Matthew **Kowalski**

Research Experience and Proposed Program

My research lies in the field of nonlinear dispersive partial differential equations (PDEs), with a particular emphasis on completely integrable systems and dispersion-managed models. As dispersion is inherently a frequency-dependent phenomenon—describing waves whose frequency determines their velocity—my work relies heavily on harmonic analysis to analyze the fine structure and evolution of solutions.

The analysis of dispersive PDEs hinges on the competition between dispersion and nonlinear interactions with the medium. My completed work has focused on understanding the long-time dynamics of these equations depending on which of these effects, if either, dominates. First, for several fundamental models, I demonstrated that *dispersive decay* occurs [31, 33, 34]: solutions to nonlinear dispersive PDEs devolve into radiation and exhibit the strong quantitative decay of their linear counterparts, indicating regimes where dispersion fully dominates nonlinear effects. In contrast, for an integrable model of recent interest, I constructed solutions exhibiting *turbulent behavior* through frequency cascades [23], leading to unbounded growth in size and roughness over time and revealing regimes where nonlinear interactions prevail. These works are detailed in the Dispersive Decay and Turbulent Behavior sections.

Building on these insights, my ongoing research program investigates the regime where dispersion and nonlinear effects are engineered to be balanced. These *dispersion-managed models* describe pulse propagation in long-haul optical fibers, where the fiber material is alternated to support stable, soliton-like pulses. Despite their central role in modeling modern communication systems, the mathematical theory of dispersion-managed models remains underdeveloped compared to classical dispersive equations. My current research advances a rigorous mathematical theory for these dispersion-managed models—building on the discoveries of my recent work [32]—and focuses on well-posedness, soliton properties and formation, and long-time dynamics through scattering. This research program is introduced in the Dispersion-Managed Models section, with explicit objects developed in Research Objectives.

Dispersive Decay

Prior to the advent of Strichartz estimates, our understanding of the long-time behavior of dispersive PDEs rested on quantitative, pointwise-in-time bounds that paralleled the dispersive decay of the underlying linear models. For example, the linear Schrödinger equation exhibits the pointwise decay:

$$|e^{it\Delta_x} f(x)| \lesssim |t|^{-d/2} \int_{\mathbb{R}^d} |f(y)| dy. \tag{1}$$

Historically, analogous estimates on nonlinear equations required high regularity initial data, often with additional spatial decay; see [22, 29, 30, 36, 43] and references therein. My work sought to recover this level of decay in nonlinear settings under the minimal assumption that solutions exist globally and scatter.

In the series of papers [31, 33, 34], I investigated the nonlinear wave equation (NLW), nonlinear Schrödinger equation (NLS), and the generalized Korteweg–de Vries equation (gKdV), respectively. In each work, I showed that the decay of the linear model survives in the presence of the nonlinearity. In particular, for the energy-critical (NLS), I proved that any scattering solution ψ in the energy space inherits the full decay of the linear flow [33]:

$$|\psi(t,x)| \le C\big(\|\psi_0\|_{\dot{H}^1}\big)|t|^{-d/2} \int_{\mathbb{R}^d} |\psi_0(y)| dy.$$

Individually, each study uniquely hinged on refined estimates of the linear propagators and on the development of detailed bounds on the fine structure of solutions. Together, they cover the three most-studied dispersive models and demonstrate dispersive decay for a wide class of scattering solutions.

For (NLS), the key idea of the publication [33] was to refine the classic spacetime estimates for (NLS) to control the singularity of (1) using Lorentz spaces—a refinement of the typical Lebesgue L^p spaces that allows for the independent treatment of the amplitude and spread of functions. Building on prior work

[16], I extended these tools to quasi-normed spaces (e.g., $L^{p,q}$ for $0 < q < 1$) and developed frequency decompositions that disentangle the oscillatory and slowly varying components of a solution. Together, these estimates provide the flexibility needed to capture sharp, pointwise decay even for rough data.

For NLW, these Lorentz methods no longer suffice due to the large number of derivatives needed to ensure decay of the linear wave equation. In [31] on NLW, I surmounted this by building estimates in Besov spaces, which decompose functions into individual frequencies and measure each frequency component before summing. These estimates are well-suited for NLW as the linear wave equation is better behaved for frequency-localized pieces and gives a decay which, in contrast to (1), is integrable in t. This circumvents the need for Lorentz spaces and yields dispersive decay for rough data.

