and $E \in \mathcal{B}$ s.t. $\mu(E) > 0$ and $\nu \geq \varepsilon\mu$ on E (equiv. $\nu(A) \geq \varepsilon\mu(A)$ for all $A \in \mathcal{B}$ with $A \subseteq E$ equiv. E is positive for $\nu - \varepsilon\mu$)

__Proof__: Let $X = P_n \cup N_n$ be a Hahn decomp.

⑩ for $\nu - \frac{1}{n}\mu$. Define $P = \bigcup_{n\in\mathbb{N}} P_n$ and
$N = P^c = \bigcap_{n\in\mathbb{N}} P_n^c = \bigcap_{n\in\mathbb{N}} N_n$.

Then $N \subseteq N_n$ and N_n is neg. for $\nu - \frac{1}{n}\mu$.
So
$0 \ge \nu(N) - \frac{1}{n}\mu(N)$ equiv. $0 \le \nu(N) \le \frac{1}{n}\mu(N) \longrightarrow 0$ as $n \to \infty$.

So $\nu(N) = 0$. If $\mu(P) = 0$, then $\nu \perp \mu$.

Otherwise, $\mu(P) > 0$, then $\mu(P_n) > 0$ for some $n \in \mathbb{N}$ and P_n is pos. for $\nu - \frac{1}{n}\mu$. So take $E = P_n$, $\varepsilon = \frac{1}{n}$. $\square$

<u>Proof of the Radon-Nikodym theorem:</u>
<u>Existence</u> of Lebesgue decomposition

<u>Case I</u>: μ and ν are finite pos. measures.

$$\mathcal{F} = \left\{ f: X \longrightarrow [0,\infty] \text{ meas.} : \int_E f\,d\mu \le \nu(E) \text{ for all } E \in \mathcal{A} \right\}.$$

$\mathcal{F} \ne \emptyset$, since $0 \in \mathcal{F}$.
If $f, g \in \mathcal{F}$, then $h = \max(f,g) \in \mathcal{F}$;
indeed, let $A = \{x \in X : f(x) > g(x)\} \in \mathcal{A}$.
Then for $E \in \mathcal{A}$ we have

$$\int_E h\,d\mu = \int_{E\cap A} h\,d\mu + \int_{E\cap A^c} h\,d\mu = \int_{E\cap A} f\,d\mu + \int_{E\cap A^c} g\,d\mu$$

$$\le \nu(E\cap A) + \nu(E \cap A^c) = \nu(E).$$

(11) If $f_n \in \mathcal{F}$ and $f_n \nearrow f$, then $f \in \mathcal{F}$: follows from MCT.

Let $a := \sup\{\int f\,d\mu : f \in \mathcal{F}\}$
$\le \nu(X) < \infty$.

Pick $f_n \in \mathcal{F}$ s.t. $\int f_n\,d\mu \longrightarrow a$.

Let $g_n = \max(f_1, \dots, f_n) \in \mathcal{F}$.

Then $\int f_n\,d\mu \le \int g_n\,d\mu \le a$; so $\int g_n\,d\mu \longrightarrow a$.

Obviously, $g_n \nearrow$. Define $f(x) = \lim_{n\to\infty} g_n(x)$ for $x \in X$.

Then $g_n \nearrow f \in \mathcal{F}$ and by MCT

$$\int f\,d\mu = \lim_{n\to\infty} \int g_n\,d\mu = a < \infty.$$

By changing f on a set of μ-meas. 0 we may assume that f takes values in $[0,\infty)$.

Consider the finite meas. ρ defined as
$d\rho = f\,d\mu$.

Since $f \in \mathcal{F}$, the measure $\lambda = \nu - \rho$ is a pos. meas. and $\nu = \lambda + \rho$.

Here $\rho \ll \mu$.

We claim that $\lambda \perp \mu$.

Otherwise, by the lemma there ex. $E \in \mathcal{A}$ and $\varepsilon > 0$ s.t. $\mu(E) > 0$ and $\lambda = \nu - \rho \ge \varepsilon\mu$ on E equiv. $\nu(A) \ge \varepsilon\mu(A) + \int_A f\,d\mu$ for $A \in \mathcal{A}$ with $A \subseteq E$

equiv. $\nu(A) \ge \int_A (\varepsilon\chi_E + f)\,d\mu$ for $A \in \mathcal{A}$.

Hence $f + \varepsilon\chi_E \in \mathcal{F}$, but $\int (f + \varepsilon\chi_E)\,d\mu = a + \varepsilon\mu(E) > a$;

(12) contradiction. So indeed, $\lambda \perp \mu$.
The existence of the Lebesgue decomp. follows.

<u>Case II:</u> μ and ν are σ-finite pos. measures.
Then there ex. a countable partition of X into measurable sets of finite μ-measure.
The same is true for ν.
By considering a common refinement we can find a partition of X into meas. sets A_n, $n \in \mathbb{N}$, s.t. $\mu(A_n), \nu(A_n) < \infty$.
We now apply Case I to the restriction
$\mu_n = \mu | A_n$, $\nu_n = \nu | A_n$
$(\mu_n(B) = \mu(B)$ for $B \in \mathcal{A}$, $B \subseteq A_n)$.

Then we get Lebesgue decomp.
$\nu_n = \lambda_n + \rho_n$, where
$\lambda_n \perp \mu_n$, $d\rho_n = f_n \, d\mu_n$, $f_n : A_n \to [0, \infty)$ integr. w.r.t. μ_n

Define
$$\lambda(B) := \sum_{n \in \mathbb{N}} \lambda_n(A_n \cap B) \quad \text{for } B \in \mathcal{A},$$
$$f(x) = f_n(x) \quad \text{if } x \in A_n, \ n \in \mathbb{N},$$
and
$$d\rho = f \, d\mu.$$
Then
$$\nu = \lambda + \rho, \quad \lambda \perp \mu, \quad \rho \ll \mu. \quad \text{(exercise!)}$$

<u>Case III:</u> General case. Consider the Jordan decomp. $\nu = \nu^+ - \nu^-$ of ν.