Geometry Qualifying Exam Solutions and Notes

Harris Khan

September 6, 2019

1 Spring 2010

Problem 1: Consider $F : M_n(\mathbb{R}) \to S_n(\mathbb{R})$ given by $F(A) = AA^t - I$.

a) Show 0 is a regular value of F.

Solution: To show 0 is a regular value of F, we must check that dF_B has full rank for all $B \in F^{-1}0$. Let $B \in F^{-1}0$ be arbitrary. Then $BB^t = I$, since $F(B) = 0$.

Notice dF_B is a linear map from $T_B(M_n(\mathbb{R})) = M_n(\mathbb{R})$ to $T_0(S_n(\mathbb{R})) = S_n(\mathbb{R})$. (These are vector spaces over $\mathbb{R}$ and hence have a single global chart making them into a manifold; moreover, for V a vector space and $x \in V$, $T_xV = V$. If you are interested in proving this, set up a linear isomorphism $\phi : V \to \mathbb{R}^k$ and observe $d\phi_x = \phi$).

To say dF_B has full rank is to say it is surjective, since $\dim(M_n(\mathbb{R})) > \dim(S_n(\mathbb{R}))$. Thus, we need to check the map $dF_B : M_n(\mathbb{R}) \to S_n(\mathbb{R})$ is surjective.

We may compute:

$$\begin{aligned} dF_B(A) = \lim_{t\to 0} \frac{F(B+tA) - F(B)}{t} &= \lim_{t\to 0} \frac{(B+tA)(B+tA)^t - I - 0}{t} \\ &= \lim_{t\to 0} \frac{BB^t + tBA^t + tAB^t + t^2AA^t - I}{t} \\ &= \lim_{t\to 0} \frac{tBA^t + tAB^t + t^2AA^t}{t} \\ &= BA^t + AB^t + \left(\lim_{t\to 0} t\right) AA^t = BA^t + AB^t \end{aligned}$$

Hence $dF_B : M_n(\mathbb{R}) \to S_n(\mathbb{R})$ is the map that sends A to $BA^t + AB^t$. This is indeed surjective. To see this, let $C \in S_n(\mathbb{R})$ be an arbitrary symmetric matrix. Let $A = \frac{1}{2}CB$. Then

$$dF_B(A) = BA^t + AB^t = \frac{1}{2}B(CB)^t + \frac{1}{2}CBB^t = \frac{1}{2}BB^tC^t + \frac{1}{2}C = C$$

where we used $BB^t = I$ and $C^t = C$.

Since $C \in S_n(\mathbb{R})$ was arbitrary, we conclude dF_B is surjective, and hence of full rank. Since $B \in F^{-1}0$ was arbitrary, we conclude 0 is a regular value of F.

b) Deduce $O_n(\mathbb{R}) \subset M_n(\mathbb{R})$ is a submanifold.

Solution: By the regular value theorem, $F^{-1}0 = \{B \in M_n(\mathbb{R}) : BB^t = I\} = O_n(\mathbb{R})$ is a submanifold of $M_n(\mathbb{R})$.

c) Find the dimension of $O_n(\mathbb{R})$ and compute $T_I(O_n(\mathbb{R}))$ as a subspace of $T_I(M_n(\mathbb{R})) = M_n(\mathbb{R})$.

Solution: The regular value theorem also tells us, moreover, that the codimension of $\{0\}$ in $S_n(\mathbb{R})$ is equal to the codimension of $F^{-1}0 = O_n(\mathbb{R})$ in $M_n(\mathbb{R})$. The codimension of $\{0\}$ in $S_n(\mathbb{R})$ is the dimension of $S_n(\mathbb{R})$ minus the dimension of 0, which is $\frac{n^2+n}{2} - 0 = \frac{n^2+n}{2}$. Hence the codimension of $O_n(\mathbb{R})$ in $M_n(\mathbb{R})$ is $\frac{n^2+n}{2}$. Meanwhile, $\dim(M_n(\mathbb{R})) = n^2$, so that $O_n(\mathbb{R})$ has dimension $n^2 - \frac{n^2+n}{2} = \frac{n^2-n}{2}$.

In fact, we even know by regular value theorem that $T_B(O_n(\mathbb{R})) \subset T_B(M_n(\mathbb{R})) = M_n(\mathbb{R})$ is just the kernel of $dF_B : M_n(\mathbb{R}) \to S_n(\mathbb{R})$. Hence, $T_I(O_n(\mathbb{R})) = \ker(dF_I)$. Recall by part A that $dF_I(A) = IA^t + AI^t = A^t + A$. Hence, $\ker(dF_I) = \{A \in M_n(\mathbb{R}) : A^t + A = 0\}$, i.e. the skew-symmetric matrices. Thus, $T_I(O_n(\mathbb{R}))$ is the set of skew-symmetric matrices.

Problem 2: Show $T^2 \times S^n$ is parallelizable for any $n \geq 1$.

Recall an n-manifold M is parallelizable if and only if the tangent bundle is trivial, i.e. $TM \cong M \times \mathbb{R}^n$ as vector bundles.

Fact: A k-dimensional bundle E over M (with $\pi : E \to M$ the projection map) is trivial if and only if there exist vector fields (sections of the vector bundle) $V_1, ..., V_k : M \to E$ with $\{V_i(p)\}_{i=1}^k$ linearly independent in $E_p = \pi^{-1}p$ for each $p \in M$.

Lemma: S^1 is parallelizable.

Proof: View $S^1 \subset \mathbb{C} \cong \mathbb{R}^2$ as a submanifold. By $G\&P$'s definition, we may view for $z = e^{it_0} \in S^1 \subset \mathbb{C}$ and $\mathbb{R} \supset V_z \xrightarrow{\phi} U_z \subset S^1 \subset \mathbb{C} = \mathbb{R}^2$ some local parameterization around z (say $\phi(t) = e^{it}$)

$$T_z S^1 = im(d\phi_{t_0})$$

Here, $d\phi_z : \mathbb{R} \to \mathbb{R}^2$ has $d\phi_{t_0}(s) = \lim_{h\to 0} \frac{e^{i(t_0+sh)} - e^{it_0}}{h} = z \lim_{h\to 0} \frac{e^{ish}-1}{h} = z \lim_{h\to 0} sie^{ish} = (iz)s$. Hence $im(d\phi_{t_0}) = \{(iz)s : s \in \mathbb{R}\}$. Next,

$$TS^1 = \{(x, v) : v \in T_x S^1 \subset \mathbb{R}^2\} \subset S^1 \times \mathbb{R}^2$$

We have a nonvanishing tangent vector field given by $V : S^1 \to TS^1$ sending $V(z) = (z, iz) \in TS^1$ (or $V(z) = \frac{\partial}{\partial\theta}|_z$, or $V((x,y)) = (-y, x)$). Since $\{iz\}$ is a linearly independent set in $T_z S^1$ for each $z \in S^1$, we conclude TS^1 is trivial, isomorphic to $S^1 \times \mathbb{R}$. In particular, S^1 is parallelizable. $\square$

Lemma: When viewing $S^n \subset \mathbb{R}^{n+1}$, NS^n is trivial, i.e. $NS^n \cong S^n \times \mathbb{R}$.

Proof: For $x \in S^n$, we have $N_x S^n = (T_x S^n)^\perp = \{sx : s \in \mathbb{R}\} \subset \mathbb{R}^{n+1}$. Moreover,

$$NS^n = \{(x, v) : v \in N_x S^n \subset \mathbb{R}^{n+1}\} \subset S^n \times \mathbb{R}^{n+1}$$

To see NS^n is trivial, it suffices to give a nonvanishing normal vector field $V : S^n \to NS^n$. Of course, this is accomplished by $V(x) = (x, x) \in NS^n$. Since $\{x\}$ is a linearly independent set in $N_x S^n$ for each $x \in S^n$, we conclude NS^n is trivial and isomorphic to $S^n \times \mathbb{R}$. $\square$

Fact: $T(M \times N) \cong \pi_M^* TM \oplus \pi_N^* TN$ as vector bundles over $M \times N$.

Corollary: The product of parallelizable manifolds is parallelizable.

Proof: Let M, N be parallelizable n and m manifolds respectively. Then

$$T(M \times N) = \pi_M^* TM \oplus \pi_N^* TN = \pi_M^*(M \times \mathbb{R}^n) \oplus \pi_N^*(N \times \mathbb{R}^m)$$

$$= M \times N \times \mathbb{R}^n \oplus M \times N \times \mathbb{R}^m = M \times N \times \mathbb{R}^{n+m} \quad \square$$

Solution: We have $T^2 \times S^n = S^1 \times S^1 \times S^n$. Since S^1 is parallelizable, it suffices to check $S^1 \times S^n$ is parallelizable. Meanwhile,

$$T(S^1 \times S^n) = \pi_{S^1}^*(TS^1) \oplus \pi_{S^n}^*(TS^n) = (S^1 \times S^n \times \mathbb{R}) \oplus \pi_{S^n}^*(TS^n) = \pi_{S^n}^*(S^n \times \mathbb{R}) \oplus \pi_{S^n}^*(TS^n)$$

$$= \pi_{S^n}^*(NS^n) \oplus \pi_{S^n}^*(TS^n) = \pi_{S^n}^*(NS^n \oplus TS^n) = \pi_{S^n}^*(S^n \times \mathbb{R}^{n+1}) = S^1 \times S^n \times \mathbb{R}^{n+1}$$

Hence $S^1 \times S^n$ is parallelizable, and the result follows. $\square$

Remark: (i) The direct sum refers to fiber product. (ii) TS^n is trivial iff $n = 1, 3, 7$.

Problem 3: Let $\pi : M_1 \to M_2$ be a smooth map between connected manifolds such that $d\pi_p : T_pM_1 \to T_{\pi(p)}M_2$ is an isomorphism for all $p \in M_1$.

a) Show that if M_1 is compact, then π is a covering space projection.

Theorem: (Stack of Records) Suppose $f : X \to Y$ is smooth, X is compact, and $\dim(X) = \dim(Y)$. Then for all $y \in Y$ regular, y has an evenly covered neighborhood, $y \in V$ with $f^{-1}y = \{x_1, ..., x_n\}$, $x_i \in U_i$ open disjoint, $f|_{U_i}$ diffeomorphisms from U_i to V, and $f^{-1}V = \sqcup_i U_i$.

Proof: Since y is regular, each df_{x_i} is an isomorphism (by dimension considerations). Then $f^{-1}y$ is a compact 0-manifold (by codimension in X of $f^{-1}y$ equal to codimension of y in Y). Hence it is a finite set, $f^{-1}y = \{x_1, ..., x_n\}$. For the moment, assume $n > 0$.

By the inverse function theorem, we have an open neighborhood W_i of x_i such that $F(W_i)$ is open and $F|_{W_i}$ is a diffeomorphism. We may insist the W_i are disjoint; otherwise, shrink to an open subset (still containing x_i). The image will remain open, and the restriction of a diffeomorphism is again a diffeomorphism.

Let $V' = \cap_i F(W_i)$. Each $F(W_i)$ is open and contains y (and there are finitely many i) so that V' is open and contains y.

Let $U_i' = (F|_{W_i})^{-1}(V') = F^{-1}V' \cap W_i$, which is of course also open, and contains x_i. Since the W_i are disjoint, so too are the U_i'. By construction, $F|_{U_i'} : U_i' \to V'$ is a diffeomorphism (it is a further restriction of $F|_{W_i}$).

Finally, shrink one last time! Writing $U' = \cup_i U_i'$ and $Z = X \setminus U'$, we see Z is closed in X, and hence compact. So $F(Z)$ is compact and hence closed. Then $V = V' \setminus F(Z)$ is open and contains y (since $f^{-1}y$ is disjoint from Z, entirely contained in U'). Finally, set $U_i = (F|_{U_i'})^{-1}V = F^{-1}V \cap U_i'$. This is again open and contains x_i. Moreover, $F|_{U_i} : U_i \to V$ is a diffeomorphism. Each U_i is disjoint since each U_i' was. Finally, $F^{-1}V \cap Z = \emptyset$ by construction, so that $F^{-1}V \subset \cup_i U_i'$. Hence $F^{-1}V = \sqcup_i U_i$.

Finally, we address the $n = 0$ case. If $f^{-1}y$ is empty, it suffices to find a neighborhood of y whose preimage is empty. This is possible, since if every open neighborhood of y intersects with $F(X)$, then y is in the closure of $F(X)$, which is closed (since it is compact in Hausdorff space Y). □

Solution: Note π is surjective. For this, notice $\pi(M_1) \subset M_2$ is compact and hence closed in M_2. It suffices to show (by connectedness of M_2) that it is open, as it is indeed nonempty.

To see that it is open, let $y \in \pi(M_1)$ be arbitrary. Write $y = \pi(x)$ for some $x \in M_1$. Since $d\pi_x$ is an isomorphism by assumption, we have π is a local diffeomorphism, with $\pi|_U : U \to V$ a diffeomorphism (and $x \in U \subset M_1$, $y = \pi(x) \in V \subset M_2$ open). In particular, $V \subset \pi(M_1)$ is an open neighborhood of y in $\pi(M_1)$. Hence $\pi(M_1)$ is open in M_2, as desired. So $\pi(M_1) = M_2$.

Finally, applying stack of records to arbitrary $y \in M_2$, we see that y has an evenly covered neighborhood, so that π is indeed a covering map. □

b) Give an example where M_2 is compact but π is not a covering space projection.

We construct an example $\pi : \mathbb{R} \to S^1$. It suffices to have $d\pi_t : \mathbb{R} = T_t\mathbb{R} \to T_{\pi(t)}S^1$ to be an isomorphism for each $t \in \mathbb{R}$, yet for π to not be surjective (so that it cannot be a covering space projection).

Let $\pi(t) = e^{if(t)} \in S^1$ for some smooth function $f : \mathbb{R} \to \mathbb{R}$. Then $d\pi_t(1) = if'(t)e^{if(t)}$, so that $d\pi_t(s) = if'(t)e^{if(t)}s$. This is a linear map between one-dimensional spaces and hence is invertible if and only if it is nonzero. It is nonzero if and only if $f'(t) \neq 0$. Thus, we need $f'(t) \neq 0$ for any $t \in \mathbb{R}$.

Let $f(t) = \arctan(t)$. Then indeed $f'(t) = \frac{1}{1+t^2}$ is always nonzero for any $t \in \mathbb{R}$. Moreover, $f(t) \in (-\pi/2, \pi/2)$ for all $t \in \mathbb{R}$, so that $\pi(t) \subset \{e^{i\theta} \in S^1 : -\pi/2 < \theta < \pi/2\} \neq S^1$. Hence π is not surjective, and we have the desired counterexample.

Problem 4: Let $\mathcal{F}^k(M)$ denote the k-forms on M. Let $U, V \subset M$ be open.

a) Explain how the SES $0 \to \mathcal{F}(U \cup V) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V) \to 0$ arises.

Definition: A 1-form or covector is a section of T^*M. It is of the form df, an evaluation at f map for smooth function $f : M \to \mathbb{R}$ (in fact it is in bijection with these).

A k-form is a section of $\bigwedge^k T^*M = \left(\bigwedge^k TM\right)^*$ (and can be thought of as a function on M to $\bigwedge^k T^*M \subset \bigwedge T^*M$). It can be written as a sum of k-fold wedges (exterior powers) of 1-forms, and can be thought of as a function, $\omega(X_1, ..., X_k)$ returning a real number. Recall

$$(\alpha \wedge \beta)(X_1, ..., X_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} sgn(\sigma)(\alpha \otimes \beta)(X_{\sigma(1)}, ..., X_{\sigma(k+l)})$$

We have $\boxed{\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega}$ and $\boxed{d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2}$.

Note $\bigwedge^k T_p^* M$ has dimension $\binom{n}{k}$ (where $n = \dim(M)$) with basis $dx_{i_1} \wedge ... \wedge dx_{i_k}$, $i_1 < ... < i_k$.

Remark: Note that $\mathcal{F}^k(M) = \Omega^k(M)$ is a $C^\infty(M)$-module, and $\Omega(M)$ is a graded $C^\infty(M)$-algebra via wedge, but is also a cochain complex via d (dimension goes up).

Solution: We view these as cochain complexes since we are dealing with cohomology. To obtain this SES, we need an SES

$$0 \to \mathcal{F}^k(U \cup V) \xrightarrow{f_k} \mathcal{F}^k(U) \oplus \mathcal{F}^k(V) \xrightarrow{g_k} \mathcal{F}^k(U \cap V) \to 0$$

for each $0 \leq k \leq n$, such that the maps commute with d. We define

$$f_k(s) = (s|_U, s|_V)$$

$$g_k(t, w) = t|_{U \cap V} - w|_{U \cap V}$$

These are $C^\infty(M)$-linear, as is necessary. It is clear $\text{im}(f_k) \subset \ker(g_k)$. Conversely, $(t, w) \in \ker(g_k)$ may be glued since they agree on the intersection, so we have equality.

Meanwhile, anything in $\ker(f_k)$ is zero on all of $U \cup V$ and hence 0. Finally, g_k is surjective, since if $\omega \in \mathcal{F}^k(U \cap V)$, pick a partition of unity of $U \cup V$ subordinate to the open cover $\{U, V\}$. Then we may find smooth functions from $U \cup V$ to $[0, 1]$ with $\phi \prec\prec U, \psi \prec\prec V$ (compact support), and with $\phi + \psi = 1$ on $U \cap V$. Then $\phi\omega$ may be viewed as an element of $\mathcal{F}^k(U)$, and $\psi\omega$ as an element of $\mathcal{F}^k(V)$. Finally, $g_k(\phi\omega, -\psi\omega) = \omega$. Hence g_k is surjective.

As a last remark, notice d is linear and commutes with restriction, and $d^2 = 0$. So we have an SES of cochain complexes.

b) Write down the LES in de Rham cohomology associated to the SES in part a and describe explicitly how the map $H^k_{deR}(U \cap V) \xrightarrow{\delta} H^{k+1}_{deR}(U \cup V)$ arises.

This requires the Zig-Zag Lemma.

Lemma: (Zig-Zag) Given an SES of modules $0 \to A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \to 0$ of cochain complexes, we have an LES ... $\to H^k(A^*) \xrightarrow{f^\#} H^k(B^*) \xrightarrow{g^\#} H^k(C^*) \xrightarrow{\delta} H^{k+1}(A^*) \to$ ... of cohomology groups. (Recall the kth cohomology group is the kernel of the next map mod the image of the previous).

Proof: We will use d to denote the maps in each cochain, through abuse of notation, but they will be indexed occasionally by the domain, making it clear which map we are referring to.

The map $f^\#$ is induced as follows: for $x \in \ker(d_{A^k})$, $f_k(x) \in \ker(d_{B^k})$ since $d(fx) = f(dx) = 0$. So we have $\ker(d_{A^k}) \to \ker(d_{B^k}) \to H^k(B)$ by modding out by $\operatorname{im}(d_{B^{k-1}})$. But then $\operatorname{im}(d_{A^{k-1}})$ factors through, giving our desired map $H^k(A) \xrightarrow{f^\#} H^k(B)$. This justifies the well-definedness of the map $[x] \mapsto [f(x)]$. Similarly we may obtain $g^\#$.

To get the map δ, we need to diagram chase. Start with $x \in \ker(d_{C^k})$. Find $y \in B^k$ with $g_k(y) = x$. Now $dy \in B^{k+1}$ has $g_{k+1}(dy) = d(g_k y) = dx = 0$, so that $dy \in \ker(g_{k+1}) = \operatorname{im}(f_{k+1})$. So it has a unique preimage $z \in A^{k+1}$. Observe $f_{k+2}dz = d(f_{k+1}z) = d(dy) = 0$, and f_{k+2} is injective, so that $dz = 0$. Hence $z \in \ker(d_{A^{k+1}})$.

If $y' \in B^k$ also has $g_k(y') = x$, then $y - y' \in \ker(g_k) = \operatorname{im}(f_k)$, so that we have an $a \in A^k$ with $f_k(a) = y - y'$. Note z is the preimage of dy under f_{k+1}; if z' is the preimage of dy', then $z - z'$ maps to $d(y - y')$. Meanwhile, $f_{k+1}(da) = d(f_k(a)) = d(y - y')$. Hence $z - z' = da$ by injectivity of f_{k+1}. This shows $[z] = [z']$ in $H^{k+1}(A)$.

Define the map $\ker(d_{C^k}) \to H^{k+1}(A)$ via $x \mapsto [z]$. To see it factors through $\operatorname{im}(d_{C^{k-1}})$, notice for $\gamma \in C^{k-1}$, taking $x = d\gamma$, notice we may select $y' \in B^{k-1}$ with $g_{k-1}y' = \gamma$, so that $x = d\gamma = dg_{k-1}y' = g_k(dy')$. Thus, we may select $y = dy' \in B^k$ as our preimage. Then we select the unique $z \in A^{k+1}$ with $f_{k+1}z = dy = 0$. So we must select $z = 0$. Hence our map sends $d\gamma \mapsto [0]$. Thus indeed it factors through the image and we get a map $H^k(C) \xrightarrow{\delta} H^{k+1}(A)$. Explicitly, this map sends $[x]$ to $[z]$, where $z \in (f_{k+1})^{-1}(d(g_k^{-1}x))$ is arbitrary.

$C^\infty(M)$-linearity of δ is easy to verify. It is also easy to see we at least get a cochain complex, since $g^\# f^\# = (gf)^\# = 0$ and $\delta g^\#([x]) = \delta([g_k(x)]) = [z]$, where $z \in (f_{k+1})^{-1}dx$ is arbitrary, though this is a singleton since f is injective. Then $f_{k+1}z = dx = 0$ since x is closed (i.e. in the kernel of this d). Hence $z = 0$, so $\delta g^\#([x]) = [0]$, and $\delta g^\# = 0$. Finally, $f^\# \delta([x]) = f^\#[z] = [f(z)]$, where $z \in f_{k+1}^{-1} d g_k^{-1} x$ is arbitrary. Hence $f^\#\delta([x]) = [w]$ for $w \in d(g_k^{-1}x)$ arbitary. Hence $[w] = [0]$, so that $f^\#\delta = 0$.

So all the images are contained in the appropriate kernels. To check reverse containments, let $[x]$ be in $\ker(f^\#)$ (with $[x] \in H^k(A^*)$, $k > 0$). Then $f^\#[x] = [f_k(x)] = [0]$, so that $f_k(x) = d\gamma$. Then $\delta[g_{k-1}\gamma] = [x]$ (where $g_{k-1}\gamma$ is indeed closed since $dg_{k-1}\gamma = g_k f_k(x) = 0$), so $[x] \in \operatorname{im}(\delta)$. Next, if $[x] \in \ker(g^\#)$, then $g_k(x) = d\gamma$ if $k > 0$, and $g_k(x) = 0$ if $k = 0$. In the former case, select y with $g_{k-1}(y) = \gamma$. Then notice $g_k(x - dy) = 0$. Let $z = x - dy$. If $k = 0$, let $z = x$. In either case, $z \in \ker(g_k) = \operatorname{im}(f_k)$, so $f_k(w) = z$ for some w. Moreover, $f_{k+1}(dw) = dz = 0$ (in both cases), so $dw = 0$, so w is closed. So $f^\#([w]) = [z] = [x]$, so $[x] \in \operatorname{im}(f^\#)$. Finally, if $[x] \in \ker(\delta)$, then for $z \in (f_{k+1})^{-1}(d(g_k^{-1}(x)))$ arbitrary, $z = dw$ for some w. Then $d(f_k(w)) = f_{k+1}z \in d(g_k^{-1}(x))$. So there is some $\lambda \in g_k^{-1}(x)$ with $d(f_k(w)) = d\lambda$, and $g_k(\lambda) = x$. Notice $g_k(f_k(w)) = 0$, so $g_k(f_k(w) - \lambda) = x$, with $f_k(w) - \lambda$ closed (since $d(f_k(w) - \lambda) = 0$). So $[x] \in \operatorname{im}(g^\#)$ as desired. □

Solution: We can make our map a bit more explicit in our case. The map δ proceeds as follows: starting with $[\omega]$ (for $\omega \in \mathcal{F}^k(U \cap V)$), first we consider a form t on U and a form s on V via the partition of unity described in part a, taking $t = \phi\omega$, $s = -\psi\omega$. Then we apply d to get $dt = d(\phi\omega)$ and $ds = -d(\psi\omega)$. Finally, these forms are glued together to get a form η on $U \cup V$, where the compatibility condition amounts to noticing that $dt|_{U\cap V} - ds|_{U\cap V} = d\omega = 0$. Our map then has $\delta([\omega]) = [\eta]$. It sends a closed k-form on $U \cap V$ to a closed $(k+1)$-form on $U \cup V$ by splitting into two non-closed forms, applying d (so that they agree on their intersection), and gluing.

Problem 5: Let $\pi : S^n \to M$ (for $n > 1$) be a covering space projection with M orientable. Show every closed k-form on M is exact (for $0 < k < n$).

Solution: Recall

$$H^k_{dR}(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & 0 < k < n \end{cases}$$

It suffices to show each $H^k_{dR}(M) \xrightarrow{\pi^*} H^k_{dR}(S^n)$ is injective for $0 < k < n$. Fix k. We define a left inverse π_* as follows: Let ω be a k-form on S^n. For any evenly covered $U \subset M$, let $\phi_i : U \to U_i$ be the smooth inverse of $\pi|_{U_i}$ (which is a diffeomorphism from U_i to U), for $i = 1, ..., q$. Then $\eta(U, i) = \phi_i^*(\omega|_{U_i})$ is a k-form on U, and so too is $\theta(U) = \frac{1}{q}\sum_{i=1}^q \eta(U, i) = \frac{1}{q}\sum_{i=1}^q \phi_i^*(\omega|_{U_i})$.

In fact, we may find a unique k-form θ on M with $\theta|_U = \theta(U)$ for each evenly covered $U \subset M$. To do this, notice for any $p \in U$ with ϕ_i as described above, and $p_i = \phi_i(p)$, we have

$$\theta(U)_p(X_1, ..., X_k) = \frac{1}{q}\sum_{i=1}^q (\phi_i^*(\omega|_{U_i}))_p(X_1, ..., X_k) = \frac{1}{q}\sum_{i=1}^q (\omega|_{U_i})_{p_i}((d\phi_i)_p X_1, ..., (d\phi_i)_p X_k)$$

$$= \frac{1}{q}\sum_{i=1}^q \omega_{p_i}(((d(\pi|_{U_i}))_{p_i})^{-1} X_1, ..., ((d(\pi|_{U_i}))_{p_i})^{-1} X_k) = \frac{1}{q}\sum_{i=1}^q \omega_{p_i}((d\pi_{p_i})^{-1} X_1, ..., (d\pi_{p_i})^{-1} X_k)$$

which is independent of choice of U containing p (it only depends on the points in the fiber, $p_1, ..., p_q \in \pi^{-1}p$). Hence, θ given by $\theta_p = \theta(U)_p$ for some $U \ni p$ evenly covered makes θ a well-defined k-form on M, with $\theta|_U = \theta(U)$ for any evenly covered U. Such a θ is clearly unique, and we define $\pi_*\omega = \theta$ in this way. We have for any evenly covered $U \subset M$,

$$(\pi_*\omega)|_U = \frac{1}{q}\sum_{i=1}^q \phi_i^*(\omega|_{U_i})$$

Next, for ω a k-form on M, notice

$$(d\pi_*\omega)|_U = d((\pi_*\omega)|_U) = \frac{1}{q}\sum_{i=1}^q d\phi_i^*(\omega|_{U_i}) = \frac{1}{q}\sum_{i=1}^q \phi_i^*(d\omega|_{U_i}) = (\pi_* d\omega)|_U$$

Hence $d\pi_*\omega = \pi_* d\omega$, so that π_* sends closed forms to closed forms and exact forms to exact forms (if $d\omega = 0$, then $d\pi_*\omega = 0$, and if $\omega = d\eta$, then $\pi_*\omega = \pi_* d\eta = d\pi_*\eta$). Thus π_* can be viewed as a map from $H^k_{dR}(S^n)$ to $H^k_{dR}(M)$. Finally,

$$(\pi_*\pi^*\omega)|_U = \frac{1}{q}\sum_{i=1}^q \phi_i^*((\pi^*\omega)|_{U_i}) = \frac{1}{q}\sum_{i=1}^q \omega|_U = \omega|_U$$

Hence $\pi_*\pi^*\omega = \omega$. Since

$$H^k_{dR}(M) \xrightarrow{\pi^*} H^k_{dR}(S^n) \xrightarrow{\pi_*} H^k_{dR}(M)$$

composes to the identity, we see π^* is injective. Since $H^k_{dR}(S^n) = 0$ for $0 < k < n$, we conclude $H^k_{dR}(M) = 0$ for $0 < k < n$, so that every closed k-form on M is exact.

Remark: Note M is also an n-manifold: consider an evenly covered chart of M. It is diffeomorphic to open subsets of S^n, which are n-manifolds.

Note for $n > 1$, S^n is a simply connected covering space of M, and therefore must be a universal cover. Thus, deck transformations act transitively on the fibers. This gives an alternative proof using $S^n/G \cong M$.

Problem 6: Calculate the singular homology of $\mathbb{R}^n \setminus \{x_1, ..., x_l\}$.

Solution: Pick disjoint open balls $B_i \ni x_i$ (with $B_i \subset \mathbb{R}^n$). Let $U = \mathbb{R}^n \setminus \{x_1, ..., x_l\}$, and let $V = \sqcup_{i=1}^l B_i \cong \sqcup_{i=1}^l \mathbb{R}^n$, where $\cong$ here denotes homeomorphic. Notice $U \cup V = \mathbb{R}^n$, and $U \cap V = \sqcup_{i=1}^l (B_i \setminus \{x_i\}) \cong \sqcup_{i=1}^l S^{n-1}$, where $\cong$ here denotes homotopy equivalent. Applying Mayer-Vietoris for singular homology, we get an LES

$$... \to H_{k+1}(U \cup V) \to H_k(U \cap V) \to H_k(U) \oplus H_k(V) \to H_k(U \cup V) \to ... \to H_0(U \cup V) \to 0$$

For $0 < k < n-1$, notice we have $H_k(U \cup V) = H_k(\mathbb{R}^n) = 0$ since $0 < k < n$. Meanwhile, $H_k(U \cap V) = H_k(\sqcup_{i=1}^l S^{n-1}) = \oplus_{i=1}^l (H_k(S^{n-1})) = 0$ since $0 < k < n-1$. Finally, $H_k(V) = \oplus_{i=1}^k H_k(\mathbb{R}^n) = 0$. Hence our exact sequence gives

$$0 \to H_k(U) \oplus 0 \to 0$$

so that $H_k(U) = 0$ for $1 \le k < n-1$. For $k = 0$, we have $H_1(\mathbb{R}^n) = 0, H_0(\mathbb{R}^n) = \mathbb{Z}$, $H_0(\sqcup_{i=1}^l S^{n-1}) = \mathbb{Z}^l$, and $H_0(\sqcup_{i=1}^l \mathbb{R}^n) = \mathbb{Z}^l$, giving an exact sequence

$$0 \to \mathbb{Z}^l \to H_0(U) \oplus \mathbb{Z}^l \to \mathbb{Z} \to 0$$

Since $H_0(U)$ is free (with rank the number of path components), we conclude $H_0(U) = \mathbb{Z}$ (and U is path connected).

Next, for $k = n-1$, we have $H_n(\mathbb{R}^n) = 0$, $H_{n-1}(\sqcup_{i=1}^l S^{n-1}) = \mathbb{Z}^l$, $H_{n-1}(\coprod_{i=1}^l \mathbb{R}^n) = 0$ and $H_{n-1}(\mathbb{R}^n) = 0$ if $n > 1$. This gives

$$0 \to \mathbb{Z}^l \to H_{n-1}(U) \oplus 0 \to 0$$

so that $H_{n-1}(U) \cong \mathbb{Z}^l$. Finally, for $k \ge n$, we get

$$0 \to H_k(U) \oplus 0 \to 0$$

since $H_k(V) = H_k(U \cup V) = H_k(U \cap V) = 0$ for $k \ge n$. We conclude $H_k(U) = 0$ for $k \ge n$. Hence for $n > 1$,

$$H_k(\mathbb{R}^n \setminus \{x_1, ..., x_l\}) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & 0 < k < n-1 \\ \mathbb{Z}^l & k = n-1 \\ 0 & k \ge n \end{cases}$$

For $n = 1$, $\mathbb{R}^1 \setminus \{x_1, ..., x_l\}$ is the disjoint union of $l+1$ open intervals, which is homotopy equivalent to $l+1$ points, so that

$$H_k(\mathbb{R}^1 \setminus \{x_1, ..., x_l\}) = \begin{cases} \mathbb{Z}^{l+1} & k = 0 \\ 0 & k > 0 \end{cases}$$

Problem 7:

a) Explain what is meant by adding a handle to a 2-sphere for a two dimensional orientable surface in general.

> **Solution:** Adding a handle to surface M is to remove two disjoint disks from M and gluing a cylinder (with each boundary circle glued to the boundary of one of the removed disks).

b) Show that a 2-sphere with a positive number of handles attached can not be simply connected.

> **Solution:** Define $M_0 = S^2$ and $M_g = M_{g-1}$ with a handle attached to the image of M_0 in M_{g-1}. Contracting the image of the sphere gives us the usual genus g compact orientable surface.
>
> Recall $\chi(M_g) = 2 - 2g$. Notice that the orientable genus g surface can be obtained by a polygon with $4g$-sides, so this can be proved directly if desired ($\chi(M_g) = 1 - 2g + 1$ since pairs of edges are identified, and all vertices end up being identified). We will instead show it using induction. A torus can be formed using a square, and ends up having one 0-cell, two 1-cells, and one 2-cell, for $\chi(T^2) = 1 - 2 + 1 = 0$. Meanwhile, $\chi(A\#B) = \chi(A) + \chi(B) - \chi(S^n)$ for a connected sum of n-manifolds. Moreover, $\chi(S^n) = 1 + (-1)^n$. For $n = 2$, this gives
>
> $$\chi(M_0) = \chi(S^2) = 1 + (-1)^2 = 2 = 2 - 2 \cdot 0$$
>
> $$\chi(M_g) = \chi(M_{g-1}\#T^2) = \chi(M_{g-1}) + \chi(T^2) - \chi(S^2) = \chi(M_{g-1}) - 2$$
>
> If $\chi(M_{g-1}) = 2 - 2(g-1)$, it follows by the above that $\chi(M_g) = 2 - 2g$. Thus, by induction, $\chi(M_g) = 2 - 2g$ as desired.
>
> Now notice $H^0_{dR}(M_g) = \mathbb{R} = H^2_{dR}(M_g)$. To see this, notice T^2 is a connected, compact, orientable 2-manifold without boundary. Hence the same is true for its connected sums. Thus each $H^0_{dR}(M_g) = \mathbb{R}$ by connectedness, and each $H^2_{dR}(M_g) = \mathbb{R}$ by being a compact orientable 2-manifold without boundary. (Or, use Poincare duality).
>
> Then $2 - 2g = \chi(M_g) = 1 - \dim_{\mathbb{R}} H^1_{dR}(M_g) + 1 = 2 - \dim_{\mathbb{R}} H^1_{dR}(M_g)$. Thus, $H^1_{dR}(M_g) \cong \mathbb{R}^{2g}$. On the other hand, if M_g is simply connected, $\pi_1(M_g) = 0$. Then $H_1(M_g) = 0$ as it is the abelianization of $\pi_1(M_g)$.
>
> We can apply universal coefficients to get $H_1(M_g) \otimes_{\mathbb{Z}} \mathbb{R} \cong H_1(M_g; \mathbb{R})$, so $H_1(M_g; \mathbb{R}) = 0$. We can apply it again to get $H^1(M_g; \mathbb{R}) \cong (H_1(M_g; \mathbb{R}))^* = 0$. Finally, we can apply De Rham's Theorem to get $H^1(M_g; \mathbb{R}) \cong H^1_{dR}(M_g)$. Finally, $0 = H^1(M_g; \mathbb{R}) \cong H^1_{dR}(M_g) \cong \mathbb{R}^{2g}$. Hence $g = 0$. $\square$

Theorem: (Universal coefficients) For k any ring, $H_i(M;k) \cong H_i(M) \otimes k \oplus \mathrm{Tor}_1(H_{i-1}(M), k)$, and $H^i(M;k) \cong \mathrm{Hom}(H_i(M), k) \oplus \mathrm{Ext}(H_{i-1}(M), k)$. More generally, for R a PID and G an R-module, we have $H^i(M;G) \cong \mathrm{Hom}_R(H_i(M;R), G) \oplus \mathrm{Ext}_R(H_{i-1}(M;R), G)$.

Note $\mathrm{Tor}(A,B) = \mathrm{Tor}(B,A) = \mathrm{Tor}(Torsion(A), B)$, so it vanishes if either is torsion free. Moreover it commutes with limits (and direct sums and products) and $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, B) = \ker(B \xrightarrow{n} B)$. $\mathrm{Ext}(A,B)$ commutes with sums in the first entry, is 0 if A is free, and $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, B) = \mathrm{coker}(B \xrightarrow{n} B) = B/nB$. Also, $Ext_R(R/(u), B) = B/uB$. $Ext_R(A,B) = 0$ if A is projective or B is injective.

For $k = \mathbb{R}$, we see $H_i(M;\mathbb{R}) = H_i(M) \otimes \mathbb{R}$, where Tor is 0 since $\mathbb{R}$ is torsion free. Similarly, $H^i(M;\mathbb{R}) = H_i(M;\mathbb{R})^*$, since $\mathrm{Ext}_{\mathbb{R}}(\mathbb{R}^k, \mathbb{R}) = 0$ since $\mathbb{R}^k$ is free and hence projective. Same for $k = \mathbb{Q}$.

For $k = \mathbb{Z}/p\mathbb{Z}$, $H_i(M;\mathbb{F}_p) = H_i(M) \otimes \mathbb{F}_p \oplus \ker(H_{i-1}(M) \xrightarrow{p} H_{i-1}(M))$ and $H^i(M;\mathbb{F}_p) = H_i(M;\mathbb{F}_p)^*$.

Theorem: (Poincare Duality) For M a compact orientable n-manifold without boundary, $H^k(M) \cong H_{n-k}(M)$. This can also be done over any coefficient ring, but in particular, for $\mathbb{Z}/2\mathbb{Z}$, orientability is free.

Theorem: (De Rham's Theorem) For a smooth manifold, $H^k_{dR}(M) \cong H_k(M;\mathbb{R})^* \cong H^k(M;\mathbb{R})$ via $\omega \mapsto ([c] \mapsto \int_c \omega)$. The second isomorphism is just by universal coefficients.

Corollary: A compact manifold X of odd dimension n has Euler characteristic 0.

Proof: We have for any field k,

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \operatorname{rank}(H_i(X)) = \sum_{i=0}^{n} (-1)^i \dim_k(H_i(X) \cong k) = \sum_{i=0}^{n} (-1)^i \dim_k H_i(X;k)$$

Taking $k = \mathbb{Z}/2\mathbb{Z} = \mathbb{F}_2$, we get orientability and Poincare duality for free, so we get

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{F}_2} H_i(X;\mathbb{Z}/2\mathbb{Z}) = \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{F}_2} H^{n-i}(X;\mathbb{Z}/2\mathbb{Z})$$

$$= \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{F}_2} (H_{n-i}(X;\mathbb{Z}/2\mathbb{Z}))^* = \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{F}_2} H_{n-i}(X;\mathbb{Z}/2\mathbb{Z})$$

$$= (-1)^n \sum_{j=0}^{n} (-1)^{-j} \dim_{\mathbb{F}_2} H_j(X;\mathbb{Z}/2\mathbb{Z}) = (-1)^n \chi(X)$$

Hence $(1 - (-1)^n)\chi(X) = 0$. For n odd, this gives $2\chi(X) = 0 \Rightarrow \chi(X) = 0$. $\square$

Problem 8: Define the degree of a smooth map $f : S^2 \to S^2$ (and show it is well-defined if needed). Show there exists a smooth map $f : S^2 \to S^2$ of degree k for each $k \in \mathbb{Z}$.

There are a few equivalent notions of degree. We can write, for $f : X \to Y$, $\deg(f)$ as the sum of signed preimages of a regular value of f, defined as $I(f, \{y\})$ for any regular value y. Recall $I(f, Z)$ is the sum over all $x \in f^{-1}Z$ of ± 1, depending on if $df_x T_x X + T_z Z$ preserves orientation of $T_z Y$. Thus for degree purposes, we count $x \in f^{-1}y$ $+1$ or -1 depending on if $df_x T_x X = T_y Y$ preserves orientation. For X, Y equidimensional, this amounts to saying $\det(df_x)$ is positive or negative.

Alternatively, use Hatcher's definition, which says $\deg(f)$ is just the integer which gives the map on top homology, $\mathbb{Z} = H_n(X) \to H_n(Y) = \mathbb{Z}$. Over $\mathbb{R}$, this is still multiplication by an integer. For de Rham cohomology, we have $H^n(X) \cong \mathbb{R}$ and $H^n(Y) \cong \mathbb{R}$ via integration over the fundamental class (i.e. a generator of $H_n(X)$ for X compact orientable). Our map $f^* : H^n(Y) \to H^n(X)$ ends up having $\boxed{\int_X f^*\omega = \deg(f) \int_Y \omega}$.

Finally, we can use local degree with Hatcher's definition. Pick y in the image with finitely many preimage points. Pick balls near those points not containing any of the other preimage points, and look at the degree of the induced map. (If it is a homeomorphism, it is ± 1).

Solution: Hatcher's proof for a degree k map from $S^n \to S^n$ works as follows: collapse the complement of k disks in S^n to a point, leaving a wedge of k copies of S^n. Then send each copy of S^n to S^n either via the identity or via reflections. Notice in the image, each point has precisely k preimages, and it is a local homeomorphism near these points, so provided we flip all the -1 to $+1$ via reflection, this gives us degree k.

Alternatively, take $f : S^2 \to S^2$ which sends $(\theta, \phi) \to (k\theta, \phi)$. Then for $\omega = g(\theta, \phi) d\theta \wedge d\phi$, we have

$$\begin{aligned}\int_{S^2} f^*\omega = \int_{S^2} f^*(g(\theta,\phi) d\theta \wedge d\phi) &= \int_{S^2} g(\theta \circ f, \phi \circ f) d(\theta \circ f) \wedge d(\phi \circ f) \\ &= \int_{S^2} g(k\theta, \phi) k d\theta \wedge d\phi = k \int_0^{\pi} \int_0^{2\pi} g(k\theta, \phi) d\theta d\phi \\ &= \int_0^{\pi} \int_0^{2\pi k} g(u, \phi) du d\phi = k \int_0^{\pi} \int_0^{2\pi} g(u, \phi) du d\phi = k \int_{S^2} \omega\end{aligned}$$

□

Remark: See Fall 2012 Problem 4 for a generalization using a different argument.

Problem 9: Explain how Stokes Theorem gives the classical divergence theorem.

Theorem: (Stokes) Let M be a smooth oriented n-manifold with boundary, and let ω be a *compactly supported* $(n-1)$-form on M. Let $i : \partial M \to M$ be the inclusion map. Then

$$\int_{\partial M} i^* \omega = \int_M d\omega$$

Remark: Functions are 0 forms. We can think of applying d as follows: d of a 0-form gives a 1-form, and this can be thought of as the gradient. d of a 1-form gives a 2-form, and this can be thought of as curl. Finally, d of a 2-form gives a volume form, and this can be thought of as divergence.

To get Green's Theorem, let $D \subset \mathbb{R}^2$ with P, Q smooth $\mathbb{R}$-valued functions on D. To compute $\int_{\partial D} Pdx + Qdy$, we can apply Stokes to $\omega = Pdx + Qdy$ to get

$$\int_{\partial D} Pdx + Qdy = \int_D d(Pdx + Qdy) = \int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dx \wedge dy$$

Similarly, for divergence theorem, take $\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$. See Fall 2018 Problem 5 for full details.

Problem 10:

a) Show any $F : S^n \to S^1 \times ... \times S^1 := T^k$ is null-homotopic (homotopic to a constant map).

The universal cover of a product is the product of universal covers. The universal cover of S^1 is $\mathbb{R}$, so that T^k has universal cover $\mathbb{R}^k$. Recall a universal cover X is simply connected, i.e. has $\pi_1(X) = 0$. Given a map $h : Y \to X$ and a covering X' of X, then h lifts to a map $g : Y \to X'$ if and only if $h_*(\pi_1(Y)) \subset p_*(\pi_(X'))$. If the spaces aren't path connected we may care about base point, in which case we need $h_*(\pi_1(Y, y_0)) \subset p_*(\pi_(X', x'_0))$ where $x'_0 \in p^{-1}h(y_0)$.

Since X', Y are both simply connected in our case, this property is satisfied (as both $\pi_1(X') = \pi(Y) = 0$), and our map $F : S^n \to T^k$ factors through to $S^n \to \mathbb{R}^k \to T^k$. The first of these maps is homotopic to the constant map via a straight line homotopy, or using the fact that $\mathbb{R}^k$ is contractible. This then descends to a homotopy on maps from $S^n \to T^k$, making F homotopic to a constant map, as desired.

b) Show there exists a map $F : T^n := S^1 \times ... \times S^1 \to S^n$ that is not null-homotopic.

Solution: Taking Ian's solution, pick $U \subset T^n$ with $U \cong \mathbb{R}^n$, and map T^n to $T^n/(T^n \setminus U)$, which is a 1-point compactification of U and hence homeomorphic to S^n. Points in the image of U are the only ones that have finite preimage sets; near the unique preimage of such a point, the map is a local homeomorphism; hence the degree of this map is ± 1 rather than 0. (Such a map can be made smooth if needed). $\square$

c) Show that every map $F : S^n \to S^{n_1} \times S^{n_2} \times ... \times S^{n_k}$, $n_1 + ... + n_k = n$, $k \geq 2$, has degree 0. (You may take F to be smooth).

Let π_i denote the projection $S^{n_1} \times ... \times S^{n_k} \to S^{n_i}$. Let ω_i be a non-vanishing (necessarily closed) n_i-form on S^{n_i}. Take $\omega = \wedge_{i=1}^k \pi_i^* \omega_i$. Then ω is a non-vanishing closed n-form on $S^{n_1} \times ... \times S^{n_k}$. Notice, then,

$$F^*\omega = \bigwedge_{i=1}^k F^* \pi_i^* \omega_i$$

Meanwhile, $F^*\pi_i^*\omega_i$ is a closed n_i-form on S^n, and since $0 < n_i < n$, it is exact (since $H^{n_i}(S^n) = 0$). Write $F^*\pi_i^*\omega_i = d\theta_i$. Then

$$F^*\omega = \bigwedge_{i=1}^k d\theta_i = d\left(\theta_1 \wedge \bigwedge_{i=2}^k d\theta_i\right)$$

So $F^*\omega$ is exact, and hence $\int_{S^n} F^*\omega = 0$ (by Stokes or by isomorphism on top cohomology with $\mathbb{R}$). If we take the ω_i to be volume forms, then ω is a volume form, so $\int_{S^n} \omega \neq 0$. This shows F has degree 0. (Alternatively, notice ω is non-vanishing and pointwise must give a basis for the n forms on S^n, as that space is 1-dimensional). $\square$

Notes

Low Dimensional Manifolds

Compact 0-manifolds are just finite sets with discrete topology. Compact connected 1-manifolds are $[0,1]$ (if we allow for boundary), and $S^1 \cong \mathbb{RP}^1$ (which does not have boundary). Hence for M a compact 1-manifold, ∂M is finite of even size.

It turns out compact connected 2-manifolds without boundary are homeomorphic if and only if diffeomorphic if and only if homotopy equivalent. The only ones are T^2, $\mathbb{RP}^2$, and connected sums thereof. The 0th sum is S^2, and we have the relation $T^2\#\mathbb{RP}^2 \cong \mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2$ (can be checked via polygon construction). Moreover, $\mathbb{RP}^2\#\mathbb{RP}^2$ is the Klein bottle. In general, $M_g = T^2\#...\#T^2$ and $N_g = \mathbb{RP}^2\#...\#\mathbb{RP}^2$ (g times).

The polygon construction for M_g (orientable) is a $4g$-gon labeled $a_1, b_1, a_1', b_1', a_2, ..., b_g'$, with a_i, a_i' having opposite orientation and b_i, b_i' having opposite orientations. For N_g (non-orientable), we can do a $2g$-gon all clockwise with labels $a_1, a_1, a_2, a_2, ..., a_g, a_g$.

A connected sum of n-manifolds involves cutting disks D^n, on from each of the two manifolds, and two disks from S^n, and gluing these. You can mimick this construction with polygons to get the desired polygon constructions.

For M a compact connected 2-manifold with boundary, notice ∂M is a compact 1-manifold without boundary, since $\partial^2 M = \emptyset$. Hence ∂M is a disjoint union of circles. Gluing disks here removes these boundaries, so we see every compact connected 2-manifold with boundary can just be obtained from deleting disks from a compact connected 2-manifold without boundary.

Euler Characteristic

We have $\chi(A\#B) = \chi(A) + \chi(B) - \chi(S^n)$ for n-manifolds A, B. Moreover, $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$, and $\chi(A \times B) = \chi(A) \cdot \chi(B)$.

As some basics, $\chi(T^2) = 1 - 2 + 1 = 0$, $\chi(S^2) = 1 - 0 + 1 = 2$, and $\chi(\mathbb{RP}^2) = 1 - 1 + 1 = 1$. Moreover, $\chi(M_g) = 2 - 2g$, and $\chi(N_g) = 2 - g$. These can be seen directly from the polygon construction.

An alternative proof for $\chi(T^2)$ is to recall $T^2 = S^1 \times S^1$. Then $\chi(S^1) = 1 - 1 = 0$ (or use the *really* high-powered fact that it is compact and odd-dimensional). Then $\chi(S^1 \times S^1) = 0 \cdot 0 = 0$.

Recall that $X \times S^1$ has n-cells of the form (e_n, e_0) and (e_{n-1}, e_1). Hence we have $c_n + c_{n-1}$ cells, where c_n denotes the number of n-cells of X (with $c_{-1} = 0$). Hence $\chi(X \times S^1) = \sum_{i=0}^n (-1)^i (c_i + c_{i-1}) = \sum_{i=0}^n (-1)^i c_i + (-1) \cdot \sum_{i=1}^n (-1)^(i-1) c_{i-1} = \chi(X) - \chi(X) = 0$. Alternatively, $\chi(X \times S^1) = \chi(X) \cdot \chi(S^1) = 0$.

For k-fold covering spaces $\tilde{M} \twoheadrightarrow M$, we have $\chi(\tilde{M}) = k \cdot \chi(M)$.

Definitions and Useful Examples

A *closed* submanifold is an imbedded manifold with a closed image. An *imbedded* manifold is an immersed manifold via an injective map whose domain is homeomorphic to its image. An *immersion* just has df_x non-vanishing.

G&P's definition for $M \subset \mathbb{R}^N$ takes $T_pM = \operatorname{im}(d\phi_0)$, where $\phi : \mathbb{R}^n \to U \subset M \subset \mathbb{R}^N$ is a chart with $\phi(0) = p$.

A topological manifold is a locally Euclidean metric space. Alternatively, we can say it is locally Euclidean and Hausdorff. Then it is metrizable if and only if it is paracompact. We need second countable to get embedding. It is second-countable if and only if it is σ-compact, which, in the connected case, is equivalent to paracompact and hence metrizable (otherwise, we just get σ-compact $\Rightarrow$ paracompact). Recall σ-compact says union of countably many compact subspaces, paracompact says locally finite subcover, and second-countable says countable base, where a base covers and has, for each $x \in B_1 \cap B_2$ a base element $B_x \ni x$ with $B_x \subset B_1 \cap B_2$.

For a smooth manifold, we just need to give an atlas: charts that cover and whose transition functions are smooth.

Fun fact: a manifold is homotopy equivalent to its interior, and every continuous map is homotopic to a smooth map.

Algebraic Topology

The homology of a disjoint union is the sum of homologies. To get the homology of a wedge $X \vee Y$ where $p \in X$ and $q \in Y$ are glued (with $(X, p), (Y, q)$ good pairs), take the good pair $(X \sqcup Y, \{p, q\})$. Then by Hatcher 2.13, since $\widetilde{H}_i(\{p, q\}) = 0$ for all $i > 0$, we instantly see $H_i(X \vee Y) \cong H_i(X \sqcup Y)$ for $i > 1$. For $i = 1$, we get an isomorphism via abelianizing the result from Van Kampen to get $H_1(X \vee Y) \cong H_1(X) \oplus H_1(Y) = H_1(X \sqcup Y)$. This leaves $\widetilde{H}_0(X \vee Y) = \widetilde{H}_0(X) \oplus \widetilde{H}_0(Y)$.

To get top homology, note by Hatcher Theorem 3.26 that if M is closed connected, its top homology is $\mathbb{Z}$ if and only if it is orientable, and is 0 otherwise. Moreover, $H_{n-1}(M)$ is free if M is orientable, and has one $\mathbb{Z}/2\mathbb{Z}$ summand otherwise. Finally, if M is compact connected but with boundary, use Lefshetz duality with $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ coefficients to get that the top homology is zero. Alternatively, look at the double $2M$ of the manifold and see by Lee that if $2M$ is orientable, so too is the regular domain $M \subset 2M$. Then see $H_n(M) = H_{n+1}(2M, M) = 0$.