For gKdV, this narrative is further complicated by the derivative present in the nonlinearity, which requires more regularity to control than the underlying linear Airy equation. In the joint work [34], I overcame this by further refining Lorentz estimates in order to leverage the local smoothing properties of the linear Airy equation. This joint work arose after I contacted the second author with suggestions that improved the previously released results of [44] and then grew into a sustained collaboration on [34].

Together, these studies established a cohesive framework for analyzing dispersive decay in nonlinear models. The resulting analytical tools have already proven to be broadly applicable; see [45, 50], which built on [31, 33] in other models. Further, these estimates form the methodological foundation for my proposed investigation of (GT) as its nonlocal structure renders it amenable to such tools; see [25].

Turbulent Behavior

My joint publication [23] investigated the continuum Calogero–Moser model (CCM), a mass-critical variant of (NLS) with $|\psi|^p\psi$ replaced by $i\psi\partial_x|\psi|^2$ and projected onto positive frequencies. This model has attracted recent attention for its unexpected combination of complete integrability—which typically enforces global H^s bounds—and turbulent behavior in the form of frequency cascades.

Previous work [18] established that solutions to CCM with mass less than 2π remain globally bounded in H^s, while those of mass 4π can behave turbulently. The joint work [23], closed this gap and established the true mass threshold for turbulent behavior by constructing solutions $\psi(t,x)$ with mass $2\pi+\varepsilon$ (for any $\varepsilon > 0$) whose Sobolev norms grow like $\|\psi(t,x)\|_{H^s} \gtrsim |t|^s$ for all $s > 0$.

The proof was built on two complementary ingredients. The first ingredient was the development of a theory of near-soliton dynamics, exploiting the integrable structure of CCM and the spectral properties of its Lax operator to establish an orbital stability result: Solutions that start near a soliton must remain close to a soliton (up to rescaling) as they evolve. The second ingredient built on an explicit formula for CCM (in the sense of Gérard) to prove a new form of dispersive decay:

$$|\psi(t,z)| \lesssim \mathrm{Im}(z)^{-1}|t|^{-1/2} \quad \text{for} \quad \mathrm{Im}(z) > 0.$$

Here, the holomorphic extension to the upper half plane is enabled by the chiral structure of CCM.

These ingredients combine to produce turbulent behavior. Working with small perturbations of the soliton, stability ensures that solutions remain spatially localized, preventing a collapse to low frequencies. The dispersive decay then concentrates the solution to increasingly fine scales, driving the transfer of energy to high frequencies. As a result, small perturbations of the soliton—which has mass 2π—will grow without bound in all Sobolev norms H^s for $s > 0$, establishing the mass threshold for such behavior.

Dispersion-Managed Models

Modern communication relies crucially on optical fibers, whose ability to transmit high-traffic data forms the backbone of global networks. Within a fiber, pulse transmission is primarily modeled by the cubic *nonlinear Schrödinger equation*, whose general form reads:

$$i\partial_t\psi + \gamma\Delta_x\psi + |\psi|^p\psi = 0, \quad \psi(0,x) = \psi_0(x), \quad \psi : (-T,T)\times\mathbb{R}^d_x \to \mathbb{C}, \tag{NLS}$$

where ψ is the complex modulation of a quasi-monochromatic carrier wave, γ is the fiber's group velocity dispersion (GVD), and the roles of t and x are flipped: t is the *distance* along the fiber and x is a *retarded* *time*, traveling with the carrier wave. In a typical fiber, dispersion dominates the nonlinear effects and causes pulses to broaden. This limits bandwidth and transmission distance as pulses overlap and interact.

A common technique to mitigate these effects, introduced in [35], is *dispersion-management*: concatenating fiber segments with opposite GVD so that higher-frequency components propagate faster in one segment and slower in the next. This corresponds to $\gamma = \gamma(t)$ alternating periodically along the fiber and results in the *dispersion-managed nonlinear Schrödinger equation* (DM-NLS).

In modern applications, *strong dispersion-management* is often employed, alternating quickly between extreme positive and negative GVD. This balances dispersive and nonlinear effects to permit stable, soliton-like pulses. In this regime, the one-dimensional cubic *Gabitov–Turitsyn equation* emerges as the natural model of the large scale dynamics [1, 17, 39]. Its general form is:

$$i\partial_t \psi + \langle\gamma\rangle \Delta_x \psi + \int_{\mathbb{R}} e^{-i\sigma\Delta_x}\left[\left|e^{i\sigma\Delta_x}\psi\right|^p \cdot e^{i\sigma\Delta_x}\psi\right] d\mu(\sigma) = 0, \quad \psi : (-T,T)\times\mathbb{R}^d_x \to \mathbb{C}, \qquad \text{(GT)}$$

where $e^{\pm i\sigma\Delta_x}$ is the linear Schrödinger propagator and $\langle\gamma\rangle$ is the average GVD. The integral is taken over a probability measure μ, dictated by fiber properties and the fine details of the dispersion management.

