Turbulent threshold and dispersive decay for the continuum Calogero–Moser model

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April 9, 2024

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Continuum Calogero–Moser Model

We recall that the focusing continuum Calogero–Moser model is given by

$$iu_t + u_{xx} - 2iu\Pi^+\partial_x|u|^2 = 0 \qquad \text{(CCM)}$$

with $\Pi^+ : L^2 \to L^2_+ = \{f \in L^2 : \operatorname{supp} \hat{f} \subset [0, \infty)\}$.

We recall the Lax pair formulation

$$\mathcal{L}_u = -i\partial_x - u\Pi^+\overline{u}$$
$$\mathcal{P}_u = i\partial_x^2 + 2u\Pi^+\partial_x\overline{u}$$

Turbulent Threshold

Theorem

For any sufficiently small $\varepsilon > 0$, there exist initial data $u_0 \in H^\infty_+(\mathbb{R})$ with $M(u_0) = 2\pi + \varepsilon$, a time $T \in (0, \infty]$, and a solution $u(t)$ to (CCM) *such that for all $s > 0$, $u \in C_t H^s_x([0, T) \times \mathbb{R})$ and*

$$\lim_{t \nearrow T} \|u(t)\|_{H^s} = +\infty.$$

In particular, if $T = \infty$, then we have the bounds

$$\|u(t)\|_{H^s} \gtrsim t^s.$$

Theorem (Dispersive decay)

Given a set of initial data $U \subset H^{\infty}_{+}$ which is bounded and equicontinuous in L^2_+ and satisfies $\langle x \rangle u_0 \in L^2$ for all $u_0 \in U$,

$$|u(t,z)| \lesssim |t|^{-\frac{1}{2}} \|u_0\|_{L^1} \left[1 + M(u)(1 + (\operatorname{Im} z)^{-1})\right] \tag{1}$$

uniformly for $\operatorname{Im} z > 0$, $u_0 \in U$, and all times of existence t.

Explicit formula

Due to Killip-Laurens-Vişan, see [2], we recall

Theorem (Explicit formula)

For any $H^\infty_+(\mathbb{R})$ *solution* $u(t)$ *to* (CCM) *with initial data satisfying* $\langle x \rangle u_0 \in L^2$,

$$u(t,z) = \frac{1}{2\pi i} I_+ \left\{ (X + 2t\mathcal{L}_{u_0} - z)^{-1} u_0 \right\}$$

for all $\operatorname{Im} z > 0$ *and all times of existence* t.

Here,

$$\mathbf{I}_+(f) := \widehat{f}(0+) = \int_{\mathbb{R}} f(x)dx$$

$$\widehat{\mathbf{X}f}(\xi) := i\frac{d\widehat{f}}{d\xi}(\xi)$$

Proof.

Let

$$A(t, z; f) = (X + 2t\mathcal{L}_f - z)^{-1}, \quad A_0(t, z) = A(t, z; 0).$$

Then by the explicit formula and a resolvent identity,

$$\begin{aligned} 2\pi i u(t, z) &= I_+ A(t, z; u_0) u_0 \\ &= I_+ A_0(t, z) u_0 + 2t I_+ A_0(t, z) u_0 \Pi^+ \overline{u_0} A(t, z; u_0) u_0. \end{aligned}$$

We focus on the first term, $I_+ A_0(t, z)$.

Dispersive decay

Proof continued.

The first term can be expressed as

$$\begin{aligned} I_+ A_0(t,z)u_0 &= I_+\big\{(X - 2it\partial_x - z)^{-1}u_0\big\} \\ &= I_+\big\{(e^{-it\Delta}Xe^{it\Delta} - z)^{-1}u_0\big\} \\ &= I_+\big\{e^{-it\Delta}(X - z)^{-1}e^{it\Delta}u_0\big\} \\ &= I_+\big\{(X - z)^{-1}[e^{it\Delta}u_0]\big\} \\ &= \frac{1}{2\pi i}[e^{it\Delta}u_0](z). \end{aligned}$$

Dispersive decay (equiv. Poisson integral) yields

$$\big|I_+ A_0(t,z)u_0\big| \lesssim |t|^{-1/2}\|u_0\|_{L^1}.$$

Dispersive decay

Proof continued.