2 Fall 2010

Problem 1: Let M be a connected smooth manifold. Show that for any two non-zero tangent vectors $v_1 \in T_{x_1}M$ and $v_2 \in T_{x_2}M$, there is a diffeomorphism $\phi : M \to M$ such that $\phi(x_1) = x_2$ and $d\phi(v_1) = v_2$.

Solution: We do this in two steps: first find a (compactly supported) diffeomorphism of M sending arbitrary $x \in M$ to arbitrary $y \in M$. Then find a diffeomorphism of M which fixes arbitrary $x \in M$ and sends arbitrary nonzero $w_1 \in T_xM$ to arbitrary nonzero $w_2 \in T_xM$. (Observe that in the first step, our original v_1 may be sent to a different vector, but that vector will still be nonzero since the derivative map of a diffeomorphism is a linear isomorphism).

We have an equivalence relation on M, where $x \sim y$ if there is a (compactly supported, i.e. identity outside of some compact set) diffeomorphism of M sending x to y. It is clear this is an equivalence relation. It suffices to show that the equivalence classes are open, as then M may be written as a disjoint union of open sets. Since M is connected, it will follow that there is only one equivalence class.

Let $S \subset M$ be an equivalence class, and let $x \in S$ be arbitrary. Pick a chart $\phi : U \to \mathbb{R}^n$ with $x \in U \subset M$ and $\phi(x) = 0$. Let $y \in U$ be arbitrary. Then $\phi(y) = (c_1, ..., c_n) = c \in \mathbb{R}^n$ is nonzero. Consider $X = \sum_{i=1}^n c_i \frac{\partial}{\partial x_i}$, the constant vector field pointing in the direction of c. Take a bump function ψ on $\mathbb{R}^n$ which is 1 on $\overline{B(0,|c|)}$ and 0 outside of $B(0, 2|c|)$. Then ψX is a compactly supported vector field on $\mathbb{R}^n$, and hence on U (pushforward via $(\phi^{-1})_*$). We will be a bit sloppy and just say ψX itself is a vector field on U. It is 0 outside of some compact subset of U, and hence can be globally extended to be 0 outside of this set. Thus we have a compactly supported global vector field Y on M. This gives us a global flow $\Phi : \mathbb{R} \times M \to M$ with $\Phi_t = \Phi(t, -)$ a diffeomorphism for each $t \in \mathbb{R}$.

Note that geometrically, Y is the same as X at and near $x, y \in \phi^{-1}\overline{B(0,|c|)}$. Now $\gamma(t) = \phi^{-1}(tc)$ gives a curve in M with $\gamma(0) = x, \gamma(1) = y$. In fact, its image is entirely contained in $\overline{B(0,|c|)}$. Notice $\phi(\gamma(t)) = tc$ has constant derivative of c, so it is integral to X. But since its image is entirely in $B(0, |c|)$, it is integral to Y. Hence it must be equal to $\Phi(-, x)$. So $y = \gamma(1) = \Phi(1, x) = \Phi_1(x)$. So we have a diffeomorphism of M, namely Φ_1, sending x to y.

Since $y \in U$ was arbitrary, we conclude $x \in U \subset S$. In particular, $x \in S$ was arbitrary and we found an open neighborhood $x \in U$ contained in S. Hence S is open. By previous remarks, we conclude $S = M$.

For the second step, it suffices to do the following: give a flow on $\mathbb{R}^n$ sending arbitrary nonzero $v_1 \in \mathbb{R}^n$ to arbitrary nonzero $v_2 \in \mathbb{R}^n$ at $t = 1$, but fixing the origin throughout. This corresponds to a vector field on $\mathbb{R}^n$, which may be bumped to be compactly supported and hence globally extended, but locally giving the same flow as long as the relevant integral curve is in the compact subset which we are bumping (which may easily be arranged, since the integral curve itself is compact).

To get the desired flow, it suffices to consider $t \mapsto e^{tX}$, for $X \in M_n(\mathbb{R})$, which is a Lie group homomorphism from $\mathbb{R}$ to $GL_n(\mathbb{R})$. This gives us a flow on $\mathbb{R}^n$ via $(t, v) \mapsto e^{tX}v$. If A is a matrix sending v_1 to v_2 such that $A = e^B$, then the flow $(t, v) \mapsto e^{tB}v$ has the desired properties, since $(1, v_1) \mapsto e^B v_1 = Av_1 = v_2$. So it suffices to show that we may find such a matrix A.

In fact, $sl_n(\mathbb{R})$, the set of skew symmetric matrices, surjects onto $SO_n(\mathbb{R})$ via $B \mapsto e^B$. (Notice that if $B^T = -B$, then B and B^T commute, so that $e^B e^{B^T} = e^{B+B^T} = e^0 = I$).

In this way, we may send v_1 to w, which differs from v_2 by a *positive* scalar multiple, via a matrix exponential by just using a matrix in $SO_n(\mathbb{R})$. For $n > 1$, this is always possible via rotation on a plane containing v_1 and v_2. Finally, we may send w to $\alpha w = v_2$ (for $\alpha > 0$) via the matrix exponential $e^{\log(\alpha)I}$. (This proof doesn't work in the $n = 1$ case when v_1 is a negative multiple of v_2).

Finally, in the $n = 1$ case, the only connected 1-manifolds are (up to diffeomorphism) $(0, 1), [0, 1), [0, 1]$ or S^1. In each case we can explicitly write down an orientation reversing diffeomorphism fixing a point. We can, via the first step, assume WLOG that the point fixed is in the interior. In the first 3 cases, $f(x) = 1 - x$ suffices, fixing $x = 1/2$. In the last case, take a reflection followed by a rotation sending $(1, 0)$ back to $(1, 0)$. □

Problem 2: Let X, Y be submanifolds of $\mathbb{R}^n$. Prove that for almost every $a \in \mathbb{R}^n$, the translate $X + a$ intersects Y transversely.

Theorem: Let N, M be manifolds and $X \subset M$ an embedded submanifold. Take $\{F_s : s \in S\}$ a smooth family of maps $F_s : N \to M$ (in the sense $F : N \times S \to M$ given by $F(s, x) = F_s(x)$ is smooth). If F is transverse to X, then F_s is transverse to X for almost all $s \in S$.

Solution: Consider $F : X \times \mathbb{R}^n \to \mathbb{R}^n$ with $F(x, a) = x + a$ (where $X \subset \mathbb{R}^n$). We claim F is transverse to Y. For this, it suffices to show for each $(x_0, a_0) \in F^{-1}Y$,

$$dF_{(x_0,a_0)}T_{(x_0,a_0)}(X \times \mathbb{R}^n) + T_yY = T_y\mathbb{R}^n = \mathbb{R}^n$$

where $y = F(x_0, a_0) = x_0 + a_0$.

In fact, this will hold trivially, as we claim the first term already gives all of $\mathbb{R}^n$. Notice

$$dF_{(x_0,a_0)} : T_{x_0,a_0}(X \times \mathbb{R}^n) \to T_y\mathbb{R}^n = \mathbb{R}^n$$

is a linear map between vector spaces. Observe for $a_1 \in \mathbb{R}^n$ arbitrary, we have $\gamma(t) = (x_0, a_0 + a_1t) \in X \times \mathbb{R}^n$ is a curve going through (x_0, a_0) at time $t = 0$. Hence $\gamma'(0) = (0, a_1)$ is a tangent vector in $T_{x_0,a_0}(X \times \mathbb{R}^n)$. (The tangent space of a product is the product of tangent spaces). To get its image $dF_{(x_0,a_0)}(0, a_1)$, we may compute $\frac{d}{dt}|_{t=0}(F \circ \gamma)(t)$. Since $(F \circ \gamma)(t) = x_0 + a_0 + a_1t$, we see this gives $a_1 \in \mathbb{R}^n$. Since a_1 was arbitrary, we conclude $dF_{(x_0,a_0)}$ is always surjective (so that F is a submersion, though we don't need this). Hence F intersects Y transversally.

By the theorem, we conclude $F_a : X \to \mathbb{R}^n$ intersects Y transversally for almost every $a \in \mathbb{R}^n$. Notice F_a is the restriction of an automorphism $T_a : \mathbb{R}^n \to \mathbb{R}^n$ to X (namely, $T_a(x) = x + a$). Of course, $F_a(X) = T_a(X) = X + a$. Then F_a can be thought of as the composition of the diffeomorphism $\phi : X \xrightarrow{\sim} X + a$ followed by the inclusion $i : X + a \hookrightarrow \mathbb{R}^n$. Then $d(F_a)_{x_0}T_{x_0}X = di \circ d\phi T_{x_0}X = di(T_{x_0+a}(X + a))$. Hence the condition of transversality of F_a to Y is equivalent to transversality of $X + a$ to Y (which is to say $i : X + a \to \mathbb{R}^n$ is transverse to Y). $\square$

Problem 3:

a) Show $SL_n(\mathbb{R})$ is a smooth submanifold.

Solution: It suffices to show 1 is a regular value of $\det : M_n(\mathbb{R}) \to \mathbb{R}$. Since the tangent spaces of these manifolds are themselves, we have

$$d(\det)_A : M_n(\mathbb{R}) \to \mathbb{R}$$

We claim this has full rank for all $A \in \det^{-1}(1)$. Notice

$$d(\det)_A(B) = \lim_{h\to 0} \frac{\det(A+hB) - \det(A)}{h} = \lim_{h\to 0} \frac{\det(A)(\det(I+hA^{-1}B) - 1)}{h}$$
$$= \lim_{h\to 0} \frac{\det(I+hA^{-1}B) - 1}{h}$$

Taking $B = kA$ for $k \in \mathbb{R}$, we get

$$d(\det)_A(kA) = \lim_{h\to 0} \frac{\det((1+hk)I) - 1}{h} = \lim_{h\to 0} \frac{(1+hk)^n - 1}{h} = \lim_{h\to 0} kn(1+hk)^{n-1} = kn$$

Since $k \in \mathbb{R}$ was arbitrary, we conclude $d(\det)_A$ is surjective, and hence of full rank.

By the regular value theorem, $SL_n(\mathbb{R}) = \det^{-1}(1)$ is a codimension 1 submanifold of $M_n(\mathbb{R})$. $\square$

b) Identify its tangent space at the identity matrix.

Solution: By the regular value theorem, $T_I SL_n(\mathbb{R})$ is the kernel of $d(\det)_I : M_n(\mathbb{R}) \to \mathbb{R}$. Taking $A = I$ in the calculation above, we see

$$d(\det)_I(B) = \lim_{h\to 0} \frac{\det(I+hB) - 1}{h} = tr(B)$$

To see the last inequality, let $\lambda_1, ..., \lambda_n$ be the (generalized) eigenvalues of B. Then $I + hB$ has eigenvalues $1 + h\lambda_i$ for $i = 1, ..., n$. Hence, $\det(I + hB) = \prod_{i=1}^{n}(1 + h\lambda_i) = 1 + h \cdot tr(B) + h^2 \cdot p(h)$, where $p(h)$ is a polynomial in h. From this, the limit is clear.

Finally, we have $T_I SL_n(\mathbb{R}) = \ker(d(\det)_I) = \ker(B \mapsto tr(B)) = \{B \in M_n(\mathbb{R}) : tr(B) = 0\}$. $\square$

c) Show $SL_n(\mathbb{R})$ has trivial Euler characteristic.

Theorem: Poincare-Hopf: The Euler characteristic of a compact, connected orientable manifold is 0 if and only if it has a non-vanishing vector field.

Solution: Let $r : SL_n(\mathbb{R}) \to SO_n(\mathbb{R})$ be given by sending each matrix to its orthogonal matrix in polar decomposition. (Writing $A = UP$ with $U \in O_n(\mathbb{R})$ and P positive definite, we see $\det(A) = 1$ so $\det(U)\det(P) = 1$. However, $\det(P) > 0$ and $\det(U) = \pm 1$, so $\det(U) = \det(P) = 1$, and $U \in SO_n(\mathbb{R})$).

In fact, r is a retract, with $r|_{SO_n(\mathbb{R})} = id_{SO_n(\mathbb{R})}$. Even more is true! It turns out this is a deformation retract! Take the straight line homotopy:

$$H : SL_n(\mathbb{R}) \times [0,1] \to SL_n(\mathbb{R})$$

given by $(A,t) \mapsto \frac{(1-t)A+tU}{\det((1-t)A+tU)}$, where $U = r(A)$. Notice the denominator is never 0, since for $A = UP$, we have

$$(1-t)A + tU = U((1-t)P + tI)$$

Note that U is invertible since it is orthogonal, and $(1-t)P + tI$ has eigenvalues $(1-t)\lambda_i + t \cdot 1$, where $\lambda_i > 0$ are the eigenvalues of P. Since this is a convex combination of positive values, it is never 0.

Next, observe $(A,0) \mapsto \frac{A}{\det(A)} = A$, and $(A,1) \mapsto \frac{U}{\det(U)} = U$. Hence, this is a homotopy between the identity on $SL_n(\mathbb{R})$, and the retract $i \circ r : SL_n(\mathbb{R}) \to SL_n(\mathbb{R})$ (where $i : SO_n(\mathbb{R}) \to SL_n(\mathbb{R})$ is the inclusion).

Hence $SL_n(\mathbb{R})$ deformation retracts to $SO_n(\mathbb{R})$. The latter is compact and connected: it is certainly closed and connected since it is a connected component of the preimage of 0 under $A \mapsto AA^T - I$ (see Spring 2010 Problem 1); it is bounded since each entry has norm at most 1. By Poincare-Hopf, it suffices to show $SO_n(\mathbb{R})$ has a nonvanishing vector field.

In fact, every Lie group has a nonvanishing vector field. To see this, let G be a Lie group, and let $0 \neq v \in T_I G$ be a nonzero vector. Then since $m_g : G \to G$ given by $m_g(h) = gh$ is a diffeomorphism, we have $d(m_g)|_I$ is an isomorphism between $T_I G$ and $T_g G$. Define X a vector field on G via $X(g) = d(m_g)|_I(v)$. It is indeed nonvanishing (since $v \neq 0$ and $d(m_g)|_I$ is bijective).

Hence by Poincare-Hopf, $SO_n(\mathbb{R})$ has Euler characteristic 0 (since it is compact connected and has a non-vanishing vector field). Since $SL_n(\mathbb{R})$ is homotopic to $SO_n(\mathbb{R})$, we conclude that $SL_n(\mathbb{R})$ has Euler characteristic 0 as well. $\square$

Problem 4:

a) Let $f_0, f_1 : M \to N$ be smooth. Define the notion of a chain homotopy between f_0^* and f_1^* (induced maps on the cochain complexes $\Omega^*(N) \to \Omega^*(M)$).

A cochain homotopy between f_0^* and f_1^* is a collection of linear maps

$$h_n : \Omega^n(N) \to \Omega^{n-1}(M)$$

with $\boxed{f_1^* - f_0^* = dh + hd}$.

If we have such maps h, then $f_1^* = f_0^*$ as maps on cohomology, as follows: if ω is a closed form, $d\omega = 0$, so that $f_1^*\omega - f_0^*\omega = d(h(\omega))$ is exact, and so $[f_1^*\omega] = [f_0^*\omega]$.

b) Let X be a smooth vector field on compact manifold M. Let $\phi_t : M \to M$ be the flow generated by X, i.e. the solution to $\frac{d\phi_t}{dt}(x) = X(\phi_t(x))$ with initial condition $\phi_0(x) = x$. Find an explicit chain homotopy between ϕ_0^* and ϕ_1^*. *Hint*: Recall Cartan's magic formula: $\mathcal{L}_X\omega = d \circ i_X\omega + i_X \circ d\omega$.

Recall $(\mathcal{L}_X\omega)_p = \lim_{h\to 0} \frac{(\phi_h^*\omega)_p - \omega_p}{h} = \frac{d}{dt}|_{t=0}(\phi_t^*\omega)_p$. (Pointwise, these things can be thought of as row vectors in T_p^*M). In particular, notice

$$(\mathcal{L}_x(\phi_s^*(\omega)))_p = \frac{d}{dt}\Big|_{t=0}(\phi_t^*\phi_s^*\omega)_p = \frac{d}{dt}\Big|_{t=0}(\phi_{t+s}^*\omega)_p = \frac{d}{dt}|_{t=s}(\phi_t^*\omega)_p$$

Then

$$\begin{aligned}(\phi_1^*\omega)_p - (\phi_0^*\omega)_p &= \int_0^1 (\frac{d}{dt}|_{t=s}(\phi_t^*\omega)_p)ds \\ &= \int_0^1 (\mathcal{L}_X(\phi_s^*\omega))_p ds \\ &= \int_0^1 (d \circ i_X(\phi_s^*\omega))_p ds + \int_0^1 (i_X \circ d(\phi_s^*\omega))_p ds \\ &= \left(d\int_0^1 i_X(\phi_s^*\omega)ds\right)_p + \int_0^1 (i_X \circ (\phi_s^*(d\omega))_p ds\end{aligned}$$

The equality on the first term holds as follows: the form $i_X\phi_s^*\omega := \eta(t)$ may be written in local coordinates as a sum of terms of the form $\alpha_t(x_1, ..., x_n)dx_{i_1} \wedge ... \wedge dx_{i_k}$. Taking d, we get terms of the form $\frac{\partial \alpha_t}{\partial x_j}dx_j \wedge dx_{i_1} \wedge ... \wedge dx_{i_k}$. To integrate with respect to t, we may do this for each coefficient separately. On the other hand, if we first integrate and then apply d, we end up taking partials $\frac{\partial \int_0^1 \alpha_t dt}{\partial x_j}$. For continuously differentiable functions and constant bounds, we have the analytic property

$$\int_0^1 \frac{\partial f_t(\vec{x})}{\partial x_j}dt = \frac{\partial}{\partial x_j}\int_0^1 f_t(\vec{x})dt$$

So these terms indeed commute.

This computation lets us define $h : \Omega^n(M) \to \Omega^{n-1}(M)$ via

$$h\omega = \int_0^1 (i_X\phi_t^*\omega)dt$$

Then notice

$$\phi_1^*\omega - \phi_0^*\omega = d(h\omega) + h(d\omega)$$

so that $\phi_1^* - \phi_0^* = dh + hd$ as desired. □

Remark: In particular, we have shown $\phi_1^* = \phi_0^*$ as maps on cohomology.

Problem 5: Let $\omega = \sum_{i=0}^n dx_{2i-1} \wedge dx_{2i}$ be a 2-form on $\mathbb{R}^{2n}$. We have S^1 acts on $\mathbb{R}^{2n}$ via $e^{it} \in S^1$ corresponds to the linear map $g_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with block diagonal matrix of n copies of CCW rotation by angle t. Define $X(x) = \frac{dg_t(x)}{dt}|_{t=0}$ for any $x \in \mathbb{R}^{2n}$.

a) Compute $\mathcal{L}_X \omega$ and find a function f on $\mathbb{R}^{2n}$ with $df = i_X \omega$.

First, write X in more standard notation. By their definition, for $p = (x_1, ..., x_{2n})$, we have $X_p = \frac{d}{dt} g_t(p)|_{t=0}$, so

$$X_p(x_{2j-1}) = \frac{d}{dt}(\cos(t)x_{2j-1}(p) - \sin(t)x_{2j}(p))|_{t=0} = -x_{2j}(p)$$

$$X_p(x_{2j}) = \frac{d}{dt}(\sin(t)x_{2j-1}(p) + \cos(t)x_{2j}(p)) = x_{2j-1}(p)$$

So

$$X = \sum_{i=1}^n \left(-x_{2j} \frac{\partial}{\partial x_{2j-1}} + x_{2j-1} \frac{\partial}{\partial x_{2j}} \right)$$

Notice

$$(i_X\omega)(Y) = \omega(X,Y) = \sum_{i=1}^n (dx_{2i-1} \wedge dx_{2i})(X,Y) = \sum_{i=1}^n X(x_{2i-1})Y(x_{2i}) - X(x_{2i})Y(x_{2i-1})$$

$$= \sum_{i=1}^n -x_{2i}Y(x_{2i}) - x_{2i-1}Y(x_{2i-1})$$

For $f : \mathbb{R}^{2n} \to \mathbb{R}$, we have

$$(df)(Y) = (\sum_{i=1}^{2n} f_{x_i} dx_i)(Y) = \sum_{i=1}^{2n} f_{x_i} Y(x_i)$$

Thus we seek a function f with $f_{x_i} = -x_i$. Taking $f(x_1, ..., x_{2n}) = -\frac{1}{2}\sum_{i=1}^{2n} x_i^2$, we see

$$(df)(Y) = (i_X\omega)(Y)$$

for all Y, so that $df = i_X\omega$. Finally,

$$\mathcal{L}_X\omega = d(i_X\omega) + i_X(d\omega) = d(df) + i_X(d\omega) = i_X(d\omega) = 0$$

where the last observation comes from the fact that

$$d\omega = d(\sum_{i=1}^n dx_{2i-1} \wedge dx_{2i}) = 0$$

□

b) The S^1 action induces an action on S^{2n-1}. Let $\mathbb{P}^{n-1}$ be the quotient space of S^{2n-1} by this S^1 action. Show that the quotient space has a natural structure of a smooth manifold, and that the tangent space at a point $\underline{x} \in \mathbb{P}^{n-1}$ (i.e. the orbit of a point $x \in S^{2n-1}$) is the quotient of the tangent space $T_x S^{2n-1}$ by the line spanned by $X(x)$, for any $x \in \underline{x}$.

First, $G = S^1 \cong SO(2)$ is a compact Lie group. By Lee Theorem 21.10, the Quotient Manifold Theorem, if G is a Lie group acting smoothly, freely and properly on a smooth manifold M, then M/G is a topological manifold of dimension $\dim M - \dim G$, and has a unique smooth structure with the property that the quotient map is a smooth submersion. By Corollary 21.6, every continuous action by a compact lie group on a manifold is proper. Hence, we get proper for free. To see $G = S^1$ acts freely on $M = S^{2n-1}$, notice $g_t(x) = x$ for all $x \iff t = 0$ (this is even true for $n = 1$, and $n > 1$ is stronger).

Hence M/G is a manifold of dimension $2n-2$. In particular, notice for any point $x \in S^{2n-1}$, $d\pi_x : T_xM \to T_{\pi(x)}(M/G)$ is surjective, so $d\pi_x(T_xM) = T_{\pi(x)}(M/G)$. In particular, $T_x(M/G) \cong T_xM/\ker(d\pi_x)$. It suffices to compute $d\pi_x$.

But notice $\pi(x) \in M/G$ is a regular value of π, so by the regular value theorem, $\pi^{-1}\pi(x) = G.x$ is a 1-manifold in M, and $T_x(G.x) = \ker(d\pi_x)$.

Finally, $T_x(G.x)$ can be computed as follows: notice $\gamma : \mathbb{R} \to G.x$ via $\gamma(t) = g_t(x)$ is a curve in $G.x$ with $\gamma(0) = x$. Then $\gamma'(0) = X_x$ is a tangent vector in $T_x(G.x)$. Notice it is nonzero since X is non-vanishing on S^{2n-1}, as seen from its coordinate expression. Hence, $T_x(G.x)$ is spanned by X_x, so that $T_{\pi(x)}(M/G) = T_xM/span(X_x)$, as desired.

c) Show ω descends to a well-defined 2-form θ on $\mathbb{P}^{n-1}$ and that the 2-form is closed.

First, we view ω as a 2-form on S^{2n-1} by pulling back via $i : S^{2n-1} \to \mathbb{R}^{2n}$. Write $\eta = i^*\omega$.

For $g \in G$ (corresponding to angle t), $m_g : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a diffeomorphism. Notice then that

$$m_g^*\omega = \sum_{i=1}^{n} d(x_{2i-1} \circ m_g) \wedge d(x_{2i} \circ m_g)$$

$$= \sum_{i=1}^{n} d(\cos(t)x_{2i-1} - \sin(t)x_{2i}) \wedge d(\sin(t)x_{2i-1} + \cos(t)x_{2i}) = \sum_{i=1}^{n} dx_{2i-1} \wedge dx_{2i} = \omega$$

Moreover, let $h_g = \tilde{m_g} : S^{2n-1} \to S^{2n-1}$ be the restriction to S^{2n-1}. Then notice $i \circ h_g = m_g \circ i$. Hence,

$$h_g^*\eta = h_g^* i^*\omega = (i \circ h_g)^*\omega = (m_g \circ i)^*\omega = i^* m_g^*\omega = i^*\omega = \eta$$

So we see η is G-invariant.

Next, note that geometrically, X can be restricted to a vector field on S^{2n-1}, since it is orthogonal to p at each point $p \in S^{2n-1}$. Let Y be this vector field (with $(i_*Y)_p = X_p$ for all $p \in S^{2n-1}$).

Next, define θ a form on M/G as follows: $\theta(X_1, X_2)_p = \eta(W, Z)_q$, where $\pi(q) = p$, $\pi_*W = X_1$, $\pi_*Z = X_2$. To see the well-definedness, first, fix choice of q. It suffices to check (by the fact that η is an alternating form) that θ is well-defined regardless of choice of W. Thus, we must check $\eta(W_1, Z)_q = \eta(W_2, Z)_q$, where $\pi_*W_1 = \pi_*W_2$. Then notice $(W_1 - W_2)_q$ is in the span of Y_q, so it suffices to check $\eta(Y, Z)_q = 0$ for any Z. But

$$\eta(Y, Z)_q = (\omega)(X, i_*Z)_q = (i_X\omega)(i_*Z)_q = (df)(i_*Z)_q = (i^*df)(Z)_q = (d(f \circ i))(Z)_q = 0$$

since $f \circ i$ is constant.

Hence η is well-defined independent of choice of vectors. To see it is independent of choice of q, let q_1, q_2 have $\pi(q_i) = p$. Pick $g \in G$ with $g.q_1 = q_2$. Moreover, $\pi \circ h_g = \pi$. Hence, if $W_i, Z_i \in T_{q_i}(M)$ map to $X_1, X_2 \in T_p(M/G)$ respectively via $(d\pi)_{q_i}$, then since η is G-invariant, we have

$$\eta(W_1, Z_1)_{q_1} = (h_g^*\eta)(W_1, Z_1)_{q_1} = \eta((h_g)_*W, (h_g)_*Z)_{q_2} = \eta(W_2, Z_2)_{q_2}$$

by the independence of the choice of vectors. So θ is well-defined. Moreover,

$$(\pi^*\theta)_q(X, Y) = \theta_{\pi(q)}(\pi_*X, \pi_*Y) = \eta_q(X, Y)$$

by definition of θ. Hence $\pi^*\theta = \eta$, as desired.

Moreover, θ is unique, since π^* is injective on forms, as follows: if $\pi^*\lambda = 0$, then $(\pi^*\lambda)(Y_1, ..., Y_k)_p = 0$ for all Y_i, so that $\lambda(d\pi Y_1, ..., d\pi Y_k) = 0$. Since $d\pi$ is surjective, we conclude $\lambda = 0$.

Now $d\eta = d(i^*\omega) = i^*(d\omega) = 0$, so that $0 = d(\pi^*\theta) = \pi^*d\theta$. Since π^* is injective, $d\theta = 0$, and θ is closed.

d) Is θ exact?

Skip!

Problem 6: If $f : S^n \to S^n$ has degree not equal to $(-1)^{n+1}$, show f has a fixed point.

Suppose f does not have a fixed point. Then write

$$H : [0,1] \times S^n \to S^n$$

$$H(t,x) = \frac{(1-t)f(x) + t(-x)}{|(1-t)f(x) + t(-x)|}$$

Notice H is well-defined since $(1-t)f(x) + t(-x) = 0 \iff (1-t)f(x) = tx$. Taking norms of both sides, we see $1 - t = t$, so $t = 1/2$, and $f(x) = x$. Since f has no fixed points, we see $(1-t)f(x) + t(-x) \neq 0$ for any t, x. Hence H is a homotopy between f and the antipodal map $S^n \to S^n$ via $x \mapsto -x$, which has degree $(-1)^{n+1}$ as it is a composition of $n+1$ reflections. Hence $\deg(f) = (-1)^{n+1}$. $\square$

Problem 7:

a) Let G be a finitely presented group. Show that there is a topological space with fundamental group $\pi_1(X) \cong G$.

> Create a wedge of circles, one for each generator of G. Attach a 2-cell via each relation. (Each relation gives a loop and hence a map from S^1 to X_1, the 1-skeleton).
>
> Notice that attaching this two cell makes the corresponding loop null-homotopic, as that loop can be brought up through the 2-cell to make it nullhomotopic (in the disk). No other loops are in the kernel - see Proposition 1.26 in Hatcher.

b) Give an example of X in the case of $G = \mathbb{Z} * \mathbb{Z}$.

> Take $X = S^1 \vee S^1$.

c) How many connected, 2-sheeted covering spaces does the space X from (b) have?

> There are two ways to do this problem. First, we may use the correspondence that connected covering spaces (keeping track of base-point) correspond to subgroups of $\pi_1(X)$ (with the fundamental group of the covering space equaling that subgroup), and conjugacy classes of subgroups correspond to ignoring the base point. The index corresponds to the number of sheets. In this case, we are seeking index 2 subgroups of G. Since they are normal, they are conjugate if and only if they are equal. Moreover, quotienting out by the subgroup gives a surjective group homomorphism to $\mathbb{Z}/2\mathbb{Z}$, and each index 2 subgroup appears precisely once as the kernel of such a morphism. So we simply count surjective homomorphisms to $\mathbb{Z}/2\mathbb{Z}$, and there are $2 * 2 - 1 = 3$ such morphisms (one of either $a, b \in G = \langle a, b \rangle$ must go to $1 \in \mathbb{Z}/2\mathbb{Z}$).
>
> Alternatively, we may use Hatcher's correspondence for covering spaces. 2-sheeted connected covering spaces of $S^1 \vee S^1$ correspond to connected graphs on 2 vertices, with each vertex having 4 edges, 2 incoming and 2 outgoing, with one incoming edge a, one outgoing edge a, one incoming edge b, and one outgoing edge b. (A loop, thus, counts as both incoming and outgoing). It is easy to see there are only 3 such graphs.
>
> Finally, the remark that connects these two constructions is the following observation: if $X = X_G$ is the Cayley-complex for a group G (i.e. X is constructed as in part a), we may construct the universal cover $\tilde{X}$ for X as follows: let the vertices of $\tilde{X}$ be the elements of g. Let there be directed edges from each $g \in G$ to $gg_\alpha \in G$ for each generator g_α. Attach a 2-cell for each loop determined by a relation (starting at any vertex in the graph).
>
> Notice G acts on $\tilde{X}_G$ by left multiplication, and this gives all of the deck transformations. Moreover, $\tilde{X}_G/G = X_G$.
>
> To get any other covering space, take $\tilde{X}_G/H$, where $H \subset G$ is the corresponding subgroup.
>
> In the case of $G = \mathbb{Z} * \mathbb{Z}$, we have $X_G = S^1 \vee S^1$, and the universal cover is an infinite bipartite tree with each vertex having degree 4, with directed edges via right multiplication. If we mod out by a finite index subgroup, we get the corresponding connected finite graph with the properties described in the first paragraph.

Problem 8: Let G be a connected topological group. Show that $\pi_1(G)$ is abelian.

This requires the Eckmann-Hilton argument. Let X be a set with two binary operations $(\cdot, \times)$, both unital, and with

$$(a \cdot b) \times (c \cdot d) = (a \times c) \cdot (b \times d)$$

for all $a, b, c, d \in X$. Then $\cdot = \times$ and both are commutative and associative.

This can be used to show that the group objects in the category of groups are precisely the abelian groups. Then we may use the fact that a functor sending terminal objects to terminal objects and products to products sends group objects to group objects. Since a topological group is a group object in the category of topological spaces, but also in the category of pointed topological spaces by taking the point to be the identity, and π_1 sends products to products (and in fact is a right adjoint) and sends the terminal object, a one point space, to 0, the terminal object in Group, we conclude π_1 must send topological groups to abelian groups.

Here is an alternative proof: we may apply Eckman-Hilton directly to define a second product on $\pi_1(G, e)$. Define

$$[\gamma] \times [\alpha] = [\gamma(t) \cdot \alpha(t)]$$

where $\cdot$ is the multiplication in G. Notice the RHS is still a loop, since $\gamma(0)\alpha(0) = e^2 = e = \gamma(1)\alpha(1)$ in G. Moreover, we may multiply homotopies pointwise to check that this is indeed well-defined. We will apply the Eckman-Hilton argument to see $\times = \circ$, where $\circ$ is the composition operation in $\pi_1(G, e)$, and hence that $\pi_1(G, e)$ is abelian.

Notice $\times$ is unital, since the constant map $\gamma(t) = e$ for all $t \in [0, 1]$ serves as a two-sided unit. Moreover,

$$([a] \times [b]) \circ ([c] \times [d]) = [a(t) \cdot b(t)] \circ [c(t) \cdot d(t)] = ([a] \circ [c]) \cdot ([b] \circ [d])$$

where the last step follows from the fact that $\circ$ follows the first path at twice the speed from $t = 0$ to $t = 1/2$, and then follows the second path at twice the speed from $t = 1/2$ to $t = 1$. So both of the last two loops follow $[a(t) \cdot b(t)]$ for $t = 0$ to $1/2$ (at twice the speed) and $[c(t) \cdot d(t)]$ for $t = 1/2$ to $t = 1$ (at twice the speed).

Here is a direct proof that bypasses Eckman-Hilton: by construction, we see both operations have the same unit, the constant map at e. Moreover,

$$[a] \circ [b] = ([e] \times [a]) \circ ([b] \times [e]) = ([e] \circ [b]) \times ([a] \circ [e]) = [b] \times [a]$$

$$= ([b] \circ [e]) \times ([e] \circ [a]) = ([b] \times [e]) \circ ([e] \times [a]) = [b] \circ [a]$$

Hence $(\pi_1(G, e), \circ)$ is abelian, as desired.

Problem 9: If $\mathbb{R}^m$ and $\mathbb{R}^n$ are homeomorphic, then $m = n$.

$\mathbb{R}^1$ can be distinguished from the rest since deleting a point leaves it disconnected. So assume $n, m > 1$. Remove a point from each space and deformation retract them to S^{m-1} and S^{n-1}, and then take the $(n-1)$st homology to see $H_{n-1}(S^{m-1}) = \mathbb{Z}$, so that, since $n-1 \neq 0$, $n-1 = m-1$, and $n = m$.

Alternatively, take their one-point compactifications to see $S^n \cong S^m$, so that $H_n(S^m) = \mathbb{Z}$, so that $n = m$ (since $n \neq 0$).

Problem 10: Let N_g be the genus g non-orientable surface, i.e. the connected sum of g copies of $\mathbb{RP}^2$. Calculate $\pi_1(N_g)$ and the homology groups of N_g.

Recall the polygon construction of N_g involving $2g$ sides and oriented edges $a_1, a_1, a_2, a_2, ..., a_g, a_g$ all oriented CCW, with a 2-cell attached via the word $a_1^2...a_g^2$. From this we see

$$\pi_1(N_g) = \langle a_1, ..., a_g | a_1^2 ... a_g^2 \rangle$$

Abelianizing, we see that in our chain complex, we will get $\partial F = 2a_1 + 2a_2 + ... + 2a_g$. From this, it is easy to see $H_2(N_g) = 0$ since the corresponding map $\mathbb{Z} = C_2 \to C_1 = \mathbb{Z}^g$ via $F \mapsto \sum_{i=1}^g 2a_i$ is injective.

Moreover, $C_1 \to C_0$ is the 0-map since there is only one vertex in the polygon construction, so each edge maps to $v - v = 0$. Hence,

$$H_1(N_g) = \mathbb{Z}^g / \langle (2, ..., 2) \rangle = \mathbb{Z}^{g-1} \times \mathbb{Z}/2\mathbb{Z}$$

Finally, $H_0(N_g) = C_0/0 = C_0 = \mathbb{Z}$, and $H_k(N_g) = 0$ for $k > 2$ since $C_k = 0$ for $k > 2$.

In short,

$$H_k(N_g) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}^{g-1} \times \mathbb{Z}/2\mathbb{Z} & k = 1 \\ 0 & k > 1 \end{cases}$$

3 Spring 2011

Problem 1: If V is a smooth vector field on an n-manifold M and $V_p \neq 0$ for some $p \in M$, show that we may find a chart (U, x) around p with $V = \frac{\partial}{\partial x_1}$.

Since we only care about a local property, it suffices to prove this for $M = \mathbb{R}^n$, $p = 0$ and $X_0 = \frac{\partial}{\partial t^1}|_0$ (where we may get the last property by rotating and rescaling to get X_0 to match as needed).

In general,

$$X_p = \sum_j f_j(p) \frac{\partial}{\partial t^j}|_p$$

Note $f_j(0) = \delta_{1,j}$ since $X_0 = \frac{\partial}{\partial t^1}|_0$.

Let ϕ_t be a local flow corresponding to X near the origin. That is, find some $U \ni 0$ and $I = (-\epsilon, \epsilon)$ with $\phi : I \times U \to U$ where for each $p \in U$, we have

$$\phi_p : I \to U$$

$$\phi_p(0) = p$$

$$\phi_p'(t) = X_{\phi_p(t)}$$

Define

$$\psi(a^1, ..., a^n) := \phi(a^1, (0, a^2, ..., a^n))$$

For notational simplicity, we do the remainder of the proof for $n = 2$, but it easily generalizes.

We have $X = f_1(x, y)\frac{\partial}{\partial x} + f_2(x, y)\frac{\partial}{\partial y}$, with $f = (f_1, f_2)$ having $f(0, 0) = (1, 0)$.

Now define $\psi(x, y) = \phi(x, (0, y))$ on some open set. Notice

$$\frac{\partial}{\partial x}\psi(x, y) = f(\psi(x, y))$$

To see this, notice

$$\frac{\partial}{\partial x}\psi(x, y) = \frac{\partial}{\partial x}\phi_{(0,y)}(x) = X_{\phi_{(0,y)}(x)} = X_{\psi(x,y)} = f(\psi(x, y))$$

Next, we claim $(d\psi)_0 = id$. To see this, notice the first column of $d\psi_0$ is $\frac{\partial}{\partial x}\phi(x, (0, y))|_{0,0}$ and the second column is $\frac{\partial}{\partial y}\phi(x, (0, y))|_{0,0}$. From previous remarks, the first is

$$f(\psi(x, y))|_{(0,0)} = f(\psi(0, 0)) = f(\phi(0, (0, 0))) = f(0, 0) = (1, 0)$$

Meanwhile for $\frac{\partial}{\partial y}\phi(x, (0, y))|_{(0,0)}$ fixes $x = 0$, and $\phi(0, (0, y)) = (0, y)$, so this equals $\frac{\partial}{\partial y}(0, y)|_{0,0} = (0, 1)$. Hence, we see $(d\psi)_0 = id$.

By the inverse function theorem, ψ is invertible in some neighborhood of 0. Define $(z, w) = \psi^{-1}(x, y)$ as a new coordinate system around 0. (In general, write $\vec{z} = \psi^{-1}(\vec{x})$).

Notice

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y}$$

Meanwhile, $(x, y) = \psi(z, w)$, so $(\frac{\partial x}{\partial z}, \frac{\partial y}{\partial z}) = \frac{\partial}{\partial z}\psi(z, w) = f(\psi(z, w)) = (f_1(x, y), f_2(x, y))$ Thus we see

$$\frac{\partial}{\partial z} = f_1(x, y)\frac{\partial}{\partial x} + f_2(x, y)\frac{\partial}{\partial y} = X$$

as desired. $\square$

Problem 2:

a) Show Cartan's magic formula: $\mathcal{L}_X = di_X + i_X d$.

It suffices to work locally. Moreover, by linearity, it suffices to consider forms of the form $f dx_1 \wedge ... \wedge dx_n$.

First, we will show this holds for 0-forms. Then, we will show that if it holds for $k-1$-forms, then it holds for exact 1-forms wedged with $k-1$-forms. Since each form $f dx_1 \wedge ... \wedge dx_n = dx_1 \wedge (f dx_2 \wedge ... \wedge dx_n)$, the result will follow.

To see this holds for 0-forms f, notice $i_X f = 0$, so it suffices to check $\mathcal{L}_X f = i_X df = df(X) = X(f)$. Recall

$$\mathcal{L}_X(f) = \lim_{h\to 0} \frac{\phi_h^* f - f}{h} = \lim_{h \to 0} \frac{f \circ \phi_h - f}{h}$$

Where ϕ_h is the flow corresponding to X. That is,

$$(\mathcal{L}_X(f))_p = \lim_{h \to 0} \frac{f(\phi_h(p)) - f(p)}{h}$$

To compute $(Xf)_p$, one must find a curve γ going through p at $t=0$ with $\gamma'(0) = X_p$. Then, $(Xf)_p = (f \circ \gamma)'(0)$. Taking $\gamma(t) = \phi_t(p)$, the result follows.

Next, suppose that the formula holds for all $k-1$-forms. Consider $dx \wedge \eta$, where η is a $k-1$-form. Then using the fact that Lie derivative commutes with exterior derivative and that the Lie derivative of a wedge follows product rule, we get

$$\mathcal{L}_X(dx \wedge \eta) = \mathcal{L}_X(dx) \wedge \eta + dx \wedge \mathcal{L}_X(\eta) = d(\mathcal{L}_X(x)) \wedge \eta + dx \wedge (i_X d\eta + di_X \eta)$$

$$= d(X(x)) \wedge \eta + dx \wedge i_X d\eta + dx \wedge di_X \eta$$

Meanwhile, i_X of a wedge follows the signed power rule, and d of a wedge does as well.

$$(i_X d + di_X)(dx \wedge \eta) = (i_X d)(dx \wedge \eta) + (di_X)(dx \wedge \eta) = (i_X)(-dx \wedge d\eta) + (d)(i_X(dx \wedge \eta))$$

$$= -(i_X dx) \wedge d\eta + dx \wedge (i_X d\eta) + (d)((i_X dx) \wedge \eta - dx \wedge (i_X \eta)))$$

$$= -(i_X dx) \wedge d\eta + dx \wedge (i_X d\eta) + d((i_X dx) \wedge \eta) - d(dx \wedge (i_X \eta)))$$

$$= -(i_X dx) \wedge d\eta + dx \wedge (i_X d\eta) + (di_X dx) \wedge \eta + (i_X dx) \wedge (d\eta) + dx \wedge (di_X \eta)$$

$$= dx \wedge (i_X d\eta) + d(X(x)) \wedge \eta + dx \wedge (di_X \eta)$$

$\square$

b) Use this to show that a vector field X on $\mathbb{R}^3$ has local flows preserving volume if and only if it has divergence 0.

Note that the flow preserves volume if and only if each pullback of the volume form $\omega = dx \wedge dy \wedge dz$ is equal to the volume form itself, i.e. $\phi_t^*\omega = \omega$ for small t. In particular, this implies $\mathcal{L}_X(\omega) = 0$ from the limit definition. Conversely, if $\mathcal{L}_X(\omega) = 0$, then $\phi_{t_0}^*\mathcal{L}_X(\omega) = 0$, so $\mathcal{L}_X(\phi_{t_0}^*\omega) = 0$ (where we may commute the $\phi_{t_0}^*$ with the limit by continuity). Meanwhile, $\mathcal{L}_X(\phi_{t_0}^*\omega) = \frac{d}{dt}|_{t=t_0}\phi_t^*\omega$. Since t_0 was arbitrary, we see $\phi_t^*\omega$ is constant, so that $\phi_t^*\omega = \phi_0^*\omega = \omega$, as desired. (Do this argument while fixing a point p; this holds for each point p).

Hence we see X preserves volume if and only if $\mathcal{L}_X\omega = 0$. By Cartan's magic formula, $\mathcal{L}_X\omega = di_X\omega + i_X d\omega$. However, $d\omega = 0$ since ω is a volume form (and hence closed, since $d\omega$ would be a 4-form on a 3-dimensional space). Thus, $\mathcal{L}_X\omega = di_X\omega$, and we have for $X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z}$,

$$X \text{ preserves volume} \iff di_X\omega = 0 \iff d(X(x)dy \wedge dz - X(y)dx \wedge dz + X(z)dx \wedge dy) = 0$$

Note that $i_X\omega$ is a 2-form, so we merely needed to solve for the coefficients of the basis vectors $dx \wedge dy, dy \wedge dz, dx \wedge dz$, which can be done by plugging in the appropriate basis vectors into $i_X\omega$. Using the expression for X, we see

$$X \text{ preserves volume} \iff d(f dy \wedge dz - g dx \wedge dz + h dx \wedge dy) = 0 \iff f_x\omega + g_y\omega + h_z\omega = 0$$

$$\iff (f_x + g_y + h_z)\omega = 0 \iff f_x + g_y + h_z = 0 \iff div(X) = 0$$

Problem 3:

a) Explain why there is a closed 2-form on $\mathbb{R}^3 - \{0\}$ which is not exact.

Since $\mathbb{R}^3 - \{0\} \cong S^2$ and $H^2(S^2) \cong \mathbb{R}$, the result follows.

b) For ϕ a form as in part a, show $\frac{\int_{S^2} f^*\phi}{\int_{S^2}\phi}$ is the degree of f. Include an explanation why the denominator is nonzero.

First, write $\Delta^n \subset \mathbb{R}^n$ as the subset $\{(x_1, ..., x_n) : \sum_{i=1}^n x_i \leq 1\}$. For $\sigma : \Delta^n \to M$, define

$$\int_\sigma \omega := \int_{\Delta^n} \sigma^*\omega$$

for closed n-forms ω. Note $\sigma^*\omega$ is an n-form on Δ^n, and we may integrate via usual integration on n-dimensional subspaces of $\mathbb{R}^n$.

This is independent of choice of $\omega \in [\omega]$ and $\sigma \in [\sigma]$.

Next, note $f_* : H_n(M) \to H_n(N)$ sends $[\sigma] \mapsto [f \circ \sigma]$. Now

$$\int_\sigma f^*\omega = \int_{\Delta^n} \sigma^* f^*\omega = \int_{\Delta^n} (f \circ \sigma)^*\omega = \int_{f_*\sigma = f\circ\sigma} \omega$$

This holds for linear combinations of singular simplices (i.e. chains) as well, by linearity of the integral. Next, for M, N closed connected orientable n-manifolds, $H_n(M) = H_n(N) = \mathbb{Z}$. Let $[M] \in H_n(M)$ and $[N] \in H_n(N)$ denote a generator of that group (called a *fundamental class*, which corresponds to a choice of orientation).

Then $f_*[M] = k[N]$ for some $k \in \mathbb{Z}$. By definition, $k = \deg(f)$.

Then notice by the above computation

$$\int_{[M]} f^*\omega = \int_{f_*[N]} \omega = \int_{k[N]} \omega = k \int_{[N]} \omega = \deg(f) \int_{[N]} \omega$$

Finally, a compact oriented embedded manifold admits a smooth triangulation, i.e. $\sigma_i : \Delta^n \to M$ with disjoint interiors, preserving orientation, and whose union is all of M. It turns out $\sum_i \sigma_i = [M]$ gives the fundamental class corresponding to this orientation. Finally, it turns out through this choice of triangulation,

$$\int_{[M]} \omega = \int_M \omega$$

where the RHS is in the usual sense. So we see for all ω,

$$\int_M f^*\omega = \int_{[M]} f^*\omega = \deg(f) \int_{[N]} \omega = \deg(f) \int_N \omega$$

Moreover, recall we have an isomorphism $\int_M : H^n(M) \to \mathbb{R}$ with $\omega \mapsto \int_M \omega$. (This is weaker than de Rham's Theorem). Hence if ω is not closed, $\int_{[N]} \omega$ is nonzero. Alternatively, use problem 4 for the specific case of S^2.

Remark: We have an induced map $H_2(S^2) \xrightarrow{f_*} H_2(S^2)$ which is multiplication by $\deg(f)$. Tensoring with $\mathbb{R}$ gives, by universal coefficient, $H_2(S^2; \mathbb{R}) \xrightarrow{f_*} H_2(S^2; \mathbb{R})$ is also the multiplication by k map ($m_k \otimes id$). We get $H^2_{dR}(S^2) \cong H_2(S^2; \mathbb{R})^*$ from de Rham's theorem via $\omega \mapsto ([c] \mapsto \int_c \omega)$. In this way we get two induced maps on $H_2(S^2; \mathbb{R})^* \to H_2(S^2; \mathbb{R})^*$: one by dualizing the multiplication by k map (which is again a multiplication by k map), and the other by going through $H_2(S^2; \mathbb{R})^* \cong H^2_{dR}(S^2) \xrightarrow{f^*} H^2_{dR}(S^2) \cong H_2(S^2; \mathbb{R})^*$. Our argument shows these induced maps are the same.

Problem 4: Show without deRham's Theorem that a 2-form on the sphere S^2 that has integral 0 is exact.

Lemma: (Poincare Lemma) Closed forms on contractible manifolds are exact.

Take $A = S^2 \setminus N$ and $B = S^2 \setminus S$ where N and S are the north and south pole respectively. Take U to be the southern hemisphere including the equator, and V to be the northern hemisphere including the equator. Note $U \subset A, V \subset B$.

Let ω have $\int_{S^2} \omega = 0$. We get for free that $d\omega = 0$, since ω is a top form.

Note $\omega|_A = i_A^* \omega$ is closed on $A = S^2 \setminus N \cong \mathbb{R}^2$ (since pullback commutes with exterior derivative; or because it is a top form). Hence, it is exact. So write $i_A^*\omega = d\eta$.

Similarly, $i_B^*\omega = d\gamma$.

Next, by $U \cap V = S^1$ (a 1-manifold with measure 0 on S^2) and $U \cup V = S^2$, we have

$$\int_{S^2} \omega = \int_U \omega|_U + \int_V \omega|_V = \int_U (d\eta)|_U + \int_V (d\gamma)|_V$$

Note that if S^2 is oriented with outward facing normal, then U, V both have opposite orientations on $\partial U = \partial V = S^1$. By Stokes,

$$\int_U (d\eta)|_U = \int_U d(\eta|_U) = \int_{\partial U} (\eta|_U)|_{\partial U} = \int_{-S^1} \eta|_{S^1} = -\int_{S^1} \eta_{S^1}$$

$$\int_V (d\gamma)|_V = \int_{S^1} \gamma|_{S^1}$$

Hence

$$\int_{S^2} \omega = \int_{S^1} (\gamma|_{S^1} - \eta|_{S^1})$$

So $\gamma|_{S^1} - \eta|_{S^1}$ is exact by the S^1 case of this result, i.e. that $\int_{S^1} : H^1(S^1) \to \mathbb{R}$ is an isomorphism.

Moroever, $A \cap B$ deformation retracts to $U \cap V = S^1$, and so i^* induces an isomorphism on cohomology (with inverse r^*). Since $i^*(\gamma|_{A\cap B} - \eta|_{A\cap B}) = \gamma|_{S^1} - \eta|_{S^1}$ is exact, we conclude $\gamma|_{A\cap B} - \eta|_{A\cap B}$ is exact.

Write $\gamma|_{A\cap B} - \eta|_{A\cap B} = df$ for $f : A \cap B \to \mathbb{R}$. Pick a partition of unity $\rho_A << A, \rho_B << B$ with $\rho_A + \rho_B = 1$ on $A \cup B = S^2$. Define

$$\theta = \begin{cases} \gamma - d(f \cdot \rho_A) & \text{on } A \\ \eta + d(f \cdot \rho_B) & \text{on } B \end{cases}$$

(Note A, B are open). Then on $A \cap B$, since $\gamma|_{A\cap B} - \eta_{\cap B} = df$, we have

$$\gamma|_{A\cap B} - d(f \cdot \rho_A)|_{A\cap B} - \eta|_{A\cap B} - d(f \cdot \rho_B)|_{A\cap B} = d(f - f \cdot \rho_A - f \cdot \rho_B)|_{A\cap B} = 0$$

since $\rho_A + \rho_B = 1$. Hence θ is well-defined. It is easy to see $d\theta = \omega$, since this holds on open sets A and B, with $A \cup B = S^2$. Thus, ω is exact, as desired. $\square$

Problem 5: Let $U = \mathbb{R}^3 \setminus \{p_1, ..., p_n\}$, where $|p_i| < 1$ (i.e. they are strictly inside the unit sphere). Suppose $V : U \to S^2$ is a smooth map, considered as a unit vector field on U. Explain from basic facts why the degree of $V|_{S^2} : S^2 \to S^2$ is equal to the sum of the indices of the vector field at each point $p_1, ..., p_n$.

Recall the index of p_i is the degree of the map $V|_{S_i} : S_i \to S^2$, where $S_i = \partial D_i$, and $D_i \ni p_i$ is a closed disk contained inside S^2, containing p_i but not containing p_j for $j \neq i$. This degree is independent of choice of D_i.

Let $W = D^2 \setminus \cup_i D_i$. Then to give ∂W an outward pointing normal, we get S^2 disjoint union with each $\partial D_i = S_i$, where the normal vector points outside for S^2 and inside for each S_i (since inside D_i is outside W).

Since degree is just a signed sum of preimages of a regular value, and ∂W is a disjoint union of S^2 and $S_1, ..., S_n$ (oriented in the opposite way), we see

$$\deg(V|_{\partial W}) = \deg(V|_{S^2}) - \sum_{i=1}^{n} \deg(V|_{S_i})$$

where we subtract the usual degree $\deg(V|_{S_i})$ to get their degree in the ∂W signed preimage calculation.