In recent years, fiber optic techniques have undergone a dramatic shift: *digital signal processing* (DSP), enabled by coherent detection, has emerged as a competitor to dispersion-management. DSP sidesteps physical compensation by propagating pulses unmanaged and digitally reversing ill-effects at the receiver. This hinges on the long-time dynamics of (NLS). However, DSP has its own challenges, particularly at high bit rates (see [49]) and many transoceanic systems already have built-in dispersion-management. Recent research and industry perspectives have proposed hybrid dispersion-management and DSP approaches; see [2, 37]. This makes a thorough understanding of (GT), and indeed of DM-NLS, essential for improving and future-proofing these systems.

Mathematically, (GT) belongs to the nonlinear Schrödinger family as a highly nonlocal variant of the monomial (NLS). While (NLS) is one of the most thoroughly analyzed models in dispersive PDEs—with a complete local theory [6, 9, 10, 27], scattering results [13, 15], and, in the cubic case, an integrable structure that leads to explicit soliton dynamics and improved well-posedness [14, 21, 51]—the (GT) model remains mathematically nascent. Paradoxically, the averaging over σ in (GT) both dampens the effects of the nonlinearity—suggesting improved results—and introduces novel mathematical challenges: it lacks a scaling symmetry, admits no known explicit solutions, and its solitons have unresolved properties. As a result, many standard methods for (NLS) fail in the setting of (GT).

As a foray into dispersion-managed models, my proposed research program seeks to provide a rigorous mathematical theory for (GT) and targets interconnected objectives within the existence and long-time dynamics of solutions and the properties of solitons. Progress on (GT) promises not only concrete insight into dispersion-managed systems, but also broader advances in the analysis of dispersive PDEs.

Research Objectives

My proposed research program centers on three interconnected themes:

Theme I. *Well-posedness of* (GT) *at optimal regularities.*
Theme II. *Structure and properties of dispersion-managed solitons in application to fiber bandwidth.*
Theme III. *Long-time dynamics and scattering of solutions to* (GT) *with connections to DSP.*

These themes are tightly interwoven. The well-posedness theory identifies the natural spaces for solutions, providing a foundation for the rigorous study of solitons and scattering. Solitons and scattering, in turn, represent opposing long-time behaviors: solitons capture coherent, localized structures while scattering describes solutions which asymptotically disperse and decay. Moreover, these explicit behaviors provide solutions that sharpen the well-posedness theory. Together, these themes mirror the applications of (GT): soliton properties dictate bandwidth limits and long-time dynamics build towards efficient DSP.

Underpinning all three themes is a deliberate methodology: rather than naively emulating the (NLS) theory and analyzing approximate solutions—inaccessible due to the nonlocal structure of (GT)—I plan to leverage structural identities. This approach naturally yields tools and insights that extend broadly to dispersive PDEs, where structural identities are ubiquitous, but approximate solutions are absent. In this way, I aim to use (GT) as a lens to revisit and enrich the study of dispersive models.

My ultimate goal is to conduct this program under minimal assumptions on the measure μ. Importantly, this further allows for the consideration of fiber loss and correcting amplification (see [8]). To build a framework for this broader theory, I focus on a model case of (GT) where μ is uniformly distributed on $[0, 1]$. This corresponds to a lossless fiber without amplification but with steady dispersion-management.

Theme I: Sharp well-posedness at critical regularities

The well-posedness theory for (NLS) hinges on the Galilean symmetry, which boosts to moving reference frames, and the scaling symmetry. These symmetries identify two critical thresholds for well-posedness in the Sobolev H^s scale: $s = 0$ and $s_c = \frac{d}{2} - \frac{2}{p}$. When $s > \max(s_c, 0)$, (NLS) is well-posed in H^s, [6]: there exists a unique, continuous map from initial data in H^s to solutions evolving continuously in H^s. In contrast, for $s < s_c$ and $s \neq 0$, instantaneous norm inflation occurs in H^s, [5, 9, 10, 28]: initially small data can evolve instantly to become arbitrarily large in H^s. Finally, and most subtly, for $s < 0$, close initial data can quickly become decoherent and separate in H^s, [26].