The second term is treated by appeal to A_0:

$$\begin{aligned}|2tI_+A_0(t,z)u_0\Pi^+\overline{u_0}A(t,z;u_0)u_0| &\lesssim |t|^{1/2}\left\|u_0\Pi^+\overline{u_0}A(t,z;u_0)u_0\right\|_{L^1}\\ &\lesssim |t|^{1/2}\|u_0\|_{L^2}^2\|A(t,z;u_0)u_0\|_{L^\infty}.\end{aligned}$$

We write $A(t,z;u_0)$ as a series expansion about $A_0(t,z)$, to find

$$\|A(t,z;u_0)u_0\|_{L^\infty} \lesssim |t|^{-1}\left[1+(\operatorname{Im} z)^{-1}\right]\|u_0\|_{L^1}$$

uniformly for an L^2-equicontinuous family U. This completes the proof of the theorem. □

Theorem (Dispersive decay)

Given a set of initial data $U \subset H^\infty_+$ *which is bounded and equicontinuous in* L^2_+ *and satisfies* $\langle x \rangle u_0 \in L^2$ *for all* $u_0 \in U$,

$$|u(t,z)| \lesssim |t|^{-\frac{1}{2}} \|u_0\|_{L^1} \left[1 + M(u)(1 + (\operatorname{Im} z)^{-1})\right] \qquad (2)$$

uniformly for $\operatorname{Im} z > 0$, $u_0 \in U$, *and all times of existence* t.

Solitons

We recall that CCM admits a stationary soliton solution

$$Q(x) = \frac{\sqrt{2}}{x+i},$$

which is the unique nonzero minimizer of the energy and has mass

$$M(Q) = 2\pi.$$

Stability

Theorem

Fix $c > 0$. For all $\varepsilon > 0$ there exists $\delta > 0$ such that if $u_0 \in H^1$ satisfies

$$\|u_0 - Q\|_{H^1} < \delta$$

and the maximal lifespan solution $u(t)$ satisfies $\|u(t)\|_{\dot{H}^1} \geq c$, then

$$\inf_{\lambda > 0;\ \theta, y \in \mathbb{R}} \|u_{\lambda,\theta,y}(t) - Q\|_{H^1} < \varepsilon$$

for all times of existence t.

Lemma (Uniform lower bound in $\dot{H}^1$)

Suppose there exist $\varepsilon > 0$ and $u_0 \in H_+^\infty$ such that $\|u_0\|_{L^2}^2 \leq 2\pi + \varepsilon$ and such that $\mathcal{L}_{u_0}$ has an eigenvalue in $(-\infty, -c\varepsilon]$ for some $c > 0$. Let u denote the corresponding maximal lifespan solution to (CCM). *Then*

$$\|u(t)\|_{\dot{H}^1} \gtrsim c$$

uniformly for all times of existence t.

Lemma (Existence of negative eigenvalue)

For any $\varepsilon > 0$, the Lax operator $\mathcal{L}_{(1+\varepsilon)Q}$ has a negative eigenvalue in $(-\infty, -\varepsilon]$.

Turbulent Threshold

Theorem

For any sufficiently small $\varepsilon > 0$*, there exist initial data* $u_0 \in H_+^\infty(\mathbb{R})$ *with* $M(u_0) = 2\pi + \varepsilon$*, a time* $T \in (0, \infty]$*, and a solution* $u(t)$ *to* (CCM) *such that for all* $s > 0$*,* $u \in C_t H_x^s([0, T) \times \mathbb{R})$ *and*

$$\lim_{t \nearrow T} \|u(t)\|_{H^s} = +\infty.$$

In particular, if $T = \infty$*, then we have the bounds*

$$\|u(t)\|_{H^s} \gtrsim t^s.$$

Turbulent Threshold

Proof.