On the other hand, recall

Theorem: Extension Theorem: $f : X \to Y$ a map between k-manifolds, $X = \partial W$. If f can be extended to W, then $\deg(f) = 0$.

Trivially, $V|_{\partial W}$ may be extended to all of W via $V|_W$. Hence, $\deg(V|_{\partial W}) = 0$.

Thus,

$$\deg(V|_{S^2}) = \sum_{i=1}^{n} \deg(V|_{S_i}) = \sum_{i=1}^{n} ind_{p_i}(V)$$

Problem 6: Explain how an SES of chain complex gives rise to an LES of homology.

See Spring 2010 Problem 4.

Problem 7:

a) Define $\mathbb{CP}^n$.

We have $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\}) / \sim$, where $\sim$ is the equivalence relation on $\mathbb{C}^{n+1}$ via $(z_0, ..., z_n) \sim \lambda(z_0, ..., z_n)$ for any $\lambda \neq 0$ in $\mathbb{C}$.

b) Compute the homology and cohomology in $\mathbb{Z}$-coefficients. If you use cell complexes, explain the attaching maps.

Give $\mathbb{CP}^n$ a cell structure with one cell in each even dimension 0 through $2n$. For $n = 0$, we get a point which is also $\mathbb{CP}^0$. Suppose we can construct $\mathbb{CP}^{n-1}$ in this way. Create $\mathbb{CP}^n$ by attaching a $2n$-cell $e^{2n} = D^{2n}$ to $\mathbb{CP}^{n-1}$ as follows:

$$\phi : S^{2n-1} \to \mathbb{CP}^{n-1}$$

$$\phi(z_0, ..., z_{n-1}) = [z_0, ..., z_{n-1}]$$

Call the resulting space $X = \mathbb{CP}^{n-1} \cup_\phi D^{2n}$. We show $X \cong \mathbb{CP}^n$.

Note we may recognize $\mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$ via $[z_0, ..., z_{n-1}] \mapsto [z_0, ..., z_{n-1}, 0]$. Moreover, we have a map

$$f : D^{2n} \to \mathbb{CP}^n$$

$$f(z_0, ..., z_{n-1}) = \left[z_0, ..., z_{n-1}, \sqrt{1 - \sum_{i=0}^{n-1} |z_i|^2}\right]$$

Note that the inclusion and f are each injective, where the injectivity of f follows from the fact that if

$$\left[z_0, ..., z_{n-1}, \sqrt{1 - \sum_{i=0}^{n-1} |z_i|^2}\right] = \left[w_0, ..., w_{n-1}, \sqrt{1 - \sum_{i=0}^{n-1} |w_i|^2}\right]$$

then there exists $\lambda \in \mathbb{C} \setminus \{0\}$ with

$$\left(z_0, ..., z_{n-1}, \sqrt{1 - \sum_{i=0}^{n-1} |z_i|^2}\right) = \left(\lambda w_0, ..., \lambda w_{n-1}, \lambda\sqrt{1 - \sum_{i=0}^{n-1} |w_i|^2}\right)$$

From the last coordinate we see $\lambda \in \mathbb{R}^+$, and from the norm squared of both sides, we see $|\lambda| = 1$, so $\lambda = 1$ and $(z_0, ..., z_{n-1}) = (w_0, ..., w_{n-1})$.

Together, the inclusion and f induce a map from the disjoint union $\mathbb{CP}^{n-1} \sqcup D^{2n} \xrightarrow{i \sqcup f} \mathbb{CP}^n$. We may factor through to a map on X if $i \circ \phi = f|_{S^{2n-1}}$, i.e. if the points glued between $\mathbb{CP}^{n-1}$ and D^{2n} map to the same points in $\mathbb{CP}^n$. This holds true: $f|_{S^{2n-1}}(z_0, ..., z_{n-1}) = [z_0, ..., z_{n-1}, 0] = i \circ \phi(z_0, ..., z_{n-1})$, since $\sum_{i=0}^{n-1} |z_i|^2 = 1$.

Hence we get a map $X \xrightarrow{g} \mathbb{CP}^n$ with $(i \sqcup f) = g\pi$ for $\pi : \mathbb{CP}^{n-1} \sqcup D^{2n} \to X$ the projection. Note g is injective, since if $g(x) = g(y)$, then write $x = \pi(a), y = \pi(b)$ for $a, b \in \mathbb{CP}^{n-1} \sqcup D^{2n}$ (this is possible since π is surjective). Then $g\pi(a) = g\pi(b)$, so $(i \sqcup f)(a) = (i \sqcup f)(b)$. If a, b are both in $\mathbb{CP}^{n-1}$ or both in D^{2n}, the injectivity of i and f respectively will imply $a = b$, so $x = \pi(a) = \pi(b) = y$. WLOG, assume $a \in \mathbb{CP}^{n-1}$ and $b \in D^{2n}$. Then $i(a) = f(b)$. In particular, notice $b \in S^{2n-1}$ since $f(b) = i(a)$ has last homogenous coordinate 0, and hence $\sum_{i=0}^{n-1} |b_i|^2 = 1$. Then since $b \in S^{2n-1}$, $f(b) = i \circ \phi(b)$, and $i \circ \phi(b) = f(b) = i(a)$. By injectivity of i, $\phi(b) = a$, so that $b \sim \phi(b) = a$, and $x = \pi(a) = \pi(b) = y$.

Thus g is injective. Next, it is surjective, since the image of g is equal to the image of i union with the image of f. If $[z_0, ..., z_n] \in \mathbb{CP}^n$, we either have $z_n = 0$, in which case it is in the image of i, or we may divide through by z_n and get an equivalent point $[y_0, ..., y_{n-1}, 1] \in \mathbb{CP}^n$. Dividing through by $\sqrt{1 + \sum_{i=0}^{n-1} |y_i|^2}$, we get an equivalent point $[w_0, ..., w_{n-1}, t]$, with $t > 0$ and $t^2 + \sum_{i=0}^{n-1} |w_i|^2 = 1$, so $t = \sqrt{1 - \sum_{i=0}^{n-1} |w_i|^2}$ and this point is in the image of f.

Finally, we have a continuous bijection from a compact space (X is compact since $\mathbb{CP}^{n-1}, D^{2n}$ are and π is surjective) to a Hausdorff space $\mathbb{CP}^n$. Hence it is a homeomorphism. $\square$

From this the homology is clear since all maps are the 0-map, so $H_k(\mathbb{CP}^n) = \mathbb{Z}$ if $0 \leq k \leq 2n$ is even, and $H_k(\mathbb{CP}^n) = 0$ otherwise. Similarly, dualizing, we see the same complex, so we get $H^k(\mathbb{CP}^n) = H_k(\mathbb{CP}^n)$ for all k.

Problem 8:

a) Find the $\mathbb{Z}$ coefficient homology of $\mathbb{RP}^2$.

We will do the case of $\mathbb{RP}^n$ for any n. Similarly to the previous problem, we can give $\mathbb{RP}^n$ a CW-structure via one cell in each dimension. We attach an n-cell to $\mathbb{RP}^{n-1}$ via the map $S^{n-1} \xrightarrow{\phi} \mathbb{RP}^{n-1}$ the double cover. We can get a homeomorphism $\mathbb{RP}^{n-1} \cup_\phi D^n \cong \mathbb{RP}^n$ in this way. Moreover, the cellular boundary formula tells us that the boundary of this n-cell (its coefficient in the unique $(n-1)$-cell) is the degree of the map $S^{n-1} \xrightarrow{\phi} \mathbb{RP}^{n-1} \xrightarrow{\pi} \mathbb{RP}^{n-1}/\mathbb{RP}^{n-2} = S^{n-1}$, where we crush all other cells to a point. To compute the degree of this, we notice the preimage of a point under π is a single point (and in fact this is a local homeomorphism near that point), provided we do not choose the image of $\mathbb{RP}^{n-2}$ under π. Moreover, the preimage of that point under ϕ is then two antipodal points in S^{n-1}. The degree of the antipodal map on S^{n-1} is $(-1)^n$. Hence if n is even, these points have the same orientation, and if n is odd, they have opposite orientation. Counting signed preimages, we see the degree of this map is 2 if n is even and 0 if n is odd. So we get a chain complex

$$0 \to \mathbb{Z} \to ... \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

If n is even, the top map is 2, so $H_n(\mathbb{RP}^n) = 0$ since this is injective. Otherwise, the map is 0, so $H_n(\mathbb{RP}^n) = \mathbb{Z}$ if n is odd. Meanwhile, notice $H_0(\mathbb{RP}^n) = \mathbb{Z}$ in both cases, and for $0 < k < n$, we have

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & 0 < k < n \text{ odd} \\ 0 & 0 < k < n \text{ even} \end{cases}$$

In summary,

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2\mathbb{Z} & 0 < k < n \text{ odd} \\ 0 & 0 < k < n \text{ even} \\ \mathbb{Z} & k = n \text{ odd} \\ 0 & k = n \text{ even} \end{cases}$$

In particular, $\mathbb{RP}^2$ has homology groups $\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, 0$.

Remark: One may do the simpler case of $\mathbb{RP}^2$ via the polygon constraction.

b) Explain (without Kunneth) how a nonzero element of the 3-homology with $\mathbb{Z}$ coefficients of $\mathbb{RP}^2 \times \mathbb{RP}^2$ arises.

Write $\mathbb{RP}^2 = e_0 \cup e_1 \cup e_2$ and $\mathbb{RP}^2 = f_0 \cup f_1 \cup f_2$ as the cell decompositions of the two copies of $\mathbb{RP}^2$. By previous remarks, $\partial e_1 = 0$, $\partial e_2 = 2e_1$, and similarly for f_1, f_2.

Then $\mathbb{RP}^2 \times \mathbb{RP}^2$ has cells $e_i \times f_j$ of dimension $i + j$ with boundary $\partial(e_i \times f_j) = \partial e_i \times f_j + (-1)^{\dim e_i} e_i \times \partial f_j$. In our case, there is one 0-cell $e_0 \times f_0$, two 1-cells $e_1 \times f_0, e_0 \times f_1$, three 2-cells $e_1 \times f_1, e_0 \times f_2, e_2 \times f_0$, two 3-cells $e_1 \times f_2, e_2 \times f_1$ and one 4-cell $e_2 \times f_2$.

As we are concerned with $H_3(\mathbb{RP}^2 \times \mathbb{RP}^2) = \ker(\partial_3)/\operatorname{im}(\partial_4)$, we notice

$$\partial_3(e_1 \times f_2) = -2e_1 \times f_1$$

$$\partial_3(e_2 \times f_1) = 2e_1 \times f_1$$

So $\ker(\partial_3) = \{(x, x) \in \mathbb{Z}^2 : x \in \mathbb{Z}\}$.

Similarly,

$$\partial_4(e_2 \times f_2) = 2e_1 \times f_2 + 2e_2 \times f_1$$

so $\operatorname{im}(\partial_4) = span((2, 2)) = \{(x, x) \in \mathbb{Z}^2 : x \in 2\mathbb{Z}\}$. So we see $H_3(\mathbb{RP}^2 \times \mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$.

In particular, our nonzero element is $[e_1 \times f_2 + e_2 \times f_1]$ (i.e. $(1, 1) \in \ker(\partial_3)$), which has boundary 0 but is not itself a boundary.

Problem 9:

a) State the Lefshetz Fixed Point Theorem.

Theorem: (Lefshetz Fixed Point Theorem) If $f : X \to X$ is a smooth function on a compact orientable manifold with $L(f) \neq 0$, then f has a fixed point.

b) Show that the Lefshetz number of any map from $\mathbb{CP}^{2n}$ to itself is nonzero and hence that every map from $\mathbb{CP}^{2n}$ to itself has a fixed point. (*Hint*: The cohomology ring is generated by the 2^{nd} cohomology).

Definition: $L(f) := I(\Delta, \Gamma(f))$, where $\Gamma(f) \subset X \times X$ is the graph of f, and $\Delta = \Gamma(id)$ is the diagonal.

Definition: For $f : X \to Y$ and $Z \subset Y$ with $f \pitchfork Z$, we have $I(f, Z) = \sum_{x \in f^{-1}Z} o(x)$, where $o(x) = \pm 1$ is the orientation number of x, which is $+1$ if $df_x(T_xX) \oplus T_zZ = T_zY$ (equality follows from transversality) gives the correct orientation on T_zY, and -1 otherwise. If f is not transverse to Z, find $g \cong f$ homotopic with $g \pitchfork Z$. This is always possible.

Then $L(f) = \sum_{x : f(x) = x} L_x(f)$, where $L_x(f) = \pm 1$ is $+1$ if $df_x - I$ preserves orientation on $T_x(X)$, and -1 otherwise. It is the degree of the map $g : \partial B \to S^{n-1}$ sending $z \to \frac{f(z)-z}{|f(z)-z|}$, where B is a disk neighborhood of x not containing any other fixed points.

Remark: $L(id)$ is the Euler characteristic.

For our purposes, here is an alternative more useful definition:

Definition: For $f : X \to X$,

$$L(f) = \sum_{k \geq 0} (-1)^k tr(f_* : H_k(X; \mathbb{Q}) \to H_k(X; \mathbb{Q})) = \sum_{k \geq 0} (-1)^k tr(f^* : H^k(X; \mathbb{Q}) \to H^k(X; \mathbb{Q}))$$

where the equality follows from the universal coefficient theorem.

Solution: Note the cup product gives a graded ring structure on $H^*(\mathbb{CP}^{2n}) = \mathbb{Z}[x]/(x^{2n+2})$ (where x has grading 2), so the generator $x \in H^2(\mathbb{CP}^{2n})$ in fact generates the whole ring, with $x \cup x$ generating $H^4(\mathbb{CP}^{2n})$, and so on. (Note for $0 \leq k \leq 4n$, $H^k(\mathbb{CP}^{2n}) = \mathbb{Z}$ if k is even and 0 otherwise).

So it suffices to know for our ring homomorphism $f^* : H^*(\mathbb{CP}^{2n}) \to H^*(\mathbb{CP}^{2n})$ what the image of x is. Note $f^*x = kx \in H^2(\mathbb{CP}^{2n})$. Hence $f^*(x^r) = k^r x^r$, and we have the trace of the map $f^* : H^m(X; \mathbb{Q}) \to H^m(X; Q)$ for even $m > 0$ is the trace of the multiplication by k^m map $\mathbb{Q} \to \mathbb{Q}$. is k^m. For odd m it is 0, and for $m = 0$, since $f^*(1) = 1$, it has trace 1. Hence,

$$L(f) = 1 + \sum_{m=1}^{2n} tr(f^* : H^{2m}(\mathbb{CP}^{2n}) \to H^{2m}(\mathbb{CP}^{2n})) = 1 + \sum_{m=1}^{2n} k^m = \frac{k^{2n+1} - 1}{k - 1}$$

if $k \neq 1$, and $L(f) = 2n + 1$ if $k = 1$. We see $L(f) = 0 \to k \neq 1$ and $k^{2n+1} = 1$, so $k = 1$ (since $k \in \mathbb{Z}$ is a an odd root of unity). Hence, we see by contradiction that $L(f) \neq 0$ for arbitrary f. Hence, f has a fixed point.

Problem 10: Compute explicitly the *simplicial* homology, with $\mathbb{Z}$ coefficients, of the surface of a tetrahedron, thus obtaining the homology of the 2-sphere.

Recall a delta complex is a union of simplices glued via some gluing rules. In our case, we may write the tetrahedron with vertices v_0, v_1, v_2, v_3 as

$$X = [v_0, v_1, v_2] \cup [v_0, v_1, v_3] \cup [v_0, v_2, v_3] \cup [v_1, v_2, v_3]$$

i.e. the union of its faces. So we have four 2-simplices, six 1-simplies, and four 0-simplies (or vertices). We may write $C_2 \cong \mathbb{Z}^4$ via the ordered basis $[v_0, v_1, v_2], [v_0, v_1, v_3], [v_0, v_2, v_3], [v_1, v_2, v_3]$, $C_1 \cong \mathbb{Z}^6$ via the ordered basis $[v_0, v_1], [v_0, v_2], [v_0, v_3], [v_1, v_2], [v_1, v_3], [v_2, v_3]$ and $C_0 \cong \mathbb{Z}^4$ with the obvious basis $[v_0], [v_1], [v_2], [v_3]$.

Recall the boundary formula

$$\partial[v_0, ..., v_k] = \sum_{i=0}^{k} (-1)^i [v_0, ..., \widehat{v_i}, ..., v_k]$$

We have our cell complex

$$0 \to C_2 = \mathbb{Z}^4 \xrightarrow{\partial_2} C_1 = \mathbb{Z}^6 \xrightarrow{\partial_1} C_0 = \mathbb{Z}^4 \to 0$$

Notice ∂_1 has the 4×6 matrix

$$\begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

This has image $(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4$ with $x_1 + x_2 + x_3 + x_4 = 0$. Note $\mathbb{Z}^4 \to \mathbb{Z}$, the augmentation map sending $(x_1, x_2, x_3, x_4) \mapsto x_1 + x_2 + x_3 + x_4$, is surjective and has kernel precisely $im(\partial_1)$, so $H_0(X) = \mathbb{Z}^4 / im(\partial_1) \cong \mathbb{Z}$.

∂_2 has the 6×4 matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

It remains to check $\ker(\partial_2) \cong \mathbb{Z}$, and $\ker(\partial_1) = im(\partial_2)$, so that $H_1(X) = 0$ and $H_2(X) = \mathbb{Z}$ (and $H_k(X) = 0$ for $k > 2$).

Row reduction shows $\ker(\partial_2) = span((-1, 1, -1, 1)) \cong \mathbb{Z}$.

To see $im(\partial_2) \supset \ker(\partial_1)$, we need to check each basis vector of $\ker(\partial_1)$ is in the image of ∂_2 (the reverse containment always holds). SKIP!

4 Fall 2011

Problem 1: Let M be a compact smooth n-manifold. Show there exists an $N \in \mathbb{N}$ such that M can be smoothly embedded into $\mathbb{R}^N$.

Since M is compact, we may cover it with finitely many charts $U_i \subset M$ with $x_i : U_i \to \mathbb{R}^n$ diffeomorphisms for $i = 1, ..., k$. By the Shrinking Lemma, we may refine this open cover to an open cover $\cup_{i=1}^k V_i$, with $\overline{V_i} \subset U_i$.

Pick $V_i \leq \psi_i << U_i$ bump functions $\psi_i : M \to \mathbb{R}$. Write

$$f = (\psi_1 \cdot x_1, ..., \psi_k \cdot x_k, \psi_1, ..., \psi_k)$$

Note each x_i is a map from U into $\mathbb{R}^n$ (so it itself has n components), and bumping allows us to view it as a map $M \to \mathbb{R}^n$ (with agrees with x_i on V_i). Then we have $f : M \to R^{nk+k}$ in this way.

For $p \in M$, since $p \in V_i$ for some i, we have $\psi_i = 1$ in a neighborhood of P, so that locally, $\psi_i x_i = x_i$. Then $(df)_p = (d(\psi_1 x_1)_p, ..., d(\psi_k x_k)_p, d(\psi_1)_p, ..., d(\psi_k)_p)$ is injective, since $d(\psi_i x_i)_p = d(x_i)_p$ is injective since x_i is a diffeomorphism. Since this holds for arbitrary $p \in M$, we get f is an immersion.

Suppose $f(p) = f(q)$ for $p, q \in M$. Since $p \in V_i$ for some i, we have $\psi_i(p) = 1$, so $\psi_i(q) = 1$ since $f(p) = f(q)$. Hence $q \in U_i$ since ψ_i has support in U_i. Then looking in a different component, we see $(\psi_i \cdot x_i)(p) = (\psi_i \cdot x_i)(q)$. Since $p, q \in U_i$, we have $\psi_i(p)x_i(p) = \psi_i(q)x_i(q)$, and since $\psi_i(p) = \psi_i(q) = 1$, we have $x_i(p) = x_i(q)$. So $p = q$ by injectivity of x_i. Hence f is injective.

Finally, an embedding is an injective immersion whose image is homeomorphic to the domain. In this case, since M is compact, $f : M \to f(M)$ is a bijection from a compact space to a Hausdorff space, and hence a homeomorphism. In general, from a compact space, it suffices to be an injective immersion. Hence, f is an embedding, as desired.

Problem 2: Prove $\mathbb{RP}^n$ is a smooth manifold of dimension n.

Solution: Recall if a Lie group G acts on a manifold M freely, properly and smoothly, then M/G is a manifold. (See Spring 2012 Problem 9 for the finite G case). Taking $G = \{\pm 1\}$ which is a discrete Lie group acting on $M = S^n$ via the identity and antipodal map (1 and -1 respectively), it is clear this action is free and smooth. Moreover, for compact G, properness is free. Hence $M/G = \mathbb{RP}^n$ is a smooth manifold. $\square$

Alternative solution: We can give $\mathbb{RP}^n$ charts as follows: take $U_i = \{[x_0, ..., x_i = 1, ..., x_n] : x_j \in \mathbb{R}\} \subset \mathbb{RP}^n$. Its preimage in S^n is $V_i = \{x \in S^n : x_i \neq 0\}$, which is open. Hence, U_i is open in the quotient topology.

Write $U_i \to \mathbb{R}^n$ via $[x_0, ..., x_n] \mapsto (\frac{x_0}{x_i}, ..., \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, ..., \frac{x_n}{x_i})$. It is clear this is well-defined as it is invariant under scaling. It is bijective, continuous, and its inverse is all continuous. The transition maps are all smooth: we send $(v_0, ..., \hat{v_i}, ..., v_n) \in \mathbb{R}^n$ to $[v_0, ..., 1, ..., v_n] \in \mathbb{RP}^n$ to $(\frac{v_0}{v_j}, ..., \frac{v_{j-1}}{v_j}, \frac{v_{j+1}}{v_j}, ..., \frac{v_n}{v_j})$, where $v_i := 1$. This is clearly smooth (since $v_j \neq 0$ on the intersection $U_i \cap U_j$). $\square$

Problem 3: Let M be a compact simply-connected n-manifold. Prove there is no smooth immersion $f : M \to T^n$, where $T^n = S^1 \times ... \times S^1$ (n times).

Solution: If $f : M \to T^n$ is an immersion, it is a local diffeomorphism by dimension counting. By the Stack of Records Theorem, Spring 2010 Problem 3, a local diffeomorphism from a compact to a connected n-manifold is a covering map. Since M is simply connected, we conclude M is the universal cover of $T^n = S^1 \times ... \times S^1$, which is $\mathbb{R} \times ... \times \mathbb{R} = \mathbb{R}^n$. Hence $M \cong \mathbb{R}^n$, which is not compact. By contradiction, no such immersion exists. $\square$

Alternative solution: Since M is simply connected, any $f : M \to T^n$ satisfies the lifting criterion $f_* \pi_1(M) \subset p_* \pi_1(T^n)$, where $p : \mathbb{R}^n \to T^n$ is the projection from the universal cover. Hence we have a lift $g : M \to \mathbb{R}^n$ with $pg = f$. If f is an immersion, since $df = dp \circ dg$, it follows g is an immersion, and hence by dimension reasons, a local diffeomorphism. Hence it is an open map, so $g(M) \subset \mathbb{R}^n$ is open and compact. By contradiction, no such immersion f may exist. $\square$

Problem 4: Give a topological proof of the fundamental theorem of algebra: every nonconstant single variable polynomial with complex coefficients has at least one complex root.

Suppose $p(z) = z^m + \sum_{i=0}^{m-1} a_i z^i$, $m > 0$, has no roots in $\mathbb{C}$. Notice for $t \in [0,1]$

$$p(z) \cdot t + (1-t) \cdot z^m = z^m + t\left(\sum_{i=0}^{m-1} a_i z^i\right) = z^m\left(1 + t\left(\sum_{i=0}^{m-1} \frac{a_i}{z^{m-i}}\right)\right)$$

Select $r > 0$ with $\sum_{i=0}^{m-1} \frac{|a_i|}{r^{m-i}} < \frac{1}{2}$ (this is possible since this sum tends to 0 as r tends to infinity). Then notice for z with $|z| = r$, we have

$$|p(z) \cdot t + (1-t) \cdot z^m| = \left|z^m\left(1 + t\left(\sum_{i=0}^{m-1} \frac{a_i}{z^{m-i}}\right)\right)\right| = r^m\left|1 + t\left(\sum_{i=0}^{m-1} \frac{a_i}{z^{m-i}}\right)\right|$$

$$\geq r^m\left(1 - t\left|\left(\sum_{i=0}^{m-1} \frac{a_i}{z^{m-i}}\right)\right|\right) \geq r^m(1 - t\sum_{i=0}^{m-1} \frac{a_i}{r^{m-i}}) = r^m(1 - t/2) \geq r^m/2 > 0$$

So we see $p(z) \cdot t + (1-t) \cdot z^m$ is nonzero for $|z| = r$ and all $t \in [0,1]$. Write $S_r = \{z \in \mathbb{C} : |z| = r\}$. Then

$$H : [0,1] \times S_r \to S^1$$

$$H(t,z) = \frac{p(z) \cdot t + (1-t) \cdot z^m}{|p(z) \cdot t + (1-t) \cdot z^m|}$$

is well-defined since the denominator is never 0 for $t \in [0,1], z \in S_r$. Notice

$$H(0,z) = z^m/r^m = (z/r)^m$$

$$H(1,z) = p(z)/|p(z)|$$

Since $S_r \to S^1$ via $z \mapsto (z/r)^m$ has degree m, so too does $p(z)/|p(z)|$. However, $p(z)/|p(z)|$ can be extended to $W = \{z \in \mathbb{C} : |z| \leq r\}$ since $p(z)$ has no roots. So degree of $p(z)/|p(z)| : S_r \to S^1$ is 0 by the Extension Theorem. Hence $m = 0$. But $m > 0$ by assumption, so we get a contradiction, and conclude no such $p(z)$ can exist. $\square$

Problem 5: Let $f : M \to N$ be smooth. Let α be a p-form on N. Show $d(f^*\alpha) = f^*(d\alpha)$.

First, let α be a 0-form $\alpha = g$. Then $d(f^*\alpha)(X) = d(g \circ f)(X) = X(g \circ f)$, while $f^*(d\alpha)(X) = (d\alpha)(f_*X) = (f_*X)(g) = X(g \circ f)$. This holds for any vector field X, so that $d(f^*\alpha) = f^*(d\alpha)$ as desired.

Next, suppose this holds for $(k-1)$-forms. Let $\alpha = dg \wedge \eta$, where η is a $k-1$ form and g is a function. Then

$$d(f^*\alpha) = d(f^*(dg \wedge \eta)) = d(f^*dg \wedge f^*\eta) = d(f^*dg) \wedge f^*\eta - f^*dg \wedge d(f^*\eta)$$

$$= d(df^*g) \wedge f^*\eta - f^*dg \wedge f^*d\eta = -f^*dg \wedge f^*d\eta$$

Meanwhile,

$$f^*(d\alpha) = f^*(d(dg \wedge \eta)) = f^*(-dg \wedge d\eta) = -f^*dg \wedge f^*d\eta$$

So $d(f^*\alpha) = f^*(d\alpha)$.

Finally, observe every k-form can locally be written as a sum of terms of the form $gdx_1 \wedge ... \wedge dx_k = dx_1 \wedge \eta$ for a $(k-1)$ form η, it follows by linearity (and the fact that it is enough to show this locally) that $f^*d\omega = d(f^*\omega)$ for every form.

Problem 6:

a) What are the de Rham cohomology groups of a smooth manifold.

We have $\Omega^i(M) \xrightarrow{d} \Omega^{i+1}(M)$ giving us a cochain complex, where $\Omega^i(M)$ is the vector space of smooth i-forms on M. Then $H^i_{dR}(M) = \ker(\Omega^i(M) \xrightarrow{d} \Omega^{i+1}(M)) / \operatorname{im}(\Omega^{i-1}(M) \xrightarrow{d} \Omega^i(M))$ is simply the cohomology of this cochain.

b) State de Rham's Theorem.

For M a smooth manifold, $H^i_{dR}(M) \cong H_i(M;\mathbb{R})^*$ via $\omega \mapsto \left([c] \mapsto \int_c \omega\right)$

Problem 7: Consider $\omega = (x^2 + x + y)dy \wedge dz$ on $\mathbb{R}^3$. Let $i : S^2 \to \mathbb{R}^3$ be the inclusion map.

a) Calculate $\int_{S^2} i^*\omega$.

We may apply Stokes Theorem, which applies to compact orientable manifolds, to see that for $B \subset \mathbb{R}^3$ given by $B = \{x \in \mathbb{R}^3 : |x| \leq 1\}$, we have

$$\int_{S^2} i^*\omega = \int_B d\omega = \int_B (2x+1)dx \wedge dy \wedge dz = \int_B (2x+1)dxdydz = 2\overline{x}V + V = V$$

where $V = 4\pi/3$ is the volume of B, and $\overline{x}$ is the average value of x on the ball, which is 0 by symmetry. Hence $\int_{S^2} i^*\omega = 4\pi/3$. $\square$

b) Construct a closed form α on $\mathbb{R}^3$ such that $i^*\alpha = i^*\omega$, or show that such an α does not exist.

Suppose α is a closed form on $\mathbb{R}^3$ which has $i^*\alpha = i^*\omega$. Then $4\pi/3 = \int_{S^2} i^*\omega = \int_{S^2} i^*\alpha = \int_B d\alpha = \int_B 0 = 0$. By contradiction, no such closed form exists. $\square$

Problem 8:

a) Let M be a Mobius band. Using homology, show that there is no retraction from M to ∂M.

We use the LES for relative homology, taking $(M, \partial M)$, which is a good pair by the Collar Neighborhood Theorem, which ensures there is a neighborhood of the boundary which retracts to it. Hence $H_n(M, \partial M) = \widetilde{H}_n(M/\partial M)$.

Of course, $\partial M \cong S^1$ is just a circle. Moreover, $M/\partial M \cong \mathbb{RP}^2$, as is clear from the polygon construction of each space. We have

$$0 \to H_1(\partial M) \xrightarrow{i_*} H_1(M) \to H_1(\mathbb{RP}^2) \to H_0(\partial M) \xrightarrow{i_*} H_0(M)$$

If M retracts onto its boundary, then $r \circ i = id$ for some $r : M \to \partial M$, so that $r_* \circ i_* = id$, and i_* is injective. From this we may simplify to get an SES

$$0 \to H_1(\partial M) \xrightarrow{i_*} H_1(M) \to H_1(\mathbb{RP}^2) \to 0$$

since the map prior to the injective map i_* on H_0 must have image 0. Thus we have an SES

$$0 \to \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

where we can compute $H_1(\mathbb{RP}^2)$ from the polygon construction, and $H_1(M) \cong H_1(S^1)$ because it deformation retracts onto its central circle (see remark below). However, $r_* : H_1(M) \to H_1(\partial M)$ then provides a splitting, so that this SES splits, and $\mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, a contradiction. So no such retract exists. $\square$

Remark: To see M deformation retracts onto its central circle, write $M = [0,1]^2/\sim$, where $(x, 0) \sim (1 - x, 1)$ for each $x \in [0, 1]$. Write $H : M \times [0, 1] \to M$ via $H((x, y), t) = t(x, y) + (1 - t)(1/2, y)$. Observe for fixed t, these are well-defined maps from the Mobius strip, since $(x, 0) \mapsto (tx + (1 - t)/2, 0) \sim (1 - tx + (t - 1)/2, 1) = (-tx + t/2 + 1/2, 1)$, and $(1 - x, 1) \mapsto (t(1 - x) + (1 - t)/2, 1) = (t/2 - tx + 1/2, 1)$, so it is well-defined regardless of choice of representative from $(x, 0) \sim (1 - x, 1)$.

We see $H(1, (x, y)) = (x, y)$ and $H(0, (x, y)) = (1/2, y)$. Moreover, $H(t, (1/2, y)) = (1/2, y)$ for all t. Hence this is a deformation retraction of M onto the subspace $\{1/2\} \times [0, 1]/\sim$, where $(1/2, 0) \sim (1/2, 1)$. Thus it is a deformation retraction onto the central circle S^1, as desired.

b) Let K be the Klein bottle. Show that there exist homotopically nontrivial simple closed curves γ_1, γ_2 on K such that K retracts to γ_1 but does not retract to γ_2.

First, notice the Klein bottle is actually two copies of the Mobius band glued together at the boundary circle. To see this, take $K = [0,1]^2/\sim$ where we have $(x,0) \sim (1-x,1)$ for all $x \in [0,1]$, and $(0,y) \sim (1,y)$ for all $y \in [0,1]$. Then notice $[1/4,3/4] \times [0,1]/\sim$ is a Mobius band, as is $([0,1/4] \cup [3/4,1]) \times [0,1]/\sim$, and these are glued along their boundary circles $\{1/4\} \times [0,1] \cup \{3/4\} \times [0,1]$. For this problem, it suffices to consider just one of these Mobius strips; lets consider the first copy.

This is indeed a circle, as follows: write $\gamma(t) = (1/4, 2t)$ for $0 \le t \le 1/2$ and $\gamma(t) = (3/4, 2-2t)$ for $1/2 \le t \le 1$. Since $\gamma(1/2) = (1/4,1) \sim (3/4,0)$, this is well-define. Since $\gamma(0) = (1/4,0) \sim (3/4,1) \sim \gamma(1)$, we see this is a loop in K.

If K deformation retracts onto γ (the boundary circle of each Mobius strip), then so too does the Mobius strip (by simply restricting $r : K \to \gamma$ to the subspace). By part a, this cannot happen.

Meanwhile, the Klein bottle does retract onto its "central circle" $\gamma_2 = \{1/2\} \times [0,1]$. Write

$$r : K \to \gamma_2$$

$$r(x,y) = (1/2, y)$$

By the same computation as in part a, we see $r(x,0) = (1/2,0) \sim (1/2,1) = r(1-x,1)$. Moreover, $r(0,y) = (1/2,y) = r(1,y)$. Hence this is a well-defined map on the Klein bottle. Moreover, $r(1/2,y) = (1/2,y)$ for each y, so that it is a retract. Note that our deformation retract from part a would not have factored through to a deformation retract for K.

Finally, we observe γ, γ_2 are nontrivial loops in K. We get the non-triviality of γ_2 for free, since $i_* : \pi_1(\gamma_2) \to \pi_1(K)$ is injective (due to $r_* i_* = id$).

To get the non-triviality of γ, notice that under the map $r_* : \pi_1(K) \to \pi_1(\gamma_2)$, it maps to $r \circ \gamma$ which is a curve as follows: $r \circ \gamma(t) = (1/2, 2t)$ for $0 \le t \le 1/2$ and $r \circ \gamma(t) = (1/2, 2-2t)$ for $1/2 \le t \le 1$. Thus $r \circ \gamma$ goes to $2\gamma_2$, and hence is homotopically nontrivial in $\pi_1(\gamma_2)$. Thus, γ is homotopically nontrivial in $\pi_1(K)$, as desired.

Problem 9: Let X be the topological space corresponding to a pentagon with edges a, a, a, a, a all oriented CCW, and a 2-cell attached via a^5. Compute the homology and cohomology groups of X with $\mathbb{Z}$ coefficients.

We have cell complex

$$0 \to C_2 \cong \mathbb{Z} \to C_1 \cong Z \to C_0 \cong Z \to 0$$

with maps

$$\partial_2 F = 5a$$

$$\partial_1 a = v - v = 0$$

That is, we have the chain complex

$$0 \to \mathbb{Z} \xrightarrow{5} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

so that $H_2(X) = 0$, $H_1(X) = \mathbb{Z}/5\mathbb{Z}$, and $H_0(X) = \mathbb{Z}$. Of course, $H_k(X) = 0$ for $k > 2$.

For cohomology, we dualize the chain complex. Alternatively, apply universal coefficient theorem. Dualizing the chain complex gives

$$0 \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z} \xrightarrow{0} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z} \xrightarrow{5} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z} \to 0$$

which gives $H^0(X) = \mathbb{Z}$, $H^1(X) = 0$, $H^2(X) = \mathbb{Z}/5\mathbb{Z}$, and $H^k(X) = 0$ for $k > 2$. $\square$

Remark: Universal coefficient would give $H^i(X) \cong \mathrm{Hom}(H_i(X), \mathbb{Z}) \oplus Ext(H_{i-1}(X), \mathbb{Z})$ (with $H_{-1} = 0$), and we would get the same result.

Problem 10: Let X, Y be topological spaces and $f, g : X \to Y$ two continuous maps. Consider the space Z obtained from the disjoint union $Y \sqcup (X \times [0,1])$ by indentifying $(x,0) \sim f(x)$ and $(x,1) \sim g(x)$ for all $x \in X$. Show that there is a long exact sequence of the form

$$... \to H_n(X) \to H_n(Y) \to H_n(Z) \to H_{n-1}(X) \to ...$$

We consider two long exact sequences for relative homology:

$$\begin{tikzcd}
... \arrow[r, "\partial"] & H_n(X \times \partial I) \arrow[r, "i_*"] \arrow[d, "q_*"] & H_n(X \times I) \arrow[r, "\phi"] \arrow[d, "q_*"] & H_n(X \times I, X \times \partial I) \arrow[r, "\partial"] \arrow[d, "q_*"] & ... \\
... \arrow[r, "\partial"] & H_n(Y) \arrow[r, "j_*"] & H_n(Z) \arrow[r, "\psi"] & H_n(Z, Y) \arrow[r, "\partial"] & ...
\end{tikzcd}$$

For the top sequence, we consider the good pair $(X \times I, X \times \partial I)$, which is indeed a good pair, since a neighborhood of ∂I in I retracts onto ∂I, so that a neighborhood of $X \times \partial I$ in $X \times I$ retracts onto $X \times \partial I$. Here $i : X \times \partial I \to X \times I$ is the inclusion map.

For the bottom sequence, we consider the good pair (Z, Y), which is indeed a good pair, since we may take $U \subset X \times I$ which deformation retracts to $X \times \partial I$, and consider the image of $Y \sqcup U$ in Z, the quotient of $Y \sqcup (X \times I)$ via the given equivalence relation. Then this deformation retracts to the image of $Y \sqcup (X \times \partial I)$, which is just Y.

Let q denote the inclusion followed by the quotient in $X \times I \hookrightarrow Y \sqcup (X \times I) \to Z$. Then notice $q : X \times I \to Z$ induces a map on homology. Moreover, $q|_{X \times \partial I}$ maps entirely to $Y \subset Z$ (where by this we mean the image of Y in Z, via the inclusion followed by quotient, which is just homeomorphic to Y). Hence q also induces maps on homology from $X \times \partial I$ to Y and from the relative pairs.

In fact, notice $X \times I/(X \times \partial I) \xrightarrow{q} Z/Y$ is a homeomorphism. Since these are good pairs, we have $H_n(X \times I, X \times \partial I) \xrightarrow{q_*} H_n(Z, Y)$ is an isomorphism (the terms may be replaced with the reduced homology off the quotient spaces, and q_* gives an isomorphism between them).

Next, notice $X \times I$ deformation retracts to $X \times \{0\}$ and to $X \times \{1\}$ (since I deformation retracts to $0, 1$ respectively). Hence each $X \times \{0\} \hookrightarrow X \times I$ and $X \times \{1\} \to X \times I$ give isomorphisms on homology. Since $H_n(X \times \partial I) = H_n(X \times \{0\}) \oplus H_n(X \times \{1\}) = H_n(X) \oplus H_n(X)$, we have the top map

$$i_* : H_n(X \times \partial I) = H_n(X) \oplus H_n(X) \to H_n(X \times I) \cong H_n(X)$$

is surjective, with $i_*(a, b) = a + b$. Since i_* is surjective, we get $\phi = 0$, and ∂ in the top row is injective. Thus $H_n(X \times I, X \times \partial I)$ is isomorphic to its image in $H_{n-1}(X \times \partial I)$ via δ, and its image is the kernel of i_*. Meanwhile, the kernel of i_* is $\{(a, -a) \in H_{n-1}(X) \oplus H_{n-1}(X) : a \in H_{n-1}(X)\} \cong H_{n-1}(X)$.

Stringing together our isomorphisms, we see $H_n(Z, Y) \cong H_n(X \times I, X \times \partial I) = \ker(i_*) \cong H_{n-1}(X)$, we see our bottom long exact sequence is the desired long exact sequence. Moreover, notice the map $H_n(Z, Y) \cong H_{n-1}(X) \to H_{n-1}(Y)$ can be computed instead by going through the top row via our isomorphism q_*. The top composition then gives $q_* : H_{n-1}(X \times \partial I) = H_{n-1}(X) \oplus H_{n-1}(X) \to H_{n-1}(Y)$ restricted to $\ker(i_*) = \{(a, -a) : a \in H_{n-1}(X)\} \cong H_{n-1}(X)$.

The map $q_* : H_n(X \times \partial I) = H_n(X) \oplus H_n(X) \to H_n(Y)$ is just the sum of the two maps $H_n(X \times \{0\}) \to H_n(Y)$ and $H_n(X \times \{1\}) \to H_n(T)$. The first of these maps is f_* and the second is g_*, since this is how $X \times \{0\}$ and $X \times \{1\}$ get mapped to $Y \subset Z$ respectively. Thus the map $\ker(i_*) \to H_n(Y)$ just maps $(a, -a) \to f_*(a) + g_*(-a) = (f_* - g_*)(a)$. Hence, we get a long exact sequence

$$... \to H_n(X) \xrightarrow{f_* - g_*} H_n(Y) \xrightarrow{j_*} H_n(Z) \to ...$$

where $j : Y \to Z$ is the inclusion. This gives the desired long exact sequence. $\square$

5 Spring 2012

Problem 1: Explain from the viewpoint of transversality theory why the sum of the indices of a vector field with isolated zeros on a compact orientable manifold is independent of the choice of vector field.

See *G&P* page 134-137 for further discussion.

Theorem: (Poincare-Hopf) Let M be a compact orientable manifold. If X is a vector field on M with only finitely many zeroes, then the sum of the indices of the zeroes is the Euler characteristic of M.

Proof: We string together some black box results to prove this. Let $\phi : \mathbb{R} \times M \to M$ be the (global) flow corresponding to the vector field X.

First, for $|t|$ sufficiently small and nonzero, the fixed points of ϕ_t will correspond precisely to the zeroes of X.

Next, since ϕ is a flow, it already gives us a homotopy between ϕ_t and $\phi_0 = id$ for any t. Hence, since Lefshetz number is homotopy invariant, we see $L(\phi_t) = L(\phi_0) = L(id) = \chi(M)$ for any t.

Moreover, take ϕ_t for t small enough. By previous remarks, its fixed points correspond to the zeroes of X, which are isolated. Hence, its Lefshetz number is the sum of its local Lefshetz numbers at each fixed point (this is true provided ϕ_t is a Lefshetz map,). We have

$$L(\phi_t) = \sum_{p \in M : \phi_t(p) = p} L_p(\phi_t) = \sum_{p \in M : X_p = 0} L_p(\phi_t)$$

Finally, for p a fixed point of ϕ_t, we have $L_p(\phi_t) = ind_p(X)$.

From this we see $\chi(M) = L(id) = L(\phi_t) = \sum_{p \in M : X_p = 0} L_p(\phi_t) = \sum_{p \in M : X_p = 0} ind_p(X)$, as desired. $\square$

Problem 2: Define the Euler characteristic of a compact orientable manifold as the index sum from the previous problem. Show (directly from this definition) that $\chi(M_g) = 2 - 2g$, where M_g is the genus g compact orientable surface, a 2-sphere with g handles attached.

The genus g compact orientable surface admits a vector field with one source, one sink, and $2g$ saddles by *G&P* page 125, which can be thought of as the oozing trajectory of liquid on a g-holed donut. The source is at the top, the sink at the bottom, and a saddle at the top and bottom of each hole. The index of a source is $+1$. To see this, notice that in a small ball around the source, we essentially obtain a map $S^1 \to S^1$ with $(x, y) \mapsto (x, y)$. For a sink, we obtain $(x, y) \mapsto (-x, -y)$, and for a saddle, we obtain maps of the form $(x, y) \mapsto (-x, y)$. Thus their indices are $+1, +1, -1$ respectively. From this, we see the sum of the indices is $2 - 2g$, as desired.

As an alternative approach to define source, sink and saddle is to look at the local Lefshetz numbers of the flow ϕ_t for small t. Since the local Lefshetz number is ± 1 depending on if $d\phi_t - I$ preserves or reverses orientation, it suffices to consider the sign of its determinant. This corresponds to how the two eigenvalues of $d\phi_t$ compare to 1. Note that for a sink, all vectors contract towards the origin, so all eigenvalues are less than 1, so $\det((d\phi_t - I)_p)$ is positive. Similarly, for a source, all eigenvalues are larger than 1, so the determinant is again positive. Finally, for a saddle, some vectors are contracting and some are expanding, so that there is one eigenvalue larger than one and one eigenvalue smaller than one, and the determinant is negative.

Problem 3: Suppose M is a triangulated compact orientable manifold (i.e. with a finite simplicial complex structure).

a) Show that the alternating sum of the betti numbers $\sum_{k=0}^{n}(-1)^k b_k$ is also equal to the alternating sum $\sum_{k=0}^{n}(-1)^k c_k$, where c_k is the number of k-simplices.

Let ∂_i denote the map $C_i \to C_{i-1}$ in the chain complex with $\mathbb{R}$ coefficients, where we define $C_{-1} = 0$. Then we have $C_i/\ker(\partial_i) \cong \operatorname{im}(\partial_i)$ as vector spaces, so that $\dim_{\mathbb{R}} C_i = \dim_{\mathbb{R}} \ker(\partial_i) + \dim_{\mathbb{R}} \operatorname{im}(\partial_i)$. Meanwhile, $\dim_{\mathbb{R}} C_i = c_i$, since the rank does not change when using $\mathbb{R}$-coefficients vs $\mathbb{Z}$-coefficients.

Now

$$\sum_{i=0}^{n}(-1)^i c_i = \sum_{i=0}^{n}(-1)^i(\dim_{\mathbb{R}} \ker(\partial_i) + \dim_{\mathbb{R}} \operatorname{im}(\partial_i))$$

Meanwhile, $H_i(X;\mathbb{R}) = \ker(\partial_i)/\operatorname{im}(\partial_{i+1})$, so $\dim_{\mathbb{R}} H_i(X;\mathbb{R}) = \dim_{\mathbb{R}} \ker(\partial_i) - \dim_{\mathbb{R}} \operatorname{im}(\partial_{i+1})$. Moreover, $\dim_{\mathbb{R}} H_i(X;\mathbb{R}) = b_i$, since by universal coefficient, $H_i(X;\mathbb{R}) = H_i(X) \otimes \mathbb{R}$, so that the rank does not change. Thus

$$\sum_{i=0}^{n}(-1)^i b_i = \sum_{i=0}^{n}(-1)^i(\dim_{\mathbb{R}} \ker(\partial_i) - \dim_{\mathbb{R}} \operatorname{im}(\partial_{i+1}))$$

$$= \sum_{i=0}^{n}(-1)^i \dim_{\mathbb{R}} \ker(\partial_i) + \sum_{i=1}^{n}(-1)^i \dim_{\mathbb{R}} \operatorname{im}(\partial_i)$$

$$= \sum_{i=0}^{n}(-1)^i \dim_{\mathbb{R}} \ker(\partial_i) + \sum_{i=0}^{n}(-1)^i \dim_{\mathbb{R}} \operatorname{im}(\partial_i)$$

$$= \sum_{i=0}^{n}(-1)^i(\dim_{\mathbb{R}} \ker(\partial_i) + \dim_{\mathbb{R}} \operatorname{im}(\partial_i))$$

so the two are equal, as desired. □

b) Show that there exists a vector field with the sum of its indices equal to the number described in part a. Do not worry about smoothness.

It suffices to exhibit such a vector field for Δ^n, as then we can glue these vector fields as we glue the simplices to get a vector field on M.

Define the vector field X on $\Delta^n \setminus \partial\Delta^n$ to be a vector field pointing towards the center of the interior of Δ^n. This will make the center an n-dimensional sink, with corresponding index $(-1)^n$ (we may insist the vector field near the center is just $p = (x_1, ..., x_n) \mapsto X_p = (-x_1, ..., -x_n)$).

Since the boundary is the union of $(n-1)$-simplices, we can repeat this process inductively, describing what to do on the interior of each k-simplex. Each simplex will then contribute a fixed point of index $(-1)^k$, so we will get the sum of the indices to be the sum of $(-1)^k$ times the number of k-simplices, as desired.

Problem 4: Suppose V is a smooth vector field on $\mathbb{R}^3$ that is nonzero at $(0,0,0)$. A vector field is said to be gradient-like at $(0,0,0)$ if there exists a nowhere zero function $\lambda(x,y,z)$ on that neighborhood such that $\lambda V = \nabla f$ for some smooth function f.

a) Write $V = (P, Q, R)$. Show that there exist functions P, Q, R such that V is not gradient-like in a neighborhood of $(0,0,0)$ (despite still being nonvanishing at that point). (*Hint:* The orthogonal complement of V taken at each point would have to be an integrable 2-plane field.)

Write $\omega = Pdx + Qdy + Rdz$. If $V = (P, Q, R)$ is gradient-like, then $\lambda\omega = df$ for some nonzero function λ and some function f. Then $\omega = \frac{1}{\lambda}df$, and $d\omega = d\left(\frac{1}{\lambda}\right)\wedge df$, and $\omega\wedge d\omega = \lambda df \wedge d\left(\frac{1}{\lambda}\right)\wedge df = 0$.

Take $P = -y, Q = x, R = 1$. Then notice $\omega = -ydx + xdy + dz$, $d\omega = 2dx \wedge dy$, and $\omega \wedge d\omega = 2dx \wedge dy \wedge dz \neq 0$. Hence, ω is not gradient-like.

b) Derive a general differential condition on (P, Q, R) which is necessary and sufficient for V to be gradient-like in a neighborhood of $(0,0,0)$.

$V = (P, Q, R)$ will be a gradient-like vector field if and only if $\omega = Pdx + Qdy + Rdz$, the dual of V, has $\omega\wedge d\omega = 0$, and this will happen if and only if V is orthogonal to $curl(V)$ (at each point).

To get the first equivalence, note by part a that if V is gradient-like, $\omega \wedge d\omega = 0$. Conversely, suppose $\omega \wedge d\omega = 0$. Since ω is a non-vanishing 1-form on a 3-manifold (some open subset of $\mathbb{R}^3$ containing the origin), we see by Fall 2013 Problem 5 that $\ker(\omega)$ is integrable. Thus, there exist submanifolds whose tangent space is $\ker(\omega)$, and hence whose normal space at each point is V. By Spivak's version of the Frobenius Theorem, found on page 192, we may even select a new coordinate system on some open set U containing 0 which sends 0 to 0 and has the integral manifolds to $\ker(\omega)$ as $\{q \in U : z(q) = a\}$ for each fixed a appropriately small. In particular, this means the normal vector fields are parallel to the z-axis in this coordinate system, so that in this coordinate system, $V = f\frac{\partial}{\partial z}$. Since it is non-vanishing, we have f is nonzero, so taking $\lambda = \frac{1}{f}$, we see $\lambda V = \frac{\partial}{\partial z}$, which is the gradient of $g(x,y,z) = (0,0,z)$. Hence V is gradient-like, as desired.

Finally, note that for f a 0-form, df is a 1-form whose dual vector field is the gradient of f. For ω a 1-form, $d\omega$ is a 2-form whose dual is the curl of the dual of ω. If ω is a 2-form, then $d\omega$ corresponds to the divergence of the dual of ω. From this correspondence, we see $\omega \wedge d\omega = 0 \iff V \perp curl(V)$. (This can also be done just by writing out the coefficient of $dx \wedge dy \wedge dz$ in $\omega \wedge d\omega$ and identifying it as $V \cdot curl(V)$).

Problem 5:

a) Define carefully the boundary map which defines the map from H_n to H_{n-1} that arises in the long exact sequence arising from an SES of chain complexes.

b) Prove that the kernel of the boundary map is equal to the image of the map into the H_n.

> See Spring 2010 Problem 4.

Problem 6: Compute the homology of $\mathbb{RP}^n$ for each $n > 1$.

> See Spring 2011 Problem 8a.

Problem 7:

a) Define $\mathbb{CP}^n$.

b) Show that $\mathbb{CP}^n$ is compact.

c) Show that $\mathbb{CP}^n$ has a cell decomposition with one cell in each dimension $0, 2, 4, ..., 2n$.

> See Spring 2011 Problem 7. The compactness follows from either the finite CW structure or the observation that we may restrict our quotient $\mathbb{C}^{n+1}/\sim$ to S^{2n+1} (which is compact) and still get $\mathbb{CP}^n$.

Problem 8: Suppose a compact real manifold M has a finite cell decomposition with only even dimensional cells. Is M necessarily orientable? Justify your answer.

If M only has even cells, then $H_1(M) = 0$. Thus, $\pi_1(M)$ cannot have a subgroup of index 2, since otherwise we would have a surjection to $\mathbb{Z}/2\mathbb{Z}$ which is abelian but not a quotient of the abelianization. Thus, M cannot have any connected 2-sheeted covering spaces. In particular, the orientation double cover of M must not be connected. Thus M is orientable.