This narrative breaks for (GT), where the nonlinearity fixes an inherent time scale and destroys any genuine scaling symmetries. In spite of this, I uncovered a third 'critical' scaling, $s_i = \frac{d}{2} - \frac{4}{p}$. In [32], I demonstrated that together, s_i, s_c, and $s = 0$ demarcate the well-posedness theory of (GT).

In [32], I first proved that (GT) is analytically well-posed in H^s for $s \geq \max(s_c, 0)$ via a power-series representation; see also [25]. I then sharpened this result by showing that the data-to-solution map fails to be smooth in H^s for $s < s_c$, employing ideas from [4, §6].

For $s < s_i$, I conjecture that norm inflation occurs. In [32], I resolved this for $s < \min(s_i, 0)$ building on the methods of [42]. In contrast, for positive regularities, methods from (NLS) fail, both from the lack of approximate solutions and the lack of a scaling symmetry. To surmount this, I built on a virial identity for (GT) and demonstrated that suitable solutions undergo *energy equipartition*: solutions with initially low kinetic energy and high potential energy rapidly redistribute and equalize, forcing the growth of kinetic energy. This leads naturally to norm inflation of (GT) in H^s for $1 \leq s < s_i$.

Building on the framework from these results, a number of concrete objectives arise that I will investigate. The first three naturally inform one another, as they all address energy equipartition.

Objective 1. *Prove norm inflation for* (GT) *in* H^s *for* $0 \leq s < \min(s_i, 1)$.
Objective 2. *Prove global existence of solutions at the 'critical' regularity* s_c.
Objective 3. *Prove norm inflation for* (NLS) *in* H^s *for* $s < s_c$ *by leveraging energy equipartition.*

In pursuit of Objective 1, my preliminary results suggest that solutions undergoing energy equipartition uniformly shift to high frequencies, which would extend norm inflation to H^s for $0 < s < \min(s_i, 1)$. The most direct path towards formalizing this is through precise upper bounds on high regularity norms of solutions—refining the classic methods of [3, 46]—together with novel lower bounds on low regularity norms. In particular, I have shown that solutions to (GT) (and (NLS)) that grow in H^1 must also exhibit growth in H^s for s above a certain threshold, providing a first step towards norm inflation.

To complement these Grönwall-type methods for Objective 1, I have constructed an almost conserved energy, with a kinetic term that behaves like H^s; see [11, 12, 47, 48] for the (NLS) theory. Once this quantity is sufficiently controlled, energy equipartition would imply that potential energy is converted into H^s norm, driving norm inflation. The use of almost conservation laws to drive growth is novel. Usually, they are used to give upper bounds on solutions. With this method, I have lowered the regularity required for global existence of solutions to (GT) below that of the energy. Further analysis will strengthen these results and push the threshold for global existence towards the 'critical' regularity of Objective 2.

Finally, these methods extend beyond (GT) and already prove norm inflation of (NLS) in H^s for $1 \leq s < s_c$, laying the groundwork for Objective 3. Success in expanding this method for (GT) will inform the development of a unified framework for proving norm inflation across dispersive PDEs.

Objective 4. *Prove that* (GT) *is ill-posed in* H^s *for* $s_i \le s < s_c$.

Although my work [32] proved that modern contraction mapping arguments fail in H^s for $s_i \le s < s_c$, a true ill-posedness mechanism has remained elusive. In pursuit of Objective 4, I will investigate local well-posedness in Fourier–Lebesgue spaces; these extended the cubic (NLS) theory beyond $s = 0$ by leaving the Sobolev hierarchy [20]. I expect this method to yield stronger local results for (GT), extending norm inflation into $s_i \le s < 0$ with my arguments in [32]. This will clarify the true threshold for well-posedness for (GT) and further the sharp well-posedness theory envisioned in Theme 1.

Theme II: Structure and properties of dispersion-managed solitons

Solitons are central to the physical relevance of (GT) as they capture the properties needed to encode bits as stable optical pulses. The existence of such *dispersion-managed solitons* has been rigorously justified for non-negative average GVD [7, 24]. In the cubic case, they are further known to decay exponentially in space and frequency [19]. However, the nonlocal structure of (GT) forces these results to formulate solitons as constrained energy minimizers, leaving their profiles, dynamical properties, and widths opaque.