We construct initial data $u_0^\varepsilon \to Q$ in H^1 as $\varepsilon \to 0$ satisfying

- $M(u_0^\varepsilon) \leq 2\pi + \varepsilon$
- $\mathcal{L}_{u_0^\varepsilon}$ has an eigenvalue $\lesssim -\varepsilon \quad$ (equiv. $\|u^\varepsilon(t)\|_{\dot{H}^1} \gtrsim 1$)
- $u_0^\varepsilon \in \mathcal{S}(\mathbb{R})$
- $\{u_0^\varepsilon\}$ is L^2-equicontinuous

Fix ε and consider the corresponding (local) solution $u^\varepsilon(t)$.

Turbulent Threshold

Proof continued.

We know that for all t, there exists $\lambda(t), \theta(t), y(t)$ such that

$$\left\| u^{\varepsilon}_{\lambda(t),\theta(t),y(t)}(t) - Q \right\|_{H^1} < \varepsilon.$$

Then $\lambda(t)$ is the characteristic width of $u^{\varepsilon}(t)$.

We want $\lambda(t) \to 0$ and already know $\lambda(t) \lesssim 1$.

Turbulent Threshold

Proof continued.

Define

$$q(t,x) = e^{-i\theta(t)}\lambda(t)^{-1/2}Q(\lambda^{-1}(x-y(t))) = \frac{e^{-i\theta(t)}\lambda(t)^{-1/2}\sqrt{2}}{\lambda(t)^{-1}(x-y(t))+i}.$$

The dispersive decay implies

$$\begin{aligned}|q(t,z)| &\leq |q(t,z)-u^\varepsilon(t,z)| + |u^\varepsilon(t,z)| \\ &\lesssim (\operatorname{Im} z)^{-1/2}\|q(t)-u^\varepsilon(t)\|_{L^2} + |t|^{-1/2}\|u_0^\varepsilon\|_{L^1}\big[1+(\operatorname{Im} z)^{-1}\big] \\ &\leq (\operatorname{Im} z)^{-1/2}\varepsilon + |t|^{-1/2}\|u_0^\varepsilon\|_{L^1}\big[1+(\operatorname{Im} z)^{-1}\big].\end{aligned}$$

Let $z(t) = y(t) + i\lambda(t)$. Then

$$\lambda(t)^{-1/2} \lesssim \lambda(t)^{-1/2}\varepsilon + |t|^{-1/2}\|u_0^\varepsilon\|_{L^1}\big[1+\lambda(t)^{-1}\big].$$

Proof continued.

Taking ε sufficiently small,

$$\begin{aligned} \lambda(t)^{-1/2} &\lesssim |t|^{-1/2}\|u_0^\varepsilon\|_{L^1}\left[1+\lambda(t)^{-1}\right] \\ |t|^{1/2} &\lesssim \|u_0^\varepsilon\|_{L^1}\left[\lambda(t)^{1/2}+\lambda(t)^{-1/2}\right] \\ |t|^{1/2} &\lesssim \|u_0^\varepsilon\|_{L^1}\left[1+\lambda(t)^{-1/2}\right]. \end{aligned}$$

Therefore $\lambda(t) \lesssim |t|^{-1}$ for sufficiently large t. Since u^ε is either global or blows up in finite time, this concludes the proof. □

Corollary

For all $c, \varepsilon, s > 0$ there exists initial data $u_0 \in H_+^\infty$ such that $M(u_0) < 2\pi$ and $\|u_0 - Q\|_{L^2} < \varepsilon$ for which the global solution $u(t)$ satisfies

$$\inf_{t\in\mathbb{R}} \|u(t)\|_{\dot{H}^s} \leq c.$$

Proof of Corollary

Proof.

We recall that in the main theorem, we considered initial data $u_0^\varepsilon \to Q$ in H^1 such that

- $\|u^\varepsilon(t)\|_{\dot{H}^1} \gtrsim 1$
- $u_0^\varepsilon \in \mathcal{S}(\mathbb{R})$
- $\{u_0^\varepsilon\}$ is L^2-equicontinuous

Below mass 2π, we have a priori H^s-bounds [1]. If we assume that a lower bound exists, then we reach a contradiction by the previous argument. □

Thank you!

References

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