Alternative Solution: By Hatcher Theorem 3.26, if M is a connected, closed and R-orientable n-manifold, then $H_n(M;R) \cong R$, and if not, then $H_n(M;R) \subset R$ is the subset $\{r \in R : 2r = 0\}$. Note an an orientable manifold is R-orientable for any ring R, and a non-orientable manifold is R-orientable if and only if R has characteristic 2.

By Hatcher Corollary 3.28, for M a connected closed n-manifold, $H_{n-1}(M)$ is free if M is orientable, and is the direct sum of a free abelian group and $\mathbb{Z}/2\mathbb{Z}$ if M is not orientable.

If M is connected, compact, orientable and with boundary, then by Lefshetz duality, $H^n(M) = H_0(M, \partial M) = H_0\widetilde{(M/\partial M)} = 0$ since M connected implies the quotient $M/\partial M$ is connected.

Solution: Let $n = \dim M$. WLOG, M is connected, since otherwise, we may consider each connected component separately.

The cell complex for M makes it so that all maps are 0. Hence $H_i(M) = C_i$, which is a free abelian group generated by all the i-cells.

Note that we must have at least one cell in the top dimension, as the top dimensional cell of dimension k will have interior homeomorphic to $\mathbb{R}^k$. Thus, we must have at least one n-cell. (In particular, since M only has even cells, n is even.) This shows $H_n(M) = C_n$ is not only free, but has rank at least one.

Moreover, M must be without boundary, as otherwise, ∂M would be an $(n-1)$-manifold, requiring an odd $(n-1)$ dimensional cell. So M is connected and closed. By Hatcher Theorem 3.26, $H_n(M) = \mathbb{Z}$ if M is orientable, and $H_n(M) = 0$ if M is non-orientable. (So these may be promoted to if and only if). Hence $H_n(M)$ has rank at most one.

So $H_n(M)$ has rank exactly one, and $H_n(M) \cong \mathbb{Z}$. From the above cases, we see M must be orientable. $\square$

Problem 9: Suppose that a finite group G acts smoothly on a compact manifold M and the action is free, i.e. $g.x = x \iff g = e$.

a) Show M/G is a manifold.

Note that this more generally holds for infinite Lie groups acting freely, properly and smoothly on a manifold. Moreover, if the Lie group is compact, the action is automatically proper. In general, $\dim(M/G) = \dim(M) - \dim(G)$.

Solution: We show part b simultaneously. Let $y \in M/G$, and write $f^{-1}y = \{g_1x, ..., g_nx\}$ (where $G = \{g_1, .., g_n\}$). These are all distinct by the freeness of the action. Pick charts on disjoint open sets $U_1, ..., U_n \subset M$ with $U_i \ni g_jx$ if and only if $i = j$. Set $W_i = \cap_{j=1}^n g_i g_j^{-1} U_j$. Notice the W_i are still disjoint by the disjointness of the U_i and still contain just g_ix, but with the added benefit that $g_k g_i^{-1} W_i = W_k$. That is, if $g_1 = e$ is the identity, we have $W_i = g_iW_1$. Pick a further open set $V_1 \subset W_1$ so that V_1 is diffeomorphic to $\mathbb{R}^n$. Then set $V_i = g_iV_1$. These are still disjoint, still contain g_ix, and we have the added benefit of $\phi_i : V_i \to \mathbb{R}^n$ diffeomorphisms.

Then notice $V = \pi(V_1) = \pi(g_iV_1) = \pi(V_i) \subset M/G$ is independent of choice of i, since $\pi(gx) = \pi(x)$ for any $x \in M$, $g \in G$. Moreover, $\pi^{-1}V = \sqcup_{i=1}^n V_i$, which is open. Hence, V is open in M/G.

Finally, notice $\pi|_{V_i} : V_i \to V$ is a homeomorphism for each V. It is clearly surjective. To see it is injective, suppose $\pi(x) = \pi(y)$ for $x, y \in V_i = g_iU$. Then there exists $g_j \in G$ with $g_jx = y$. Hence $y \in g_jV_i = (g_jg_i)U = V_k$, where $g_jg_i = g_k$. Then $y \in V_k \cap V_i$, so that we must have $k = i$ by the fact that these sets are disjoint. Hence $g_jg_i = g_i$, and $g_j = e$. Hence $x = g_jx = y$. So π is injective.

Thus $\pi|_{V_i} : V_i \to V$ is bijective. It is open, since for $W \subset V_i$ open, $\pi^{-1}\pi(W) = \sqcup_{i=1}^n g_iW$ is open, so that $\pi(W)$ is open in the quotient topology. Hence $\pi(W) \subset V$ is open, so $\pi|_{V_i}$ is an open map. Thus $\pi|_{V_i}$ is a homeomorphism.

Note $y = \pi(x) \ni V$, and $V \cong V_i \xrightarrow{\phi_i} \mathbb{R}^n$ is a homeomorphism. This makes M/G a manifold of the same dimension as M, as we may find a chart for each point in M/G. In fact, note that our choice of neighborhood V is also an evenly covered neighborhood, so that π is a $|G|$-sheeted covering space projection. Finally, G acts on M as deck transformations of M over M/G. $\square$

b) Show $M \to M/G$ is a covering space.

See the previous part.

c) If $H^k_{dR}(M) = 0$ for some $k > 0$, is $H^k_{dR}(M/G)$ necessarily 0? Prove your answer.

Lemma: Let G be a group (possibly infinite), and let $H \subset G$ be a finite index subgroup. Then there exists a subgroup $K \subset H \subset G$ with $[G : K] < \infty$ and $K \trianglelefteq G$.

Proof: Write $n = [G : H]$. G acts on its cosets G/H via left multiplication. This gives us a homomorphism $\phi : G \to Sym(G/H) \cong S_n$. Its kernel is the intersection of all the stabilizers, $K = \ker(\phi) = \cap_{g \in G} stab(gH)$. In particular, notice the stabilizer of H under this action is precisely H, since $g.H = H \iff g \in H$. Hence, $K \subset H$. Moreover, $G/K \cong \mathrm{im}(\phi) \subset S_n$, so that $[G : K] = |G/K| \leq n!$. $\square$

Corollary: Let $\widetilde{N} \to N$ be a finite-sheeted covering map. Then there exists a covering $M \to \widetilde{N} \to N$ with $M \to N$ finite sheeted and regular.

Proof: There is a Galois correspondence between covering spaces over N and subgroups of $G = \pi_1(N)$. In one direction, apply $p_*\pi_1$ to the covering map $M \xrightarrow{p} N$ to get the corresponding subgroup. In the reverse direction, given a subgroup $H \subset G$, the corresponding cover is $N'/H \to N$, where N' is the universal cover of N. This correspondence reverses the lattice. The index of the subgroup corresponds to the number of sheets of the cover. Finally, normal subgroups correspond to regular covers.

If $\widetilde{N} \to N$ is a finite-sheeted cover, then $\widetilde{N}$ corresponds to a finite index subgroup $\underline{H} \subset G$. Then by the lemma, we may find $K \subset H \subset G$ with $K \trianglelefteq G$ and $[G : K] < \infty$. Thus, $K \subset H \subset G$ corresponds to covers $M \to \widetilde{N} \to N$ with $M \to N$ regular and finite-sheeted. $\square$

Proposition: Finite-sheeted covering maps induce an injection on de Rham cohomology.

Proof: It suffices to consider finite regular covers $M \to M/G$, as follows: let $\widetilde{N} \xrightarrow{p} N$ be a finite-sheeted covering map. By the corollary above, we may find a cover $M \to \widetilde{N} \to N$ with M a finite sheeted regular cover over N. Then we have $M \xrightarrow{\pi} \widetilde{N} \xrightarrow{p} N$. On cohomology, we get $H^k_{dR}(N) \xrightarrow{p^*} H^k_{dR}(\widetilde{N}) \xrightarrow{\pi^*} H^k_{dR}(M)$. If the composition $H^k_{dR}(N) \xrightarrow{\pi^* p^*} H^k_{dR}(M)$ is injective, then p^* is also injective. Thus it suffices to show that finite regular covers induces an injection on de Rham cohomology. Thus we consider covers of the form $M \to M/G$ with G finite. (Each regular cover may be written in this way).

Let $\pi : M \to M/G$ denote the covering space projection. To see π^* is injective on de Rham cohomology, we construct a one-sided inverse to π^*, a map $\pi_* : H^k_{dR}(M) \to H^k_{dR}(M/G)$ with $\pi_*\pi^* = id$. We may follow the construction of π_* as in Spring 2010 Problem 5. Here is an alternative (arguably better) construction. First, we construct $\pi_* : \Lambda^k(M) \to \Lambda^k(M/G)$ a map on forms. Then we show it commutes with d, so that it gives us an induced map on cohomology. For $\omega \in \Lambda^k(M)$, consider the G-invariant form $\alpha = \frac{1}{G}\sum_{g \in G} g^*\omega \in \Lambda^k(M)$. Note $d\pi_q : T_pM \to T_p(M/G)$, for $p = \pi(q)$, is an isomorphism since π is a local homeomorphism and $M, M/G$ are manifolds of the same dimension. Define

$$(\pi_*\omega)_p(X_1, \ldots, X_k) = \alpha_q((d\pi)_q^{-1}X_1, \ldots, (d\pi)_q^{-1}X_k)$$

where $q \in \pi^{-1}p$ is arbitrary. This is well-defined independent of choice of q, since from $\alpha = g^*\alpha$ for $g \in G$, we get

$$\alpha_q((d\pi)_q^{-1}X_1, \ldots, (d\pi)_q^{-1}X_k) = (g^*\alpha)_q((d\pi)_q^{-1}X_1, \ldots, (d\pi)_q^{-1}X_k)$$

$$= (\alpha)_{gq}((dg)_q(d\pi)_q^{-1}X_1, \ldots, (dg)_q(d\pi)_q^{-1}X_k)$$

Notice $\pi \circ g = \pi$, so that $(d\pi)_{gq}(dg)_q = d(\pi \circ g)_q = (d\pi)_q$, and $(dg)_q(d\pi)_q^{-1} = d(\pi)_{gq}$. Thus

$$(\alpha)_q((d\pi)_q^{-1}X_1, \ldots, (d\pi)_q^{-1}X_k) = (\alpha)_{gq}((dg)_q(d\pi)_q^{-1}X_1, \ldots, (dg)_q(d\pi)_q^{-1}X_k)$$

$$= (\alpha)_{gq}((d\pi)_{gq}^{-1}X_1, \ldots, (d\pi)_{gq}^{-1}X_k) = (\alpha)_{q'}((d\pi)_{q'}^{-1}X_1, \ldots, (d\pi)_{q'}^{-1}X_k)$$

where $q' = gq \in \pi^{-1}p$. Ranging over $g \in G$, we see that our definition of $\pi_*\omega$ was indeed independent of q.

Let $\eta \in \Lambda^k(M/G)$ be a form. Then notice $\omega = \pi^*\eta \in \Lambda^k(M)$ is already G-invariant, since $g^*\omega = g^*\pi^*\eta = (\pi \circ g)^*\eta = \pi^*\eta = \omega$. Notice then

$$(\pi_*\pi^*\eta)_p(X_1, \ldots, X_k) = (\pi^*\eta)_q((d\pi)_q^{-1}X_1, \ldots, (d\pi)_q^{-1}X_k) = \eta_p(X_1, \ldots, X_k)$$

Hence π_* gives a left inverse to $\pi^* : \Lambda^k(M/G) \to \Lambda^k(M)$. So the latter is injective as a map on forms.

Meanwhile, notice

$$(\pi^*\pi_*\omega)_q(Y_1, \ldots, Y_k) = (\pi_*\omega)_p((d\pi)_qY_1, \ldots, (d\pi)_qY_k) = \alpha_q(Y_1, \ldots, Y_k) = \frac{1}{|G|}\sum_{g \in G}(g^*\omega)_q(Y_1, \ldots, Y_k)$$

or in short

$$\pi^*\pi_* = \frac{1}{|G|}\sum_{g \in G} g^*$$

Now

$$\pi^*d\pi_* = d\pi^*\pi_* = d\left(\frac{1}{|G|}\sum_{g \in G} g^*\right) = \frac{1}{|G|}\sum_{g \in G} dg^* = \frac{1}{|G|}\sum_{g \in G} g^*d = \left(\frac{1}{|G|}\sum_{g \in G} g^*\right)d = \pi^*\pi_*d$$

Since π^* is injective on forms and from the above computation,n $\pi^*d\pi_* = \pi^*\pi_*d$, we conclude $d\pi_* = \pi_*d$.

Thus π_* induces a map $\pi_* : H^k_{dR}(M) \to H^k_{dR}(M/G)$ on cohomology, still with $\pi_*\pi^* = id$. Hence π^* is also injective on cohomology (not just forms), as desired. $\square$

Solution: Applying the proposition, for $p : M \to M/G$, we have $H^k_{dR}(M/G) \xhookrightarrow{p^*} H^k(M) = 0$, so that $H^k_{dR}(M/G) = 0$. $\square$

Remark: The proof also shows $\Lambda^k(M/G)$ is in bijection with G-invariant forms in $\Lambda^k(M)$.

Problem 10: Let $M = \mathbb{RP}^2 \times \mathbb{RP}^2$. Homology elements of a product manifold can arise as a product of a cycle in one factor and a cycle in the other. Show that there is an element (and find it explicitly) of $H_3(M)$ that does not arise in this way.

See Spring 2011 Problem 8b.

6 Fall 2012

Problem 1:

a) Show $SL_2(\mathbb{R})$ is diffeomorphic to $S^1 \times \mathbb{R}^2$.

Solution: To each matrix $A \in SL_2(\mathbb{R})$, we have a unique polar decomposition $A = OP$ where O is orthogonal and P is positive semidefinite. Since $1 = \det(A) = \det(O)\det(P)$, $\det(O) = \pm 1$ and $\det(P) \geq 0$, we see $\det(O) = 1$ and $\det(P) = 1$. In particular, $O \in SO_2(\mathbb{R})$, and P is positive definite.

Note $SO_2(\mathbb{R}) \cong S^1$ since the special orthogonal 2 by 2 matrices correspond to rotations by an angle θ (so we have the bijection mapping to $e^{i\theta} \in S^1$).

Meanwhile, if P is positive definite of determinant 1, we have

$$P = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

with $ad - b^2 = 1$. In particular, a, d must be nonzero (and in fact positive, since PSD matrices have nonnegative diagonal entries). Hence we may always write $d = \frac{1+b^2}{a}$. In fact, by Sylvester's criterion, any such matrix with $a > 0$ and $ad - b^2 > 0$ must be positive definite. Hence the positive definite 2 by 2 determinant matrices P are in bijection with ordered pairs (a, b) with $a > 0$ and $b \in \mathbb{R}$ arbitrary (where we just selecct $d = \frac{1+b^2}{a}$ to construct the corresponding matrix). Hence if $SPD_2(\mathbb{R})$ is the set of positive definite 2 by 2 matrices of determinant 1,

$$SPD_2(\mathbb{R}) \cong \{(a,b) \in \mathbb{R}^2 : a > 0\} = (0, \infty) \times \mathbb{R} \cong \mathbb{R}^2$$

Conversely, for each $O \in SO_2(\mathbb{R})$ and $P \in SPD_2(\mathbb{R})$, it is clear $OP \in SL_2(\mathbb{R})$. Thus by polar decomposition we get

$$SL_2(\mathbb{R}) \cong SO_2(\mathbb{R}) \times SPD_2(\mathbb{R}) \cong S^1 \times \mathbb{R}^2$$

as desired. $\square$

Remark: Regarding smoothness of polar decomposition, note that we pick $P = \sqrt{A^T A}$, and this is continuous with respect to the entries of $A \in SL_2(\mathbb{R})$ since eigenvalues vary continuously and square roots are continuous. Moreover, we then have $O = AP^{-1}$, so that this also varies continuously in the entries of A.

Alternative Solution: For $A \in SL_2(\mathbb{R})$, Gram-Schmidt and QR decomposition gives $A = QR$, with $Q \in O_2(\mathbb{R})$ and R upper triangular with nonnegative diagonal entries. By the same argument as above, we see $\det(Q) = \det(R) = 1$. Hence $Q \in SO_2(\mathbb{R}) \cong S^1$ and R is of the from

$$R = \begin{bmatrix} r & s \\ 0 & 1/r \end{bmatrix}$$

with $r > 0$ and $s \in \mathbb{R}$. Conversely, every $Q \in SO_2(\mathbb{R})$ paired with any such R give $QR \in SL_2(\mathbb{R})$. Hence we again get $SL_2(\mathbb{R}) \cong SO_2(\mathbb{R}) \times (0, \infty) \times \mathbb{R} \cong S^1 \times R^2$. $\square$

Remark: This gives an alternative proof (to Fall 2010 Problem 3c) for the fact that $SL_2(\mathbb{R})$ has trivial Euler characteristic: $\chi(X \times S^1) = 0$ for any CW complex X.

b) Show $SL_2(\mathbb{C})$ is diffeomorphic to $S^3 \times \mathbb{R}^3$.

Via polar decomposition, for each $A \in SL_2(\mathbb{C})$ we may write $A = UP$ for unique U unitary and P positive semidefinite and Hermitian. Note then $\det(U) = \det(P) = 1$, so $U \in SU_2(\mathbb{C})$, and P is positive definite Hermitian of determinant 1. On the one hand, we have

$$SU_2(\mathbb{C}) \cong S^3$$

$$\begin{bmatrix} a & -\overline{b} \\ b & \overline{a} \end{bmatrix} \leftrightarrow (a,b) \in S^3 \subset \mathbb{C}^2$$

where we notice every matrix in $SU_2(\mathbb{R})$ with first column $(a,b) \in \mathbb{C}^2$ has second column $\overline{(-b,a)}$, and has $|a|^2 + |b|^2 = 1$ (from the determinant condition). Conversely, any such matrix is in $SU_2(\mathbb{R})$. Hence we get the above diffeomorphism.

Meanwhile, if P is positive definite Hermitian of determinant 1, we have

$$P = \begin{bmatrix} a & \overline{b} \\ b & c \end{bmatrix}$$

This has determinant $ac - |b|^2 = 1$. In particular we have a, c nonzero and $c = \frac{1+|b|^2}{a}$. Meanwhile, by Sylvester's criterion, such a matrix is positive definite if and only if $a > 0$ and $ac - |b|^2 > 0$. Hence we have the positive definite Hermitian matrices of determinant 1 are in bijection with ordered pairs (a,b) with $a > 0$ and $b \in \mathbb{C}$. So we have

$$SL_2(\mathbb{C}) \cong SU_2(\mathbb{C}) \times (0,\infty) \times \mathbb{C} \cong S^3 \times \mathbb{R}^3$$

as desired. $\square$

Problem 2: For $n \geq 1$, construct a nowhere vanishing smooth vector field on $\mathbb{RP}^{2n-1}$.

Definition: Vector fields X on M and Y on N are F-related for $F : M \to N$ if $F_* X_p = Y_{F(p)}$ for each $p \in M$.

Theorem: (Lee, Proposition 8.23) For $N \subset M$ a submanifold, $i : N \overset{i}{\hookrightarrow} M$ the inclusion map, and X a vector field on M with $X_p \in T_pN \subset T_pM$ for each $p \in N$, there exists a vector field Y on N which is i-related to X, i.e. has $Y_p = i_* X_p = X_p \in T_pN$ for each $p \in N$.

Solution: Notice the vector field $X_p = ip$ on $\mathbb{C}^n = \mathbb{R}^{2n}$ is tangent to $S^{2n-1} \subset \mathbb{C}^n$, since p is orthogonal to ip, so that $ip \in T_pS^{2n-1}$. Hence we have a vector field Y on S^{2n-1} with $Y_p = ip$ for $p \in S^{2n-1}$. The vector field Y corresponds to a section of the tangent bundle, $S^{2n-1} \xrightarrow{Y} TS^{2n-1}$.

Let $\pi : S^{2n-1} \to \mathbb{RP}^{2n-1}$ denote the projection. Then we have a morphism $TS^{2n-1} \xrightarrow{d\pi} T(\mathbb{RP}^{2n-1})$.

The composition gives $S^{2n-1} \xrightarrow{Y} TS^{2n-1} \xrightarrow{d\pi} T(\mathbb{RP}^{2n-1})$. Write $Z = d\pi \circ Y$. Then $Z_p = d\pi_p Y_p$. We show $Z_p = Z_{-p}$, i.e. $d\pi_p Y_p = d\pi_{-p} Y_{-p}$, so that this map factors through to a map $V : \mathbb{RP}^{2n-1} \to T(\mathbb{RP}^{2n-1})$ with $V \circ \pi = Z$. Then since $V_{\pi(p)} = Z_p = d\pi_p Y_p \in T_{\pi(p)}(\mathbb{RP}^{2n-1})$, we will get V is a vector field on $\mathbb{RP}^{2n-1}$. Since Y is nonvanishing and $d\pi_p$ is injective (in fact, bijective, since it is a local diffeomorphism), we will get V is a nonvanishing vector field on $\mathbb{RP}^{2n-1}$ as desired.

Thus it just remains to check $d\pi_p Y_p = d\pi_{-p} Y_{-p}$. Letting $f : S^{2n-1} \to S^{2n-1}$ be the antipodal map $f(p) = -p$, since $\pi \circ f = \pi$, we see

$$d\pi_p(Y_p) = d(\pi \circ f)_p(Y_p) = d\pi_{f(p)} df_p Y_p = d\pi_{-p} df_p Y_p$$

Thus it just remains to check $df_p Y_p = Y_{-p}$. Since $Y_p = i_* X_p = di_p X_p$, we have $df_p Y_p = d(i \circ f)_p X_p$. Writing $g : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ as $g(x) = -x$, we have $i \circ f = g|_{S^{2n-1}} = g \circ i$, so that $d(i \circ f)_p X_p = d(g \circ i)_p X_p = dg_p di_p X_p = dg_p Y_p$. However, $g(x) = -x$ is linear so $dg_p = g$. So $dg_p Y_p = -Y_p = -ip = Y_{-p}$, as desired. $\square$

Problem 3: Let $M \subset \mathbb{R}^n$ be a smooth submanifold of dimension $m < n-2$. Show $\mathbb{R}^n \setminus M$ is connected and simply-connected.

Theorem: Extension Theorem: Let $Z \subset Y$ be a closed submanifold, and $C \subset X$ a closed set. Let $f : X \to Y$ have $f|_C \pitchfork Z$. Then there exists a $g : X \to Y$ homotopic to f, with $g \pitchfork Z$, and $g = f$ on a neighborhood of C.

Solution: Let $p, q \in \mathbb{R}^n \setminus M$. Select a path in $\mathbb{R}^n$ from $\gamma : [0,1] \to \mathbb{R}^n$ from p to q. Taking $X = [0,1]$, $C = \{0,1\} \subset X$ closed, $Y = \mathbb{R}^n$, $Z = M \subset \mathbb{R}^n$ a closed submanifold. Notice $f|_C : \{0,1\} \to \mathbb{R}^n$ has $f(0) = p \notin Z$, $f(1) = q \notin Z$. Hence $f|_C$ trivially intersects Z transversally. By the extension theorem, we may find $g : [0,1] \to \mathbb{R}^n$ with $g(0) = p, g(1) = q$ and $g \pitchfork Z = M$.

If $g(x) \in Z$ for some $x \in X$, we must have $dg_x T_x X \oplus T_{g(x)} Z = T_{g(x)} \mathbb{R}^n$. By dimension considerations, we see the LHS has dimension at most $m + 1 < n$, so this is impossible. Hence, we must have $g(x) \notin Z$ for any x. Hence, $g : [0,1] \to \mathbb{R}^n \setminus M$ does not intersect $M = Z$. Thus it is a path from p to q in $\mathbb{R}^n \setminus M$. Since these points were arbitrary, we conclude $\mathbb{R}^n \setminus M$ is path-connected.

Let $\gamma : [0,1] \to \mathbb{R}^n \setminus M$ be a loop, with $\gamma(0) = \gamma(1) = p$. Select $H : [0,1] \times [0,1] \to \mathbb{R}^n$ be a path homotopy between γ and the constant map (since γ is nullhomotopic in $\mathbb{R}^n$). That is, we have for all $x \in [0,1]$ and $t \in [0,1]$,

$$H(0,x) = \gamma(x)$$

$$H(1,x) = p$$

$$H(t,0) = H(t,1) = p$$

Take $X = [0,1] \times [0,1]$, $C = \{0,1\} \times [0,1] \cup [0,1] \times \{0,1\}$ closed, $Y = \mathbb{R}^n$, $Z = M \subset Y$ a closed submanifold. Notice $H|_C$ does not intersect $Z = M$, since γ is a path in $\mathbb{R}^n \setminus M$ and $p \notin M$. Hence it trivially intersects transversally. By the extension theorem, we have

$$G : [0,1] \times [0,1] \to \mathbb{R}^n$$

with $G = H$ on C and $G \pitchfork M$. By dimension considerations, we see $dG_(t,x) T_{(t,x)}[0,1] \times [0,1] \oplus T_{G(t,x)} Z$ has dimension at most $m + 2 < n$, so it cannot intersect $Z = M$ at all. Hence G maps to $\mathbb{R}^n \setminus M$. Thus we have

$$G : [0,1] \times [0,1] \to \mathbb{R}^n \setminus M$$

$$G(0,x) = H(0,x) = \gamma(x)$$

$$G(1,x) = H(1,x) = p$$

$$G(t,0) = H(t,0) = p$$

$$G(t,1) = H(t,1) = p$$

We conclude G is a path homotopy between γ and the constant map in $\mathbb{R}^n \setminus M$. Since γ was arbitrary, we see $\pi_1(\mathbb{R}^n \setminus M) = 0$ and $\mathbb{R}^n \setminus M$ is simply connected. $\square$

Problem 4:

a) Show that for $n \geq 1$ and $k \in Z$, there exists a continuous map $f : S^n \to S^n$ of degree k.

One could attempt to generalize Spring 2010 Problem 8. We instead use Hatcher's argument from page 138.

Let B_i, $i = 1, ..., k$ be disjoint open disks on S^n. Set $B = \sqcup_{i=1}^k B_i$. Then $S^n/(S^n \setminus B) \cong \vee_{i=1}^k S^n$. To see this, note $\overline{B_i}/\partial B_i \cong S^n$. Each $\overline{B_i} \subset S^n$ then maps homeomorphically to S^n under this quotient, so that B_i maps to S^n with a point (the image of $\partial B_i \subset S^n \setminus B$) removed. Thus each B_i maps to a copy of $S^n \setminus p$, and the remaining $S^n \setminus B$ maps to the missing point p, giving a wedge of spheres.

Next, map $\vee_{i=1}^k S^n \to S^n$ via mapping each copy of S^n to S^n either via the identity or via a reflection (i.e. a degree -1 map), insisting p maps to p in both cases (so p is on the hyperplane of reflection). The choice of which map to use for each copy is specified shortly.

Thus we have a map $S^n \to S^n/(S^n \setminus B) \to S^n$. To compute its degree, select some $y \in S^n$ in the codomain not equal to p (the point where $S^n/(S^n \setminus B) = \vee_{i=1}^k S^n$ is wedged). Then notice its preimage under the first map consists of one point in each summand of the wedge. The second map is a local homeomorphism near each preimage point, so that each preimage point contributes a degree of ± 1. The unique preimage of each point via the first map contributes ± 1 as well, since it is also a local homeomorphism. Thus in the composition, each of the k preimage points contribute ± 1. In the above construction, we map either via identity or reflection to ensure all of these local degrees are $+1$, so that the degree is k. Alternatively, we may choose them all to be -1 to get a degree $-k$. $\square$

b) Let X be a compact, oriented n-manifold. Show that for any $k \in Z$, there exists a map $f : X \to S^n$ of degree k.

Let $U \subset X$ be an open set diffeomorphic to $\mathbb{R}^n$. Then note $X/(X \setminus U) \cong S^n$ (it is compact and contains a homeomorphic copy of $U \cong \mathbb{R}^n$ with one extra point, so it must be the one-point compactification).

This gives us a map $X \to X/(X \setminus U) \cong S^n$ which is a local homeomorphism for any point in U, so that the degree of this map is ± 1. Then we may compose this with a map $S^n \to S^n$ of degree $\pm k$ to get a degree k map, as desired. $\square$

Problem 5: Assume that $\Delta = span\{X_1, ..., X_k\}$ is a k-dimensional distribution spanned by vector fields on open set $\Omega \subset M$ in an n-manifold M. For each open $V \subset \Omega$, define

$$\mathcal{Z}_V = \{u \in C^\infty(V) | X_i u = 0, i = 1, ..., k\}$$

Show Δ is integrable if and only if for each $x \in \Omega$, there exists an open neighborhood $x \in V \subset \Omega$ and $n-k$ functions $u_1, ..., u_{n-k}$ on $\mathcal{Z}_V$ such that the differentials $du_1, ..., du_{n-k}$ are linearly independent at each point in V.

Suppose Δ is integrable. For each $p \in \Omega$, we can find a chart $(x, V \ni p)$ with

$$x : V \to (-\epsilon, \epsilon)^n$$

$$x(p) = 0$$

and the integral manifolds being of the form $N = \{x^{k+1}(q) = a^{k+1}, ..., x^n(q) = a^n\}$, for each fixed $a^i \in (-\epsilon, \epsilon)$. Note $T_q N = span(\frac{\partial}{\partial x^i}|_q, i = 1, ..., k) = \Delta_q = span((X_1)_q, ..., (X_k)_q)$.

Then notice since $\frac{\partial}{\partial x^i} x^j = 0$ for $i \neq j$, we have $x^{k+1}, ..., x^n$ vanish on all functions in $T_q N = \Delta_q$. Thus, we must have $(X_i)(x^j) = 0$ for $j = k+1, ..., n$. Hence $x^j \in \mathcal{Z}_V$ for $j = k+1, ..., n$.

Moreover, since these are coordinate functions, we have $dx^{k+1}, ..., dx^n$ are linearly independent at each point in V. Setting $u_i = x_{k+i}$, we get the desired functions.

Conversely, suppose we have $u_1, ..., u_{n-k}$ on $\mathcal{Z}_V$ such that $du_1, ..., du_{n-k}$ are linearly independent at each point in V.

By Lee Theorem 19.7, it suffices to check that for any η a smooth 1-form which annihilates Δ, $d\eta$ also annihilates Δ.

By the independence of the du_i, $i = 1, ..., n-k$, we see they span $\mathcal{Z}_V$ at each point, so that if η annihilates Δ on V, we have

$$\eta = \sum_{i=1}^{n-k} f_i du_i$$

for smooth functions f_i. Then

$$d\eta = \sum_{i=1}^{n-k} df_i \wedge du_i$$

For $X, Y \in \Delta$, we have $(df_i \wedge du_i)(X, Y) = df_i(X) du_i(Y) - df_i(Y) du_i(X) = 0 - 0 = 0$ since du_i annihilates Δ. Thus, we see $d\eta(X, Y) = 0$. We conclude if η is a 1-form annihilating Δ, so too is $d\eta$. By Lee's Theorem 19.7, we conclude Δ is integrable. $\square$

Problem 6: Define $(n-1)$-forms on $\mathbb{R}^n \setminus \{0\}$ via

$$\sigma = \sum_{i=1}^{n} (-1)^{i-1} x^i dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^n$$

$$\omega = \frac{1}{|x|^n}\sigma$$

a) Let $i : S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ be inclusion and $r : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ the retraction $r(x) = \frac{x}{|x|}$. Show $\omega = r^* i^* \sigma$.

First, we have to make the nontrivial observation that

$$\sigma_p(X_1, ..., X_{n-1}) = \det\left(\begin{bmatrix} \vec{p} & (X_1)_p & \dots & (X_{n-1})_p \end{bmatrix}\right)$$

To see this, it suffices to plug in basis vectors $\frac{\partial}{\partial x^1}, ..., \widehat{\frac{\partial}{\partial x^i}}, ..., \frac{\partial}{\partial x^n}$. Then the RHS becomes

$$\det\begin{bmatrix} \vec{p} & e_1 & ... & \widehat{e_i} & ... & e_n \end{bmatrix} = (-1)^{i-1} \det\begin{bmatrix} e_1 & ... & e_{i-1} & \vec{p} & e_{i+1} & ... & e_n \end{bmatrix} = (-1)^{i-1} x^i(p)$$

which agrees with the coefficients of σ. Next, to see $r^*i^*\sigma = \omega$, it suffices to check pointwise, i.e. $(r^*i^*\sigma)_p = \omega_p$ for each $p \in \mathbb{R}^n \setminus \{0\}$. For fixed p, it suffices again to check by plugging in basis vectors. This time, we select a basis of $T_p\mathbb{R}^n = T_pS_p^{n-1} \oplus N_pS_p^{n-1}$ by selecting a basis of each component, where S_p^{n-1} is the unique $(n-1)$-sphere containing p. For $N_pS_p^{n-1}$, we may simply pick the basis $\{\vec{p}\}$.

First, we check that if any $(X_i)_p \in N_p(S_p^{n-1})$, then both $(r^*i^*\sigma)_p(X_1, ..., X_{n-1})$ and $\omega_p(X_1, ..., X_{n-1})$ are 0. Write $(X_i)_p = \lambda\vec{p}$. Then by the determinant form of σ above, we see immediately $\sigma_p(X_1, ..., X_{n-1}) = 0$. Then $\omega_p(X_1, ..., X_{n-1}) = \frac{1}{|x|^n}\sigma_p(X_1, ..., X_{n-1}) = 0$ as well. Meanwhile,

$$(r^*i^*\sigma)_p(X_1, ..., X_{n-1}) = \sigma_{\frac{p}{|p|}}(d(i \circ r)_p X_1, ..., d(i \circ r)_p X_{n-1})$$

Taking $\gamma(t) = t\vec{p}$, we see $\gamma'(1) = \vec{p}$. Meanwhile, $r \circ \gamma = \vec{p}/|\vec{p}|$ is constant, so that $(dr)_p\vec{p} = 0$. Then $d(i \circ r)_p(X_i)_p = (di)_{p/|p|} dr_p(X_i)_p = 0$ for $(X_i)_p = \lambda\vec{p}$. From this we see

$$\sigma_{\frac{p}{|p|}}(d(i \circ r)_p X_1, ..., d(i \circ r)_p X_{n-1}) = 0$$

Hence $\omega_p(X_1, ..., X_{n-1}) = (r^*i^*\sigma)_p(X_1, ..., X_{n-1}) = 0$ whenever any $(X_i)_p \in N_pS_p^{n-1}$.

By the above remarks, it only remains to check on the basis vectors in $T_pS_p^{n-1}$. However, if $X_p \in T_pS^{n-1}$, we may write it as $(dj_p)_pY_p$, where $j_p : S_p^{n-1} \to \mathbb{R}^n \setminus \{0\}$ is inclusion, for some vector field Y on S_p^{n-1}. Thus it remains to check

$$(r^*i^*\sigma)_p((dj_p)_p(Y_1)_p, ..., (dj_p)_p(Y_{n-1})_p) = \omega_p((dj_p)_p(Y_1)_p, ..., (dj_p)_p(Y_{n-1})_p)$$

However, for this, it simply suffices to check $j_p^*(r^*i^*\sigma) = j_p^*(\omega)$.

Note $i \circ r \circ j_p = x/|p|$ is just multiplication by $1/|p|$, so that $(i \circ r \circ j_p)^*\sigma$ is $\frac{1}{|p|^n}\sigma|_{S_p^{n-1}}$, where we gain a $1/|p|$ factor from each x_i and dx_i term.

Meanwhile, $j_p^*\omega = j_p^*(\frac{1}{|x|^n}\sigma) = \frac{1}{|x|^n \circ j_p} j_p^*\sigma = \frac{1}{|p|^n} j_p^*\sigma = \frac{1}{|p|^n}\sigma|_{S_p^{n-1}}$, since $|x|^n \circ j_p = |p|^n$ is constant. $\square$

b) Show σ is not closed.

We have

$$d\sigma = \sum_{i=1}^{n}(-1)^{i-1}dx^i \wedge dx^1 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^n = \sum_{i=1}^{n} dx^1 \wedge ... \wedge dx^i \wedge ... \wedge dx^n = n(dx^1 \wedge ... \wedge dx^n) \neq 0$$

so σ is not closed.

c) Show ω is closed but not exact.

We have $\omega = r^*i^*\sigma$, so $d\omega = d(r^*i^*\sigma) = r^*i^*(d\sigma) = 0$, since $d\sigma$ is an n-form, so that $i^*d\sigma = 0$ as it is an n-form on S^{n-1}. Hence ω is closed.

Meanwhile, note we have $i^*\omega = i^*r^*i^*\sigma = (r \circ i)^*i^*\sigma = i^*\sigma$, since $r \circ i = id$. It is clear from the expression for σ that we have a form $\hat{\sigma}$ on $\mathbb{R}^n$ with $\sigma = j^*\hat{\sigma}$ for $j : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ the inclusion.

It is also clear $d\hat{\sigma} = n(dx^1 \wedge ... \wedge dx^n)$ from the same calculation as part b. Then by Stokes we have

$$\int_{S^{n-1}} i^*\omega = \int_{S^{n-1}} i^*\sigma = \int_B d\hat{\sigma} = \int_B n(dx^1 \wedge ... \wedge dx^n) = n \cdot vol(B) > 0$$

where $B = \{x \in \mathbb{R}^n : |x| \leq 1\}$ is the closed ball. On the other hand, if ω were exact, we would have $\omega = d\theta$ so that $i^*\omega = i^*d\theta = d(i^*\theta)$ is exact and hence integrates to 0. By contradiction, we must have ω is not exact.

Remark: Stokes does not apply directly to $\int_{S^{n-1}} i^*\omega$, as ω cannot be extended to the entire ball B due to the norm squared term in its expression.

Problem 7: Let M be a compact orientable smooth manifold of dimension $4n+2$. Show $\dim H^{2n+1}(M;\mathbb{R})$ is even.

Note that via the cup product (which, in the case of de Rham cohomology is just the wedge product), we have

$$H^{2n+1}(M;\mathbb{R}) \times H^{2n+1}(M;\mathbb{R}) \xrightarrow{\wedge} H^{4n+2}(M;\mathbb{R}) \cong \mathbb{R}$$

Of course, $\wedge$ is bilinear and has $\omega \wedge \alpha = (-1)^{2n+1}\alpha \wedge \omega = -\alpha \wedge \omega$, so that $\wedge$ is alternating in this case. Hence we have an alternating bilinear form on $H^{2n+1}(M;\mathbb{R}) \cong \mathbb{R}^k$ (corresponding to some matrix k by k matrix A) via

$$\mathbb{R}^k \times \mathbb{R}^k \xrightarrow{A} R$$
$$(v, w) \mapsto v^T Aw$$

Moreover, this matrix A must be invertible. To see this, suppose $Aw = 0$ for some w. Then $v^T Aw = 0$ for each v. In other words, we have some $2n+1$ form ω with $\alpha \wedge \omega = 0$ for all $2n+1$-forms α. It suffices to see $\omega = 0$ locally. To see this, write out ω in some coordinate system, and let α vary between each $dx_{i_1} \wedge ... \wedge dx_{i_{2n+1}}$ to see that each corresponding coefficient of $dx_{j_1} \wedge ... \wedge dx_{j_{2n+1}}$ is zero (where $\{x_{i_1}, ..., x_{i_{2n+1}}, x_{j_1}, ..., x_{j_{2n+1}}\}$ are all $4n+2$ coordinates). Thus indeed if $Aw = 0$, then $w = 0$, so that A is invertible.

Since $(w, v) \mapsto w^T Av = (v^T A^T w)^T = -(v^T Aw)$, we must have $A^T = -A$, so that A is skew symmetric. Taking determinants, we see $\det(A^T) = \det(-A) = (-1)^k \det(A)$. On the other hand, $\det(A^T) = \det(A)$. Thus, $(-1)^k \det(A) = \det(A)$. Since $\det(A) \neq 0$, we must have k is even, as desired. $\square$

Problem 8: Show that there is no compact 3-manifold M with $\partial M \cong \mathbb{RP}^2$.

Proposition: The Euler characteristic of an odd dimensional closed manifold is zero.

Proof: See here.

Proposition: Let M be a compact manifold with boundary. Construct the double of the manifold, $2M$, as the adjunction space $M \cup_\phi M$, where $\phi : \partial M \to M$ is the inclusion. Equivalently, $2M = (M \times \{0,1\})/\sim$, where the equivalence relation has $(x,0) \sim (x,1)$ for all $x \in \partial M$. Then $2M$ is a closed manifold of the same dimension as M.

Proof: If $x \notin \partial M$, then $x \in M$ is contained in some chart $x \in U \subset M$ with $U \cong \mathbb{R}^n$ (and in particular we may insist $U \cap \partial M = 0$ by shrinking). Then $(x,0) \in (U,0)$ and $(x,1) \in (U,1)$ both have charts. Each (U,i) is open and homeomorphic to U, and thus to $\mathbb{R}^n$.

Meanwhile, if $x \in \partial M$, then pick some $V \subset M$ open with $V \cong H^n$ the upper half of $\mathbb{R}^n$. Then $(x,0) = (x,1) \in 2M$ has neighborhood $(V,0) \cup (V,1)$ which is homeomorphic to two upper half planes glued together at the boundary, i.e. $\mathbb{R}^n$ itself.

Since every point has a neighborhood homeomorphic to $\mathbb{R}^n$, $2M$ is a manifold without boundary with the same dimension as M. Its compactness is clear as it is the quotient of a compact space $M \times \{0,1\}$. $\square$

Remark: The proof also makes it clear that if M is connected, so too is $2M$, since each point has a path to its own component ($(M,0)$ or $(M,1)$), and points in opposite components may first travel to $(\partial M, 0) = (\partial M, 1)$ to cross.

See also Lee 9.29 and 9.30. $2M$ is a smooth manifold without boundary, is compact if M is, and is connected if M is. In fact, if $2M$ is orientable, then the regular domain $M \subset 2M$ is also orientable.

Remark: Note $\partial M \subset M$ is closed. So if M is compact, so too is ∂M.

Proposition: For M an odd dimensional compact manifold with boundary, $\chi(\partial M) = 2\chi(M)$.

Proof: Let $U \subset M$ be a collar neighborhood of $\partial M \subset M$. Then U deformation retracts onto ∂M. In $2M$, take $A = (U,0) \cup (M,1)$ and $B = (U,1) \cup (M,0)$. Then since U deformation retracts to the boundary, we have A deformation retracts to $(M,1)$ and B to $(M,0)$. Moreover, $A \cap B = (U,0) \cup (U,1)$ deformation retracts to $(\partial M, 1) = (\partial M, 2) \cong \partial M$, and $A \cup B = 2M$. We have a LES by Mayer-Vietoris:

$$... \to H_k(A \cap B) \to H_k(A) \oplus H_k(B) \to H_k(A \cup B) \to ...$$

which, by our deformation retracts is equivalent to the following LES:

$$... \to H_k(\partial M) \to H_k(M) \oplus H_k(M) \to H_k(2M) \to ...$$

Note that the alternating sum of ranks of abelian groups in an exact sequence add to 0. Thus we have (for $n = \dim M$)

$$\sum_{i=0}^{n} (-1)^i \left(\text{rank}(H_i(\partial M) - 2 \cdot \text{rank}(H_i(M)) + \text{rank}(H_i(2M))\right) = 0$$

so that

$$\chi(\partial M) - 2\chi(M) + \chi(2M) = 0$$

Meanwhile, $2M$ is, by previous proposition, a closed n-manifold, so that when n is odd, we have $\chi(2M) = 0$ by the above proposition. Thus in this case we have $\chi(\partial M) = 2\chi(M)$ is even. $\square$

Solution: Note $\chi(\mathbb{RP}^2) = 1 - 1 + 1 = 1$ from its cell construction, which is odd, so that by the previous proposition $\mathbb{RP}^2$ is not the boundary of an odd-dimensional compact manifold. $\square$

Problem 9: Let $L_i \subset \mathbb{R}^n$ be the coordinate axes $L_i = \{x \in \mathbb{R}^n : x_j = 0 \text{ for all } j \neq i\}$. Calculate the homology groups of $\mathbb{R}^n \setminus (L_1 \cup ... \cup L_n)$.

The deformation retract of $\mathbb{R}^n \setminus \{0\}$ to S^{n-1} via

$$H : [0,1] \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$$

$$H(t,x) = (1-t)x + \frac{tx}{||x||}$$

restricts to

$$H : [0,1] \times \mathbb{R}^n \setminus \{L_1 \cup ... \cup L_n\} \to \mathbb{R}^n \setminus \{L_1 \cup ... \cup L_n\}$$

since $x \notin L_i$ for any i if and only if there are two indices $i \neq j$ with x_i, x_j both nonzero. If x has $x_i \neq 0$, then $(1-t)x_i + tx_i/||x|| = x_i((1-t) + t/||x||)$ is also nonzero, noting for $t > 0$ $(1-t) + t/||x|| > 1 - t \geq 0$, and for $t = 0$, $(1-t) + t/||x|| = 1 > 0$.

Then notice this restriction of H gives a deformation retract of $\mathbb{R}^n \setminus \{L_1 \cup .... \cup L_n\}$ to $S^{n-1} \setminus p_1, ..., p_n, q_1, ..., q_n$, where $\{p_i, q_i\} = S^{n-1} \cap L_i$.

Geometrically, the deformation retraction of $\mathbb{R}^n \setminus \{0\}$ to the sphere S^{n-1} sends the line L_i, and only the line L_i, to $p_i, q_i \in S^{n-1}$ (depending on if $x_i > 0$ or $x_i < 0$). Each point in the deformation retract simply follows a straight line to the sphere. Thus restricting this deformation retraction simply avoids those points.

Next, notice $S^{n-1} \setminus \{p_1\}$ for some $p \in S^{n-1}$ is homeomorphic to $\mathbb{R}^{n-1}$ via stereographic projection. This homeomorphism then restricts to a homeomorphism sending $S^{n-1} \setminus \{p_1, ..., p_n, q_1, ..., q_n\}$ to $\mathbb{R}^{n-1} \setminus \{\pi(p_2), ..., \pi(p_n), \pi(q_1), ..., \pi(q_n)\}$, i.e. $\mathbb{R}^{n-1}$ with $2n-1$ points removed.

Now this is Spring 2010 Problem 6. By that problem, we have For $n > 2$,

$$H_k(\mathbb{R}^n \setminus \{L_1, ..., L_n\}) = H_k(\mathbb{R}^{n-1} \setminus \{x_1, ..., x_{2n-1}\}) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & 0 < k < n-2 \\ \mathbb{Z}^{2n-1} & k = n-2 \\ 0 & k \geq n \end{cases}$$

For $n = 2$, we have

$$H_k(\mathbb{R}^2 \setminus \{L_1, L_2\}) = H_k(\mathbb{R}^1 \setminus \{x_1, x_2, x_3\}) = \begin{cases} \mathbb{Z}^4 & k = 0 \\ 0 & k > 0 \end{cases}$$

For $n = 1$, the reader may compute $H_k(\emptyset)$ via any desired method. $\square$

Problem 10:

a) Let X be a finite CW complex. Explain how the homology groups of X are related to those of $X \times S^1$ (without using Kunneth, of course).

Let X have k-cells $e_1^k, ..., e_{n_k}^k$ for each $k = 0, ..., N$. Give S^1 a CW structure with one 0-cell v and one 1-cell $e = [0, 1]$, gluing both endpoints to v.

The product $X \times S^1$ then has cells which are products of cells of X and cells of S^1. Thus we have k-cells $e_i^k \times v$ and $e_j^{k-1} \times e$ for $i = 1, ..., n_k$ and $j = 1, .., n_{k-1}$. We have a product rule for boundaries (similar to exterior derivative of a wedge) which gives

$$\partial(e_i^k \times v) = \partial e_i^k \times v + (-1)^k e_i^k \times \partial v = \partial e_i^k \times v$$

$$\partial(e_j^{k-1} \times e) = \partial e_j^{k-1} \times e + (-1)^{k-1} e_j^{k-1} \times \partial e = \partial e_j^{k-1} \times e$$

since $\partial v = 0$ and $\partial e = v - v = 0$. Consider the chain complex for X:

$$0 \to C_N \to ... \to C_i \xrightarrow{\partial_i} C_{i-1} \to ... \to C_0 \to 0$$

Then the chain complex for $X \times S^1$ is

$$... \to C_i \oplus C_{i-1} \xrightarrow{(\partial_i, \partial_{i-1})} C_{i-1} \oplus C_{i-2} \to ...$$

Notice $\ker(\partial_i, \partial_{i-1}) \subset C_i \oplus C_{i-1}$ is simply $\ker(\partial_i) \oplus \ker(\partial_{i-1})$, and $\mathrm{im}(\partial_{i+1}, \partial_i) \subset C_i \oplus C_{i-1}$ is $\mathrm{im}(\partial_{i+1}) \oplus \mathrm{im}(\partial_i)$. Taking quotients, we see

$$H_i(X \times S^1) = H_i(X) \oplus H_{i-1}(X)$$

where $H_{-1}(X) = 0$. $\square$

b) For each $n \geq 0$, give an example of a compact smooth manifold of dimension $2n + 1$ with $H_i(X) = \mathbb{Z}$ for $i = 0, ..., 2n + 1$.

Take $X = \mathbb{CP}^n \times S^1$. By the previous problem, $H_i(X) = H_i(\mathbb{CP}^n) \oplus H_{i-1}(\mathbb{CP}^n) \cong \mathbb{Z}$ for all $i = 0, ..., 2n + 1$, since the homology $H_i(\mathbb{CP}^n)$ of $\mathbb{CP}^n$ is $\mathbb{Z}$ if and only if $0 \leq i \leq 2n$ is even, and 0 otherwise. $\square$

7 Spring 2013

Problem 1:

a) Show $S \subset M_{m\times n}(\mathbb{R})$ the subset of rank 1 matrices is a submanifold of dimension $m+n-1$.

For the special case of rank 1 matrices, we have a very short proof: for the open set of rank 1 matrices A with $A_{ij} \neq 0$, we have a chart simply sending A to the ith row and jth column. There are n real entries in the ith row, m in the jth column, and A_{ij} appears as the duplicate, and so only needs to be included once. This thus gives us a map into $\mathbb{R}^{m+n-1}$. More specifically, it is a map into the open set $\mathbb{R}^{m+n-2} \times (\mathbb{R} \setminus \{0\})$, since $A_{ij} \neq 0$. Note that the matrix is entirely determined by this row and column since it has rank 1 (and by $A_{ij} \neq 0$ we know what to multiply each row or column by to get the other rows and columns). Thus this map is injective. Moreover, this same process can easily be reversed to get a rank 1 matrix upon fixing the given entries, so that this map is actually a bijection. It is clear both directions are continuous (in fact, smooth), since it just involves projection onto entries of the matrix. Thus we get a homeomorphism of this open set of rank 1 matrices to an open set in $\mathbb{R}^{m+n-1}$, thus giving a manifold structure as desired. $\square$

b) Show that the subset $T \subset M_{m\times n}(\mathbb{R})$ of rank k matrices form a submanifold of dimension $k(m+n-k)$.

Each rank k matrix has an invertible k by k minor. WLOG, we assume this is the top left k by k minor; otherwise, we may permute the rows and columns to allow this to happen.

For $A \in M_{m\times n}(\mathbb{R})$ of any rank with the top k by k minor invertible, write

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$$

where B is k by k with $\det(B) \neq 0$ (hence the openness of this condition). Then notice

$$\begin{bmatrix} B & C \\ D & E \end{bmatrix} \begin{bmatrix} I_{k\times k} & -B^{-1}C \\ 0 & I_{(n-k)\times(n-k)} \end{bmatrix} = \begin{bmatrix} B & 0 \\ D & -DB^{-1}C + E \end{bmatrix}$$

Since $\begin{bmatrix} I & -B^{-1}C \\ 0 & I \end{bmatrix}$ is an invertible n by n matrix (it is upper triangular with all 1's in the diagonal), we see

$$\text{rank}\left(\begin{bmatrix} B & C \\ D & E \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} B & 0 \\ D & -DB^{-1}C + E \end{bmatrix}\right)$$

so that the rank k matrices of the above form are precisely those with $-DB^{-1}C + E = 0$ (since B already has k independent columns).

Define a map from the open subset of $M_{m\times n}(\mathbb{R})$ whose top k by k minor is invertible to $M_{(m-k)\times(n-k)}(\mathbb{R})$ via $\begin{bmatrix} B & C \\ D & E \end{bmatrix} \mapsto -DB^{-1}C + E$. It suffices to show that $0 \in M_{(m-k)\times(n-k)}(\mathbb{R})$ is a regular value of this map. Then by the preimage theorem (viewing this as a map from U, the open set of matrices of rank at least k with top k by k minor invertible), those matrices of rank precisely k and whose top k by k minor is invertible will be a manifold of codimension $(m-k)(n-k)$, and hence of dimension $mn - (m-k)(n-k) = nk - k^2 + mk = k(m+n-k)$. Thus each such matrix will have a chart to $\mathbb{R}^{k(m+n-k)}$. Via permutation of rows and columns (which is a diffeomorphism on $M_{m\times n}(\mathbb{R})$), we will thus get a chart for an arbitrary rank k matrix, as desired.