The widths of dispersion-managed solitons are of particular importance to optical fibers, where narrow solitons allow for high bit rates. For wide solitons, one expects a resemblance to the solitons of (NLS) as the profile is unable to disperse in the averaging interval of (GT); see [40]. In contrast, narrow solitons must contend with the full nonlocal structure of (GT), which leaves their existence uncertain.

My investigation into the existence and stability of narrow solitons gives three clustered objectives:

Objective 5. *Prove the existence and stability of arbitrarily narrow solitons.*
Objective 6. *Formalize the small-scale limit of* (GT) *in* H^{s_c}.
Objective 7. *Establish a lack of uniform continuity in* H^s *for* $s < 0$.

In pursuit of Objective 5—and given recent progress on the large spatial scale limit of (GT), [40]—it is natural to investigate the small scale limit of (GT) to determine which structural features dominate. I expect that this will yield a version of (GT) with the nonlinearity extended to $[0, \infty)$, resolving Objective 6. By surmounting the limitations of [7, 19, 24] in this limit, a soliton with arbitrarily narrow width would be obtained, formalizing Objective 5. This has the dual purpose of illuminating new ill-posedness methods for (GT), connecting back to Objective 4, as ill-posedness is typically a small-scale phenomenon.

Building on Objective 5, one can boost narrow solitons so they decohere quickly, breaking uniform continuity and completing Objective 7 for positive GVD; see [26]. In the case of negative GVD, where solitons are not expected (see Objective 10), a promising alternative is to leverage the virial identity for (GT) to control the spread of solutions under boosts. Like energy equipartition, this would give a unifying method for proving a lack of uniform continuity without approximate solutions.

Theme III: Long-time dynamics and scattering

Scattering is a central theme in dispersive models, describing solutions which fully succumb to dispersion and devolve into radiation. For (NLS), scattering has a rich history, with, in particular, a complete theory for critical models; see [13, 15]. In contrast, for (GT), the theory remains relatively stark. Current results have established scattering for (GT) in the 'inter-critical' ranges ($0 < s_c < 1$) [25], but no results are available for large data at the critical endpoints: $s_c = 0$ and $s_c = 1$.

My overarching goal is to develop a scattering theory for (GT) that parallels, and potentially improves upon, the rich theory for (NLS). This leads to a preliminary objective for scattering and two objectives that connect back to Theme 2 as well as my prior work on dispersive decay [31, 33, 34]:

Objective 8. *Prove scattering for large initial data in the 'mass-critical'* ($s_c = 0$) *case.*
Objective 9. *Prove dispersive decay for scattering solutions to* (GT).
Objective 10. *Prove non-existence of dispersion-managed solitons for negative average GVD.*

In pursuit of Objective 8, I will investigate interaction Morawetz estimates for (GT); these structural identities were crucial in the critical scattering theory of (NLS), [15]. For (GT), I expect this story to be cleaner than (NLS) because the existence time depends only on the size of the initial data, even in critical spaces, [25, 32]. With scattering established in a critical space, I can then expand my prior work [31, 33, 34] and prove dispersive decay for scattering solutions to (GT), completing Objective 9.

A remarkable feature of (GT) is that the existence of solitons is known for vanishing GVD [24] and unresolved for negative GVD: numerics suggest that stable pulses persist for extreme distances [38, 41], while physics studies suggest that pulses bleed radiation [52]. Though rigorous analysis of this case is complicated by the unboundedness of the constrained energy from below [52], scattering offers a path towards resolution. In particular, progress towards Objective 8 negates the existence of solitons, proving a step towards Objective 10. This connects scattering back to Theme 2 and the soliton theory for (GT).

Undergraduate Mentorship and Research

A growing focus of mine has been the mentorship of undergraduates and their research. As a graduate student, this has primarily been through the Directed Reading Program (DRP), which offers undergraduates the opportunity to engage in independent study under the guidance of graduate mentors. Within the DRP, I have mentored eight projects, on topics ranging from Fourier analysis to quantum computing. I have also served as an organizer, pairing students with mentors and coordinating the student-led colloquium.

Looking forward, I would like to extend this mentorship to undergraduate research. In particular, I hope to work with students on numerical simulations of (GT) using split-step methods, with applications to DSP in dispersion-managed fibers. This project is well-suited for undergraduates, providing an introduction to dispersive PDEs and the opportunity to contribute meaningfully to ongoing research.

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