To see $F : U \to M_{(m-k)\times(n-k)}(\mathbb{R})$ via $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix} \mapsto -DB^{-1}C + E$ indeed has 0 as a regular value, we show F is actually a submersion. Notice for $X \in M_{(m-k)\times(n-k)}$,

$$dF_A\left(\begin{bmatrix} 0 & 0 \\ 0 & X \end{bmatrix}\right) = \lim_{t\to 0} \frac{F\left(A + t\begin{bmatrix} 0 & 0 \\ 0 & X \end{bmatrix}\right) - F(A)}{t}$$

$$= \lim_{t\to 0} \frac{-DB^{-1}C + E + tX - (-DB^{-1}C + E)}{t} = X$$

Since X is arbitrary, we see dF_A is surjective for any A. Hence F is a submersion, 0 is a regular value, and the result follows from above remarks. $\square$

Problem 2: Let ω be a 1-form on a smooth manifold M.

a) Define $\int_c \omega$ for piece-wise smooth curves $c : [0,1] \to M$.

For $c : [0,1] \to M$ piecewise smooth, with each $\gamma_i = c|_{[t_{i-1},t_i]} : [t_{i-1}, t_i] \to M$ smooth for $0 = t_0 \leq t_1 \leq ... \leq t_n = 1$. Define

$$\int_c \omega = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \gamma_i^* \omega$$

b) Show that $\omega = df$ for a smooth function $f : M \to \mathbb{R}$ if and only if $\int_c \omega = 0$ for all closed curves $c : [0,1] \to M$.

Suppose $\omega = df$ for some $f : M \to \mathbb{R}$. Let $c : [0,1] \to M$ be piecewise smooth with smooth pieces $\gamma_i = c|_{[t_{i-1},t_i]}$, $i = 1, .., n$, and closed, i.e. with $c(0) = c(1)$ (so $\gamma_1(0) = \gamma_n(1) = p$). By the above definition, we have

$$\int_c \omega = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \gamma_i^* \omega = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \gamma_i^* df = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} d(\gamma_i^* f) = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} d(f \circ \gamma_i) = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (f \circ \gamma_i)'(t)dt$$

$$= \sum_{i=1}^n (f \circ \gamma_i)(t_i) - (f \circ \gamma_i)(t_{i-1}) = (f \circ \gamma_n)(1) - (f \circ \gamma_1)(0) = f(p) - f(p) = 0$$

as desired.

Conversely, suppose $\int_c \omega = 0$ for each closed piecewise smooth curve. WLOG assume M is connected, as it suffices to show ω is exact on each component. Fix $x_0 \in M$ and define $f : M \to \mathbb{R}$ via $f(x) = \int_{\gamma_x} \omega$, where γ_x is any smooth path from x_0 to x. To see this is is indeed well-defined, suppose ρ_x is another smooth path from x_0 to x. Define $c : [0,1] \to M$ via $c(t) = \gamma_x(2t)$ for $0 \leq t \leq 1/2$, and $c(t) = \rho_x(2 - 2t)$ for $1/2 \leq t \leq 1$. Then notice c is well-defined at $t = 1/2$ since $\gamma_x(1) = \rho_x(1)$. Moreover, c is piecewise smooth and closed, with $c(0) = \gamma_x(0) = x_0 = \rho_x(0) = c(1)$. Thus by assumption, $\int_c \omega = 0$. On the other hand, by definition,

$$0 = \int_c \omega = \int_0^{1/2} (\gamma_x(2t))^* \omega + \int_{1/2}^1 (\rho_x(2-2t))^* \omega = \int_0^1 (\gamma_x)^* \omega - \int_0^1 (\rho_x)^* \omega$$

Thus, $\int_0^1 (\gamma_x)^* \omega = \int_0^1 (\rho_x)^* \omega$, and f is well-defined.

From this, we notice $f(x_0) = 0$, since we may take the constant path from x_0 to x_0. Next, notice if c is a piecewise smooth curve from x_0 to x, the above computation shows

$$\int_c df = (f \circ \gamma_n)(1) - (f \circ \gamma_1)(0) = f(x) - f(x_0) = f(x)$$

Hence, $\int_c df = \int_c \omega$ for any piecewise smooth curve c. Finally, we show if $\int_c \eta = 0$ for every piecewise smooth curve c, then $\eta = 0$. Since $\omega - df$ has this property, we will conclude $\omega = df$, as desired.

To see this fact, suppose $\int_c \eta = 0$ for every piecewise smooth curve c. Let $p \in M$ be arbitrary. It suffices to show $\eta_p = 0$.

Select a chart (x, U) with $p \in U$ mapping to $x(p) = 0$. Write $\eta = \sum_{i=1}^k g_i dx_i$ for some smooth functions g_i on U. It suffices to show each $g_i(p) = 0$. Fix i and define a map $\gamma_i : [-\epsilon, \epsilon] \to M$ via $\gamma_i(t) = x^{-1}(te_i)$, where e_i is the ith basis vector in $\mathbb{R}^n$ Notice then

$$0 = \int_{\gamma_i} \eta = \sum_{j=1}^k \int_{-\epsilon}^{\epsilon} \gamma_i^*(g_j dx_j) = \sum_{j=1}^k \int_{-\epsilon}^{\epsilon} (g_j \circ \gamma_i) d(x_j \circ \gamma_i)$$

But $x(\gamma_i(t)) = x(x^{-1}te_i) = te_i$, so that $x_j \circ \gamma_i = t$ if $i = j$ and 0 otherwise. This leaves

$$0 = \int_{\gamma_i} \eta = \int_{-\epsilon}^{\epsilon} (g_i \circ \gamma_i) dt = \int_{-\epsilon}^{\epsilon} g_i(x^{-1}te_i) dt$$

Notice by FTC that

$$\lim_{h \to 0} \frac{\int_0^h g_i(\gamma_i(t))dt}{h} = \frac{d}{dh}|_{h=0} \int_0^h g_i(\gamma_i(t))dt = g_i(\gamma_i(0)) = g_i(x^{-1}0e_i) = g_i(p)$$

Similarly, $\lim_{h\to 0} \frac{\int_{-h}^0 g_i(\gamma_i(t))dt}{h} = g_i(p)$. Then

$$0 = \lim_{\epsilon \to 0} \frac{\int_{-\epsilon}^{\epsilon} g_i(x^{-1}te_i)dt}{\epsilon} = 2g_i(p)$$

so that $g_i(p) = 0$. By previous remarks, $\eta = 0$, so that $\omega = df$, as desired. □

Problem 3: Let $S_1, S_2 \subset M$ be smooth embedded submanifolds.

a) Define what it means for S_1, S_2 to be transversal.

We say $S_1 \pitchfork S_2$ in M if for all $x \in S_1 \cap S_2$, we have $T_xS_1 \oplus T_xS_2 = T_xM$. Equivalently, the inclusion map $i : S_1 \hookrightarrow M$ has $i \pitchfork S_2$, where for $f : S_1 \to M$ and $S_2 \subset M$, we say $f \pitchfork S_2$ if for each $x \in S_1$ with $f(x) \in S_2$, we have $df_xT_xS_1 \oplus T_{f(x)}S_2 = T_{f(x)}M$.

b) Show that if $S_1, S_2 \subset M$ are transversal then $S_1 \cap S_2 \subset M$ is a smooth embedded submanifold of dimension $\dim S_1 + \dim S_2 - \dim M$.

Solution: This will follow from the preimage theorem, proved below, applied to $i : S_1 \hookrightarrow M$ which is transversal to S_2. Then $i^{-1}S_2 = S_1 \cap S_2$ will be a submanifold of S_1 and hence of M with codimension in S_1 equal to the codimension of S_2 in M, so that it has dimension $\dim(S_1) - (\dim(M) - \dim(S_2)) = \dim(S_1) + \dim(S_2) - \dim(M)$. $\square$

Lemma: Let $Z \subset Y$ be a submanifold. Then for all $x \in Z$, we may find an open set $U \ni x$ with $\phi : U \to \mathbb{R}^l$, $l = \operatorname{codim}_Y(Z)$ a submersion, i.e. $d\phi_y : T_yU \to \mathbb{R}^l$ surjective for all $y \in U$. Thus 0 is a regular value of ϕ. Then $\phi^{-1}0 = U \cap Z$.

Remark: This can be thought of as the converse to regular value theorem. Locally, submanifolds are just preimages of regular values.

Proof: The inclusion map $Z \hookrightarrow Y$ locally looks like the inclusion $(a_1, ..., a_k) \hookrightarrow (a_1, ..., a_k, 0, ..., 0)$. This gives us the desired ϕ via $(a_{k+1}, ..., a_n)$. $\square$

Corollary: (Preimage Theorem) Let $F : X \to Y$, $F \pitchfork Z$, with $dF_XT_x(X) \oplus T_{F(x)}Z = T_{F(x)}Y$ for all $x \in F^{-1}Z$. Then $F^{-1}Z$ is a submanifold of X with $\operatorname{codim}_X F^{-1}Z = \operatorname{codim}_Y Z$.

Proof: For $p \in Z$, find (U, ϕ) with $p \in U \subset Y$ such that $\phi : U \to \mathbb{R}^{\operatorname{codim}_Y(Z)} = R^l$ is a submersion, as in the lemma. Then $U \cap Z = \phi^{-1}0$. We have

$$T_xZ \hookrightarrow T_xY \twoheadrightarrow \mathbb{R}^l$$

but this composition $(d\phi_x \circ di_x) = d(\phi \circ i)_x = 0$, since $\phi \circ i$ is constant at 0. In particular, we have $T_xZ \subset \ker(d\phi_x)$ for all $x \in Z$.

We claim $\phi \circ F : V \to \mathbb{R}^l$ for $V = F^{-1}U \subset X$ has 0 as a regular value. To see this, notice for $x \in (\phi \circ F)^{-1}0 = F^{-1}\phi^{-1}0 = F^{-1}(U \cap Z) = V \cap F^{-1}Z$, we have the composition

$$T_xX \xrightarrow{dF_x} T_{F(x)}Y \xrightarrow{d\phi_{F(x)}} \mathbb{R}^l$$

By transversality, we have $T_{F(x)}Y = dF_xT_xX \oplus T_{F(x)}Z$. Meanwhile, by the above remarks, $T_{F(x)}Z$ is in the kernel of $d\phi_{F(x)}$. Since $d\phi_{F(x)}$ is surjective, it must then be that dF_xT_xX surjects onto $\mathbb{R}^l$. Thus, the composition $d\phi_{F(x)}dF_x = d(\phi \circ F)_x$ is surjective. We conclude 0 is a regular value of $\phi \circ F$.

Hence by the regular value theorem, we conclude $(\phi \circ F)^{-1}0 = V \cap F^{-1}Z$ is a submanifold of X with codimension $l = \operatorname{codim}_Y(Z)$. Notice our construction had $p \in Z$ arbitrary, $p \in U$, and $V = F^{-1}U$, so that this gives us charts for arbitrary $F^{-1}p \in F^{-1}Z$. Hence all of $F^{-1}Z$ is a manifold, of the same dimension as each $V \cap F^{-1}Z$. $\square$

Problem 4: Let $S \subset M$ be given as $f^{-1}c$ for $f = (f_1, ..., f_k) : M \to \mathbb{R}^k$ and $c \in \mathbb{R}^k$ a regular value. If $g : M \to \mathbb{R}$ is smooth, show that its restriction $g|_S$ has a critical point at $p \in S$ if and only if there are constants $\lambda_1, ..., \lambda_k$ with

$$dg_p = \sum_{i=1}^{k} \lambda_i (df^i)_p$$

Let $S \subset M$ have $S = f^{-1}0$ for $f : M \to \mathbb{R}^k$. For $g : M \to \mathbb{R}$ and $i : S \hookrightarrow M$, we have $d(g \circ i)_p : T_pS \to \mathbb{R}$ is not of full rank if and only if $d(g \circ i)_p = 0$, or equivalently, $dg_p \circ di_p = 0$.

If $dg_p = \sum_{i=1}^k \lambda_i (df_i)_p$, then

$$dg_p \circ di_p = \sum_{i=1}^{k} \lambda_i (df_i)_p \circ di_p = \sum_{i=1}^{k} \lambda_i d(f_i \circ i)_p = 0$$

since each $f_i \circ i : S \to M$ is constant at c_i. Thus by the above, in this case, we see g has a critical point at $p \in S$.

Since c is a regular value of f, we have $df_p : T_pM \to \mathbb{R}^k$ has full rank for each $p \in S = f^{-1}c$. We may view df_p as a matrix of full rank. This is equivalent to having its rows $(df_1)_p, ..., (df_k)_p$ linearly independent (since the number of rows k is fewer than $\dim M$).

On the other hand, each $f_i \circ i$ is constant, so each $(df_i)_p : T_pM \to \mathbb{R}$ factors through $(di)_p T_pS = T_pS \subset \ker(df_i)_p$, giving unique maps

$$T_i : T_pM/T_pS \to \mathbb{R}$$

with $T_i = f_i \circ \pi$, where $\pi : T_pM \to T_pM/T_pS$ is the projection.

The linear independence of the $(df_i)_p$ then implies the linear independence of the T_i, since if $\sum_{i=1}^k \lambda_i T_i = 0$, then writing $T_pM \cong T_pS \oplus T_pM/T_pS$, notice $\sum_{i=1}^k \lambda_i (df_i)_p (x + y) = \sum_{i=1}^k \lambda_i (0 + T_i(y)) = 0$, so that $\sum_{i=1}^k \lambda_i (df_i)_p = 0$. Thus each $\lambda_i = 0$, and the T_i are linearly independent.

Thus by dimension counting, we see that the T_i must span the dual space $(T_pM/T_pS)^*$. Suppose $g|_S$ has a critical point at $p \in S$. Then by the above, $d(g \circ i)_p = 0$, so that dg_p also factors through to a map $t : T_pM/T_pS \to \mathbb{R}$. Hence $t = \sum_{i=1}^k \lambda_i T_i$, so that $dg_p = \sum_{i=1}^k \lambda_i (df_i)_p$, as desired. $\square$

Problem 5: Let M be a smooth compact orientable manifold with boundary. Show that there is no smooth retract $r : M \to \partial M$.

Let $r : M \to \partial M$ be a smooth retract. Let $c \in \partial M$ be a regular value of r. This is always possible by Sard. Then by the regular value theorem, $r^{-1}c$ is a submanifold of M of dimension 1. It is compact since it is closed and M is compact. Moreover, $\partial(r^{-1}c) = r^{-1}c \cap \partial M = (\partial r)^{-1}c$, where $\partial r = r \circ i = id_{\partial M}$, where $i : \partial M \to M$ is the inclusion. Then

$$\partial(r^{-1}c) = (\partial r)^{-1}c = \{c\}$$

On the other hand, the boundary of a compact 1-manifold must have an even number of points. By contradiction, no such retract exists. □

Alternative Solution: Since M is compact orientable with boundary, use Lefshetz duality to get $H_n(M) = H^0(M, \partial M) = \widetilde{H_0(M/\partial M)} = 0$ (since $M/\partial M$ is connected), and $H_n(M, \partial M) = H^0(M) = H_0(M)^*$. On the other hand, the LES for relative homology gives

$$0 \to H_n(M, \partial M) \to H_{n-1}(\partial M) \xrightarrow{i_*} H_{n-1}(M)$$

Since $r \circ i = id$ gives $r_* \circ i_* = id$, so that i_* is injective. On the other hand, its kernel from the above exact sequence is isomorphic to $H_n(M, \partial M)$. Thus, $H_n(M, \partial M) = 0$. By contradiction, we see no such retract can exist.

Problem 6: Let $A \in GL_{n+1}(\mathbb{C})$.

a) Show that A defines a smooth map $A : \mathbb{CP}^n \to \mathbb{CP}^n$.

Since A is invertible, we may restrict $A : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ to a map $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}^{n+1} \setminus \{0\}$, which we also call A. We have $\mathbb{C}^{n+1} \setminus \{0\} \xrightarrow{A} \mathbb{C}^{n+1} \setminus \{0\} \twoheadrightarrow \mathbb{CP}^n$, where the second map is the canonical projection map $q : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ via $(z_0, ..., z_n) \mapsto [z_0, ..., z_n]$. Note for $\lambda \in \mathbb{C} \setminus \{0\}$, we have $qA(\lambda(y_0, ..., y_n)) = q(\lambda A(y_0, ..., y_n)) = q(A(y_0, ..., y_n))$, since $q(\vec{z}) = q(\lambda\vec{z})$ for any $\lambda \neq 0$. Thus by universal property of quotients, qA factors through to a map $\overline{A} : \mathbb{CP}^n \to \mathbb{CP}^n$ via $\overline{A}[x_0, ..., x_n] = qA(x_0, ..., x_n)$. In short, $\boxed{\overline{A}[x_0, ..., x_n] = [A(x_0, ..., x_n)]}$. Hereafter we refer to A interchangeably as the matrix or this induced map $\overline{A}$. □

b) Show that the fixed points of $A : \mathbb{CP}^n \to \mathbb{CP}^n$ correspond to eigenvectors of the original matrix.

Suppose $A[x_0 : .... : x_n] = [x_0 : ... : x_n]$. Then $[A(x_0, ..., x_n)] = [x : 0 : ... : x_n]$, so that $A(x_0, ..., x_n)$ is a nonzero complex multiple of $(x_0, ..., x_n)$, which is precisely when $(x_0, ..., x_n)$ is a nonzero eigenvector of A.

c) Show that $A : \mathbb{CP}^n \to \mathbb{CP}^n$ is a Lefshetz map if the eigenvalues of A all have multiplicity 1.

Definition: For $f : X \to X$ and fixed point $x \in X$, we have x is a Lefshetz fixed point if $df_x - I : T_xX \to T_xX$ is invertible. A map is Lefshetz if each fixed point is Lefshetz. Equivalently, $\Gamma(f) \pitchfork \Delta = \Gamma(id)$ in $X \times X$.

Solution: If every eigenvalue of A has algebraic multiplicity 1, then A is diagonalizable. WLOG, we deal with the case $A = diag(\lambda_0, ..., \lambda_n)$, as otherwise we may change basis accordingly. Since A is invertible, we can of course take $\lambda_i \neq 0$ for all i. The fixed points of the corresponding map from $\mathbb{CP}^n \to \mathbb{CP}^n$ are thus $[e_i]$ for $i = 0, ..., n$, where $e_i \in C^{n+1}$ is the ith standard basis vector (indexing from 0 to n).

Let $U_i = \{[x_0 : .... : x_{i-1} : 1 : x_{i+1} : ... : x_n]\} \cong \mathbb{C}^n$. Note $A|_{U_i} : U_i \to U_i$ maps U_i to U_i, since $A([x_0 : ... : 1 : ... : x_n]) = [\lambda_0 x_0 : ... : \lambda_i : ... : \lambda_n x_n] = [\lambda_0 x_0/\lambda_i : ... : 1 : ... : \lambda_n x_n/\lambda_i]$.

Thus $dA_{[e_i]} = diag(\lambda_0/\lambda_i, ..., \lambda_{i-1}/\lambda_i, \lambda_{i+1}/\lambda_i, ..., \lambda_n/\lambda_i)$. Since each $\lambda_j \neq \lambda_i$ for $j \neq i$, we see 1 is not an eigenvalue of $dA_{[e_i]}$, so that $dA_{[e_i]} - I$ is invertible. Since this works for arbitrary i, we see each fixed point is a Lefshetz fixed point and A is a Lefshetz map, as desired. $\square$

d) Show that the Lefshetz number $A : \mathbb{CP}^n \to \mathbb{CP}^n$ is $n + 1$. You may use the fact that $GL_{n+1}(\mathbb{C})$ is connected.

Select $\gamma : [0, 1] \to GL_{n+1}(\mathbb{C})$ with $\gamma(0) = A$ and $\gamma(1) = I$. Then define $H : [0, 1] \times \mathbb{CP}^n \to \mathbb{CP}^n$ via $H(t, x) = \gamma(t)(x)$, where by $\gamma(t) \in GL_{n+1}(\mathbb{C})$ applied to $x \in \mathbb{CP}^n$, we mean the map induced as in part a. This shows $\gamma(0) = A : \mathbb{CP}^n \to \mathbb{CP}^n$ is homotopic to $\gamma(1) = I : \mathbb{CP}^n \to \mathbb{CP}^n$, which is just the identity map. Note Lefshetz number is homotopy invariant, and $\mathbb{CP}^n$ only has even homology groups, so that

$$L(A) = L(id) = \chi(\mathbb{CP}^n) = \sum_{i=0}^{2n} (-1)^i \operatorname{rank} H_i(\mathbb{CP}^n) = \sum_{k=0}^{n} (-1)^{2k} \operatorname{rank} H_{2k}(\mathbb{CP}^n) = \sum_{k=0}^{n} \operatorname{rank} \mathbb{Z} = n+1$$

as desired.

Problem 7: Let $f : S^n \to S^n$ be a continuous map.

a) Define $\deg(f)$ and show that when f is smooth, $\deg(f) \int_{S^n} \omega = \int_{S^n} f^*\omega$ for all $\omega \in \Lambda^n(S^n)$.

See Spring 2011 Problem 3b.

b) Show that if f has no fixed points, then $\deg(f) = (-1)^{n+1}$.

See Fall 2010 Problem 6.

Problem 8: Let $f : S^{n-1} \to S^{n-1}$ be a continuous map, and let D^n be the disk with $\partial D^n = S^{n-1}$.

a) Define the adjunction space $D^n \cup_f D^n$.

For spaces X, Y and subspace $A \subset X$ with function $f : A \to Y$, the adjunction space $X \cup_f Y$ is the quotient space $(X \sqcup Y)/\sim$, where $\sim$ is the equivalence relation given by $x \sim f(x)$ for each $x \in A$.

b) Let $\deg(f) = k$ and compute the homology groups $H_i(D^n \cup_f D^n, \mathbb{Z})$.

In $D^n \cup_f D^n$, we are attaching two n-cells to S^{n-1}: one via the identity map $S^{n-1} \xrightarrow{id} S^{n-1}$ and the other via $S^{n-1} \xrightarrow{f} S^{n-1}$. (The attaching maps are maps from the boundary of the n-cell D^n to the $(n-1)$-skeleton, which in this case we take to be S^{n-1}).

We may give S^{n-1} a cell structure of a single 0-cell and one $(n-1)$-cell. Thus we have in $D^n \cup_f D^n$ two n-cells e_1^n (attached via id), e_2^n (attached via f), an $(n-1)$-cell e^{n-1}, and a 0-cell e^0.

By the cellular boundary formula, we may compute the coefficient of ∂e_1^n in the unique $(n-1)$ cell e^{n-1} by computing the degree of the map $\partial D^n \xrightarrow{id} S^{n-1} \to S^{n-1}/e^0 = S^{n-1}$, where the last map is just the identity map as it just quotients by all cells of dimension less than or equal to $n-1$, except for e^{n-1} itself. Of course, this is just the identity map and so has degree 1. Thus, $\partial(e_1^n) = e^{n-1}$. Similarly, $\partial(e_2^n) = ke^{n-1}$, since the coefficient is the degree of $\partial D^n \xrightarrow{f} S^{n-1} \to S^{n-1}/e^0 = S^{n-1}$.

Meanwhile, $\partial(e^{n-1}) = 0$ (which is clear if $n-1 > 1$ as there are no $n-2$ cells, and if $n-1 = 1$, then $\partial(e^{n-1}) = e^0 - e^0 = 0$). It is clear $\partial(e^0) = 0$.

For $n-1 > 1$, we get the chain complex for $X = D^n \cup_f D^n$ given by

$$0 \to C_n(X) = \mathbb{Z}^2 \xrightarrow{\partial_n} C_{n-1}(X) = \mathbb{Z} \to 0 \to ... \to 0 \to C_0(X) = \mathbb{Z} \to 0$$

Notice $H_n(X) = \ker(\partial_n) = \{(x,y) \in \mathbb{Z}^2 : x + ky = 0\} \cong \mathbb{Z}$ via $(x,y) \mapsto y$. Meanwhile, ∂_n is clearly surjective, so that $H_{n-1}(X) = 0$. It is clear $H_i(X) = 0$ for $0 < i < n-1$, and $H_0(X) = \mathbb{Z}$.

For $n-1 = 1$, we get a slightly different chain complex, with

$$0 \to C_2(X) \to C_1(X) \to C_0(X) \to 0$$

But we still have ∂_2 surjective, ∂_1 is zero, and $\ker(\partial_2) \cong \mathbb{Z}$, so that $H_2(X) \cong \mathbb{Z}$, $H_1(X) = 0$, and $H_0(X) = \mathbb{Z}$. In all cases we see

$$H_i(D^n \cup_f D^n) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & 0 < i < n \\ \mathbb{Z} & i = n \\ 0 & i > n \end{cases}$$

Remark: This is the same as the homology of S^n, and is independent of the choice of f!

c) Assume that f is a homeomorphism. Show $D^n \cup_f D^n$ is homeomorphic to S^n.

Write $g : D^n \to S^n$ with $g(x) = (x, \sqrt{1-|x|^2})$. (Note $x \in \mathbb{R}^n$ so $(x, \sqrt{1-|x|^2}) \in \mathbb{R}^{n+1}$, with the first entry contributing n components and the last contributing one.)

Similarly, write $h : D^n \to S^n$ via $h(x) = (f(x/|x|) \cdot |x|, -\sqrt{1-|x|^2})$, with $h(0) := (0, ..., 0, -1)$. Note that h is continuous (in particular at 0), since if $x_n \in D^n \setminus \{0\}$ have $x_n \to 0$, then $|f(x_n/|x_n|)|$ is fixed at 1, and $|x_n| \to 0$, so that $|f(x_n/|x_n|) * |x_n|| \to 0$ and $-\sqrt{1-|x_n|^2} \to -1$.

Hence we have two maps $h, g : D^n \to S^n$, inducing a map $D^n \sqcup D^n \xrightarrow{h \sqcup g} S^n$. Next, recalling our map $f : \partial D^n \to S^{n-1} \subset D^n$, which we regard as a map from the first copy of D^n to the second, we see $x \sim f(x)$ for each $x \in S^{n-1}$ in the first copy. Then notice $h(x) = (f(x/|x|), -\sqrt{1-|x|^2}) = (f(x), 0)$, since $|x| = 1$. Meanwhile, $g(f(x)) = (f(x), -\sqrt{1-|f(x)|^2}) = (f(x), 0)$. Hence, for $x \sim f(x)$, we have $h(x) = g(f(x))$. So we get a well-defined map

$$D^n \cup_f D^n \xrightarrow{\phi = (h \sqcup g)/\sim} S^n$$

It suffices to check this is a bijection, since $D^n \cup_f D^n$ is compact (a quotient of compact space $D^n \sqcup D^n$), and S^n is Hausdorff.

To see this map is bijective, suppose $\phi(x) = \phi(y)$. If x, y are both in the image of the first copy of D^n in $D^n \sqcup D^n \twoheadrightarrow D^n \cup_f D^n$, then $\phi(x) = h(x)$ and $\phi(y) = h(y)$. From $h(x) = \phi(x) = \phi(y) = h(y)$, we get $\sqrt{1-|x|^2} = \sqrt{1-|y|^2}$, so that $|x| = |y|$. Thus either both points are zero and $x = y$, or else $f(x/|x|) = f(y/|y|)$. But f is bijective, so $x/|x| = y/|y|$, so that $x = y$ (since $|x| = |y|$).

Similarly, if both x, y come from the image of the second copy of D^n, then $\phi(x) = \phi(y) \Rightarrow g(x) = g(y)$, so that $x = y$ by looking at the first component of g.

Finally, suppose x comes from one copy of D^n and y from the other. Then $h(x) = \phi(x) = \phi(y) = g(y)$, so that from the last component, we get $|x| = |y| = 1$, as otherwise, we would have a strictly negative $-\sqrt{1-|x|^2}$ equal the strictly positive $\sqrt{1-|y|^2}$. Then from the first component, we get $f(x/|x|) * |x| = y$, so that $f(x) = y$ (since $|x| = |y| = 1$). But then $x \sim y$ in $D^n \cup_f D^n$, so that $x = y$ in this space.

Hence we see ϕ is injective. To see it is surjective, let $p \in S^n$. If the last coordinate of p is nonnegative, write $p = (q, \sqrt{1-|q|^2})$, and notice $p = g(q)$. If the last coordinate of p is negative, write $p = (q, -\sqrt{1-|q|^2})$. If $q = 0$, then notice $h(0) = p$. Otherwise, we may assume $q \neq 0$. Take the unique $z \in S^{n-1}$ with $f(z) = q/|q|$. Then set $r = z * |q|$. Notice, then, that $h(r) = (f(z) * |q|, -\sqrt{1-|q|^2}) = (q, -\sqrt{1-|q|^2}) = p$.

Thus we see ϕ is surjective. Since it is a continuous bijection from a compact to Hausdorff space, we conclude it is a homeomorphism, as desired. $\square$

Problem 9: Let $f : M \to N$ be a finite covering map between closed manifolds. Prove or find a counterexample:

a) Do M, N have the same fundamental groups?

Take $S^2 \twoheadrightarrow \mathbb{RP}^2$. Then $\pi_1(S^2) = 0, \pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$.

b) Do M, N have the same de Rham cohomology groups?

With the same example, $H^2_{dR}(S^2) \cong \mathbb{R}$ and $H^2_{dR}(\mathbb{RP}^2) = 0$, since the first is orientable and the second is not.

c) When M is simply connected, do M, N have the same singular homology groups?

With the same example, $H_1(S^2) = 0$ and $H_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$

Problem 10: Let $A \subset X$ be a subspace. Define the relative singular homology groups $H_i(X, A)$ and show there is a long exact sequence

$$... \to H_i(A) \to H_i(X) \to H_i(X, A) \to H_{i-1}(A) \to ...$$

We have a short exact sequence of chain complexes

$$0 \to C_n(A) \xrightarrow{i_*} C_n(X) \xrightarrow{q} C_n(X)/C_n(A) \to 0$$

where q is the quotient map $C_n(X) \twoheadrightarrow C_n(X)/C_n(A)$. Note that the $C_n(X)/C_n(A)$ give a chain complex with boundary $\partial([\sigma]) = [\partial\sigma] \in C_{n-1}(X)/C_{n-1}(A)$. This is well-defined, since we have the composition $C_n(A) \xrightarrow{i_*} C_n(X) \xrightarrow{\partial} C_{n-1}(X) \twoheadrightarrow C_{n-1}(X)/C_{n-1}(A)$ is 0, as $\partial \circ i_* = i_* \circ \partial$, so that this is the same as the composition $C_n(A) \xrightarrow{\partial} C_{n-1}(A) \xrightarrow{i_*} C_{n-1}(X) \to C_{n-1}(X)/C_{n-1}(A)$ which is indeed 0. From this it also follows $\partial \circ q = q \circ \partial$, so that q is a chain map.

To this SES of chain complexes, we apply the Zig Zag Lemma from Spring 2010 Problem 5.
$\square$

8 Fall 2013

Problem 1: Let $f : M \to N$ be a non-singular smooth map between connected manifolds of the same dimension.

a) Is f necessarily injective/surjective?

> We have $\mathbb{R} \twoheadrightarrow S^1$ via $t \mapsto e^{it}$ is not injective, and $(a, b) \hookrightarrow \mathbb{R}$ is not surjective, even though both are local diffeomorphisms between connected manifolds.

b) Is f necessarily a covering map when N is compact?

> Consider $(a, b) \hookrightarrow [a, b]$.

c) Is f necessarily an open map?

> Since f must be a local diffeomorphism, it is a local homeomorphism and hence open. Take $V \subset M$ nonempty open, and $y \in f(V)$ arbitrary. Write $y = f(x)$ for $x \in V$. Since f is a local homeomorphism, select open set $U \ni x$ such that $f|_U : U \to f(U)$ is a homeomorphism with $f(U)$ open in N. In particular, $f|_U$ is open, so $y \in f|_U(U \cap V) = f(U \cap V) \subset f(V)$ is an open neighborhood of $y \in f(V)$, and $f(V)$ is open as desired.

d) Is f necessarily a closed map?

> Take $(a, b) \hookrightarrow \mathbb{R}$, which has image $(a, b) \subset \mathbb{R}$ which is not closed.

Problem 2: Let M be a connected, compact manifold with non-empty boundary. Show that there is no retract $M \to \partial M$.

> See Spring 2013 Problem 5. We don't have orientability for the homology solution, but we can drop that assumption by working in $\mathbb{Z}/2\mathbb{Z}$ coefficients.

Problem 3: Let $M, N \subset \mathbb{R}^{p+1}$ be two compact, smooth, oriented submanifolds of dimension m and n respectively, with $m + n = p$. Suppose that $M \cap N = \emptyset$. Consider the linking map $\lambda : M \times N \to S^p$ by $\lambda(x, y) = \frac{x-y}{\|x-y\|}$. Write $l(M, N) = \deg(\lambda)$.

a) Show that $l(M, N) = (-1)^{(m+1)(n+1)} l(N, M)$.

Note by definition that $l(N, M)$ is the degree of the map $\mu : N \times M \to S^p$ via $\mu(y, x) = \frac{y-x}{||y-x||}$. We write this as a composition of λ with other maps as follows:

$$N \times M \xrightarrow{T} M \times N \xrightarrow{\lambda} S^p \xrightarrow{\phi} S^p$$

where $T(y, x) = (x, y)$ is the "swapping" map, and $\phi(z) = -z$ is the antipodal map. Since $\mu(y, x) = -\lambda(x, y) = \phi(\lambda(T(y, x)))$, we have $\mu = \phi \circ \lambda \circ T$, and $\deg(\mu) = \deg(\phi) \deg(\lambda) \deg(T)$.

Since ϕ is the antipodal map from S^p to S^p, it has degree $(-1)^{p+1}$. Meanwhile, note that $T : N \times M \to M \times N$ is clearly a diffeomorphism. Each point $(x, y) \in M \times N$ has precisely one preimage, $(y, x) \in N \times M$, and locally this map looks the same as

$$\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^n$$

$$(y_1, ..., y_n, x_1, ..., x_m) \mapsto (x_1, ..., x_m, y_1, ..., y_n)$$

This map can be thought of as the composition of mn transpositions, and each transposition has degree -1 (as is clear from the determinant of the corresponding Jacobian), so that $\deg(T) = (-1)^{mn}$.

Hence we conclude $\deg(\mu) = (-1)^{p+1}(-1)^{mn} \deg(\lambda) = (-1)^{m+n+mn+1} \deg(\lambda)$, giving $l(N, M) = (-1)^{(m+1)(n+1)} l(M, N)$, as desired.

b) Show that if M is the boundary of an oriented submanifold $W \subset \mathbb{R}^{p+1}$ disjoint from N, then $l(M, N) = 0$.

We add the assumption that N is boundariless. Note that λ may be extended to $W \times N$, since $W \cap N = \emptyset$. Write $\overline{\lambda} : W \times N \to S^p$ via $\overline{\lambda}(x, y) = \frac{x-y}{||x-y||}$. Clearly, $\overline{\lambda}$ extends λ. Moreover, $\partial(W \times N) = (\partial W \times N) \sqcup (W \times \partial N) = M \times N \sqcup (W \times \emptyset) = M \times N$. Hence, $M \times N$ is the boundary of a manifold $W \times N$, with $\lambda : M \times N \to S^p$ able to be extended to all of $W \times N$. By the extension theorem, $l(M, N) = \deg(\lambda) = 0$.

Problem 4: Show that a 1-form ω on a connected manifold M is exact if and only if $\int_c \omega = 0$ for all piecewise smooth curves ω.

See Spring 2013 Problem 2.

Problem 5: Let ω be a smooth nonvanishing 1-form on a 3-dimensional manifold M.

a) Show that $\ker(\omega)$ is integrable if and only if $\omega \wedge d\omega = 0$.

We have $\ker(\omega)$ is integrable if and only if for any two vector fields $X, Y \in \ker(\omega)$, we have $[X, Y] \in \ker(\omega)$. Notice for 1-forms ω, we have

$$\omega([X, Y]) = X(\omega(Y)) - Y(\omega(X)) - d\omega(X, Y)$$

Hence if $X, Y \in \ker(\omega)$, then $[X, Y] \in \ker(\omega)$ if and only if $d\omega(X, Y) = 0$ (the above formula would give $\omega([X, Y]) = -d\omega(X, Y)$, since $\omega(X) = \omega(Y) = 0$).

So we have $\ker(\omega)$ is integrable if for every $X, Y \in \ker(\omega)$, we have $d\omega(X, Y) = 0$. Next, for any $p \in M$, $\ker(\omega)_p = \ker(\omega_p : \mathbb{R}^3 \to \mathbb{R})$ is 2-dimensional since ω_p is nonzero. Pick a basis of $\ker(\omega)_p$ and extend it to a basis of T_pM. Let this basis be X, Y, Z with $Y, Z \in \ker(\omega)_p$ a basis. Then notice

$$(\omega \wedge d\omega)_p(X, Y, Z) = \omega_p(X)(d\omega)_p(Y, Z) - \omega_p(X)(d\omega)_p(Z, Y) = 2\omega_p(X)(d\omega)_p(Y, Z)$$

since the $\omega_p(Y)$ and $\omega_p(Z)$ terms always vanish. Moreover, $\omega_p(X)$ is nonzero, as X is not in $\ker(\omega_p)$.

If $\ker(\omega)$ is integrable, then $(d\omega)_p(Y, Z) = 0$ by the above remarks for each p. Then $\omega \wedge d\omega$ is locally zero on the basis X, Y, Z, so that it is identically zero locally, and hence globally. So $\omega \wedge d\omega = 0$. Conversely, if $\omega \wedge d\omega = 0$, we see by the above that $(d\omega)_p(Y, Z) = 0$ for the local basis vectors Y, Z of $\ker(\omega)$, so that this is true for any two vectors in $\ker(\omega)$, and $\ker(\omega)$ is integrable by the above equivalence.

We conclude for ω a nonvanishing 1-form on a 3-manifold, $\ker(\omega)$ is integrable if and only if $\omega \wedge d\omega = 0$.

b) Give an example of a codimension 1 distribution on $\mathbb{R}^3$ that is not integrable.

Take $\omega = -ydx + xdy + dz$. Then $\omega \wedge d\omega = (-ydx + xdy + dz) \wedge (2dx \wedge dy) = 2dx \wedge dy \wedge dz \neq 0$, so that $\ker(\omega)$ is not integrable.

Problem 6: Let $f : \mathbb{R}^n \to \mathbb{R}$ a smooth function.

a) Define the gradient ∇f as a vector field dual to the differential df.

The dual $(dx_i)^* = \frac{\partial}{\partial x_i}$ gives rise to

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

$$\nabla f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}$$

b) Define the Hessian $H_f(X, Y)$ as a symmetric $(0, 2)$-tensor.

To say the Hessian is a $(0, 2)$-tensor is to say it is the tensor of 0 tangent vectors and 2 cotangent vectors. Define

$$H_f = \sum_{1 \le i,j \le n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \otimes dx_j$$

We can think of H_f as a bilinear form. That is, writing down a matrix with $(H_f)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$, we have $H_f(X, Y) = X^T H_f Y$.

This matrix H_f and hence the corresponding $(0, 2)$ tensor is symmetric since the mixed partials commute.

c) If the usual Euclidean inner product is denoted $g_p(X, Y) = X \cdot Y$, show that $H_f(X, Y) = \frac{1}{2}(\mathcal{L}_{\nabla f} g)(X, Y)$.

Note that we may write $g_p = \sum_{i=1}^{n} dx_i \otimes dx_i$, with associated matrix of the identity, so that $g_p(X, Y) = X^T I Y = X^T Y = X \cdot Y$. Now

$$\mathcal{L}_{\nabla f}(g_p) = \mathcal{L}_{\nabla f} \left(\sum_{i=1}^{n} dx_i \otimes dx_i \right)$$

$$= \sum_{i=1}^{n} \mathcal{L}_{\nabla f}(dx_i \otimes dx_i) = \sum_{i=1}^{n} (\mathcal{L}_{\nabla f} dx_i) \otimes dx_i + \sum_{i=1}^{n} dx_i \otimes (\mathcal{L}_{\nabla f} dx_i)$$

Recall $\mathcal{L}_X \alpha = i_X d\alpha + d i_X \alpha$. Thus, we get

$$\mathcal{L}_{\nabla f} dx_i = i_{\nabla f} 0 + d i_{\nabla f} dx_i = d\left((\nabla f)(x_i)\right) = d\left(\frac{\partial f}{\partial x_i}\right) = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j$$

Continuing our computation from above, we see

$$\mathcal{L}_{\nabla f}(g_p) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j \right) \otimes dx_i + \sum_{i=1}^{n} dx_i \otimes \left(\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \otimes dx_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \otimes dx_j = 2H_f$$

as desired. $\square$

Problem 7: Let $M = T^2 \setminus D^2$ be the complement of a disk in the torus. Determine all connected surfaces that can be 3-fold covers of M.

Note that $T^2 \setminus D^2$ deformation retracts to $S^1 \vee S^1$. One may see this by viewing T^2 a the usual quotient of the unit square. Deleting a disk from the center of the square, we see the rest deformation retracts to the boundary of the square, which, upon gluing, gives $S^1 \vee S^1$. Now we may use the usual construction of covering spaces of $S^1 \vee S^1$. The algebraic method counts index 3 subgroups of $G = \pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$ in order to get 3-fold connected covering spaces, keeping track of base point. If $H \subset G$ is index 3, then G acts on the cosets G/H transitively, giving a homomorphism $G \to S(G/H) \cong S_3$ whose image is a transitive subgroup. In our isomorphism $S(G/H) \cong S_3$, we insist on sending the coset H to $1 \in \{1, 2, 3\}$, but we may send the other two cosets to either $2, 3$ in any order. Meanwhile, any such homomorphism into S_3 lets us recover H by taking the stabilizer of $1 \in \{1, 2, 3\}$.

There are $6 * 6 = 36$ homomorphisms $G \to S_3$, sending each of the generators to any element of S_3. If the image has order 2, note that there are four homomorphisms $G \to \mathbb{Z}/2\mathbb{Z}$, and of these, one is not surjective, so that three are. Since S_3 has three order 2 subgroups, we have 9 homomorphisms $G \to S_3$ with image of order 2. We have one $G \to S_3$ with trivial image, leaving behind 26 homomorphisms $G \to S_3$ with image A_3 or S_3 (both of which are transitive). Note then that H appears as the stabilizer in precisely two such homomorphisms, as we may swap $2, 3$ in $\{1, 2, 3\}$ without affecting the stabilizer of 1. This gives that there are thirteen subgroups $H \subset G$ of order 3, so that there are thirteen 3-fold connected covers of $S^1 \vee S^1$, keeping track of base point.

If we did not keep track of base point, we would need to consider subgroups up to conjugacy. This can also be accomplished. Refine our count a bit further to notice that we have 26 homomorphisms $G \to S_3$ with image A_3 or S_3. Since $A_3 \cong \mathbb{Z}/3$ and we have 9 homomorphisms $G \to A_3$, of which only one is not surjective, we must have 8 of the 26 homomorphisms $G \to S_3$ whose image is A_3, so that the remaining 18 have image S_3. Note that H contains the kernel since the kernel is the intersection of the stabilizers. So if $G \to S_3$ has image A_3 (of order 3), it must have kernel precisely H, so that H is normal. Thus, it is equal to all of its conjugates. By the above argument, H still occurs as the stabilizer of 1 in two such homomorphisms, so these homomorphisms contribute $8/2 = 4$ subgroups up to conjugacy.

Finally, the 18 surjective homomorphisms $G \to S_3$ have the stabilizer of 1 is H, but the stabilizers of $2, 3$ are conjugates of H. Permuting $\{1, 2, 3\}$ in any of the 6 possible ways, we still get H and conjugates of H. Conversely, any conjugate gHg^{-1} necessarily is the stabilizer of some gH. Thus these homomorphisms contribute $18/6 = 3$ subgroups up to conjugacy. In total, we get $4 + 3 = 7$ subgroups up to conjugacy, so that we have seven 3-fold connected covers of $S^1 \vee S^1$, ignoring basepoint.

Now, we do this graphically. Recall that graphically, the 3-fold connected covers will correspond to connected directed graphs on 3-vertices with each vertex having 4 edges: one incoming and one outgoing edge of each of type a and b.

From this we get the following graphs. We can be sure we have exhaustively listed them all by casing on how many loops we have, noting the possible number are $3, 2, 1, 0$ (to maintain connectedness).

1. (3 loops) Vertices $1, 2, 3$ with edges $b = (1, 2)$, $b = (2, 3)$, $b = (3, 1)$, and edges $a = (1, 1)$, $a = (2, 2)$, $a = (3, 3)$. We get another such graph by swapping all a's and b's. If we want to keep track of base point, note the base points are all indistinguishable here. So this contributes 2 to both counts (keeping track of basepoint vs not keeping track of basepoint).
2. (2 loops) Vertices $1, 2, 3$ with edges $a = (1, 2), a = (2, 1), a = (3, 3)$, $b = (1, 1), b = (2, 3), b = (3, 2)$. This time, swapping a and b changes nothing. However, all 3 vertices are distinguishable. Thus we contribute 1 to the count ignoring base point, and 3 to the count not ignoring base point.
3. (1 loop) Vertices $1, 2, 3$ with edges $b = (1, 1)$, $b = (2, 3)$ and $b = (3, 2)$, along with $a = (1, 2), a = (2, 3), a = (3, 1)$. Again we may swap all a's and b's. This contributes 2 to not keeping track of basepoint, but 6 if we are keeping track, as all 3 vertices are distinguishable.
4. (0 loops) Vertices $1, 2, 3$ with edges $a = (1, 2), a = (2, 3), a = (3, 1)$, $b = (1, 2), b = (2, 3), b = (3, 1)$. Here swapping a and b does nothing, and all vertices are indistinguishable, so we add 1 to both counts. Similarly, we get vertices $1, 2, 3$ with edges $a = (1, 2), a = (2, 3), a = (3, 1), b = (2, 1), b = (3, 2), b = (1, 3)$, which is the same as the previous example but with one set of edges going in the opposite orientation as the other. Again, a and b being swapped changes nothing, and vertices are indistinguishable, so we add 1 to both counts.

In total, we see there are 7 connected 3-fold covers if we ignore base point, and 13 if we keep track of base point, as desired.

Finally, to get coverings of M, attach 2-cells minus a disk to the boundary words $aba^{-1}b^{-1}$, one for each vertex. This does not affect any counts.

Problem 8: Let $n > 0$ and let A be a finitely presented abelian group. Show there is a topological space X with $H_n(X) \cong A$.

Over Noetherian rings, a module is finitely presented if and only if it is finitely generated. So we have A is a finitely generated abelian group. Then by FTFGAG, we have $A \cong \mathbb{Z}^r \oplus \mathbb{Z}/m_1\mathbb{Z} \oplus ... \oplus \mathbb{Z}/m_k\mathbb{Z}$. Since $H_n(X \sqcup Y) = H_n(X) \oplus H_n(Y)$, it suffices to find spaces with homology groups $\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z}$ for $m \in \mathbb{Z}_{>0}$.

Of course, $H_n(S^n) = \mathbb{Z}$. To get X with $H_n(X) = \mathbb{Z}/m\mathbb{Z}$, construct X by attaching an $n+1$ cell $e^{n+1} = D^{n+1}$ to S^n via $\partial D^{n+1} = S^n \xrightarrow{f} S^n$ any function of degree m. Give S^n the usual cell structure of one 0-cell and one n-cell. By the cellular boundary formula, we get ∂e^{n+1} is the degree of $S^n \to fS^n \to S^n/e^0 = S^n$, where in the last step we quotient out by all other cells, which does nothing. By construction the degree of this is m, so that $\partial e^{n+1} = me^n$. Then $H_n(X)$ is the kernel of ∂_{n-1} mod the image of ∂_n. Note $\partial_{n-1} = 0$ and there is only one n-cell, so that $H_n(X) = \mathbb{Z}/im(\partial_n) = \mathbb{Z}/m\mathbb{Z}$, as desired. □

Problem 9: Compute the homology groups and fundamental group of $S^3 \setminus H$, where H is the Hopf-link, i.e. two linked circles.

Note $S^3 \setminus H$ is the same as first deleting $p \in H$ then deleting $H \setminus \{p\}$. That is, $S^3 \setminus H = (S^3 \setminus \{p\}) \setminus (H \setminus \{p\})$. WLOG, $p = \infty$ is the point at infinity, so that $S^3 \setminus \{p\} \cong \mathbb{R}^3$. Meanwhile, $H \setminus \{p\}$ leaves behind a circle and a line going through it. WLOG, we may take this to be the unit circle on the xy-plane and the z-axis.

On the other hand, $\mathbb{R}^3$ minus the unit circle and z-axis deformation retracts to a torus. To see this, notice that each half plane $\{(r, \theta, z) \in R^3 : r \in (0, \infty), \theta = \theta_0, z \in \mathbb{R}\} \cong (0, \infty) \times \mathbb{R} \cong \mathbb{R}^2$, when deleting the z-axis and unit circle, leaves behind $(0, \infty) \times \mathbb{R} \setminus \{(\cos(\theta_0), \sin(\theta_0), 0)\}$, which is homeomorphic to $\mathbb{R}^2$ minus a point, which deformation retracts to a circle. In this way, each half plane deformation retracts to a circle, and $\mathbb{R}^3$ minus the z-axis and unit circle deformation retracts to a torus (as we get a circle for each $\theta = \theta_0$ in a continuous fashion). Thus, the problem reduces to computing the fundamental group and homology groups of the torus. □

Problem 10: Let H be the quaternion algebra over $\mathbb{R}$, with $i^2 = j^2 = -1$, $ij = -ji = k$. The quotient space $\mathbb{HP}^n = (\mathbb{H}^{n+1} \setminus \{0\})/\mathbb{H}^*$ is called quaternionic projective space. Compute $H_k(\mathbb{HP}^n)$.

We mimic the construction of $\mathbb{RP}^n$ and $\mathbb{CP}^n$ (from Spring 2011 Problem 7 and Spring 2011 Problem 8) to give $\mathbb{HP}^n$ a cell structure with a cell in every dimension which is a multiple of 4, up to $4n$. Of course, $\mathbb{HP}^0$ is a point, which is a single 0-cell. Next, given the cell structure on $\mathbb{HP}^{n-1}$, we can get the cell structure on $\mathbb{HP}^n$ by attaching a $(4n)$-cell with

$$S^{4n-1} \xrightarrow{\phi} \mathbb{HP}^{n-1}$$

$$(\alpha_0, ..., \alpha_{n-1}) \mapsto [\alpha_0 : ... : \alpha_{n-1}]$$

where we view $S^{4n-1} \subset \mathbb{R}^{4n} \cong \mathbb{H}^n$.Then $D^{4n} \cup_\phi \mathbb{HP}^{n-1} \cong \mathbb{HP}^n$ via

$$\mathbb{HP}^{n-1} \hookrightarrow \mathbb{HP}^n$$

$$[\alpha_0 : ... : \alpha_{n-1}] \mapsto [\alpha_0 : ... : \alpha_{n-1} : 0]$$

$$D^{4n} \xrightarrow{f} \mathbb{HP}^n$$

$$(\alpha_0, ..., \alpha_{n-1}) \mapsto \left[\alpha_0 : ... : \alpha_{n-1} : \sqrt{\left(1 - \sum_{i=0}^{n-1} |\alpha_i|^2\right)}\right]$$

The same argument as in the previous exercises shows that these maps factor through and give a bijective map $D^{4n} \cup_\phi \mathbb{HP}^{n-1} \to \mathbb{HP}^n$ from a compact space to a Hausdorff space, so that it is a homeomorphism, as desired.

The cell complex has all maps are 0, since there are no cells of adjacent dimension. Thus, we must have $H_k(\mathbb{HP}^n) = C_k(\mathbb{HP}^n)$ the free abelian group on the k-cells. So we see

$$H_k(\mathbb{HP}^n) = \begin{cases} \mathbb{Z} & 4|k \text{ and } 0 \leq k \leq 4n \\ 0 & \text{otherwise} \end{cases}$$

9 Spring 2014

Problem 1: Let $\Gamma \subset \mathbb{R}^2$ be the graph of the function $y = |x|$.

a) Construct a smooth function $f : \mathbb{R} \to \mathbb{R}^2$ whose image is Γ.

Pick $\phi : \mathbb{R} \to \mathbb{R}$ a bump function with $0 \leq \phi \leq 1$, with $\phi = 1$ on $K = [-1, 1]$ and $\phi = 0$ on $U = (-2, 2)$. Set $\psi = 1 - \phi$. Then $\psi \geq 0$ is 0 on $[-1, 1]$ and 1 outside of $(-2, 2)$.

Set $g : \mathbb{R} \to \mathbb{R}^2$ via $g(x) = (\psi(x)x, |\psi(x)x|) = (\psi(x)x, \psi(x)|x|)$.

It is clear g is smooth at $x \neq 0$. Meanwhile, note $\psi(x)x \equiv 0$ in a neighborhood of $x = 0$. Hence g is smooth at $x = 0$.

Finally, notice $g(0) = (0, 0)$, $g(x) = (x, |x|)$ for $|x| \geq 2$, and the image of g is connected. Thus it must contain all points in between by IVT, so that g has image precisely the graph of $y = |x|$, as desired. □

b) Can f be an immersion?

Any such map must have $f : \mathbb{R} \to \mathbb{R}^2$ given by $f(x) = (g(x), |g(x)|)$. Suppose f is an immersion and $f(x_0) = (0, 0)$. Then df_{x_0} is injective. Then we need $g'(x_0) \neq 0$ and $\frac{d}{dx}|_{x=x_0}|g(x)|$ to exist and be nonzero.

But on the other hand,

$$g'(x_0) = \lim_{h\to 0} \frac{g(x_0 + h) - g(x_0)}{h} = \lim_{h\to 0^+} \frac{g(x_0 + h)}{h}$$

Then

$$|g'(x_0)| = \lim_{h\to 0^+} \frac{|g(x_0 + h)|}{h} = \lim_{h\to 0^+} \frac{|g(x_0 + h)| - |g(x_0)|}{h} = \frac{d}{dx}|_{x=x_0}|g(x)|$$

Similarly,

$$|g'(x_0)| = \lim_{h\to 0^-} \frac{|g(x_0 + h)|}{|h|} = \lim_{h\to 0^-} \frac{-|g(x_0 + h)| - |g(x_0)|}{h} = -\frac{d}{dx}|_{x=x_0}|g(x)|$$

Thus, $\frac{d}{dx}|_{x=x_0}|g(x)| = |g'(x_0)| = -\frac{d}{dx}|_{x=x_0}|g(x)|$, so that these are all 0. By contradiction, such an f may not exist. □

Problem 2: Let W be a smooth manifold with boundary, and $f : \partial W \to \mathbb{R}^n$ a smooth map for some $n \geq 1$. Show that there is a smooth map $F : W \to \mathbb{R}^n$ such that $F|_{\partial W} = f$.

Pick a collar neighborhood of $\partial M \subset M$. That is, select some neighborhood $U \subset M$ with $\partial M \subset U$ and $U \cong [0, 1) \times \partial M$ (with ∂M corresponding to $\{0\} \times \partial M$ in the correspondence). Write $\pi : U \to \partial M$ as the projection of U onto ∂M. Pick a bump function $\rho(x)$ on M that is 1 on ∂M (which is closed) and 0 outside of U.

Write $g : M \to \mathbb{R}^n$ via $g(x) = \rho(x)f(\pi(x))$. Note $\pi(x)$ is only defined for $x \in U$, but $\rho(x)$ is zero outside of U, so g is well-defined.

Finally, notice for $x \in \partial M \subset U$, we have $\rho(x) = 1$, and $\pi(x) = x$, so $g(x) = f(x)$, as desired.

Problem 3: Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere. Determine the values of $n \geq 0$ for which the antipodal map $S^n \to S^n$ is isotopic to the identity.

Note that when n is even, $x \mapsto -x$ and $x \mapsto x$ have different degrees, so we only concern ourselves with the case when n is odd. Now $S^n \subset \mathbb{R}^{n+1} \cong \mathbb{C}^k$, where $k = (n + 1)/2$. Write $H : [0, \pi] \times S^{2k-1} \to S^{2k-1}$ via $H(t, x) = e^{it}x$. Then $H(0, x) = x$ and $H(\pi, x) = -x$, so that H is a homotopy between $x \mapsto x$ and $x \mapsto -x$. Thus, for n even, these are not homotopic, and for n odd, these are homotopic.

Remark: In general, two maps between $S^n \to S^n$ are homotopic if and only if they have the same degree.

Problem 4: Let $\omega_1, \ldots, \omega_k$ be 1-forms on a smooth n-dimensional manifold M. Show that $\{\omega_i\}$ are linearly independent if and only if $\omega_1 \wedge \ldots \wedge \omega_k \neq 0$.

First suppose the ω_i are dependent. WLOG, we may write $\omega_n = \sum_{j=1}^{n-1} c_j\omega_j$. Then

$$\bigwedge_{k=1}^{n} \omega_k = \left(\bigwedge_{k=1}^{n-1} \omega_k\right) \wedge \sum_{j=1}^{n-1} c_j\omega_j = \sum_{j=1}^{n-1} \left(c_j \left(\bigwedge_{k=1}^{n-1} \omega_k\right) \wedge \omega_j\right) = 0$$

since each $\omega_j \wedge \omega_j = 0$ for 1-forms ω_j, so that each term in the final sum above is 0.

Conversely, suppose the $\omega_1, ..., \omega_n$ are independent. Locally, then, they correspond to some dual basis $v_1, ..., v_n$ (take one more dual of T_p^*M to get T_pM), with $\omega_i(v_j) = \delta_{ij}$. Then $(\omega_1 \wedge ... \wedge \omega_n)(v_1, ..., v_n) = 1$ is nonzero, as desired.

Problem 5: Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be the two-dimensional torus, L the line $3x = 7y$ in $\mathbb{R}^2$, and $S = \pi(L) \subset M$, where $\pi : \mathbb{R}^2 \to M$ is the projection map. Find a differential form on M which represents the Poincare dual of S.

Let θ be a 1-form on S^1 with $\int_{S^1} \theta = 1$. Then $dx := \pi_1^*\theta, dy := \pi_2^*\theta$ are 1-forms on $S^1 \times S^1$. In fact they are independent, and we have $[dx], [dy]$ form a basis of $H^1_{dR}(M) = \mathbb{R}^2$. (Alternatively, define dx, dy as the pushforwards of the $G = \mathbb{Z}^2$-invariant forms dx, dy on $\mathbb{R}^2$.) Note $\int_M dx \wedge dy = 1$.

Note $S = \pi(L)$ defines a cycle and hence an element of $H_1(M; \mathbb{R})$. By the Poincare dual of S, with associated inclusion map $i : S \to M$, we seek $\omega \in H^1(M; \mathbb{R})$ with

$$\int_S i^*\eta = \int_M \eta \wedge \omega$$

for all 1-forms $\eta \in H^1(M; \mathbb{R})$. That is, we have two isomorphisms: $H_1(M; \mathbb{R}) \to (H^1(M; \mathbb{R})^*)$ via $L \mapsto [\eta \mapsto \int_L i_L^*\eta]$ and $H^1(M; \mathbb{R}) \cong (H^{2-1}(M; \mathbb{R})^*)$ via $\omega \mapsto [\eta \mapsto \int_M \eta \wedge \omega]$. Then the Poincare dual of S corresponds to the 1-form ω which maps to the same element of $H^1(M; \mathbb{R})^*$.

Next, since $[\eta] \in H^1(M; \mathbb{R})$ and $[dx], [dy]$ give an $\mathbb{R}$-basis of this, we have $[\eta] = a[dx] + b[dy]$ for some $a, b \in \mathbb{R}$. So up to exact form, we may simply take $\eta = a \cdot dx + b \cdot dy$ for constants $a, b \in \mathbb{R}$.

Finally, it suffices to check $\int_S i^*\eta = \int_M \eta \wedge \omega$ for the basis, $\eta = dx$ and $\eta = dy$, by linearity. For $\eta = dx$, we have

$$\int_S i^*dx = \int_S i^*\pi_1^*\theta == \int_S (\pi_1 \circ i)^*\theta = \deg(\pi_1 \circ i : S \to S^1) \int_{S^1} \theta = \deg(\pi_1 \circ i : S \to S^1) = 7$$

where we notice that $\pi_1 \circ i : S \to S^1$ is a 7-fold cover of S^1, since S is a loop from $(0, 0)$ to $(7, 3)$. Similarly,

$$\int_S i^*dy = \deg(\pi_2 \circ i) = 3$$

Meanwhile, we need

$$\int_S i^*dx = \int_M dx \wedge (a \cdot dx + b \cdot dy) = \int_M bdx \wedge dy = b$$

$$\int_S i^*dy = \int_M dy \wedge (a \cdot dx + b \cdot dy) = \int_M -adx \wedge dy = -a$$

Thus, we need $b = 7, a = -3$, so that $\omega = -3dx + 7dy$ gives the cohomology class of the Poincare dual of S.

Problem 6: Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere, with the round metric g_S (the restriction of the usual metric on $\mathbb{R}^{n+1}$). Consider $H = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ equipped with the Euclidean metric g_H. Any line passing through the north pole p and another point $A \in S^n$ will intersect this hyperplane in a point A'. The Map $\Psi : S^n \setminus \{p\} \to H$, defined by $\Psi(A) = A'$ is called the stereographic projection. Show that Ψ is conformal, i.e. for any $x \in S^n \setminus \{p\}$, there bilinear form $(g_S)_x$ is a multiple of $\Psi^*((g_H)_{\Psi(x)})$.

We have $\phi^{-1} : \mathbb{R}^n \to S^n \setminus \{p\}$ via

$$\phi^{-1}(u) = \left(\frac{2u}{|u|^2+1}, \frac{|u|^2-1}{|u|^2+1} \right)$$

and $i : S^n \setminus \{p\} \to \mathbb{R}^{n+1}$ the inclusion.

The metric g_S is $i^*\omega$, where $\omega = dx_1 \otimes dx_1 + ... + dx_{n+1} \otimes dx_{n+1}$ is the standard metric on $\mathbb{R}^{n+1}$.

The metric g_H is $\eta = dx_1 \otimes dx_1 + ... + dx_n \otimes dx_n$.

To show ϕ is conformal, we would like to show $i^*\omega$ and $\phi^*\eta$ differ by a positive function, i.e. that there is some function λ with $i^*\omega = \lambda^2 \phi^* \eta$.

It is enough to show $(\phi^{-1})^* i^* \omega = \mu^2 \eta$ for some function μ, as then $i^*\omega = (\phi^*\mu^2)\phi^*\eta = (\phi^*\mu)^2\phi^*\eta$.

Now

$$(\phi^{-1})^* i^* \omega = (i \circ \phi^{-1})^* \omega = (i \circ \phi^{-1})^* \sum_{i=1}^{n+1} (dx_i \otimes dx_i)$$

$$= \left(\sum_{i=1}^{n} d\left(\frac{2u_i}{|u|^2+1} \right) \otimes d\left(\frac{2u_i}{|u|^2+1} \right) \right) + d\left(\frac{|u|^2-1}{|u|^2+1} \right) \otimes d\left(\frac{|u|^2-1}{|u|^2+1} \right)$$

It suffices to check that this is $\mu^2\eta$ for some smooth function μ. For illustration purposes, the $n = 1$ case gives

$$(\phi^{-1})^* i^* \omega = d\left(\frac{2x}{x^2+1} \right) \otimes d\left(\frac{2x}{x^2+1} \right) + d\left(\frac{x^2-1}{x^2+1} \right) \otimes d\left(\frac{x^2-1}{x^2+1} \right)$$

$$= \left(\frac{d}{dx}\left(\frac{2x}{x^2+1} \right) \right)^2 dx \otimes dx + \left(\frac{d}{dx}\left(\frac{x^2-1}{x^2+1} \right) \right)^2 dx \otimes dx$$

$$= \frac{4}{(x^2+1)^2} dx \otimes dx$$

The calculation for $n > 1$ similarly gives a positive function times η. $\square$

Problem 7: Let X be the wedge sum $S^1 \vee S^1$. Give an example of an irregular covering space $\tilde{X} \to X$.

Regular covering spaces of X (covering spaces whose group of deck transformations act transitively on all fibers) correspond to normal subgroups of $\pi_1(X)$. Taking $\langle a \rangle \subset \langle a, b \rangle = \pi_1(S^1 \vee S^1)$, we see it is not normal, since in particular $b^{-1}ab \notin \langle a \rangle$. The corresponding covering space can be found by quotienting the universal cover by the action of this subgroup. This gives an infinite graph with vertices b^i for $i \in \mathbb{Z}$, and edges $b^i \to b^{i+1}$ labeled b, and $b^i \to b^i$ self loops labeled a.

Problem 8: For $n \geq 2$, let X_n denote the $2n$-gon (including the interior face), with opposite sides glued with parallel orientation.

a) Write down the associated cellular chain complex.

In all cases, we have one face and n-edges. Note for n even, we get 1 vertex, and for n odd, we get 2 vertices. To see this, label the vertices in the polygon with elements of $\mathbb{Z}/2n\mathbb{Z}$. Then notice the identifications allow for vertex i to be identified to vertex $i + (n-1)$. So the vertex 0 gets identified with the subgroup generated by $n-1$. If $n-1, 2n$ are relatively prime, this is the whole group. Otherwise, since $n-1, n$ are relatively prime, we have $gcd(n-1, 2n) = 2$, so $n-1$ generates a subgroup of index 2, leaving behind 2 cosets and hence two vertices in X_n. Since $n-1, n$ are relatively prime, we see $n-1, 2n$ are relatively prime if and only if $n-1$ is odd, so that n is even. Hence we see for n even, we have 1 vertex and for n odd we have two.

So we have for n even the chain complex

$$0 \to \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^n \xrightarrow{\partial_1} \mathbb{Z} \to 0$$

where $\partial_2(F)$ is the abelianization of the boundary word, $a_1 + ... + a_n - a_1 - ... - a_n = 0$, and $\partial_1(a_i) = v - v = 0$, so that both maps are 0 in this chain complex.

For the n odd case, we have

$$0 \to \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^n \xrightarrow{\partial_1} \mathbb{Z}^2 \to 0$$

where again, $\partial_2(F) = 0$, and $\partial_1(a_i) = v - w$ for v, w the two generators of $\mathbb{Z}^2 = C_0$.

b) Show that X_n is a surface, and find its genus.

Computing the homology, we see in the n even case, since all maps are 0, we have $H_0(X_n) = \mathbb{Z}$, $H_1(X_n) = \mathbb{Z}^n$, $H_2(X_n) = \mathbb{Z}$.

In the n odd case, we have still have $H_2(X_n) = \mathbb{Z}$, but this time, we have $H_1(X_n)$ is the kernel of the map $\mathbb{Z}^n \to \mathbb{Z}^2$ which sends each generator a_i to the same element $v - w = (1, -1) \in \mathbb{Z}^2$. Then notice $(x_1, ..., x_n) \mapsto (\sum_{i=1}^n x_i) \cdot (1, -1)$, so that $(x_1, ..., x_n) \mapsto 0$ if and only if $\sum_{i=1}^n x_i = 0$. This is the kernel of the augmentation map, which is isomorphic to $\mathbb{Z}^{n-1}$, as it has basis $a_1 - a_i$ for $i = 2, ..., n$. Thus $H_1(X_n) = \mathbb{Z}^{n-1}$. Finally, $H_0(X_n)$ is the quotient of $\mathbb{Z}^2$ and the image of ∂_1, which is the span of $(1, -1)$. Note $(x, y) = (x + y, 0) + (-y, y)$, so that we have an isomorphism $H_0(X_n) \to \mathbb{Z}$ via $[(x, y)] \mapsto x + y$. Hence $H_0(X_n) = \mathbb{Z}$ as well. So we see

$$H_k(X_n) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}^n & k = 1, n \text{ even} \\ \mathbb{Z}^{n-1} & k = 1, n \text{ odd} \\ \mathbb{Z} & k = 2 \\ 0 & k > 2 \end{cases}$$

Note that X_n is a surface: at each point in the interior of the face, it is clear X_n has a neighborhood homeomorphic to $\mathbb{R}^n$. For points on the edges, we get two half planes and hence a full $\mathbb{R}^n$ upon gluing. Since $H_1(M_g) = \mathbb{Z}^{2g}$, we see $g = n/2$ if n is even, or $g = (n-1)/2$ if n is odd.

Problem 9:

a) Consider the space Y obtained from $S^2 \times [0, 1]$ by identifying $(x, 0) \sim (-x, 0)$ and $(x, 1) \sim (-x, 1)$ for $x \in S^2$. Show that Y is homeomorphic to the connected sum $\mathbb{RP}^3 \# \mathbb{RP}^3$.

To get $\mathbb{RP}^3 \# \mathbb{RP}^3$, delete a 3-ball from each $\mathbb{RP}^3$, and connect with a tube $S^2 \times [0, 1]$, gluing $S^2 \times \{0\}$ to the boundary of the 3-ball in one copy of $\mathbb{RP}^3$, and $S^2 \times \{1\}$ to the boundary of the other deleted 3-ball.

From the cell structure, note that deleting a 3-cell from $\mathbb{RP}^3$ leaves behind $\mathbb{RP}^2$. Hence, we may just glue a tube of cylinders by gluing $S^2 \times \{0\}$ to one copy of $\mathbb{RP}^2$ and $S^2 \times \{1\}$ to the other copy. This is the same as the construction of Y, which is a tube of cylinders with the ends replaced with copies of $\mathbb{RP}^2$.

b) Show $S^2 \times S^1$ is a double cover of $\mathbb{RP}^3 \oplus \mathbb{RP}^3$.

We can think of $S^2 \times S^1$ as $S^2 \times [0, \pi]$ union with $S^2 \times [-\pi, 0]$, quotiented by $(x, -\pi) \sim (x, \pi)$. Then note $S^2 \times [0, \pi]$ has a map to Y (the quotient of $S^2 \times [0, 1]$) by $(x, t) \mapsto (x, t/\pi)$, with $(x, 0) \mapsto [(x, 0)] \in \mathbb{RP}^2$ and $(x, \pi) \mapsto [(x, 1)]$ both giving double covers. Similarly we get a map from the $S^2 \times [-\pi, 0]$ to Y via $(x, t) \mapsto [(x, -t/\pi)]$. Then notice each point in Y is double covered, since $[(x, 0)]$, $[(x, 1)]$ is double covered by $S^2 \times \{0\}$, $S^2 \times \{\pi\} = S^2 \times \{-\pi\}$ respectively, and each (x, t) is covered by $S^2 \times \{\pi t\}$ and $S^2 \times \{-\pi t\}$.

I apologize to the mathematical community for this proof.

Problem 10: Let X be a topological space. Describe the relation between the homology groups of X and $S(X)$, where $S(X)$ is the suspension of X, obtained by taking $X \times [0,1]$ and identifying $X \times \{0\}$ to a point and $X \times \{1\}$ to a point.

We can solve this with the generalized Mayer Vietoris (Fall 2011 Problem 10) via the maps $f, g : X \to Y = \{0,1\}$ (with trivial topology) via $f(x) = 0$ for all $x \in X$, $g(x) = 1$ for all $x \in X$. Then Z as constructed in Fall 2011 Problem 10 gives the desired space $Z = S(X)$, and we have a long exact sequence

$$\ldots \to H_n(X) \to H_n(Y) \to H_n(Z) \to \ldots$$

But since $H_n(Y) = 0$ for $n > 0$, we have for $n > 1$,

$$H_n(Y) = 0 \to H_n(Z) \to H_{n-1}(X) \to H_{n-1}(Y) = 0$$

so that $H_n(Z) \cong H_{n-1}(X)$ for $n > 1$.

Now note that Z is connected. To see this, note that each $(x, i) \in X \times I$ has a path to $(x, 0) \in X \times I$. Thus, in Z, this gives us a path from the image of (x, i) to the unique point p which is the image of $X \times \{0\}$. Since each (x, i) has a path to p, we conclude Z is connected. Thus for $n = 0$, note $H_0(Z) = \mathbb{Z}$, and we have

$$0 \to H_1(Z) \to H_0(X) \to \mathbb{Z}^2 \to \mathbb{Z} \to 0$$

shows $H_1(Z)$ is isomorphic to a subgroup of a free group $H_0(X)$, and hence is itself free. Counting rank, we see the alternating sum of the ranks is zero, so that $\operatorname{rank} H_1(Z) = \operatorname{rank} H_0(X) - 1$, so that

$$H_n(Z) = \begin{cases} H_{n-1}(X) & n > 1 \\ \mathbb{Z}^{(\#\text{ c.c. of } X) - 1} & n = 1 \\ \mathbb{Z} & n = 0 \end{cases}$$

10 Fall 2014

Problem 1: Let $f : M \to N$ be a proper immersion between connected manifolds of the same dimension. Show that f is a covering map.

Let $y \in N$. Pick $y \in K \subset N$ a compact neighborhood. Then $M_1 = f^{-1}K$ is compact since f is proper. It is a manifold since df_p is surjective for each $p \in f^{-1}K$, so that $f \pitchfork K$. Then $g = f|_{M_1} : M_1 \to N$ is an immersion between manifolds of the same dimension (hence a local diffeomorphism) with M_1 compact and N connected. By Spring 2010 Problem 3, g is a covering map. In particular, p has an evenly covered neighborhood via f, since $f = g$ on a neighborhood of p. Since $p \in N$ was arbitrary, the result follows.

Alternative solution: (Much more high-powered) Proper maps are closed, and local diffeomorphisms (more generally, local homeomorphisms) are open, so that $f(M)$ is clopen and f is surjective. Surjective proper submersions are fiber bundles by Ehresmann's Theorem. Then notice for any $y \in N$, y is a regular value of f so that $f^{-1}y$ is a (compact, by properness) 0-manifold, and hence just a discrete set of points. So M is a fiber bundle with discrete fibers, and hence a covering space.

Problem 2: Let $M^m \subset \mathbb{R}^n$ be a closed, connected submanifold of dimension m.

a) Show that $\mathbb{R}^n \setminus M^m$ is connected for $m \leq n-2$.

Repeat of Fall 2012 Problem 3.

b) When $m = n-1$, show that $\mathbb{R}^n \setminus M^m$ is disconnected by showing that the mod 2 intersection number $I_2(f, M) = 0$ for all smooth maps $f : S^1 \to \mathbb{R}^n$.

Let $p \in M$. Find a neighborhood $p \in U \subset \mathbb{R}^n$ with $U \xrightarrow{\phi} \mathbb{R}^n$ a diffeomorphism such that $U \cap M$ maps to $\phi(U \cap M) \subset \{(x_1, ..., x_n) : x_n = 0\}$, with $\phi(p) = 0$. This is possible since M has codimension 1. Select the straight line path γ in $\mathbb{R}^n$ from $(0, ..., 0, 1)$ to $(0, ..., 0, -1)$. Clearly, γ is orthogonal to the $x_n = 0$ plane. Moreover, it crosses precisely once. Thus, $\gamma \pitchfork U \cap M$ and $I_2(\gamma, U \cap M) = 1$, where we view γ as a path in the original space $U \subset \mathbb{R}^n$. (Transversality is preserved by diffeomorphisms). Finally, since intersection number can be calculated locally, $I_2(\gamma, U \cap M \subset U) = I_2(\gamma, M \subset \mathbb{R}^n)$. So $I_2(\gamma, M) = 1$.

If $\lambda : [0,1] \to \mathbb{R}^n \setminus M$ is a path between the same two points, then $I_2(\lambda, M) = 0$ simply since λ does not intersect M at all, and hence is transverse for free. However, $\lambda \cong \gamma$ as maps from $[0,1]$ to $\mathbb{R}^n$, so that $0 = I_2(\lambda, M) = I_2(\gamma, M) = 1$. By contradiction, no such λ can exist.

Problem 3: Let ω be an n-form on a closed, connected non-orientable n-manifold M and $\pi : \mathcal{O} \to M$ the orientation cover.

a) Show that $\pi^*\omega$ is exact.

Let $\alpha : \mathcal{O} \to \mathcal{O}$ be the nontrivial deck transformation, so that $\pi\alpha = \pi$. This exists because $\mathcal{O}$ must be a normal cover, as it corresponds to an index 2 subgroup.

Since α is a homeomorphism, it has degree 1. But if α had degree 1, then M would be orientable. Thus α must be reversing orientation. We have

$$\int_{\mathcal{O}} \pi^*\omega = \int_{\mathcal{O}} (\pi \circ \alpha)^*\omega = \int_{\mathcal{O}} \alpha^*\pi^*\omega = \deg(\alpha) \int_{\mathcal{O}} \pi^*\omega = -\int_{\mathcal{O}} \pi^*\omega$$

So $\int_{\mathcal{O}} \pi^*\omega = 0$. Since $\mathcal{O}$ is a closed connected orientable manifold, we have $H^n_{dR}(\mathcal{O}) \cong \mathbb{R}$ via the map $\int_{\mathcal{O}}$. Hence, we have $[\pi^*\omega] = 0$ in $H^n(\mathcal{O})$, so $\pi^*\omega$ is exact.

b) Show that ω is exact.

Note $\mathcal{O} \xrightarrow{\pi} M$ is a finite sheeted covering space, so that we get an injection on de Rham cohomology by Fall 2012 Problem 9. Since $H^n(M) \xrightarrow{\pi^*} H^n(\mathcal{O})$ has $[\omega] \mapsto [\pi^*\omega] = 0$, and this map is injective, we conclude $[\omega] = 0$, so that ω is exact, as desired.

Problem 4: Show that for $n \geq 1$, any smooth map $f : S^{n-1} \to S^{n-1}$ has a smooth extension $F : D^n \to D^n$.

See Spring 2014 Problem 2. We can extend $i \circ f : S^{n-1} \to \mathbb{R}^n$ to a map $g : D^n \to \mathbb{R}^n$, but in fact, our construction ensures that $|g(x)| \in [0, 1]$, so that g is really a map $g : D^n \to D^n$, as desired.

Problem 5: Let M be a smooth manifold and ω a nowhere vanishing 1-form on M. Show that ω is locally proportional to the differential of a function (i.e. locally $\omega = \lambda df$) if and only if $\omega \wedge d\omega = 0$.

See Spring 2012 Problem 4.

Problem 6: Show that the space of all 2×3 matrices of rank 1 forms a smooth manifold.

See Spring 2013 Problem 1.

Problem 7: A compact surface of genus g, smoothly embedded in $\mathbb{R}^3$, bounds a compact region called a handlebody H.

a) Prove that two copies of H glued together along their boundaries by the identity map produces a closed topological 3-manifold $2H$.

b) Compute the homology of $2H$.

c) Compute the relative homology of $(2H, H)$ where H is one of the two copies.

Solution: Note $2H$ is a closed manifold by the second proposition here.

We denote the double of the handlebody $2H$, the triple $3H$, and so on. We give a solution that will give the homology for any $k \cdot H$, $k > 1$.

From $\partial H = M_g$, we have

$$\widetilde{H}_i(\partial H) = \begin{cases} 0 & i = 0 \text{ or } i > 2 \\ \mathbb{Z}^{2g} & i = 1 \\ \mathbb{Z} & i = 2 \end{cases}$$

Note H homotopy equivalent to a wedge of circles, as is clear geometrically (the picture flattens out). More formally, $S^1 \times D^2$ is homotopy equivalent to S^1 since D^2 is homotopy equivalent to a point. Meanwhile, H is the connect sum of g copies of $S^1 \times D^2$. If we insist the connecting tubes all connect to previous connecting tubes, crushing the connecting tubes (which, when solid, are homotopic to a point anyway) gives a wedge of g copies of $S^1 \times D^2$, which all flatten (are homotopy equivalent to) a wedge of g copies of S^1.

Since the reduced homology of a wedge is the sum of the reduced homologies, we immediately get

$$\widetilde{H}_i(H) = \begin{cases} 0 & i = 0 \text{ or } i > 1 \\ \mathbb{Z}^g & i = 1 \end{cases}$$

We compute the relative homology groups $H_i(H, \partial H)$ below. Alternatively, use Lefshetz Duality (since H is compact orientable 3-manifold) to get

$$H_i(H, \partial H) = H^{3-i}(H) = \begin{cases} 0 & i = 0, 1 \\ \mathbb{Z}^g & i = 2 \\ \mathbb{Z} & i = 3 \\ 0 & i > 3 \end{cases}$$

Notice since $(H, \partial H)$ is a good pair by collar neighborhood, so too is (kH, H). Moreover, $kH/H \cong \bigvee_{i=1}^{k-1}(H/\partial H)$ is the $(k-1)$-fold wedge of copies of $H/\partial H$. Since reduced homology of a wedge is sum of reduced homologies, we get

$$H_i(kH, H) = \widetilde{H}_i(kH/\partial H) = \oplus_{i=1}^{k-1} \widetilde{H}_i(H/\partial H) = \oplus_{i=1}^{k-1} \widetilde{H}_i(H, \partial H) = \begin{cases} 0 & i = 0, 1 \\ \mathbb{Z}^{(k-1)g} & i = 2 \\ \mathbb{Z}^{k-1} & i = 3 \\ 0 & i > 3 \end{cases}$$

Next, we get by Hatcher 2.13 a long exact sequence of reduced homology groups

$$\ldots \to \widetilde{H}_i(H) \to \widetilde{H}_i(kH) \to H_i(kH, H) \to \widetilde{H}_{i-1}(H) \to \ldots$$

From $\widetilde{H}_i(H) = 0$ for $i \neq 1$, we instantly get $\widetilde{H}_i(kH) = H_i(kH, H)$ for $i \neq 1, 2$. This leaves

$$0 \to \widetilde{H}_2(kH) \to H_2(kH, H) \to \widetilde{H}_1(H) \xrightarrow{\sim} \widetilde{H}_1(kH) \to 0$$

where the indicated map is an isomorphism by the π_1 calculation below. Thus we get the previous map is zero, so that $\widetilde{H}_2(kH) = H_2(kH, H)$, and $\widetilde{H}_1(kH) \cong \widetilde{H}_1(H)$. In short,

$$\widetilde{H}_i(kH) = \begin{cases} H_i(kH, H) & i \neq 1 \\ \widetilde{H}_1(H) & i = 1 \end{cases} = \begin{cases} 0 & i = 0 \\ \mathbb{Z}^g & i = 1 \\ \mathbb{Z}^{(k-1)g} & i = 2 \\ \mathbb{Z}^{k-1} & i = 3 \\ 0 & i > 3 \end{cases}$$

Handlebody - Fundamental Group

We may compute $\pi_1(kH)$ by Van Kampen, as well as induced maps $\pi_1(\partial H) \to \pi_1(H)$ and $\pi_1(H) \to \pi_1(kH)$. We may select, by collar neighborhood, an open subset $\partial H \subset U \subset H$ that deformation retracts to ∂H. (So $(H, \partial H)$ is a good pair). In kH, which we may think of as a quotient of $H \times \{1, ..., k\}$, so that if W is the image of $\cup_{i=1}^{k}(U, i)$, we see W deformation retracts to $\partial H \subset kH$. Moreover, setting $A_i = (H, i) \cup W$, we see A_i deformation retracts to a copy of H. The union of the $A_1, ..., A_k$ is all of kH, and the intersection of any two or more of them is W, which deformation retracts to ∂H as previously stated.

From Van Kampen, we then have $\pi_1(kH)$ surjects onto the free product $\pi_1(A_1) * ... * \pi_1(A_k)$, with kernel generated precisely by the relations $\alpha = \beta$ for any α in the image of $\pi_1(A_i \cap A_j) \to \pi_1(A_i)$ and any β in the image of $\pi_1(A_i \cap A_j) \to \pi_1(A_j)$ (and for any i, j).

The map from $A_i \cap A_j \to A_i$, upon deformation retracting, is really just the map $\partial H \to H$. Note from the polygon construction that $\pi_1(\partial H = M_g) = \langle a_1, ..., a_g, b_1, ..., b_g | a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} ... a_g b_g a_g^{-1} b_g^{-1} \rangle$. Meanwhile, $\pi_1(H \cong S^1 \vee ... \vee S^1) = \langle c_1, ..., c_g \rangle$, since by previous remarks H is homotopy equivalent to a wedge of g circles. Moreover, the map $\pi_1(\partial H) \to \pi_1(H)$ sends a_i to c_i and b_i to 0.

To illustrate this, notice for $g = 1$ we have $\partial H = S^1 \times S^1 \to H = S^1 \times D^2$. In this case, $\pi_1(\partial H) = \langle a_1, b_1 | a_1 b_1 a_1^{-1} b_1^{-1} \rangle$, and to get H, we add a 2-cell via the relation b_1 (to make the second copy of S^1 into a D^2), as well as a 3-cell, which does not affect π_1. Then we get by Hatcher 1.26 that $\pi_1(H) = \langle a_1, b_1 | a_1 b_1 a_1^{-1} b_1^{-1}, b_1 \rangle \cong \langle c_1 \rangle$ via $a_1 \mapsto c_1$ and $b_1 \mapsto 0$. In general, to get from ∂H to H, we add 2-cells via b_i for $i = 1, ..., g$, giving $\pi_1(H) = \langle a_1, ..., a_g, b_1, ..., b_g | a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} ... a_g b_g a_g^{-1} b_g^{-1}, b_1, ..., b_g \rangle \cong \langle c_1, ..., c_g \rangle$ via $a_i \mapsto c_i$ and $b_i \mapsto 0$. From this it is also clear that the kernel of this map is $\langle b_1, ..., b_g \rangle \subset \pi_1(\partial H)$.

In particular, we get $\pi_1(\partial H) \twoheadrightarrow \pi_1(H)$ is a surjection. Completing the Van Kampen argument, writing $\pi_1(A_i) = \pi_1(H) = \langle c_1^i, ..., c_g^i \rangle$, note $\pi_1(A_i \cap A_j) \to \pi_1(A_i)$ and $\pi_1(A_i \cap A_j) \to \pi_1(A_j)$ map $a_k \in \pi_1(A_i \cap A_j) = \pi_1(\partial H)$ to $c_k^i \in \pi_1(A_i), c_k^j \in \pi_1(A_j)$, and maps b_k to zero in both. Thus we have by Van Kampen $\pi_1(kH) = \langle c_1^1, ..., c_g^1, ..., c_1^k, ..., c_g^k | c_i^j = c_i^k \text{ for all } i, j, k \rangle \cong \langle c_1, ..., c_g \rangle$. In particular, we see the map $\pi_1(H) \xrightarrow{\sim} \pi_1(kH)$ is an isomorphism. Abelianizing, we see we have a surjection $\boxed{\phi : H_1(\partial H) \twoheadrightarrow H_1(H)}$ with $\ker(\phi) = \langle b_1, ..., b_g \rangle \subset H_1(\partial H)$, so that $\boxed{\ker(\phi) \cong \mathbb{Z}^g}$. Moreover, we still have isomorphisms $\boxed{H_1(H) \xrightarrow{\sim} H_1(kH)}$ for any $k \geq 1$.

Handlebody - Relative Homology

Using Hatcher 2.13 for the LES of reduced relative homology, we have an LES

$$... \to \widetilde{H}_i(\partial H) \to \widetilde{H}_i(H) \to H_i(H, \partial H) \to \widetilde{H}_{i-1}(\partial H) \to ...$$

Since $\widetilde{H}_i(H) = \widetilde{H}_{i-1}(\partial H) = 0$ for $i = 0, i > 3$, we have $H_i(H, \partial H) = 0$ in these cases. We have exact sequences

$$0 \to H_3(H, \partial H) \to \widetilde{H}_2(\partial H) = \mathbb{Z} \to 0$$

$$0 \to H_2(H, \partial H) \to \widetilde{H}_1(\partial H) = \mathbb{Z}^{2g} \twoheadrightarrow \widetilde{H}_1(H) = \mathbb{Z}^g \to H_1(H, \partial H) \to 0$$

where we have a surjection $\phi : \widetilde{H}_1(\partial H) \to \widetilde{H}_1(H)$ by the argument above. Thus, the next map is zero, and we get $H_1(H, \partial H) = 0$ and $H_2(H, \partial H) = \ker(\phi) \cong \mathbb{Z}^g$. Thus

$$H_i(H, \partial H) = \begin{cases} 0 & i = 0, 1 \\ \mathbb{Z}^g & i = 2 \\ \mathbb{Z} & i = 3 \\ 0 & i > 3 \end{cases}$$

Problem 8: Consider the space $X = M_1 \cup M_2$ where M_1, M_2 are Mobius bands and $M_1 \cap M_2 = \partial M_1 = \partial M_2 \cong S^1$. (Here the Mobius band is the quotient space $[-1,1]^2/((1,y) \sim (-1,-y))$).

a) Determine the fundamental group of X.

A good relevant problem to look at for this is Fall 2011 Problem 8.

Note $X = 2M$ is just the standard construction turning a manifold with boundary into one without. Taking collar neighborhoods of the boundary and applying Van Kampen, we get $\pi_1(X)$ is the pushout of the diagram

$$\begin{array}{c} \pi_1(\partial M) \to \pi_1(M) \\ \downarrow \\ \pi_1(M) \end{array}$$

Note that $\partial M \cong S^1$ and M deformation retracts to its central circle. Writing $M = [0,1]^2/\sim$, with $(x,0) \sim (1-x,1)$, the retract is $r : M \to \mu$ is $r(x,y) = (1/2, y)$, where μ is the image of $1/2 \times [0,1]$, i.e. $1/2 \times [0,1]/\sim$ with $(1/2,0) \sim (1/2,1)$. The boundary ∂M, which is the path from $(0,0)$ to $(0,1) = (1,0)$ followed by the path from $(1,0)$ to $(1,1) = (0,0)$, maps to 2μ under the retract. Thus, our diagram becomes (writing $\pi(\partial M) = \mathbb{Z}$ and $\pi_1(M)$ multiplicatively)

$$\begin{array}{c} \pi_1(\partial M) \xrightarrow{1 \mapsto a^2} \pi_1(M) = \langle a \rangle \\ \downarrow (1 \mapsto b^2) \\ \pi_1(M) = \langle b \rangle \end{array}$$

so that $\pi_1(X) = \langle a, b | a^2 b^{-2} \rangle = \langle a, b | a^2 = b^2 \rangle$.

b) Is X homotopy equivalent to a compact orientable surface of genus g for some g?

Note $H_1(X) = \langle a, b | a^2 = b^2, ab = ba \rangle \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ via $a \mapsto (1,0)$, $b \mapsto (1,1)$, with inverse map $(1,0) \mapsto a$ and $(0,1) \mapsto ab^{-1}$. But $H_1(M_g) \cong \mathbb{Z}^{2g}$. By contradiction, X is not M_g for any g.

In fact, X is the Klein bottle by classification of surfaces. See Fall 2011 Problem 8 to understand why.

Problem 9: Determine all connected covering spaces of the wedge sum $\mathbb{RP}^{14} \vee \mathbb{RP}^{15}$.

Connected covering spaces (ignoring base point) will correspond to conjugacy classes of subgroups of $\pi_1(\mathbb{RP}^{14} \vee \mathbb{RP}^{15}) = \pi_1(\mathbb{RP}^{14}) * \pi_1(\mathbb{RP}^{15}) \cong \pi_1(\mathbb{RP}^2) * \pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, since π_1 only depends on the 2-skeleton. Next,

RETURN TO THIS

Problem 10: Let D be the unit disk in the complex plane and let S^1 be the unit circle. Consider $T^2 = S^1 \times S^1$ and two copies of D¡ D_1 and D_2. Let X be the quotient $T^2 \sqcup D_1 \sqcup D_2$ by $e^{i\theta} \sim (e^{ip\theta}, 1) \in T^2$ and $e^{i\phi} \sim (1, e^{iq\phi}) \in T^2$ for $e^{i\theta} \in D_1$, $e^{i\phi} \in D_2$, and $p, q > 1 \in \mathbb{Z}$. Compute the homology groups of X.

Give the torus its standard cell structure with one 0-cell v, two 1-cells a, b, and a 2-cell F_1 attached via word $aba^{-1}b^{-1}$. We attach two more 2-cells, F_2, F_3 via words a^p and b^q respectively. Now we have one 0-cell, two 1-cells, and three 2-cells, with cell complex

$$0 \to C_2(X) = \mathbb{Z}^3 \xrightarrow{\partial_2} C_1(X) = \mathbb{Z}^2 \xrightarrow{\partial_1} C_0(X) = \mathbb{Z} \to 0$$

We have $\partial_1(a) = v - v = 0$, $\partial_1(b) = 0$, $\partial_2(F_1) = a + b - a - b = 0$, $\partial_2(F_2) = p \cdot a$, $\partial_2(F_3) = q \cdot b$. Thus we have $\operatorname{im}(\partial_2) = \operatorname{span}((p, 0), (0, q)) \subset \mathbb{Z}^2$, and $\ker(\partial_2) = \operatorname{span}((1, 0, 0)) \subset \mathbb{Z}^3$. From this we see $H_2(X) = \ker(\partial_2) \cong \mathbb{Z}$, $H_1(X) = \mathbb{Z}^2/\operatorname{im}(\partial_2) = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$, and $H_0(X) = \mathbb{Z}$.

11 Spring 2015

Problem 1: Let $M(n, m, k) \subset M(n, m)$ be the space of $n \times m$ rank k matrices. Show that $M(n, m, k)$ is a smooth manifold of dimension $nm - (n - k)(m - k)$.

See Spring 2013 Problem 1.

Problem 2: Assume that $N \subset M$ is a codimension 1 properly embedded submanifold. Show that N can be written as $f^{-1}(0)$ where 0 is a regular value of a smooth function $f : M \to \mathbb{R}$ if and only if there is a vector field X on M that is transverse to N.

Definition: We say $X \pitchfork N$ if $\operatorname{span}(X_p) + T_pN = T_pM$ for all $p \in N$.

Solution: Suppose $N = f^{-1}(0)$ where 0 is a regular value of $f : M \to \mathbb{R}$. Write $X = \nabla f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}$. By Lee page 343, gradients are normal to level sets, so that $X_p \notin T_pN$ for any $p \in N$, and we have $\operatorname{span}(X_p) + T_pN = T_pM$ by dimension considerations. We conclude $X \pitchfork N$.

The backwards direction is false in general according to Lee.

Problem 3: Consider two collections of 1-forms $\omega_1, \ldots, \omega_k$ and $\phi_1, \ldots, \phi_k$ on an n-dimensional manifold M. Assume that $\omega_1 \wedge \ldots \wedge \omega_k = \phi_1 \wedge \ldots \wedge \phi_k$ never vanishes on M. Show there are smooth functions $f_{ij} : M \to \mathbb{R}$ such that $\omega_i = \sum_{j=1}^k f_{ij}\omega_j$.

Solution: A wedge of 1-forms is 0 if and only if they are linearly dependent, by Spring 2013 Problem 4. Notice $\phi_1 \wedge ... \wedge \phi_k$ is nonzero so that these are all independent, while $\phi_1 \wedge ... \wedge \phi_k \wedge \omega_i = \omega_1 \wedge ... \wedge \omega_k \wedge \omega_i = 0$, since $\omega_i \wedge \omega_i = 0$ for 1-forms ω_i. From this we see ω_i must be a linear combination of the $\phi_1, ..., \phi_k$, as desired.

Alternative Solution: It suffices to consider everything locally, so take a dual basis $X_1, , ..., X_k$ of $\omega_1, .., \omega_k$, and $Y_1, ..., Y_k$ of $\phi_1, ..., \phi_k$. Then notice

$$(\omega_1 \wedge ... \wedge \omega_k)(Z, X_2, ..., X_k) = \omega_1(Z)\omega_2(X_2)...\omega_k(X_k) = \omega_1(Z)$$

for any vector field Z, since all other permutations of the terms will give 0. Meanwhile,

$$(\phi_1 \wedge ... \wedge \phi_k)(Z, X_2, ..., X_k) = \sum_{j=1}^n f_{1j}\phi_j(Z)$$

where f_{1j} is some complicated term involving permutations of the $\phi_l(X_m)$, but these are all smooth functions. From this we see $\omega_1 = \sum_{j=1}^n f_{1j}\phi_j$. A similar argument gives $\omega_i = \sum_{j=1}^n f_{ij}\phi_j$ for each i.

Problem 4: Consider a smooth map $F : \mathbb{RP}^n \to \mathbb{RP}^n$.

a) When n is even, show that F has a fixed point.

See Spring 2011 Problem 9 for a similar problem.

Over $\mathbb{Q}$, $H^k(\mathbb{RP}^n; \mathbb{Q}) = \begin{cases} \mathbb{Q} & k = 0 \\ 0 & k > 0 \end{cases}$, since $H^k(\mathbb{RP}^n; \mathbb{Q}) \cong \operatorname{Hom}_{\mathbb{Z}}(H_k(\mathbb{RP}^n), \mathbb{Q})$ (with the ext term vanishing). Note $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}) \cong \mathbb{Q}$ and $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Q}) = 0$, so the result follows from the homology of $\mathbb{RP}^n$. Since $H_k(\mathbb{RP}^n) = 0$ or $\mathbb{Z}/2\mathbb{Z}$ for $k > 0$, we see $H^k(\mathbb{RP}^n; \mathbb{Q}) = 0$ for $k > 0$. For $k = 0$, we get $H^0(\mathbb{RP}^n; \mathbb{Q}) = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}) = \mathbb{Q}$, as desired.

Now $L(f) = \sum_{i=0}^n (-1)^i tr(f^* : H^i(\mathbb{RP}^n; \mathbb{Q}) \to H^i(\mathbb{RP}^n; \mathbb{Q})) = 1 + 0 = 1 \neq 0$, since only the $i = 0$ term survives, and $f^* : H^0(\mathbb{RP}^n; \mathbb{Q}) \to H^0(\mathbb{RP}^n; \mathbb{Q})$ is just the identity map (we have $f^*(1) = 1$ since f^* is a cohomology ring homomorphism). Since $L(f) \neq 0$, f has a fixed point, as desired.

b) When n is odd, give an example where F does not have a fixed point.

Since n is odd, write $n = 2k - 1$. Then $S^n \subset \mathbb{R}^{2k} = \mathbb{C}^k$, and we have $f : S^n \to S^n$ via $p \mapsto ip$. Then $S^n \xrightarrow{f} S^n \xrightarrow{\pi} \mathbb{RP}^n$ has $\pi(f(-x)) = \pi(-ix) = \pi(ix) = \pi(f(x))$, so that this factors through to a map $g : \mathbb{RP}^n \to \mathbb{RP}^n$, with $g([x]) = [f(x)]$. Suppose $g([x]) = [x]$. Then $[f(x)] = [x]$, so that $f(x) = x$ or $f(x) = -x$. Then $ix = x$ or $ix = -x$. Since x is nonzero, we get $i = 1$ or $i = -1$, in both cases a contradiction. Hence g has no fixed points.

Problem 5: Assume we have a codimension 1 distribution $\Delta \subset TM$.

a) Show that if the quotient bundle TM/Δ is trivial (equivalently, there is a vector field on M that never lies in Δ), then there is a 1-form ω on M such that $\Delta = \ker(\omega)$ everywhere on M.

> Since TM/Δ is trivial, set $\phi : TM/\Delta \xrightarrow{\sim} M \times \mathbb{R}$. We have a fiber-wise surjection $q : TM \twoheadrightarrow TM/\Delta$, and a projection $\pi : M \times \mathbb{R} \to \mathbb{R}$.
>
> Define $\omega_p(X) = \pi(\phi_p(q_p(X)))$. That is, $\omega_p = \pi \circ \phi_p \circ q_p$. Then note $\omega_p(X) = 0 \iff \phi_p q_p(X) \in M \times \{0\}$. Since $\phi_p : T_pM/\Delta_p \to p \times \mathbb{R}$ maps into $p \times \mathbb{R}$, we see $\omega_p(X) = 0 \iff \phi_p q_p(X) = (p, 0) \iff q_p(X) = 0 \in T_pM/\Delta_p \iff X_p \in \Delta_p$. Thus, $\Delta = \ker(\omega)$.

b) Give an example where TM/Δ is not trivial.

> Skip!

c) With ω_1 and ω_2 as in (a), show that $\omega_1 \wedge d\omega_1 = f^2 \omega_2 \wedge d\omega_2$ for a smooth function f that never vanishes.

> It suffices to show $\omega_1 = f\omega_2$ for a nonvanishing f. Suppose $\ker(\omega_1)_p = \ker(\omega_2)_p = \Delta_p$ for each $p \in M$. Select a vector field X with $X_p \notin \Delta_p$ for any p in some neighborhood U. Define $f(p) = \frac{(\omega_1)_p(X)}{(\omega_2)_p(X)}$. By construction, f is well defined and nonzero. Pick a local basis $X_2, ..., X_n$ of Δ, so that $X, X_2, ..., X_n$ are a local basis for the tangent space. It is easy to see $\omega_1 = f\omega_2$ by checking this on each basis vector. Finally, we may patch together the local choices of f via partition of unity to get the desired result.

d) If $\omega \wedge d\omega \neq 0$, show that Δ is not integrable.

> Note that the argument in Fall 2013 Problem 5 for $\ker(\omega)$ integrable implies $\omega \wedge d\omega$ generalizes. Instead of just picking two vectors Y, Z for a basis of $\ker(\omega)_p$, we may pick $X_2, ..., X_n$ a basis. The same computation shows that the 3-form $\omega \wedge d\omega$ is always zero on any 3 basis vectors, so that $\omega \wedge d\omega = 0$ as desired.

Problem 6: Let $\omega = \frac{xdy\wedge dz + ydz\wedge dx + zdx\wedge dy}{(x^2+y^2+z^2)^{3/2}}$ be a 2-form defined on $\mathbb{R}^3 \setminus \{0\}$. Compute $\int_{S^2} i^*\omega$ and $\int_{S^2} j^*\omega$ where $j : S^2 \to \mathbb{R}^3$ is $(x, y, z) \mapsto (3x, 2y, 8z)$.

First, let $\eta = xdy\wedge dz + ydz\wedge dx + zdx\wedge dy$. Since for each $p = (x, y, z) \in S^2$, we have $(x^2+y^2+z^2)^{3/2} = 1$, we see $\eta_p = \omega_p$ for each $p \in S^2$. Thus, $i^*\omega = i^*\eta$. Moreover, notice η is a form on all of $\mathbb{R}^3$, not just $\mathbb{R}^3 \setminus \{0\}$. From this, we may apply Stokes Theorem to get

$$\int_{S^2} i^*\omega = \int_{S^2} i^*\eta = \int_B d\eta = \int_B 3dV = 4\pi$$

where $B \subset \mathbb{R}^3$ is the closed unit ball (so that $\partial B = S^2$). Note we could not have applied Stokes Theorem directly to ω, since ω is not defined on all of B (in particular, it is undefined at 0 and may not be extended in a continuous way).

Remark: Notice $\eta = i_N dV$, where $dV = dx \wedge dy \wedge dz$, and N is the unit vector field normal to the sphere, so that $N = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$. Hence $i^*\eta$ is just the surface area form on S^2, so that $\int_{S^2} i^*\eta$ is the surface area of S^2, which is 4π.

Next, notice $j(S^2) = E$ is an ellipsoid, diffeomorphic to S^2 with obvious inverse map $j^{-1}(x, y, z) = (x/3, y/2, z/8)$. So j may be factored as the composition $S^2 \xrightarrow{\phi} E \overset{k}{\hookrightarrow} \mathbb{R}^3 \setminus \{0\}$, where $k : E \to \mathbb{R}^3 \setminus \{0\}$ is the inclusion map, and $S^2 \xrightarrow{\phi} E$ is the diffeomorphism given by j (its codomain has been restricted). Note that S^2 is entirely in the inside of E. Let D denote the region outside S^2 but inside E. In particular, $0 \notin D$, so ω is defined on all of D.

Notice $\partial D = S^2 \sqcup E$, but S^2 is given an inward pointing normal and E is given an outward pointing normal (as inside S^2 is outside D). The inclusion map $\partial D \overset{f}{\hookrightarrow} \mathbb{R}^3 \setminus \{0\}$ is just $i \sqcup k$. By Stokes, we have

$$\int_E k^*\omega - \int_{S^2} i^*\omega = \int_E f^*\omega - \int_{S^2} f^*\omega = \int_{\partial D} f^*\omega = \int_D d\omega = 0$$

since direct computation shows $d\omega = (r^{-3} - 3x^2r^{-5} + r^{-3} - 3y^2r^{-5} + r^{-3} - 3z^2r^{-5})dV = 0$. Thus,

$$\int_E k^*\omega = \int_{S^2} i^*\omega = 4\pi$$

Finally, since $j = k \circ \phi$ and $\phi : S^2 \to E$ is a diffeomorphism, we have

$$\int_{S^2} j^*\omega = \int_{S^2} \phi^*(k^*\omega) = \int_E k^*\omega = 4\pi$$

since pulling back via diffeomorphism preserves the value of the integral. This gives the desired result.

Problem 7: Define the de Rham cohomology groups of a manifold M and compute $H^i_{dR}(S^1)$ directly from the definition.

Note that $H^0_{dR}(S^1)$ is just the kernel of $\Lambda^0(S^1) \xrightarrow{d} \Lambda^1(S^1)$ which sends 0-forms, i.e. smooth functions $f : S^1 \to \mathbb{R}$, to 1-forms df. Note $df = f'(x)dx$ locally, so that if $df = 0$, we have f is locally constant. Since S^1 is connected, we conclude f is constant. Conversely, if f is constant, then $df = 0$. Thus, we see $H^0_{dR}(S^1)$ is in bijection with constant functions $f : S^1 \to \mathbb{R}$, which are just a choice of $x \in R$, so that $H^0_{dR}(S^1) \cong \mathbb{R}$.

Next, we consider the map $T : H^1_{dR}(S^1) \to \mathbb{R}$ via $[\omega] \mapsto \int_{S^1} \omega$. Note that this is well-defined since by Stokes Theorem,

$$\int_{S^1} d\eta = \int_{\partial S^1} \eta = \int_{\emptyset} \eta = 0$$

Moreover, it is clearly $\mathbb{R}$-linear. By Spring 2013 Problem 2, we have ω is exact if and only if for each $c : [0,1] \to S^1$ a closed curve, we have $\int_0^1 c^*\omega = 0$. Since $\pi_1(S^1) = \mathbb{Z}$, every closed curve on S^1 is homotopic to a multiple of $c : [0,1] \to S^1$ via $c(t) = e^{2\pi i t}$. But this is just restricts to the constant map $S^1 \to S^1$. Thus, we see ω is exact if and only if $\int_{S^1} \omega = 0$. Hence, T is injective.

Meanwhile, T is surjective as follows: let $\omega = i^*(-ydx + xdy)$, where $i : S^1 \to \mathbb{R}^2$ is the inclusion. Then $\int_{S^1} \omega = \int_B d(-ydx + xdy) = 2\int_B (dx \wedge dy) = 2 \cdot \text{area}(B) = 2\pi \neq 0$. Hence, T is nonzero, so that its image is at least one dimensional, and hence exactly one dimensional. So $H^1_{dR}(S^1) \cong \mathbb{R}$ via T.

The higher homotopy groups are all 0 since there are no higher dimensional forms.

Problem 8: Let X be a CW complex consisting of a vertex p, two edges a and b, and two 2-cells f_1 and f_2, where the boundaries of a, b map to p, the boundary of f_1 maps to the loop ab^2 and the boundary of f_2 is mapped to the loop ba^2.

a) Compute $\pi_1(X)$. Is this group finite?

Note a, b are both generating loops of the 1-skeleton $X_1 \cong S^1 \vee S^1$. By Hatcher Proposition 1.26, we have $\pi_1(X) = \langle a, b | ab^2, ba^2 \rangle$. Note if $ab^2 = 1$, then $a = b^{-2}$, so that $1 = ba^2 = b \cdot b^{-4} = b^{-3}$, and we have $b^3 = 1$. Thus $b = b^{-2}$, and so $a = b$. So we get a map $\langle a, b | ab^2, ba^2 \rangle \mapsto \rangle b | b^3 \rangle$ by sending both a, b to b. The inverse map sends b to $a = b$. These are both well-defined maps that give inverses to one another, so we see $\pi_1(X) = \langle b | b^3 \rangle = \mathbb{Z}/3\mathbb{Z}$, which is finite.

b) Compute $H_i(X)$ for each i.

We have the chain complex

$$0 \to C_2 = \mathbb{Z}^2 \xrightarrow{\partial_2} C_1 = \mathbb{Z}^2 \xrightarrow{\partial_1} C_0 = \mathbb{Z} \to 0$$

Notice $\partial f_1 = a + 2b$ and $\partial f_2 = 2a + b$, as these are the abelianizations of the boundary words. Meanwhile, $\partial a = \partial b = p - p = 0$. So we have $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}^2/\operatorname{im}(\partial_2)$ and $H_2(X) = \ker(\partial_2)$, with $H_i(X) = 0$ for $i > 2$. In coordinates, we have ∂_2 is the map $\mathbb{Z}^2 \to \mathbb{Z}^2$ with $(1,0) \mapsto (1,2)$ and $(0,1) \mapsto (2,1)$. We may put the matrix $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ into smith normal form to get

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

so that $H_1(X) \cong \mathbb{Z}^2/(\mathbb{Z} \times 3\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$. Alternatively, just abelianize the answer from part a to get $H_1(X) = \pi_1(X) = \mathbb{Z}/3\mathbb{Z}$.

Finally, note that the matrix for ∂_2 is invertible over $\mathbb{R}$. Thus there are no vectors $(x, y) \in \mathbb{R}^2$ with

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus in partiular there are no such vectors $(x, y) \in \mathbb{Z}^2$, and we have $H_2(X) = \ker(\partial_2) = 0$. So

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/3\mathbb{Z} & i = 1 \\ 0 & i > 1 \end{cases}$$

Problem 9: Let X, Y be topological spaces and let $f, g : X \to Y$ be two maps. Consider $Z = (X \times [0,1]) \sqcup Y/\sim$ where $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$. Show that there is a long exact sequence of the form $\ldots \to H_i(X) \xrightarrow{a} H_i(Y) \xrightarrow{b} H_i(Z) \xrightarrow{c} H_{i-1}(X) \to \ldots$ and describe the maps a, b, c.

Repeat of Fall 2011 Problem 10.

Problem 10: Let $n \geq 0$ be an integer. Let M be a compact, orientable, smooth manifold of dimension $4n + 2$. Show that $\dim H^{2n+1}(M, \mathbb{R})$ is even.

Repeat of Fall 2012 Problem 7.

12 Fall 2015

Problem 1: Let $M_n(\mathbb{R})$ be the space of $n \times n$ matrices with real coefficients.

(a) Show that $\mathrm{SL}_n(\mathbb{R})$ is a smooth submanifold of $M_n(\mathbb{R})$.

(b) Show that $\mathrm{SL}_n(\mathbb{R})$ has trivial Euler characteristic.

See Fall 2010 Problem 3.

Problem 2: Let $f, g : M \to N$ be smooth maps between smooth manifolds that are smoothly homotopic. Prove that if ω is a closed form on N, then $f^*\omega$ and $g^*\omega$ are cohomologous.

We follow Lee's Lemma 17.9. Consider $i_t : M \hookrightarrow M \times [0,1]$ via $x \mapsto (x,t)$. We show $i_0^* = i_1^*$ as maps on cohomology.

With this, we will then apply it to our case as follows: since f, g are homotopic, we have

$$H : M \times [0,1] \to N$$

with $H \circ i_0 = f$ and $H \circ i_1 = g$. Then $f^* = i_0^* H^*$ and $g^* = i_1^* H^*$. Since $i_0^* = i_1^*$ as maps on cohomology, we will get $f^* = g^*$ as maps on cohomology, as desired.

Take $\theta : \mathbb{R} \times (M \times [0,1]) \to M \times \mathbb{R}$ via $\theta(t,(x,s)) = (x, t+s)$. Note $i_t = \theta_t \circ i_0$, so it suffices to show $\theta_0^* = \theta_1^*$ as maps on cohomology.

Then from Fall 2010 Problem 4, we get $\theta_0^* = \theta_1^*$, as desired.

Problem 3: Prove that $[\mathcal{L}_X, i_Y]\omega = i_{[X,Y]}\omega$, for ω a k-form with $k \geq 1$.

Recall $\mathcal{L}_X = i_X d + di_X$, and form 1-forms ω, we have $\omega([X,Y]) = X(\omega(Y)) - Y(\omega(X)) - d\omega(X,Y)$.

Trivially, for 0-forms f, we have

$$[L_X, i_Y]f = L_X i_Y f + i_Y(L_X f) = 0 + 0 = 0 = i_{[X,Y]}f$$

since contraction of a 0-form gives 0.

For a more interesting base case, let ω be a 1-form. Then

$$[\mathcal{L}_X, i_Y]\omega = \mathcal{L}_X i_Y \omega - i_Y \mathcal{L}_X(\omega)$$

$$= (i_X d + di_X)i_Y\omega - i_Y(i_X d + di_X)\omega$$

$$= (i_X di_Y + di_X i_Y - i_Y i_X d - i_Y di_X)\omega$$

But note $i_X i_Y \omega = 0$ since ω is a 1-form. So we get

$$[\mathcal{L}_X, i_Y]\omega = (i_X di_Y - i_Y i_X d - i_Y di_X)\omega = (d(\omega(Y)))(X) - (d\omega)(X,Y) - (d(\omega(X)))(Y)$$

$$= X(\omega(Y)) - (d\omega)(X,Y) - Y(\omega(X)) = \omega([X,Y]) = i_{[X,Y]}\omega$$

By our usual trick, each k-form may locally be written as sums of exact 1-forms wedged with $(k-1)$-forms, so it suffices to show that if this formula holds for $(k-1)$-forms, then it holds for the wedge of an exact 1-form and a $(k-1)$-form. Let $\alpha = d\eta$ be an exact 1-form and θ a $(k-1)$-form. Notice that since $\mathcal{L}_X$ follows product rule and i_Y follows signed product rule, we get

$$[\mathcal{L}_X, i_Y](\alpha \wedge \theta) = \mathcal{L}_X i_Y(\alpha \wedge \theta) - i_Y \mathcal{L}_X(\alpha \wedge \theta)$$

$$= \mathcal{L}_X(i_Y(\alpha) \wedge \theta - \alpha \wedge i_Y(\theta)) - i_Y(\mathcal{L}_X(\alpha) \wedge \theta + \alpha \wedge \mathcal{L}_X(\theta))$$

$$= \mathcal{L}_X(i_Y(\alpha)) \wedge \theta + i_Y\alpha \wedge \mathcal{L}_X\theta - \mathcal{L}_X\alpha \wedge i_Y(\theta) - \alpha \wedge (\mathcal{L}_X i_Y \theta)$$

$$\cdots - i_Y \mathcal{L}_X(\alpha) \wedge \theta + \mathcal{L}_X(\alpha) \wedge i_Y\theta - i_Y\alpha \wedge \mathcal{L}_X(\theta) + \alpha \wedge i_Y \mathcal{L}_X(\theta)$$

$$= (i_{[X,Y]}\alpha) \wedge \theta - \alpha \wedge (i_{[X,Y]}\theta) = i_{[X,Y]}(\alpha \wedge \theta)$$

where we apply the fact that $[\mathcal{L}_X, i_Y] = i_{[X,Y]}$ on 1-forms, and on $(k-1)$-forms by inductive hypothesis.

We conclude this is true for all forms, so that $[\mathcal{L}_X, i_Y] = i_{[X,Y]}$, as desired.

Problem 4: Let $M = \mathbb{R}^3/\mathbb{Z}^3$ be the 3-dimensional torus, and $C = \pi(L)$ the image of the line L from $(0,1,1)$ to $(1,3,5)$. Find a differential form on M representing the Poincare dual of C.

See the similar calculation for Spring 2014 Problem 5. Set $dx = \pi_1^*\theta, dy = \pi_2^*\theta, dz = \pi_3^*\theta$ with $\int_{S^1} \theta = 1$, so that $\int_M dx \wedge dy \wedge dz = 1$. Since the cohomology classes of $dx \wedge dy, dy \wedge dz, dx \wedge dz$ form a basis of $H^2_{dR}(M)$ (by, for instance, Kunneth), it suffices to seek a Poincare dual of the form $\omega = a \cdot (dy \wedge dz) + b \cdot (dx \wedge dz) + c \cdot (dx \wedge dy)$.

Note C may be viewed as a line from $(0,0,0)$ to $(1,2,4)$. Then for $i : C \to M$, note $\pi_1 \circ i : C \to M$ is a 1-fold cover, $\pi_2 \circ i$ is a 2-fold cover, and $\pi_3 \circ i$ is a 4-fold cover. Then

$$\int_C i^* dx = \int_C i^* \pi_1^* \theta = \deg(\pi_1 \circ i) \int_{S^1} \theta = \deg(\pi_1 \circ i) = 1$$

$$\int_C i^* dy = \deg(\pi_2 \circ i) = 2$$

$$\int_C i^* dz = \deg(\pi_3 \circ i) = 4$$

Meanwhile, for $\omega = a \cdot (dy \wedge dz) + b \cdot (dx \wedge dz) + c \cdot (dx \wedge dy)$ the Poincare dual, we must have

$$1 = \int_C i^* dx = \int_M dx \wedge \omega = \int_M a(dx \wedge dy \wedge dz) = a$$

$$2 = \int_C i^* dy = \int_M dy \wedge \omega = \int_M -b(dx \wedge dy \wedge dz) = -b$$

$$4 = \int_C i^* dz = \int_M dz \wedge \omega = \int_M c(dx \wedge dy \wedge dz) = c$$

So we see $\omega = (dy \wedge dz) - 2(dx \wedge dz) + 4(dx \wedge dy)$.

Problem 5: Recall that the Hopf fibration $\pi : S^3 \to S^2$ is defined as follows: if we identify $S^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$, and $S^2 = \mathbb{CP}^1$ with homogenous coordinates, then $\pi(z_1, z_2) = [z_1, z_2]$. Show that π does not admit as smooth section, i.e. a map $s : S^2 \to S^3$ with $\pi \circ s = \mathrm{id}_{S^2}$.

Such a map would ensure $\pi_* \circ s_* = id_*$, so that s_* would be injective on homology. Thus we would get an injection $s_* : H_2(S^2) \to H_2(S^3)$, which is an injective map from $\mathbb{Z}$ to 0. This is a contradiction.

Problem 6: Let $M^n \subset \mathbb{R}^n$ be a smooth submanifold of dimension $m < n-2$. Show that its complement $\mathbb{R}^n \setminus M$ is connected and simply connected.

See Fall 2012 Problem 3.

Problem 7: Show that there exists no smooth degree 1 map $S^2 \times S^2 \to \mathbb{CP}^2$.

Note $\mathbb{C} \hookrightarrow \mathbb{CP}^1$ via $z \mapsto [z:1]$ has diffeomorphic image and misses precisely one point, namely $[1:0] \in \mathbb{CP}^1$. Hence, $\mathbb{CP}^1$ is the one-point compactification of $\mathbb{C} \cong \mathbb{R}^2$, so that $\mathbb{CP}^1 \cong S^2$.

Recall $H^*(\mathbb{CP}^n) = \mathbb{Z}[y]/(y^{n+1})$, with y having degree 2. Then the generator $y \in H^2(\mathbb{CP}^n) \cong \mathbb{Z}$ generates the entire cohomology ring.

Note that if X, Y are finite CW complexes and $H^k(X;R)$ and $H^k(Y;R)$ are free R-modules for each k, we have $H^*(X \times Y; R) = H^*(X;R) \otimes_R H^*(Y;R)$. In particular, we have

$$H^*(S^2 \times S^2) = H^*(\mathbb{CP}^1 \times \mathbb{CP}^1) = \mathbb{Z}[x_1]/(x_1^2) \otimes_{\mathbb{Z}} \mathbb{Z}[x_2]/(x_2^2) = \mathbb{Z}[x_1, x_2]/(x_1^2, x_2^2)$$

$$H^*(\mathbb{CP}^2) = \mathbb{Z}[y]/(y^3)$$

where $|x_i| = 2$ and $|y| = 2$. Letting $f : S^2 \times S^2 \to \mathbb{CP}^2$ be smooth, we see f induces a ring homomorphism

$$f^* : \mathbb{Z}[y]/(y^3) \to \mathbb{Z}[x_1, x_2]/(x_1^2, x_2^2)$$

Since $f^*(y)$ must have degree 2, we have $f^*(y) = ax_1 + bx_2$ for $a, b \in \mathbb{Z}$. (Just observe $H^2(S^1 \times S^1) = H^0(S^1) \otimes H^2(S^1) \oplus H^2(S^1) \otimes H^0(S^1)$ by Kunneth, so that there are no other degree 2 elements).

Then $f^*(y^2) = (ax_1 + bx_2)^2 = a^2x_1^2 + 2abx_1x_2 + b^2x_2^2 = 2abx_1x_2$. Note x_1x_2 is the generator of $H^4(S^1 \times S^1) = H^2(S^1) \otimes H^2(S^1)$, so that $\deg(f) = 2ab$ must be even. $\square$

Problem 8: Show that $\mathbb{CP}^{2n}$ is not a covering space of any manifold except itself.

Suppose $\mathbb{CP}^{2n}$ was a cover of some manifold X. Note $\mathbb{CP}^{2n}$ is simply connected, since $\mathbb{CP}^1 \cong S^2$ is simply connected and π_1 only depends on the 2-skeleton. Thus, $\mathbb{CP}2n$ must be the universal cover of X. In particular, the group of deck transformations would be isomorphic to $\pi_1(X)$ and it would act transitively on the fibers in the usual way, with $\gamma.y = \tilde{\gamma}(1)$, where $\tilde{\gamma}$ is a lift of γ to a path in $\mathbb{CP}^{2n}$ starting at y.

Note then that if $\gamma.y = y$, we would have $\tilde{\gamma}$ itself is a loop. Then we may view this as a map $\gamma : S^1 \to X$ which lifts to a map $\tilde{\gamma} : S^1 \to \mathbb{CP}^{2n}$. This implies $\gamma_*\pi_1(S^1) \subset p_*\pi_1(\mathbb{CP}^{2n}) = 0$, so that $\gamma_*\pi_1(S^1) = 0$. Then γ must be null homotopic in X, so that $[\gamma] = 0$. In short, we have shown more generally that the action of $\pi_1(X)$ on the universal cover of X is free.

Let $f : \mathbb{CP}^{2n} \to \mathbb{CP}^{2n}$ be the map $y \mapsto \gamma.y$ for some fixed $[\gamma] \in \pi_1(X)$. By Spring 2011 Problem 9, f must have a fixed point. So $f(y) = y$ for some $y \in \mathbb{CP}^{2n}$, so that $\gamma.y = y$. By the above, we have $[\gamma] = 0$. Since γ was arbitrary, we conclude $\pi_1(X) = 0$ and X is simply connected. Then X is its own universal cover, and $\mathbb{CP}^{2n} \cong X$.

Problem 9: Given $f : X \to Y$, define $C_f = (X \times [0,1]) \sqcup Y)/\sim$ where $(x,1) \sim f(x)$ and $(x,0) \sim (x',0)$ for all $x, x' \in X$. Show there is a long exact sequence

$$\ldots \to H_{i+1}(X) \xrightarrow{f_*} H_{i+1}(Y) \to \tilde{H}_{i+1}(C_f) \to H_i(X) \to \ldots$$

Recall the mapping cylinder $M_f = ((X \times I) \sqcup Y)/\sim$ (given by $(x,1) \sim f(x)$ for all $x \in X$) deformation retracts to $Y \subset M_f$. Moreover, $C_f = M_f/A$, where A is the image of $X \times \{0\} \subset (X \times I) \sqcup Y \to M_f$ (so that in fact, $A = X \times \{0\} \subset M_f$ is homeomorphic to X). Note (M_f, A) is a good pair since $X \times [0, \epsilon)$ deformation retracts to $X \times \{0\}$. We get by the LES for relative homology of a good pair

$$... \to H_n(A) \xrightarrow{i_*} H_n(M_f) \to \widetilde{H_n(M_f/A)} \to ...$$

Note $A = X \times \{0\} = X$, $M_f/A = C_f$. Meanwhile, the composition $X = A \subset M_f \xrightarrow{r} Y$ just sends $x \mapsto f(x) \in Y$, where $r : M_f \to Y$ is the retract. Since M_f actually deformation retracts onto Y, r is an isomorphism, and when we replace $H_n(M_f)$ with $H_n(Y)$ (which is isomorphic), we replace the map i_* with the map $r_* i_* = (r \circ i)_* = f_*$. Hence we get

$$... \to H_n(X) \xrightarrow{f_*} H_n(Y) \to \widetilde{H_n(C_f)} \to ...$$

which is the desired long exact sequence.

Problem 10: Let $\mathbb{RP}^n$ be the real projective space given by $S^n/\sim$ where $x \sim -x$.

(a) Give a CW decomposition of $\mathbb{RP}^n$.

(b) Use the cell decomposition to compute $H_k(\mathbb{RP}^n)$.

(c) For which values of $n \geq 1$ is $\mathbb{RP}^n$ orientable?

See Spring 2011 Problem 8.

13 Spring 2016

Problem 1: Consider the space of all straight lines in $\mathbb{R}^2$ (not necessarily just those passing through the origin). Explain how to give it the structure of a smooth manifold. Is it orientable?

We have a map from the space of all straight lines in $\mathbb{R}^2$ to $\mathbb{RP}^2$ via $ax+by+c=0 \mapsto [a:b:c] \in \mathbb{RP}^2$. This map is clearly well-defined and injective. Its image is precisely $\mathbb{RP}^2 \setminus \{[0:0:1]\}$, and this is an open submanifold of $\mathbb{RP}^2$.

Note that $S^2 \twoheadrightarrow \mathbb{RP}^2$ gives the orientation cover of $\mathbb{RP}^2$. Then $S^2 \setminus \{(0,0,1),(0,0,-1)\}$ (which is still orientable as it is an open subset of S^2) gives the orientation cover of $\mathbb{RP}^2 \setminus \{[0:0:1]\}$, and it is connected (homeomorphic to $\mathbb{R}^2 \setminus \{(0,0,0)\}$ which in turn is homotopic to S^1), so that $\mathbb{RP}^2 \setminus \{[0:0:1]\}$ must be non-orientable.

Remark: More generally, deleting a point from an n-manifold for $n > 1$ does not affect orientability.

Problem 2: Let X and Y be submanifolds of $\mathbb{R}^n$. Prove that for almost every $a \in \mathbb{R}^n$, that $X + a \pitchfork Y$.

Repeat of Fall 2010 Problem 2.

Problem 3: Consider the vector field $X(z) = z^{2019} + 2019z^{2018} + 2019$ on $\mathbb{C} = \mathbb{R}^2$. Compute the sum of the indices of X over all the zeroes of X.

Take $D \subset \mathbb{C}$ a compact disk containing all roots of $z^{2019} + 2019z^{2018} + 2019 = 0$. Then of course the sum of the indices of the zeros of X in $\mathbb{C}$ is equal to the sum of the indices of the zeros of X in D. The latter is $\chi(D)$ by Poincare-Hopf, and $\chi(D) = \chi(\{*\}) = 1$.

Problem 4: Let M be a compact, odd-dimensional manifold with non-empty boundary. Show that $\chi(M) = \frac{1}{2}\chi(\partial M)$.

See the third proposition here.

Problem 5: Let M be a compact oriented n-manifold with $H^1(M, \mathbb{R}) = 0$. For which integers k is there a smooth map $f : M \to T^n$ of degree k?

Follow the same argument as Spring 2010 Problem 10c.

Problem 6: Let T^2 be the torus and $p \in T^2$.

a) Compute the de Rham cohomology of $X = T^2 \setminus \{p\}$, where $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with coordinates (x, y).

Note that X deformation retracts to $S^1 \vee S^1$ as is clear from the picture of T^2 as $[0,1]^2/\sim$, where $(x,0) \sim (x,1)$ and $(0,y) \sim (1,y)$ for each $x, y \in [0,1]$. ($T^2 \setminus (1/2, 1/2)$ deformation retracts to the square identified in the specificed way, which gives $S^1 \vee S^1$).

Thus we can easily figure out singular homology groups to be $H_1(X) = \mathbb{Z}^2$ and $H_0(X) = \mathbb{Z}$ (from reduced homology of wedge is sum of reduced homologies, or from abelianizing π_1 and using that X is connected). Hence, by universal coefficient theorem, $H_0(X;\mathbb{R}) = \mathbb{R}$, $H_0(X;\mathbb{R}) = \mathbb{R}^2$. Since X is a manifold, we have by de Rham's theorem that $H^0_{dR}(X) = \mathbb{R}^* = \mathbb{R}$ and $H^1_{dR}(X) = (\mathbb{R}^2)^* = \mathbb{R}^2$ (where we use $H^i_{dR}(X) = (H_i(X;\mathbb{R}))^*$ by de Rham's Theorem). Thus

$$H^i_{dR}(X) = \begin{cases} \mathbb{R} & i = 0 \\ \mathbb{R}^2 & i = 1 \\ 0 & i > 1 \end{cases}$$

b) Is the volume form $\omega = dx \wedge dy$ exact on $X = T^2 \setminus \{p\}$?

Notice $d\omega$ is a 3-form on a 2-manifold, so $d\omega = 0$ and ω is closed. Since $H^2(X) = 0$, closed 2-forms on X are exact, so ω is exact, as desired.

Problem 7: Exhibit a space whose fundamental group is isomorphic to $\mathbb{Z}/m\mathbb{Z} * \mathbb{Z}/n\mathbb{Z}$. Find another space with fundamental group $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.

See Fall 2010 Problem 7.

Problem 8: Let L_i denote the axes of $\mathbb{R}^3$. Compute $\pi_1(\mathbb{R}^3 \setminus (L_x \cup L_y \cup L_z))$.

From the discussion of Fall 2012 Problem 9, we have $\mathbb{R}^3 \setminus (L_x \cup L_y \cup L_z)$ deformation retracts to S^2 minus 6 points, which is homeomorphic to $\mathbb{R}^2$ minus 5 points, call them $p_1 = (x_1, y_1), \ldots, p_5 = (x_5, y_5)$. WLOG, $x_i \in (2i-1, 2i)$. Set $U_1 = ((-\infty, 3) \times \mathbb{R}) \setminus \{p_1\}$, $U_5 = ((8, \infty) \times \mathbb{R}) \setminus \{p_5\}$, and $U_i = ((2i-2, 2i+1) \times \mathbb{R}) \setminus \{p_i\}$ for $i = 2, 3, 4$. Then $U_1 \cup \ldots \cup U_5 = \mathbb{R}^2 \setminus \{p_1, \ldots, p_5\}$. Moreover, each $U_i \cong \mathbb{R}^2 \setminus \{p_i\}$, which is homotopic to a circle, so that $\pi_1(U_i) = \mathbb{Z}$. Moreover, each $U_i \cap U_j$ is homeomorphic to $\mathbb{R}^2$, hence simply connected with $\pi_1(U_i \cap U_j) = 0$. Van-Kampen then gives us $\pi_1(\mathbb{R}^2 \setminus \{p_1, \ldots, p_5\}) = \pi_1(U_1 \cup \ldots \cup U_5) = \pi_1(U_1) * \ldots * \pi_5(U_5) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$, since the kernel elements are all trivial (from $\pi_1(U_i \cap U_j) = 0$). More generally, $\mathbb{R}^2$ minus k points gives us $\mathbb{Z} * \ldots * \mathbb{Z}$ (k times) by the same argument. Of course, we don't need to worry about other $\mathbb{R}^n \setminus \{x_1, \ldots, x_k\}$ (for $n > 2$) as far as fundamental groups are concerned, as those will be simply connected by Fall 2012 Problem 3.

Problem 9: Let X be a topological space and $p \in X$. The reduced suspension, ΣX is defined by $X \times [0,1]/\sim$ where $(X \times \{0,1\}) \cup (\{p\} \times [0,1])$ is contracted to a point. Describe the relation between the homology groups of X and ΣX.

Write $I = [0,1]$. We add the assumption that there is some neighborhood $U \subset X$ of p that deformation retracts to p, i.e. that (X,p) is a good pair. See Spring 2014 Problem 10. Notice $\Sigma X = S(X)/A$, where $A \subset S(X)$ is the image of $p \times I \subset X \times I \to (X \times I)/(X \times \partial I) = S(X)$. By assumption, (X,p) is a good pair, so that $(X \times I, p \times I)$ is a good pair. In fact, $(X \times I, X \times \partial I)$ is also a good pair, so that $(X \times I, (X \times \partial I) \cup (p \times I))$ is a good pair. Thus $(S(X), A)$ is a good pair.

We get by Hatcher 2.13 a long exact sequence of reduced relative homology, which is actually quite useful: $\boxed{... \to \widetilde{H_k(A)} \to \widetilde{H_k(S(X))} \to \widetilde{H_k(\Sigma(X))} \to \widetilde{H_{k-1}(A)} \to ...}$ Note A is the homeomorphic image of $p \times I$, so that A has the homotopy of a point. So each reduced $\tilde{H_k(A)} = 0$, and we have

$$\widetilde{H_k(S(X))} = \widetilde{H_k(\Sigma(X))}$$

Then $H_k(S(X)) = H_k(\Sigma(X))$ for all k.

Problem 10: Consider the 3-form $\alpha = x_1 dx_2 \wedge dx_3 \wedge dx_4 - x_2 dx_1 \wedge dx_3 \wedge dx_4 + x_3 dx_1 \wedge dx_2 \wedge dx_4 - x_4 dx_1 \wedge dx_2 \wedge dx_3$ on $\mathbb{R}^4$.

a) Compute $\int_{S^3} i^*\alpha$.

Note $d\alpha = 4dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 = 4dV$. if $B \subset \mathbb{R}^4$ is the closed unit ball, we have

$$\int_{S^3} i^*\alpha = \int_B d\alpha = \int_B 4dV = 4V(B)$$

From analysis, one might recall the volume of the closed unit ball in $\mathbb{R}^n$ is $\pi^{n/2}/\Gamma(n/2+1)$, so that $V(B) = \pi^2/\Gamma(3) = \pi^2/2$. Thus, $\int_{S^3} i^*\alpha = 2\pi^2$.

b) Let γ be the 3 form $\gamma = \frac{1}{(x_1^2+x_2^2+x_3^2+x_4^2)^k}\alpha$ for $k \in \mathbb{R}$. Find the values of k where γ is closed, and where it is exact.

Since $\gamma_p = \alpha_p$ for each $p \in S^3$, so that $i^*\gamma = i^*\alpha$ and

$$\int_{S^3} i^*\gamma = \int_{S^3} i^*\alpha \neq 0$$

If $\gamma = d\theta$, then $i^*\gamma = d(i^*\theta)$, and $\int_{S^3} d(i^*\theta) = 0$ by Stokes theorem, since $\partial S^3 = \emptyset$. Hence γ is not exact.

Meanwhile, let $R(x_1, ..., x_4) = x_1^2 + ... + x_4^2 = r^2$. Note $\gamma = \frac{1}{R^k}\alpha$, so that

$$d\gamma = d\left(R^{-k}\right) \wedge \alpha + \frac{1}{R^k} d\alpha$$

Now

$$d\left(R^{-k}\right) = -kR^{-k-1}\sum_{i=1}^4 2x_i dx_i$$

Thus

$$d\gamma = -kR^{-k-1}\sum_{i=1}^4 2x_i dx_i \wedge \alpha + \frac{4}{R^k}dV$$

$$= -kR^{-k-1}\sum_{i=1}^4 2x_i^2 dV + \frac{4}{R^k}dV = (-2kR^{-k} + 4R^{-k})dV$$

So $d\gamma = 0 \iff 2kR^{-k} = 4R^{-k} \iff 2k = 4 \iff k = 2.$

14 Fall 2016

Problem 1: Let M be a smooth manifold. Prove that for any two disjoint closed subsets A, B there is a smooth function $f : M \to \mathbb{R}$ such that $f = 0$ on A and $f = 1$ on B.

Pick a partition of unity subordinate to the cover $\{A^c, B^c\}$ of M. Write $f, g : M \to [0,1]$ with $f + g = 1$, $f << A^c, g << B^c$. Then for $x \in A$, we have $x \notin A^c$, so that $f(x) = 0$. Hence $g(x) = 1$. So we see $g(x) \equiv 1$ on A. Since g is supported in B^c, we see $g(x) \equiv 0$ on B. This gives the desired function.

Problem 2: Let $M \subset \mathbb{R}^N$ be a smooth k-dimensional submanifold. Prove that M can be immersed into $\mathbb{R}^{2k}$.

Skip!

Problem 3: Let $U_1, \ldots, U_n$ be n bounded, connected, open subsets of $\mathbb{R}^n$. Prove that there exists an $(n-1)$ dimensional hyperplane $H \subset \mathbb{R}^N$ that bisects every U_i; i.e. if A and B are the two half spaces that form $\mathbb{R}^n \setminus H$, then $\mathrm{vol}(U_i \cap A) = \mathrm{vol}(U_i \cap B)$.

Skip!

Problem 4: Show that $D = \ker(dx_3 - x_1 dx_2) \cap \ker(dx_1 - x_4 dx_2) \subset T\mathbb{R}^4$ is a smooth distribution of rank 2, and determine whether D is integrable.

Note $dx_3 - x_1 dx_2$ can be thought of as the matrix $\begin{bmatrix} 0 & -x_1 & 1 & 0 \end{bmatrix}$ and $dx_1 - x_4 dx_2$ can be thought of as $\begin{bmatrix} 1 & -x_4 & 0 & 0 \end{bmatrix}$. Here we are picking the standard basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_4}$ of $T\mathbb{R}^4$, so that these forms are, at any point, maps from $T_p\mathbb{R}^4$ to $\mathbb{R}$, and hence correspond to a 1×4 matrix at that point.

Now $X = \sum_{i=1}^4 f_i \frac{\partial}{\partial x_i} \in D$ if and only if

$$\begin{bmatrix} 0 & -x_1 & 1 & 0 \\ 1 & -x_4 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(at each point, i.e. that these functions are 0). However, notice that regardless of choice of $x_1, \ldots, x_4$, we have $\begin{bmatrix} 0 & -x_1 & 1 & 0 \\ 1 & -x_4 & 0 & 0 \end{bmatrix}$ has rank 2 from its first and third column. Hence, the kernel is pointwise a 2-dimensional vector space, so that D is indeed a smooth distribution of rank 2.

In fact, we see $X_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ (i.e. $X_1 = \frac{\partial}{\partial x_4}$) and $X_2 = \begin{bmatrix} x_4 \\ 1 \\ x_1 \\ 0 \end{bmatrix}$ (i.e. $X_2 = x_4 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}$) are always in the kernel of the matrix, and, regardless of point, are always linearly independent. So X_1, X_2 form a global basis of D.

On the other hand,

$$[X_1, X_2] = \frac{\partial}{\partial x_4}\left(x_4 \frac{\partial}{\partial x_1} + 1\frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}\right) - \left(x_4 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}\right)\left(1\frac{\partial}{\partial x_4}\right)$$

$$= 1\frac{\partial}{\partial x_1} + 0 + 0 - 0 - 0 - 0 = \frac{\partial}{\partial x_1} \notin D$$

So $X_1, X_2 \in D$ but $[X_1, X_2] \notin D$. So D is not integrable.

Problem 5:

a) Let M be a smooth, compact manifold and $N \subset M$ a smooth compact submanifold. Explain (in terms of integrals) what it means for a closed differential form ω to be the Poincare dual to N.

> If $\dim(N) = k, \dim(M) = n$, then the Poincare dual to N is the unique $(n-k)$-form ω such that for all k-forms η, we have
> $$\int_N i^*\eta = \int_M \eta \wedge \omega$$
> See the discussion in Spring 2014 Problem 5 for more details.

b) You are now free to use knowledge of homology/cohomology: let $M = T^2$ with coordinates $(x, y) \in (\mathbb{R}/\mathbb{Z})^2$. Identify a submanifold $N \subset M$ dual to the form dy, and show that they are indeed dual.

> Let $\pi_1, \pi_2 : M = S^1 \times S^1 \to S^1$ denote the two projections. Define $dx = \pi_1^*\theta$, $dy = \pi_2^*\theta$, where θ is a 1-form on S^1 with $\int_{S^1} \theta = 1$. Note then $\int_M dx \wedge dy = 1$.
>
> Take $N = S^1 \times \{p\} \subset S^1 \times S^1$ oriented CCW. Since $[dx], [dy]$ form a basis of $H^1_{dR}(M) = \mathbb{R}^2$, for any closed 1-form η, we may write $[\eta] = a[dx] + b[dy]$ for $a, b \in \mathbb{R}$. So, we may check the above formula for Poincare dual, which is well-defined regardless of representative of $[\eta]$, by simply taking η of the form $a \cdot dx + b \cdot dy$. Then $\eta \wedge dy = a \cdot dx \wedge dy$, so that
> $$\int_M \eta \wedge dy = a \int_M dx \wedge dy = a$$
> Meanwhile, for $i : N \to M$ inclusion, note $i^*dx = i^*\pi_1^*\theta = (\pi_1 \circ i)^*\theta = \theta$, since $\pi_1 \circ i : S^1 = S^1 \times \{p\} \to M \twoheadrightarrow S^1$ is just the identity map. Meanwhile, $i^*dy = (\pi_2 \circ i)^*\theta = 0$, since $\pi_2 \circ i$ is just the constant map $S^1 \mapsto p \in S^1$.
>
> Thus,
> $$\int_N i^*\eta = \int_{S^1} a(i^*dx) + b(i^*dy) = \int_{S^1} a\theta = a$$
> since $\int_{S^1} \theta = 1$. Thus we see indeed N is the Poincare dual to dy.

c) Give an example of a closed 1-form on T^2 that is not Poincare dual to any submanifold.

> For N a closed connected oriented 1-dimensional submanifold of M, notice $f : N \to S^1$ gives $\int_N f^*\omega = \deg(f) \int_{S^1} \omega$. We claim the form $\alpha = \pi dx$ has no Poincare dual. To see this, note that if N is the Poincare dual, then for $\eta = dy$, we must have
> $$\int_N i^*(dy) = \int_M dy \wedge (\pi dx) = -\pi$$
> On the other hand,
> $$-\pi = \int_N i^*(dy) = \int_N (\pi_2 \circ i)^*\theta = \deg(\pi_2 \circ i) \int_{S^1} \theta = \deg(\pi_2 \circ i)$$
> However, degree is always an integer. By contradiction, we see there can be no closed connected oriented manifold that is the Poincare dual of πdx.

Problem 6: Let M be a smooth, compact, oriented n-manifold of Euler characteristic 0.

a) Show that M admits a nowhere vanishing vector field.

> See the exercises in *G&P* from pages 144 to 146. We may find a vector field on M with finitely many zeros. By Spring 2017 Problem 1, we may move, via a diffeomorphism of M, all of the zeros into some chart $U \subset M$ with $U \cong B(0,1)$ the open unit ball in $\mathbb{R}^n$. Call this vector field X. Note that the sum of the indices of the zeros of X is $\chi(M) = 0$ by Poincare-Hopf. From Spring 2011 Problem 5, this is also the degree of the map $\partial\overline{U} \cong S^n \to S^n$ via $p \mapsto \frac{X_p}{|X_p|}$, so that this degree is zero. By the extension theorem, we get a map $g : \overline{U} \to S^n$ which extends $\frac{X_p}{|X_p|}$ on the boundary. Take the vector field Y given by
> $$Y_p = \begin{cases} g(p) & p \in \overline{U} \\ \frac{X_p}{|X_p|} & p \notin U \end{cases}$$
> This is well-defined since the two cases agree on the boundary. (We can make this smooth if necessary by taking a bump function, picking a smaller ball $V \subset U$ still containing all the zeros of X).
>
> Then note Y is nonvanishing, since $g(p) \in S^n$ is a unit vector for each p, and X_p does not vanish outside of U. Thus, Y is the desired vector field.

b) A Lorentzian metric on M is a smoothly varying, symmetric bilinear form $g_p : T_pM \times T_pM \to \mathbb{R}$ of signature $(n-1, 1)$; that is, for all $p \in M$ there is a basis $e_1, \ldots, e_n$ of T_pM such that with respect to this basis, g_p is a diagonal matrix with $n-1$ entries of 1 and one entry of -1. Prove that M admits a Lorentzian metric.

> Skip!

Problem 7: Let X be a connected CW-complex with $\pi_1(X, x)$ finite. Show that any map $f : X \to (S^1)^n$ is null-homotopic.

> Note $\pi_1((S^1)^n) = (\pi_1(S^1))^n = \mathbb{Z}^n$, since π_1 preserves products. In particular, $\pi((S^1)^n)$ is torsion-free. Note $f_*\pi_1(X) \subset \pi_1((S^1)^n)$ is a subgroup of a free group and hence free. It is also the image of a finite group and hence finite. So it is a finite free group, and hence must be zero. Thus, $f_*\pi_1(X) = 0$.
>
> Note $\mathbb{R}^n$ is the universal cover of $(S^1)^n$, so that for $p : \mathbb{R}^n \to (S^1)^n$, since $f_*\pi_1(X) = 0 \subset p_*\pi_1(\mathbb{R}^n) = 0$, we see f lifts to a map on the universal cover. So we have a map $g : X \to \mathbb{R}^n$ with $p \circ g = f$.
>
> Note g is null-homotopic in $\mathbb{R}^n$ via a straight line homotopy, or just by noting $\mathbb{R}^n$ deformation retracts to a point. If $H : X \times [0,1] \to \mathbb{R}^n$ is a homotopy between $g(x) = H(x, 0)$ and $c(x) = H(x, 1)$, where $c(x) = c$ for all $x \in X$, then $p \circ H : X \times [0,1] \to (S^1)^n$ is a homotopy between $f = p \circ g$ and $d = p \circ c$, the constant map $d(x) = p(c)$ for each $x \in X$. Hence f is null-homotopic, as desired.

Problem 8: Let $X = \mathbb{RP}^2 \vee \mathbb{RP}^2$. Let a generate π_1 of the first summand, and b of the second. For $n \geq 1$, describe the covering space $p : Y \to X$ such that $p_*(\pi_1(Y))$ is the subgroup $\langle (ab)^n \rangle$ of $\pi_1(X)$.

> See Fall 2014 Problem 9 for detailed discussion.

Problem 9: Let $S^2 \xleftarrow{q_1} S^2 \vee S^2 \xrightarrow{q_2} S^2$ be the maps that crush out one of the two summands. Let $f : S^2 \to S^2 \vee S^2$ be a map such that $q_i \circ f : S^2 \to S^2$ is a map of degree d_i. Compute the homology groups of $X = (S^2 \vee S^2) \cup_f D^3$.

We have one 3-cell, two 2-cells and one 0-cell. This gives the chain complex

$$0 \to C_3 = \mathbb{Z} \xrightarrow{\partial_3} C_2 = \mathbb{Z}^2 \xrightarrow{\partial_2} C_1 = 0 \xrightarrow{\partial_1} C_0 = \mathbb{Z} \to 0$$

Note for our face $F \in C_3$, we have $\partial_3(F) = d_1 e_1 + d_2 e_2$ by the cellular boundary formula. The other maps are necessarily 0. Hence, $H_0(X) = \mathbb{Z}$, $H_1(X) = 0$, $H_2(X) = \mathbb{Z}^2/\operatorname{im}(\partial_3)$, $H_3(X) = \ker(\partial_3)$, and $H_i(X) = 0$ for $i > 3$.

If d_1, d_2 are not both zero, ∂_3 is injective and we have $H_3(X) = 0$. Meanwhile, putting the matrix $\begin{bmatrix} d_1 & d_2 \end{bmatrix}$ in Smith normal form amounts to continually subtracting the smaller of the two numbers from the larger one and replacing the larger with this difference. Hence, we get the SNF to be $\begin{bmatrix} k = \gcd(d_1, d_2) & 0 \end{bmatrix}$. (We define $\gcd(n, 0) = n$). Thus,

$$H_2(X) = \mathbb{Z}^2/\operatorname{im}(\partial_3) \cong \mathbb{Z}^2/(\langle (k,0) \rangle) = \mathbb{Z}/k\mathbb{Z} \oplus \mathbb{Z}$$

Of course, if d_1, d_2 are both zero, we instead get $H_3(X) = \mathbb{Z}$ and $H_2(X) = \mathbb{Z}^2$. In short, we have for d_1, d_2 not both zero,

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ (\mathbb{Z}/\gcd(d_1, d_2)\mathbb{Z}) \oplus \mathbb{Z} & i = 2 \\ 0 & i > 2 \end{cases}$$

and if $d_1 = d_2 = 0$, we get

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ \mathbb{Z}^2 & i = 2 \\ \mathbb{Z} & i = 3 \\ 0 & i > 3 \end{cases}$$

Problem 10: If $f : X \to X$ is a self map, then the mapping torus of f is the quotient $T_f = (X \times [0,1])/\sim$ where $(x, 0) \sim (f(x), 1)$. Let f_n be map of degree n on S^3. Compute the homology groups of T_{f_n}.

We again use Fall 2011 Problem 10 with $X = Y = S^3$, $f = f_n : S^3 \to S^3$, $g = id : S^3 \to S^3$, so that $Z = T_{f_n}$ giving us a long exact sequence

$$\ldots \to H_k(S^3) \xrightarrow{f_* - id_*} H_k(S^3) \to H_k(T_{f_n}) \to H_{k-1}(S^3) \to \ldots$$

For $k > 4$, $H_k(S^3) = H_{k-1}(S^3) = 0$, so that $H_k(T_{f_n}) = 0$. For $k = 3$, we have the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{f_* - id_*} \mathbb{Z} \to H_3(T_f) \to 0$$

Since f has degree n and id has degree 1, we see the first map is multiplication by $n - 1$ (the degree gives the map on top homology). Thus we see $H_3(T_f) = \mathbb{Z}/(n-1)\mathbb{Z}$.

Next, for $k = 2$, since $H_k(S^3) = H_{k-1}(S^3) = 0$, we have $H_k(T_f) = 0$. In this case, T_f is the quotient of connected space $S^3 \times [0,1]$, so that $H_0(T_f) = \mathbb{Z}$. Finally, for $k = 1$ we have

$$0 \to H_1(T_f) \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0$$

Hence $H_1(T_f)$ is a subgroup of a free group and therefore free. Counting rank, we see $\operatorname{rank}(H_1(T_f)) - 1 + 1 - 1 = 0$, so that $H_1(T_f) \cong \mathbb{Z}$. Thus

$$H_k(T_f) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z} & k = 1 \\ 0 & k = 2 \\ \mathbb{Z}/(n-1)\mathbb{Z} & k = 3 \\ 0 & k > 3 \end{cases}$$

15 Spring 2017

Problem 1: Let M be a connected smooth manifold of dimension at least 2. Prove that for any $2n$ distinct points $x_1, \ldots, x_n, y_1, \ldots, y_n \in M$ that there is a diffeomorphism $f : M \to M$ such that $f(x_i) = y_i$ for all i.

First, see Fall 2010 Problem 1 for the $n = 1$ case. In fact, we can see that in this case, for any two points $x, y \in M$, we may find a compactly supported diffeomorphism $\phi : M \to M$ with $\phi(x) = y$.

Suppose for any selection of points $x_1, ..., x_k, y_1, ..., y_k$, $k \leq n-1$, in any manifold N, we may find a compactly supported diffeomorphism $\phi : N \to N$ with $\phi(x_i) = y_i$.

We are given points $x_1, ..., x_n, y_1, ..., y_n \in M$. Set $N = M \setminus \{x_n, y_n\}$. We may find a diffeomorphism $\phi : N \to N$ with $\phi(x_i) = y_i$ for $i = 1, ..., n-1$ by inductive hypothesis. Note that then in some neighborhood of x_n and y_n, ϕ is just the identity. To see this, put some metric on N (e.g. by embedding it into $\mathbb{R}^k$) and look at open sets $M \setminus \overline{B(x_n, r)} \subset N$ for $r > 0$. This gives an open cover of N, and hence finitely many of them cover the compact support of ϕ. Thus we see we may pick r small enough so that $B(x_n, r)$ is disjoint from the compact support, so that ϕ is the identity on this open neighborhood. A similar argument works for y_n. So we see ϕ may be extended to a map $\psi : M \to M$ with $\psi(x_n) = x_n, \psi(y_n) = y_n$, and $\psi(x) = \phi(x)$ for $x \in N$.

We still have ψ is a diffeomorphism, and $\psi(x_i) = y_i$ for $1 \leq i < n$. Moreover, $\psi(x_n) = x_n, \psi(y_n) = y_n$.

Similarly, set $N' = M \setminus \{x_1, ..., x_{n-1}, y_1, ..., y_{n-1}\}$ and find $\lambda : N' \to N'$ with $\lambda(x_n) = y_n$. A similar argument shows λ can be extended to $\tau : M \to M$ with $\tau(x) = \lambda(x)$ for all $x \in N'$, and $\tau(x_i) = x_i, \tau(y_i) = y_i$ for $i = 1, ..., n-1$.

Thus, $\psi \circ \tau(x_i) = y_i$ for each $i = 1, ..., n$, and this is the desired diffeomorphism.

Remark: In fact, these are all (compactly supported) isotopies, i.e. diffeomorphisms homotopic to the identity, with ϕ_t a (compactly supported) diffeomorphism for each time $t \in [0, 1]$. This is clear from the proof of the $n = 1$ case, and our generalization above would allow us to extend each ϕ_t as well, so that we would still get isotopies.

Problem 2: Let $M_{2n\times 2n}(\mathbb{R}) = \mathbb{R}^{4n^2}$. Consider the following matrix $\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Show that the subspace $S = \{A : A^T\Omega A = \Omega$ is a smooth submanifold, and compute its dimension.

Let $Sk_{2n}(\mathbb{R}) = \{A \in M_{2n}(\mathbb{R}) : A^T = -A\}$ be the set of skew symmetric matrices. It is clear from considering matrix entries that $Sk_{2n}(\mathbb{R}) \subset M_{2n}(\mathbb{R})$ is a submanifold diffeomorphic to $\mathbb{R}^{\binom{2n}{2}}$ (it is entirely determined by the (i,j) entries for $i < j$; the diagonal entries must be zero).

Define $F : M_{2n}(\mathbb{R}) \to Sk_{2n}(\mathbb{R})$ via $A \mapsto A^T\Omega A$. Note $(A^T\Omega A)^T = A^T\Omega^T A = -A^T\Omega A$, so that this is indeed skew symmetric.

Note $\gamma : \mathbb{R} \to M_{2n}(\mathbb{R})$ given by $\gamma(t) = A + tB$ is a curve through $\gamma(0) = A$ in the direction of $\gamma'(0) = B$, so that

$$dF_A(B) = (F \circ \gamma)'(0) = \lim_{t\to 0} \frac{F(A+tB) - F(A)}{t}$$

$$= \lim_{t\to 0} \frac{(A+tB)^T\Omega(A+tB) - A^T\Omega A}{t} = \lim_{t\to 0} \frac{A^T\Omega A + tB^T\Omega A + tA^T\Omega B + t^2B^T\Omega B - A^T\Omega A}{t}$$

$$= \lim_{t\to 0} B^T\Omega A + A^T\Omega B + tB^T\Omega B = B^T\Omega A + A^T\Omega B$$

If $A \in F^{-1}\Omega$, then $A^T\Omega A = \Omega$, so taking determinants, we see $\det(A)^2\det(\Omega) = \det(\Omega)$. Since $\det(\Omega) \neq 0$ as Ω is invertible, we have $\det(A)^2 = 1$ and A is invertible.

Fix $A \in F^{-1}\Omega$. Let $C \in Sk_{2n}(\mathbb{R})$ be arbitrary. Take $B = \frac{1}{2}\Omega^{-1}(A^-1)^TC$. Then notice $A^T\Omega B = \frac{1}{2}C$, and $B^T\Omega A = -(A^T\Omega B)^T = -(\frac{1}{2}C)^T = +\frac{1}{2}C$, so that $dF_A(B) = C$. Hence

$$dF_A : T_A(M_{2n}(\mathbb{R})) = M_{2n}(\mathbb{R}) \to T_\Omega(Sk_{2n}(\mathbb{R})) = Sk_{2n}(\mathbb{R})$$

is surjective. This holds for any $A \in F^{-1}\Omega$. Thus, Ω is a regular value of F, and $F^{-1}\Omega = \{A : A^T\Omega A = \Omega\}$ is a smooth submanifold of $M_{2n}(\mathbb{R})$.

Its codimension in $M_{2n}(\mathbb{R})$ is the dimension of $Sk_{2n}(\mathbb{R})$, which is $\binom{2n}{2}$ as computed above. Thus $F^{-1}\Omega$ has dimension $4n^2 - \binom{2n}{2} = 4n^2 - n(2n-1) = 4n^2 - 2n^2 + n = 2n^2 + n$.

Problem 3: Use the Poincare Hopf index theorem to calculate the Euler characteristic of the n-sphere. (Drawings are not enough!)

For odd n, write $n = 2k-1$, so $S^n \subset \mathbb{R}^{2k} = \mathbb{C}^k$. Then $X_p = ip$ gives a nonvanishing vector field on S^n (since $ip \perp p$ for each p, so that $X_p \in T_pS^n$ for each $p \in S^n$). Hence $\chi(S^n) = 0$ in this case.

For even n, write $n = 2k$. Write $S^n \subset \mathbb{R}^{2k+1} = \mathbb{C}^k \times \mathbb{R}$. Define $X_{(p,r)} = (ip, 0)$. Again, $(p,r) \perp (ip,0)$, so that $X_{(p,r)} \in T_{(p,r)}S^n$ for each point $(p,r) \in S^n$. Hence X can indeed be viewed as a vector field on S^n. Note $X_{(p,r)} = (0,0)$ if $(ip,0) = (0,0)$, so that we must have $p = 0$. Hence, $r = \pm 1$ (in order for $(p,r) \in S^n$).

Poincare-Hopf tells us the Euler characteristic of S^n will be the sum of the indices of the zeros of X. To compute the index of the zero $(p,r) = (0,1)$, pick the ball $B = \{(p,r) \in S^n : r \geq 0\}$, which contains $(0,1)$ in the interior, but does not contain $(0,-1)$ at all. Then the index is the degree of the map from $\partial B \to S^{n-1}$ via $q \mapsto \frac{X_q}{|X_q|}$. Note $\partial B = \{(p,r) \in S^n : r = 0\} = S^{n-1}$ itself. Meanwhile, the map sends

$$(p,0) \in S^{n-1} \mapsto \frac{X_{p,0}}{|X_{p,0}|} = \frac{(ip,0)}{|(ip,0)|} = (ip,0) \in S^{n-1}$$

To easily see what the degree of this map is, just notice in real coordinates this is the map from S^{n-1} to S^{n-1} sending $(x_1, ..., x_n) \mapsto (-x_2, x_1, -x_4, x_3, ..., -x_n, x_{n-1})$. (Note n is even so this pairing makes sense). Thus the degree of this map is $(-1)^{\#\text{flips}} \cdot (-1)^{\#\text{negations}} = (-1)^{n/2}(-1)^{n/2} = 1$. Hence, $ind_{(0,1)}X = 1$.

To compute the index of $(0,-1)$, we may use the ball $B' = \{(p,r) \in S^n : r \leq 0\}$. In this case, $\partial B' = \partial B$, and we get the same degree calculation, so that $ind_{(0,-1)}X = 1$.

Hence $\chi(S^n) = 1 + 1 = 2$ in this case, as desired.

Problem 4:

a) State Cartan's magic formula.

b) Use this to show that a vector field X on $\mathbb{R}^3$ has a flow (locally) that preserves volume if and only if $\text{div}(X) = 0$.

See Spring 2011 Problem 2.

Problem 5: Let $\omega = \frac{-ydx+xdy}{(x^2+y^2)^\alpha}$ be a 1-form on $\mathbb{R}^2 \setminus \{0\}$ and $\alpha \in \mathbb{R}$. Consider $\int_\gamma \omega$, where $\gamma : S^1 \to \mathbb{R}^2 \setminus \{0\}$ is a smooth map.

a) For which $\alpha \in \mathbb{R}$ do we have $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$ whenever γ_0 and γ_1 are smoothly homotopic?

First, if ω is closed, then by Lee 16.26, we have $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$ for any two γ_0, γ_1 smoothly homotopic.

Next, suppose we have $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$ for any γ_0, γ_1 smoothly homotopic. Note the circle of radius R, $S^1(R) \subset \mathbb{R}^2 \setminus \{0\}$ is homotopic to the unit circle $S^1 \subset \mathbb{R}^2 \setminus \{0\}$ (and in fact maps to it diffeomorphically under the deformation retract of $R^2 \setminus \{0\} \to S^1$). Hence, we have

$$\int_{S^1(R)} \omega = \int_{S^1} \omega$$

so that $\int_{S^1(R)} \omega$ is independent of $R > 0$. Thus, using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, note $\omega = r^{2-2\alpha} d\theta$, and

$$\int_{S^1(R)} \omega = \int_0^{2\pi} R^{2-2\alpha} d\theta = 2\pi R^{2-2\alpha}$$

Since this must be independent of $R > 0$, we see the exponent must be zero, so that $\alpha = 1$.

Finally, suppose $\alpha = 1$. Then $\omega = \frac{-ydx+xdy}{x^2+y^2}$, and direct computation shows $d\omega = 0$. So ω is closed.

We conclude $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$ for each γ_0, γ_1 smoothly homotopic if and only if ω is closed if and only if $\alpha = 1$.

b) What are the possible values of $\int_\gamma \omega$ where α is closed as in part (a)?

Since $\mathbb{R}^2 \setminus \{0\}$ deformation retracts to S^1, each loop γ is homotopic to $k \cdot S^1$, the loop which goes around S^1 k-times, for some $k \in \mathbb{Z}$. Thus, the possible values are

$$\int_\gamma \omega = \int_{k \cdot S^1} \omega = k \int_{S^1} \omega = k \int_0^{2\pi} d\theta = 2\pi k$$

for $k \in \mathbb{Z}$.

Problem 6: Let X and Y be connected CW complexes, and let $p : \tilde{X} \to X$ be a path connected covering space. Let $f : Y \to X$ be continuous. Let $f^*(\tilde{X}) = \{(y, \tilde{x}) : f(y) = p(\tilde{x})\} \subset Y \times \tilde{X}$, and consider the projection map $f^*(p) : f^*(\tilde{X}) \to Y$.

a) Show that $f^*(p)$ is a covering map.

> For notational convenience, write $\pi = f^*(p)$. Write $q : f^*(\tilde{X}) \to \tilde{X}$ with $q((y, \tilde{x})) = \tilde{x} \in X$. Note $f\pi((y, \tilde{x})) = f(y) = p(\tilde{x}) = pq((y, \tilde{x}))$, so that $f\pi = pq$.
>
> Let $y \in Y$ be arbitrary. Write $x = f(y)$. Pick an evenly covered neighborhood $U \ni x$ with $p^{-1}U = \sqcup_\alpha V_\alpha$ and $U_\alpha \cong U$ via $p_\alpha = p|_{U_\alpha} : U_\alpha \to U$.
>
> Since $f\pi = pq$, we have $\pi^{-1}f^{-1}U = q^{-1}p^{-1}U = \sqcup_\alpha q^{-1}U_\alpha$. Define open subsets $W_\alpha = q^{-1}U_\alpha \subset f^*(\tilde{X})$.
>
> Set $W = f^{-1}U$. Note $U \ni x = f(y)$, so $y \in W$. The above computation shows $\pi^{-1}W = \sqcup_\alpha W_\alpha$. Moreover, we even have $W \cong W_\alpha$ via $z \mapsto (z, p_\alpha^{-1}f(z))$. Note that since $z \in W$, $f(z) \in U$, so that $p_\alpha^{-1}f(z) \in U_\alpha$. Moreover, $f(z) = p(p_\alpha^{-1}f(z))$, so that $(z, p_\alpha^{-1}f(z)) \in f^*(\tilde{X})$, and $q((z, p_\alpha^{-1}f(z))) = p_\alpha^{-1}f(z) \in U_\alpha$, so that $(z, p_\alpha^{-1}f(z)) \in q^{-1}U_\alpha = W_\alpha$.
>
> Thus we have $W \to W_\alpha$ via $z \mapsto (z, p_\alpha^{-1}f(z))$, and $W_\alpha \xrightarrow{\pi} W$ via $(z, \tilde{x}) \mapsto z$. So we see these are inverses to one another, and $W_\alpha \cong W$ for each α.
>
> Thus for $y \in Y$ arbitrary, we have found $W \ni y$ with $\pi^{-1}W = \sqcup_\alpha W_\alpha$ and for each α, $W_\alpha \cong W$ via π. So W is an evenly covered neighborhood of Y, and π is a covering map, as desired.

b) Let $(y, \tilde{x}) \in f^*(\tilde{X})$ and let $x = f(y) = p(\tilde{x})$. If $f_*\pi_1(Y, y) \subset p_*\pi_1(\tilde{X}, \tilde{x})$, and the cover $p : \tilde{X} \to X$ is non-trivial, show that $f^*(\tilde{X})$ is disconnected.

> Skip!

Problem 7: Let $X = S^1 \times D^2$ with boundary $\partial X = S^1 \times S^1$. Compute $H_k(X, \partial X; \mathbb{Z})$ for all k.

Note D^2 deformation retracts to a point, so that $X = S^1 \times D^2$ is homotopy equivalent to S^1. Note $S^1 \subset D^2$ is a good pair, so $(X, \partial X)$ is also a good pair. From Hatcher 2.13, we get a long exact sequence of reduced homology groups

$$... \to \widetilde{H_i(\partial X)} \to \widetilde{H_i(X)} \to H_i(X, \partial X) \to ...$$

Note

$$\widetilde{H_i(\partial X)} = H_i(T \widetilde{= S^1} \times S^1) = \begin{cases} 0 & i = 0, i > 2 \\ \mathbb{Z}^2 & i = 1 \\ \mathbb{Z} & i = 2 \end{cases}$$

$$\widetilde{H_i(X)} = \widetilde{H_i(S^1)} = \begin{cases} 0 & i = 0 \text{ or } i > 1 \\ \mathbb{Z} & i = 1 \end{cases}$$

Moreover, notice the map $\partial X \hookrightarrow X$, i.e. the map $S^1 \times S^1 \to S^1 \times D^2$, is the map $a \mapsto a, b \mapsto 0$ on π_1 of these spaces, where a, b are the two generators of $\pi_1(\partial X = S^1 \times S^1)$, and a is the generator in $\pi_1(X \cong S^1)$. Since H_1 of both of these spaces is the same as their π_1, we see the map induced on H_1 is surjective.

For $i > 3$ since $\widetilde{H_i(X)} = H_{i-1}\widetilde{(\partial X)} = 0$, we see $H_i(X, \partial X) = 0$. Our long exact sequence becomes

$$0 \to H_3(X, \partial X) \to \mathbb{Z} \to 0 \to H_2(X, \partial X) \to \mathbb{Z}^2 \twoheadrightarrow \mathbb{Z} \to H_1(X, \partial X) \to 0 \to 0 \to H_0(X, \partial X) \to 0$$

So $H_3(X, \partial X) = \mathbb{Z}$, $H_0(X, \partial X) = 0$. The surjectivity of $\mathbb{Z}^2 \twoheadrightarrow \mathbb{Z}$ on H_1 gives $H_1(X, \partial X) = 0$ and a short exact sequence

$$0 \to H_2(X, \partial X) \to \mathbb{Z}^2 \to \mathbb{Z} \to 0$$

from which it becomes clear $H_2(X, \partial X)$ is a subgroup of a free abelian group and hence free. Rank counting then gives $H_2(X, \partial X) = \mathbb{Z}$. Thus

$$H_i(X, \partial X) = \begin{cases} 0 & i = 0, 1 \\ \mathbb{Z} & i = 2, 3 \\ 0 & i > 3 \end{cases}$$

Remark: From Lefshetz duality, since X is a compact connected orientable 3-manifold with boundary, $H_i(X, \partial X) = H^{3-i}(X) = H^{3-i}(S^1)$. Since the cohomology groups of S^1 are $H^k(S^1) = \mathbb{Z}$ for $k = 0, 1$ and 0 otherwise, we see this agrees with the above result.

Problem 8: Let X be a CW complex and let $\widetilde{X} \to X$ be a covering space. Let G be the group of deck transformations.

a) Show that for any k and any abelian group M, the group G naturally acts on $H_k(\widetilde{X}; M)$.

> Note G acts on $\tilde{X}$ via deck transformations, giving a homeomorphism $g : \tilde{X} \to \tilde{X}$ for each $g \in G$. These of course induce maps $g_* : H_k(\tilde{X}; M) \to H_k(\tilde{X}; M)$.

b) Show that the map $p_* : H_k(\widetilde{(X)}; M) \to H_k(X; M)$ factors through the quotient of $H_k(\widetilde{X}; M)$ by the subgroup S generated by $m - g \cdot m$ for all $m \in H_k(\widetilde{X}; M)$ and $g \in G$.

> Trivially, since $pg = p$ for any $g \in G$, we have $p_* g_* = p_*$. Thus, $p_*(g_* m - m) = 0$ for all $m \in M, g \in G$.

c) Give an example for which the induced map $H_k(\widetilde{X}; M)/S \to H_k(X; M)$ in (b) is not surjective.

> Take $p : \mathbb{R} \twoheadrightarrow S^1$ the standard covering map $p(t) = e^{it}$. For any abelian group M, by universal coefficient theorem $H_1(S; M) = H_1(S^1) \otimes M \oplus Tor(H_0(S^1), M) = (\mathbb{Z} \otimes M) \oplus Tor(\mathbb{Z}, M) = \mathbb{Z} \otimes M = M$, since Tor vanishes when one of the entries is free. Meanwhile, $H_1(\mathbb{R}; M) = 0$. Thus, $H_1(\mathbb{R}; M)/S \to H_1(S^1; M) = M$ is necessarily the zero map, so it is not surjective.

Problem 9:

a) Find the homology groups $H_k(\mathbb{RP}^2)$ for all k.

b) Describe a cell decomposition for $\mathbb{RP}^2 \times \mathbb{RP}^2$. Use this to show that $H_3(\mathbb{RP}^2 \times \mathbb{RP}^2)$ is non-trivial (without using Kunneth).

> See Spring 2011 Problem 8.

Problem 10: Let G be a finite group and X a smooth manifold on which G acts smoothly. If the action of G on X is free, then show that $X \to X/G$ is a covering map.

> See Spring 2012 Problem 9.

16 Fall 2017

Problem 1: Let M be a smooth manifold. For a 1-form ω, prove that $d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$.

It suffices to show this locally. Then, by linearity, it suffices to consider forms $\omega = f dg$, as all other 1-forms can be written locally as sums of such forms.

Then $d\omega = df \wedge dg$. Hence

$$d\omega(X,Y) = (df \wedge dg)(X,Y) = X(f)Y(g) - X(g)Y(f)$$

Meanwhile, $\omega(Y) = fY(g)$, $\omega(X) = fX(g)$, so that by product rule

$$X(\omega(Y)) = X(fY(g)) = X(f)Y(g) + fXY(g)$$

$$Y(\omega(X)) = Y(f)X(g) + fYX(g)$$

Subtracting these, we see

$$X(\omega(Y)) - Y(\omega(X)) = X(f)Y(g) - X(g)Y(f) + f[X,Y](g) = d\omega(X,Y) + [X,Y](\omega)$$

as desired.

Problem 2: Let $M_n(\mathbb{R})$ be the space of $n \times n$ matrices with real coefficients.

a) Show that $O(n)$ is a smooth submanifold of $M_n(\mathbb{R})$.

See Spring 2010 Problem 1.

b) Show that $O(n)$ has a trivial tangent bundle.

In fact, any Lie group G is parallelizable. Pick a basis $v_1, ..., v_k$ of T_eG (with $e \in G$ the identity). Define vector fields $X_i(g) = dg_e v_i$, where $g : G \to G$ is the multiplication by $g \in G$ map. Since g is a diffeomorphism, each $dg_e : T_eG \to T_gG$ is an isomorphism, so that $dg_e v_1, ..., dg_e v_k$ is a basis of T_gG. Hence, the $X_1, ..., X_k$ are k linearly independent vector fields on G, so that G is parallelizable, as desired.

Problem 3: Recall the Hopf fibration $\pi : S^3 \to S^2$ where $S^2 = \mathbb{CP}^1$ and $S^3 \subset \mathbb{C}^2$ is defined by $\pi(z_1, z_2) = [z_1 : z_2]$. There is another fibration $p : UTS^2 \to S^2$ called the unit tangent bundle, whose fiber over $x \in S^2$ consists of the unit tangent vectors in T_xS^2 (where we view it as a submanifold of $\mathbb{R}^3$ to measure length). Show that there is a covering map $f : S^3 \to UTS^2$ of degree 2 satisfying $p \circ f = \pi$.

SKIP!

Problem 4: Consider $\omega = xdy - ydx + dz$. Prove that $f\omega$ is not closed for any nowhere zero function $f : \mathbb{R}^3 \to \mathbb{R}$.

See Spring 2012 Problem 4.

Problem 5: Let x, y, z denote the standard Euclidean coordinates on $\mathbb{R}^3$, and let dA denote the standard area form on S^2. Determine the values of n for which $z^n dA$ is an exact 2-form on S^2.

Recall the standard area form on $S^n \overset{i}{\hookrightarrow} \mathbb{R}^{n+1}$ is $dA = i^* i_N dV$, where $N_p = p$, i.e. $N = \sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i}$, is the unit normal vector to the sphere, and $dV = dx_1 \wedge ... \wedge dx_{n+1}$ is the standard volume form on $\mathbb{R}^{n+1}$. A simple computation shows

$$\omega := i_N dV = \sum_{j=1}^{n+1} (-1)^{j-1} x_j dx_1 \wedge ... \wedge \widehat{dx_j} \wedge ... \wedge dx_{n+1}$$

so that $dA = i^*\omega$.

For $n = 2$, we see $\omega = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$, so $dA = i^*\omega$.

Taking $\eta = z^n\omega$, we see $z^n dA = i^*\eta$ is exact on S^2 if and only if $\int_{S^2} i^*\eta = 0$. By Stokes, we see

$$\int_{S^2} i^*\eta = \int_B d\eta$$

where B is the unit ball. Hence we see $i^*\eta$ is exact if and only if $\int_B d\eta = 0$. Meanwhile, $d\eta = nz^{n-1}dz \wedge \omega + z^n d\omega = nz^{n-1}(zdx \wedge dy) + z^n \cdot 3dV = (n+3)z^n dV$.

Now $\int_B d\eta = \int_B (n+3)z^n dV$. Note that if n is odd, $(n+3)z^n$ is an odd function, so that it integrates to zero over the unit ball (split into integrals over $z \geq 0$ and $z \leq 0$). Meanwhile, if n is even, $(n+3)z^n$ is a nonnegative function that is not identically zero on the ball, so that it must integrate to a positive value. Hence, we see $\int_B d\eta = 0$ if and only if n is odd. So $z^n dA = i^*\eta$ is exact if and only if n is odd.

Problem 6:

a) Define what it means for a manifold M to be orientable.

An n-manifold M is orientable if it admits an atlas with each transition function xy^{-1} being an orientation preserving map between open subsets of $\mathbb{R}^n$. For maps $f : U \to V$ between open subsets of $\mathbb{R}^n$, we say f is orientation preserving if $\det(df_p) > 0$ at each point $p \in U$.

b) Show that every non-orientable connected manifold M admits a connected, oriented double cover.

Define $\hat{M} = \{(p, O_p) : p \in M \text{ and } O_p \text{ is an orientation of } T_pM\}$.

We have a base for the topology on $\hat{M}$: for each ordered pair $(U, \mathcal{O})$, where $\mathcal{O}$ is an orientation of $U \subset M$ open, define $V_{(U,\mathcal{O})} = \{(p, O) : O = \mathcal{O}_p\} \subset \hat{M}$. These sets form a base for the topology on $\hat{M}$.

It is easy to check $\hat{M} \twoheadrightarrow M$ then gives a 2-sheeted covering space, which then naturally endows $\hat{M}$ with a smooth structure (as the projection is a local homeomorphism). To make $\hat{M}$ oriented, simply orient $T_{(p,\mathcal{O}_p)}\hat{M} \cong T_pM$ via $\mathcal{O}_p$, and $T_{(p,-\mathcal{O}_p)}\hat{M} \cong T_pM$ via $-\mathcal{O}_p$.

Next, notice each connected component of $\hat{M}$ is also a cover of M. Thus, $\hat{M}$ is either connected or two disjoint copies of M. Since each component of an oriented manifold is also oriented, if M is not orientable, we must have $\hat{M}$ is connected, as desired.

Problem 7: Let M be a smooth, compact, connected, orientable n-manifold (without boundary).

a) Show that if the Euler characteristic of M is 0, then M admits a nowhere vanishing vector field.

> See Fall 2016 Problem 6a.

b) If M_g is a surface of genus g, what is $\min_v(\#\text{ zeroes of } v)$, where v ranges over vector fields on M whose zeroes are isolated and have index ± 1? Give a proof.

> First, see Spring 2012 Problem 2. This gives us a construction of a vector field with $2+2g$ zeros. Of these, we have one sink, one source, and $2g$ saddles. Diffeomorph the sink, source and two saddles into an open set $U \subset M_g$ diffeomorphic to a unit ball in $\mathbb{R}^2$ so that the remaining zeroes are outside of $\overline{U}$. (See Spring 2017 Problem 1 for why we can do this).
>
> We would like to observe, as in , that the sum of the indices of the zeros of X inside $\overline{U}$ is zero, so that $\partial U \to S^2$ via $p \mapsto \frac{X_p}{|X_p|}$ has degree zero. Then it extends to a unit vector field on all of $\overline{U}$. The issue is that we cannot take a unit vector field on $\overline{U}^c$ as we did in that problem as there are zeros outside of U.
>
> To fix this, we refine our choice of U slightly. First, let $m > 0$ be the minimum value of $|X_p|$ on ∂U. We may WLOG assume one of the zeros, z, is at the center of the ball. Now for each point on the boundary, draw the radial line from that point to the center. Since $|X_p| \geq m$ and $|X_z| = 0$, by the intermediate value theorem, there is some point in between with $|X_q| = m$. Pick the q closest to p, call it $q(p)$. We do this for each $p \in \partial U$. At the end, we are left with a curve $q(p)$ around the origin. Call the region bounded inside V. Then V still contains the original four zeros, since we insisted on picking $q(p)$ closest to p, and having a zero outside of V would mean the zero was between p and $q(p)$ for some p, for which IVT would guarantee we can find an even closer point.
>
> Moreover, V is still simply connected and thus diffeomorphic to a unit ball again, by the Riemann mapping theorem. We now have the added benefit that on ∂V, X has constant norm $|X_q| = m$ for each $q \in \partial V$. (Each $q \in \partial V$ is $q(p)$ for some $p \in \partial U$). We now run through the above argument to get that X_q/m can be extended to a map $\overline{V} \to S^k$. Multiplying this map by the constant m, we see we may in fact extend X_q to $\overline{V}$, this time containing no zeros (and in fact having constant norm m on $\overline{V}$).
>
> In this way, we have a new vector field with four fewer zeros than the previous $2+2g$, so that we have $2g-2$ zeros. Meanwhile, since the sum of the indices of the zeros is $\chi(M_g) = 2-2g$, and each index is ± 1, we see we need at least $|2-2g| = 2g-2$ zeros. Thus, this is the minimum number of zeros possible.

Problem 8: Let $M = [0,1]^2/\sim$ where $(x,1) \sim (1-x,0)$ for all $x \in [0,1]$. Let $X = (M \times \{0,1\})/\sim$ where $(y,1) \sim (y,0)$ for $y \in \partial M$. Determine the fundamental group of X.

First, see Fall 2011 Problem 8b to see that M is the Mobius band and $X = K$ is the Klein-bottle. Using any polygon representation of K, e.g. $[0,1]^2/\sim$ via $(x,0) \sim (x,1)$ and $(0,y) \sim (1,1-y)$, or $[0,1]^2/\sim$ via $(x,0) \sim (1,x)$ and $(x,1) \sim (0,x)$, we see from Hatcher Proposition 1.26

$$\pi_1(K) = \langle a, b | abab^{-1} \rangle = \langle a, c | a^2c^2 \rangle$$

where a is the loop corresponding to the path from $(1,0)$ to $(1,1)$ in both cases, b is the loop from $(0,0)$ to $(1,0)$ in the first case, and c is the loop from $(1,1)$ to $(0,1)$ in the second case.

Problem 9: A compact surface (without boundary) of genus g, embedded in $\mathbb{R}^3$ in the standard way bounds a compact 3-dimensional region called a handlebody H. Let $X = H \times \{0,1,2\})/\sim$ where $(x,i) \sim (x,j)$ for all $x \in \partial H$ and $i,j \in \{0,1,2\}$. Compute the homology of X.

See Fall 2014 Problem 7.

Problem 10:

a) Let A be a single circle in $\mathbb{R}^3$. Compute $\pi_1(\mathbb{R}^3 \setminus A)$.

Let X be a bounded subset of $\mathbb{R}^3$. We can view $\mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ as a subset of its one point compactification, so that $\mathbb{R}^3 \setminus X \subset S^3 \setminus X$. We claim the induced map $\pi_1(\mathbb{R}^3 \setminus X) \to \pi_1(S^3 \setminus X)$ is an isomorphism. To see this, take $U = \mathbb{R}^3 \setminus X \subset S^3$ and V a neighborhood of $\infty \in S^3$, homeomorphic to an open ball, which is disjoint from X (this is possible since X is bounded). Then $U \cup V = S^3 \setminus X$, and $U \cap V = V \setminus \{\infty\} \cong \mathbb{R}^3 \setminus \{0\}$ (since $V \cong \mathbb{R}^3$ is homeomorphic to $\mathbb{R}^3$). By Fall 2012 Problem 3, $U \cap V$ is simply connected and $\pi_1(U \cap V) = 0$.

By Van Kampen, $\pi_1(S^3 \setminus X) = \pi_1(U \cup V) = \pi_1(U) * \pi_1(V)$, since $\pi_1(U \cap V) = 0$ says $\pi_1(U) * \pi_1(V) \twoheadrightarrow \pi_1(U \cup V)$ has no kernel. Meanwhile, $V \cong \mathbb{R}^3$, so $\pi_1(V) = 0$. Thus, $\pi_1(S^3 \setminus X) = \pi_1(U) = \pi_1(\mathbb{R}^3 \setminus X)$, as desired.

For the next part of the argument, set $X = A$ to be the circle. Notice for $S^3 \setminus A$, we can assume WLOG that $\infty \in A$. Then $S^3 \setminus A$ is exactly equal to $\mathbb{R}^3$ minus a line. WLOG, this line is the x-axis L_x. By a similar argument as Fall 2012 Problem 9, we see $\mathbb{R}^3 \setminus L_x$ deformation retracts to $S^2 \setminus \{p_1, p_2\}$. WLOG, taking $p_1 = \infty$, we see this is homeomorphic to $\mathbb{R}^2 \setminus \{p\}$. This deformation retracts to S^1. Hence, $\pi_1(\mathbb{R}^3 \setminus A) = \pi_1(S^3 \setminus A) = \pi_1(S^1) = \mathbb{Z}$.

b) Let A, B be disjoint circles in $\mathbb{R}^3$, supported in the upper and lower half space respectively. Compute $\pi_1(\mathbb{R}^3 \setminus (A \cup B))$.

WLOG, A is entirely contained in $\mathbb{R}^2 \times (1, \infty)$, and B is entirely contained in $\mathbb{R}^2 \times (-\infty, -1)$. Taking $U = (\mathbb{R}^2 \times (-\infty, 1)) \setminus B$ and $V = (\mathbb{R}^2 \times (-1, \infty)) \setminus A$, we see $U \cup V = \mathbb{R}^3 \setminus (A \cup B)$, $U \cong \mathbb{R}^3 \setminus A$, $V \cong \mathbb{R}^3 \setminus B \cong \mathbb{R}^3 \setminus A$, and $U \cap V = \mathbb{R}^2 \times (-1,1) \cong \mathbb{R}^3$. Thus, $\pi_1(U \cap V) = 0$ and by Van Kampen, $\pi_1(U \cup V) = \pi_1(U) * \pi_1(V) = \mathbb{Z} * \mathbb{Z}$.

c) If the circles become linked, how does the fundamental group change?

Notice by the remarks in part a that since $X = A \cup B$ is bounded, $\pi_1(\mathbb{R}^3 \setminus (A \cup B)) = \pi_1(S^3 \setminus (A \cup B))$. Since the circles are linked, $H = A \cup B \subset S^3$ is the Hopf-link. Now by Fall 2013 Problem 9 this will deformation retract to the torus, so that $\pi_1(\mathbb{R}^3 \setminus H) = \pi_1(S^3 \setminus H) = \pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$.

17 Spring 2018

Problem 1: Suppose that M, N are connected smooth mainfolds of the same dimension and $f : M \to N$ is a smooth submersion.

a) Prove that if M is compact, then f is onto and f is a covering map.

b) Give an example of a smooth submersion $\pi : M \to N$ such that M and N have the same dimension, N is compact, and π is onto, but π is not a covering map.

See Spring 2010 Problem 3. We need to modify the example in part b so that $\pi : \mathbb{R} \to S^1$ via $\pi(t) = e^{i \cdot f(t)}$ is surjective while still satisfying the other properties. We may adjust the example $t \mapsto e^{i \cdot \arctan(t)}$ ot the example $t \mapsto e^{i \cdot 3\arctan(t)}$. In this case, the map is surjective, but $(1, 0) \in S^1$ is only hit once while $(-1, 0)$ is hit twice. Alternatively, simply consider the restriction $(-3\pi/2, 3\pi/2) \to S^1$ via $t \mapsto e^{it}$.

Problem 2: Let $\Phi_N, \Phi_S : \mathbb{R} \times S^2 \to S^2$ be two global flows on the sphere S^2. Show that there is an $\epsilon > 0$ and a neighborhood U of the North pole, V of the South pole, and a global flow $\varphi : \mathbb{R} \times S^2 \to S^2$ such that $\Phi(t, q) = \Phi_N(t, q)$ for all $t \in (-\epsilon, \epsilon)$, $q \in U$ and $\Phi(t, q) = \Phi_S(t, q)$ for all $t \in (-\epsilon, \epsilon)$ and $q \in V$.

Definition: (Lee) A flow domain on a manifold M is an open subset $\mathcal{D} \subset \mathbb{R} \times M$ such that for fixed $p \in M$, $\{t \in \mathbb{R} : (t, p) \in \mathcal{D}\}$ is an open interval containing 0.

Let X, Y be the corresponding vector fields for Φ_N, Φ_S respectively. (That is, $X_p = \frac{d}{dt}|_{t=0}\Phi_N(t, p)$ and $Y_p = \frac{d}{dt}|_{t=0}\Phi_S(t, p)$.)

Pick neighborhoods $U' \ni N$ and $V' \ni S$ with disjoint closures. Pick a bump function $\psi : S^2 \to \mathbb{R}$ with $\psi \equiv 1$ on U' and $\psi \equiv 0$ on V'. Write $Z = \psi X + (1 - \psi)Y$. This is still a vector field as it is a linear combination of vector fields. Moreover, $Z|_{U'} = X|_{U'}$ and $Z|_{V'} = Y|_{V'}$.

Since S^2 is compact, Z induces a global flow $\Phi : \mathbb{R} \times S^2 \to S^2$.

Apply Lee's Theorem 9.12, the Fundamental Theorem on Flows, to get that since $Z|_{U'} = X|_{U'}$ is a vector field on U', there is a unique flow $\theta : \mathcal{D} \to U'$ of this vector field, where $\mathcal{D} \subset \mathbb{R} \times U'$ is a maximal flow domain on which such a flow can be defined. Meanwhile, notice $\Phi, \Phi_N : \mathbb{R} \times M \to M$ satisfy the same properties as the unique flow θ when restricted to $\mathcal{D}$. By uniqueness, $\Phi|_{\mathcal{D}} = \theta = (\Phi_N)|_{\mathcal{D}}$. Since $(0, N) \in \mathcal{D}$ and $\mathcal{D}$ is open, we may find some $\epsilon' > 0$ and some $U \ni N$ open neighborhood of N with $(0, N) \in (-\epsilon', \epsilon') \times U \subset \mathcal{D}$. Restricting further, we see $\Phi = \Phi_N$ on $(-\epsilon, \epsilon) \times U$. That is, $\Phi(t, q) = \Phi_N(t, q)$ for each $t \in (-\epsilon', \epsilon')$ and $q \in U$. A similar argument shows we can get some ϵ'' and some open neighborhood $V \ni S$ with $\Phi(t, q) = \Phi_S(t, q)$ for $t \in (-\epsilon'', \epsilon'')$ and $q \in V$. Taking $\epsilon = \min(\epsilon', \epsilon'')$ gives the desired result.

Problem 3: For $n \geq 1$, consider the subset $X \subset \mathbb{CP}^{2n}$ given by $X = \{[z_0 : \ldots : z_{2n}]\} \in \mathbb{CP}^{2n} : z_{n+1} = \ldots = z_{2n} = 0\}$.

a) Show that X is a smooth manifold.

Clearly $\mathbb{CP}^n \cong X$. By Lee Proposition 5.2, it suffices to construct an embedding (an injective proper immersion) $\phi : \mathbb{CP}^n \to \mathbb{CP}^{2n}$ whose image is precisely X. The desired map is $\phi([z_0 : ... : z_n]) = [z_0 : ... : z_n : 0 : ... : 0]$. It is clearly well-defined and injective, with image exactly X. The properness of the map follows for free from compactness of $\mathbb{CP}^n$. Thus it remains to check the map is an immersion. For this, pick $p \in \mathbb{CP}^{2n}$ arbitrary. WLOG, we may assume p has some homogenous coordinates $p = [z_0 : ... : z_n]$ with $z_0 \neq 0$ (this is true for some z_i, but the argument does not change regardless of i). Let $U_n \subset \mathbb{CP}^{2n}$ be the points $q \in \mathbb{CP}^{2n}$ with $q = [w_0 : ... : w_n]$ for some $w_i \in \mathbb{C}$ with $w_0 \neq 0$. Then $U_n \cong \mathbb{C}^n$ via the usual chart $[w_0 : ... : w_n] \mapsto \left(\frac{w_1}{w_0}, ..., \frac{w_n}{w_0}\right)$ with inverse $(x_1, ..., x_n) \mapsto [1 : x_1 : ... : x_n]$. Note $\phi(U_n) \subset U_{2n} \cong \mathbb{C}^{2n}$, i.e. U_n maps to points in $\mathbb{CP}^{2n}$ which also have a homogeneous representation $[z_0 : ... : z_{2n}]$ with $z_0 \neq 0$. We see that the map $\phi : U_n \to U_{2n}$ is, in coordinates, a map $\mathbb{C}^n \to \mathbb{C}^{2n}$ given by the composition $(x_1, ..., x_n) \mapsto [1 : x_1 : .... : x_n] \mapsto [1 : x_1 : ... : x_n : 0 : ... : 0] \mapsto (x_1, ..., x_n, 0, ..., 0)$. That is, it is just the linear injective map $\mathbb{C}^n \to \mathbb{C}^{2n}$ via $(x_1, ..., x_n) \to (x_1, ..., x_n, 0, ..., 0)$. The derivative of this map is itself and hence also injective. So ϕ is an immersion.

We conclude $X \cong \mathbb{CP}^n$ is an embedded submanifold of $\mathbb{CP}^{2n}$.

b) Calculate the mod 2 intersection number of X with itself.

Skip!

Problem 4: Suppose that N is a smoothly embedded submanifold of a smooth manifold M. A vector field on M is called tangent to N is $X_p \in T_pN \subset T_pM$ for all $p \in M$.

a) Show that if X and Y are vector fields on M both tangent to N, then $[X, Y]$ is also tangent to N.

Let $i : N \to M$ be the inclusion map. Here we view a vector field X on a manifold M as an $\mathbb{R}$-linear map $X : C^\infty(M) \to C^\infty(M)$ (via $X(f)(p) = X_p(f)$) satisfying $X(fg) = X(f)g + fX(g)$. Then note i gives a natural map $C^\infty(M) \xrightarrow{i_*} C^\infty(N)$ via $f \mapsto f \circ i$. We may view i_*X as the composition $i_* \circ X : C^\infty(M) \to C^\infty(N)$. Under this framework, we say X is tangent to N if we may find some $X' : C^\infty(N) \to C^\infty(N)$ a vector field on N such that $X' \circ i_* = i_* \circ X$.

Note that this framework also lets us view $[X, Y]$ as a map $[X, Y] : C^\infty(M) \to C^\infty(M)$ given by $[X, Y] = X \circ Y - Y \circ X$. It is an easy exercise that $[X, Y]$ is a vector field (i.e. is $\mathbb{R}$-linear and satisfies product rule).

Since X, Y are vector fields on M tangent to N, we may find vector fields X', Y' on N with $X' \circ i_* = i_* \circ X$ and $Y' \circ i_* = i_* \circ Y$. Notice

$$[X', Y'] \circ i_* = (X' \circ Y' - Y' \circ X') \circ i_* = X' \circ Y' \circ i_* - Y' \circ X' \circ i_*$$

$$= X' \circ i_* \circ Y - Y' \circ i_* \circ X = i_* \circ X \circ Y - i_* \circ Y \circ X = i_* \circ (X \circ Y - Y \circ X) = i_* \circ [X, Y]$$

Thus, the vector field $Z = [X, Y]$ on M is tangent to N, as we have the vector field $Z' = [X', Y']$ on M with $Z' \circ i_* = i_* \circ Z$.

b) Illustrate this principle by choosing two vector fields X, Y tangent to $S^2 \subset \mathbb{R}^3$, computing $[X, Y]$, and checking that it's tangent to S^2.

Take $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ and $Y = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}$. Note X, Y are both tangent to S^2. To see this, note we may view $T_pS^2 \subset T_p\mathbb{R}^3 = \mathbb{R}^3$ as the orthogonal complement of $\operatorname{span}(p) \subset \mathbb{R}^3$. Note $(-y, x, 0)$ and $(-z, 0, x)$ are both orthogonal to (x, y, z) for any x, y, z, so that both X and Y are tangent to S^2. Now

$$[X, Y](x) = XY(x) - YX(x) = X(-z) - Y(-y) = 0 - 0 = 0$$

$$[X, Y](y) = XY(y) - YX(y) = X(0) - Y(x) = 0 - (-z) = z$$

$$[X, Y](z) = XY(z) - YX(z) = X(x) - Y(0) = -y - 0 = -y$$

Hence, $[X, Y] = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$. Notice $(0, z, -y)$ is everywhere orthogonal to (x, y, z), so that $[X, Y]$ is also tangent to S^2, as desired.

Problem 5: A symplectic form on an eight diemsional manifold M is defined to be a closed 2 form ω such that $\omega \wedge \omega \wedge \omega \wedge \omega$ is a volume form. Determine which of the following manifolds admits symplectic forms: $S^8, S^2 \times S^{,}S^2 \times S^2 \times S^2$.

Note $H^2_{dR}(S^8) = 0$, so that if ω is a closed 2-form, $[\omega] = 0$. Then certainly $[\omega \wedge \omega \wedge \omega \wedge \omega] = 0 \in H^8_{dR}(S^8)$ is not a volume form.

Similarly, by Kunneth, $H^4_{dR}(S^2 \times S^6) = 0$, so that $[\omega \wedge \omega] = 0$ for any 2-form ω, and so $[\omega \wedge \omega \wedge \omega \wedge \omega] = 0$ is not a volume form.

Finally, let η be a volume form on S^2. Writing $\pi_i : S^2 \times S^2 \times S^2 \times S^2$ as projection onto the ith coordinate, we have $\pi_1^*\eta \wedge \pi_2^*\eta \wedge \pi_3^*\eta \wedge \pi_4^*\eta$ is a volume form on $S^2 \times S^2 \times S^2 \times S^2$ via Kunneth, since $H^8_{dR}(S^2 \times S^2 \times S^2 \times S^2) \cong H^2_{dR}(S^2) \otimes H^2_{dR}(S^2) \otimes H^2_{dR}(S^2) \otimes H^2_{dR}(S^2) \cong \mathbb{R}$.

Take $\omega = \pi_1^*\eta + \pi_2^*\eta + \pi_3^*\eta + \pi_4^*\eta$. We make the simple observation that $\pi_i^*\eta \wedge \pi_i^*\eta = \pi_i^*(\eta \wedge \eta) = 0$, since $\eta \wedge \eta$ is a 4-form on S^2. Then a simple computation shows

$$\omega \wedge \omega \wedge \omega \wedge \omega = 24 \cdot \pi_1^*\eta \wedge \pi_2^*\eta \wedge \pi_3^*\eta \wedge \pi_4^*\eta$$

which is a volume form, as desired.

Problem 6: Let U be a bounded open set in $\mathbb{R}^3$ with smooth boundary, and let V be a smooth vector field on $\mathbb{R}^3$. The classical divergence theorem expresses the triple integral $\iiint_U \operatorname{div}(V)\, d(\mathrm{vol})$ as a surface integral over the boundary of U. State this theorem, and show how it can be obtained as a particular case of Stokes' Theorem for differential forms.

See Fall 2018 Problem 5.

Problem 7: Let M and N be smooth, connected, orientable n-manifolds for $n \geq 3$.

a) Compute the fundamental group of $M\#N$ in terms of M and N (assume the basepoint is on the boundary of the sphere where M and N are glued).

We assume in addition that M, N are closed manifolds.

To construct the connect sum, we take open sets $U \subset M$ and $V \subset N$ diffeomorphic to the unit ball in $\mathbb{R}^n$, and glue their boundaries $\partial U \cong S^{n-1}$ and $\partial V \cong S^{n-1}$ via some diffeomorphism of S^{n-1}.

Take $A = M \setminus \overline{U}$ and $B = U'$ some larger ball. Then $A \cap B \cong S^{n-1}$, and $A \cup B = M$. Since $n > 2$, $\pi_1(A \cap B) = \pi_1(S^{n-1}) = 0$. Since B is contractible, $\pi_1(B) = 0$. Thus, $\pi_1(A) \cong \pi_1(A \cup B)$ by Van Kampen, so that $\pi_1(M \setminus \overline{U}) = \pi_1(M)$. Similarly, $\pi_1(N \setminus \overline{V}) = \pi_1(N)$. A manifold is homotopy equivalent to its interior, so that this also gives us $\pi_1(M \setminus U) = \pi_1(M)$ and $\pi_1(N \setminus V) = \pi_1(N)$. Alternatively, just notice deleting an open ball from $\mathbb{R}^n$ and deleting a closed ball from $\mathbb{R}^n$ are homotopy equivalent since they both deformation retract to a sphere. In fact, if we only deformation retract things inside the sphere and keep things outside the sphere fixed, we see one of these spaces deformation retracts to the other. Since this is locally what our picture looks like, we get $M \setminus \overline{U}$ deformation retracts to $M \setminus U'$ for some slightly bigger open set U', and similar for $N \setminus \overline{V}$. In any case, we have $\pi_1(M \setminus U) = \pi_1(M \setminus U') = \pi_1(M)$ and $\pi_1(N \setminus V) = \pi_1(N)$.

Next, in $M\#N$, take $A = M \cup U_N$ and $B = N \cup U_M$, where $U_N \subset N, U_M \subset M$ are collar neighborhoods of the boundary of $N \setminus V$ and $M \setminus U$ respectively. Then $A \cong M \setminus U$, $B \cong N \setminus V$, $A \cup B = M\#N$, and $A \cap B = U_M \cap U_N \cong S^{n-1}$. Again we have $\pi_1(A \cap B) = 0$, so that Van Kampen gives $\pi_1(A \cup B) = \pi_1(A) * \pi_1(B)$. So $\pi_1(M\#N) = \pi_1(M) * \pi_1(N)$, as desired.

b) Compute the homology groups of $M\#N$.

Note for the open unit ball B, $(\mathbb{R}^n \setminus B)/S^{n-1} \cong \mathbb{R}^n$, since we just glue the boundary $\partial\overline{B} = S^{n-1}$ to a point. Thus, $(M \setminus U)/(\partial\overline{U}) \cong M$ and $(N \setminus V)/(\partial\overline{V}) \cong N$. Hence, if we take $X = \partial\overline{U} = \partial\overline{V} \cong S^{n-1} \subset M\#N$, then $M\#N/X \cong M \vee N$ is a wedge sum. Moreover, note $(M\#N, X)$ is a good pair via $U_N \cup U_M$ from the previous part. Thus we get from the LES for relative homology:

$$\dots \to \widetilde{H}_k(S^{n-1}) \to \widetilde{H}_k(M\#N) \to \widetilde{H}_k(M \vee N) \to \widetilde{H}_{k-1}(S^{n-1}) \to \dots$$

For $k \neq n-1$, we have $\widetilde{H}_k(S^{n-1}) = 0$. Thus for $k \neq n-1, n-2$, we immediately get $\widetilde{H}_k(M\#N) = \widetilde{H}_k(M \vee N) = \widetilde{H}_k(M) \oplus \widetilde{H}_k(N)$. Then we have an LES

$$0 \to \widetilde{H}_n(M\#N) \to \widetilde{H}_n(M \vee N) \to \mathbb{Z} \to \widetilde{H}_{n-1}(M\#N) \to \widetilde{H}_{n-1}(M \vee N) \to 0$$

If M, N are closed connected orientable, then $H_n(M) = H_n(N) = \mathbb{Z}$, and $M\#N$ is also closed connected orientable, so that $H_n(M\#N) = \mathbb{Z}$. By Hatcher Corollary 3.28, for closed connected orientable n-manifolds, H_{n-1} is free. Thus, $H_{n-1}(M), H_{n-1}(N), H_{n-1}(M\#N)$ are all free. Moreover, $H_{n-1}(M \vee N) = H_{n-1}(M) \oplus H_{n-1}(N)$ is also free. We have an LES

$$0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to \widetilde{H}_{n-1}(M\#N) \to \widetilde{H}_{n-1}(M \vee N) \to 0$$

so that rank counting gives $\widetilde{H}_{n-1}(M\#N) = \widetilde{H}_{n-1}(M \vee N)$. So we see

$$\widetilde{H}_k(M\#N) = \begin{cases} \widetilde{H}_k(M \vee N) & 0 \leq k < n \\ \mathbb{Z} & k = n \\ 0 & k > n \end{cases} = \begin{cases} \widetilde{H}_k(M) \oplus \widetilde{H}_k(N) & 0 \leq k < n \\ \mathbb{Z} & k = n \\ 0 & k > n \end{cases}$$

c) For part (a), what changes if $n = 2$? Use this to describe the fundamental groups of orientable surfaces.

Let $M = M_g$ and $N = M_{g'}$ be genus g and g' orientable surfaces. Then from the polygon construction, we see $M \setminus U$ deformation retracts to the boundary of the polygon, which is a wedge of $2g$-circles generated by $a_1, ..., a_g, b_1, ..., b_g$. Similarly, $N \setminus V$ deformation retracts to a wedge of $2g'$ circles generated by $c_1, ..., c_{g'}, d_1, ..., d_{g'}$. Taking the same sets as in part a, we see $\pi_1(A \cap B) \to \pi_1(A)$, i.e. the map $\pi_1(S^1) \to \pi_1(M \setminus U) \cong \pi_1(S^1 \vee ... \vee S^1)$ just maps the loop to the boundary of the polygon, which is the loop corresponding to the boundary word. Thus, this map sends $1 \in \pi_1(S^1) = \mathbb{Z}$ to $[a_1, b_1] \cdot [a_2, b_2] \cdot ... \cdot [a_g, b_g]$. Meanwhile, if we orient the boundary of N in the opposite way, then $1 \in \pi_1(S^1)$ maps to $([c_1, d_1] \cdot ... \cdot [c_{g'}, d_{g'}])^{-1}$. Then Van Kampen tells us $\pi_1(M\#N)$ is the free product of $\pi_1(M \setminus U) = \langle a_1, ..., a_g, b_1, ..., b_g \rangle$ and $\pi_1(N \setminus V) = \langle c_1, ..., c_{g'}, d_1, ..., d_{g'} \rangle$ modulo setting the above two images of 1 equal. That is,

$$\pi_1(M\#N) = \langle a_1, ..., a_g, c_1, ..., c_{g'}, b_1, ..., b_g, d_1, ..., d_{g'} | [a_1, b_1] \cdot ... \cdot [a_g, b_g] \cdot [c_1, d_1] \cdot ... \cdot [c_{g'}, d_{g'}] \rangle$$

which is exactly the fundamental group of the genus $g + g'$ surface.

Problem 8: Determine all of the possible degrees of maps $S^2 \to S^1 \times S^1$.

See Spring 2010 Problem 10c. There is also a more direct way to do this: see that any such map lifts to a map to the universal cover, giving $S^2 \to \mathbb{R}^2 \to S^1 \times S^1$. The first of these is nullhomotopic since $\mathbb{R}^2$ is contractible; composing that homotopy with the projection $R^2 \to S^1 \times S^1$ gives a nullhomotopy for the original map.

Problem 9: Point S^2 via the south pole, and consider $S^2 \times S^2$.

a) Describe a cell structure on $S^2 \times S^2$ that is compatible with the inclusion $S^2 \vee S^2 \to S^2 \times S^2$ as those pairs where one coordinates is the south pole.

Give S^2 the cell structure with a 0-cell at the south pole, e_0, along with a 2-cell e_2, with $\partial e_2 = 0$. Similarly, the second copy of S^2 can be given a 0-cell f_0 and the 2-cell f_2, with $\partial f_2 = 0$. In this way, $S^2 \times S^2$ has cells $e_0 \times f_0, e_0 \times f_2, e_2 \times f_0$ and $e_2 \times f_2$. Each of these cells has boundary 0 since there are no cells of adjacent dimension.

Meanwhile, $S^2 \vee S^2$, if we are to glue the south poles together, gives us cells $e_0 = f_0$, e_2, f_2.

The inclusion map $S^2 \vee S^2$ into $S^2 \times S^2$ can be seen as follows: we have a map $S^2 \hookrightarrow S^2 \times S^2$ via $x \mapsto (x, f_0)$, and similarly a map $S^2 \hookrightarrow S^2 \times S^2$ via $x \mapsto (e_0, x)$. Here e_0, f_0 denote the south poles in the respective spheres. Then notice that these maps sends $e_0 \mapsto (e_0, f_0)$ and $f_0 \mapsto (e_0, f_0)$ (and this is the only shared point in the images), so that it factors through to a map $S^2 \vee S^2 \hookrightarrow S^2 \times S^2$.

It respects the cells, since $e_0 = f_0$ maps to $e_0 \times f_0$, e_2 maps to $e_2 \times f_0$, and f_2 maps to $e_0 \times f_2$. Thus we can see $S^2 \vee S^2 \subset S^2 \times S^2$ can be recognized as the subcomplex of cells $e_0 \times f_0, e_0 \times f_2$ and $e_2 \times f_0$. Thus this is the desired cell structure.

b) Let X be $(S^2 \times S^2) \cup_{S^2} D^3$, where we attach the 3-disk via the map $S^2 \xrightarrow{\phi} S^2 \vee S^2$ which crushes a great circle connecting the north and south poles. Compute the homology groups of X.

> This space has the cell structure of $S^2 \times S^2$ with an extra 3-cell F attached. Notice the boundary of the 3-cell is a $\mathbb{Z}$-linear combination of the 2-cells $e_2 \times f_0$ and $e_0 \times f_2$. By the cellular boundary formula, to compute the coefficient of ∂F in $e_2 \times f_0$, we need the degree of $S^2 \xrightarrow{\phi} S^2 \vee S^2 \to S^2$ where the second map quotients by the remaining cells. In other words, it quotients by $e_0 \times f_2$, which is the face of the second sphere. This leaves behind just one sphere. Note that this map has degree ± 1, since it is a local homeomorphism: the preimage of any point except for the wedge point is size 1, and it has a neighborhood which whose preimage is homeomorphic. Thus, we see that ∂F has a degree of ± 1 for $e_2 \times f_0$. A similar argument shows ∂F has a degree of ± 1 for $e_0 \times f_2$. We can choose an isomorphism $C_2 \cong \mathbb{Z}^2$ which ensures this is $\partial F = (1, -1)$ by just picking the two generators of C_2 as $\pm e_2 \times f_0$ and $\pm e_0 \times f_2$ so that the signs work out. Then we have a cell complex
>
> $$0 \to C_4 = \mathbb{Z} \xrightarrow{0} C_3 = \mathbb{Z} \xrightarrow{\partial_3} C_2 = \mathbb{Z}^2 \to C_1 = 0 \to C_0 = \mathbb{Z} \to 0$$
>
> where all maps except ∂_3 are zero since prior to adding the 3-cell F, everything had boundary zero. From this we see $H_4(X) = \mathbb{Z}, H_3(X) = \ker(\partial_3) = 0$, $H_2(X) = \mathbb{Z}^2/\langle(1,1)\rangle \cong \mathbb{Z}$ via the augmentation map $(x, y) \mapsto x + y$, $H_1(X) = 0, H_0(X) \cong \mathbb{Z}$. Hence
>
> $$H_i(X) = \begin{cases} \mathbb{Z} & i = 0, 2, 4 \\ 0 & \text{otherwise} \end{cases}$$

Problem 10: Let X be a semi-locally simply connected space, and let $p : \tilde{X} \to X$ be the universal cover.

a) Show that any map $\sigma : \Delta^n \to X$ lifts to a map $\tilde{\sigma} : \Delta^n \to \tilde{X}$ where Δ^n is the standard n-simplex.

> Since Δ^n is convex, it is contractible, so that $\pi_1(\Delta^n) = 0$. Hence this map satisfies the lifting criteria $\sigma_* \pi_1(\Delta^n) = 0 \subset p_* \pi_1(\tilde{X}) = 0$.

b) Show that if $f_1, f_2 : \Delta^n \to \tilde{X}$ are two lifts of σ, then there is an element g of the fundamental group of X such that $g \circ \tilde{\sigma}_1 = \tilde{\sigma}_2$, where g is viewed as an automorphism of $\tilde{X}$ via the deck transformations.

> By Hatcher Proposition 1.34, since Δ^n is connected, any two lifts which agree at a point are equal. Let $0 \in \Delta^n$ be a point. Define $x = \sigma(0)$. Note $p(f_1(0)) = p(f_2(0)) = \sigma(0) = x$. Thus $f_1(0), f_2(0)$ are both in $p^{-1}x$. Note $\tilde{X}$ is the universal cover, so that in particular, the group of deck transformations $\pi_1(X)$ acts transitively on the fibers. Hence we may find $g \in \pi_1(X)$ with $g.f_1(0) = f_2(0)$.
>
> Since $p \circ g = p$, notice $g.f_1 : \Delta^n \to \tilde{X}$ is another lift of σ, this time with $g.f_1(0) = f_2(0)$. Since these lifts agree at a point, they are equal, so that $g.f_1 = f_2$, as desired.
>
> *Remark:* The proof of the proposition just follows from showing the set of points where two lifts agree is clopen. It is clearly closed, and openness follows from the fact that $p^{-1}U$ is a disjoint union of open sets, so that the image from a connected domain can only land in one of them.

18 Fall 2018

Problem 1: Let M be a compact smooth n-manifold, and $f : M \to \mathbb{R}^n$ a smooth map. Let $S = \{p \in M : \text{rank}(df_p) < n\}$.

a) Prove that $S \neq \emptyset$.

> Suppose S were empty. Then f has full rank everywhere, so that f is a local diffeomorphism. In particular, it is an open map, so that $f(M)$ is open and compact in $\mathbb{R}^n$. Thus we get a contradiction.
>
> *Remark:* More generally, submersions are open, so we can run this argument for $f : M \to \mathbb{R}^k$ with $k \leq \dim M$.
>
> *Alternative Solution*: We need to add the assumption that M is without boundary.
>
> It suffices to show every map $g : M \to \mathbb{R}$ has a critical point, since if $p \in M$ is a critical point of $\pi_1 \circ f : M \to \mathbb{R}$, then $d(\pi_1 \circ f)_p = 0$, so that $(d\pi_1)_{f(p)} df_p = 0$. Since π_1 is linear, this gives $\pi_1 df_p = 0$. Transposing, we get $df_p^T \pi_1^T = 0$, so that $A^T = df_p^T$ has a nontrivial kernel, and so $A = df_p$ is not invertible. (Note A is square). Thus $\text{rank}(df_p) < n$ and p is a critical point of f.
>
> To see why every map $g : M \to \mathbb{R}$ has a critical point, note $g(M)$ is compact, so that g has a global max. At the global max $g(p)$, we have $dg_p = 0$ by a standard analysis argument, so that p is a critical point, as desired.

b) Prove that $f(S) \subset \mathbb{R}^n$ has empty interior.

> The set of critical values has measure 0, so cannot contain any open sets (which have positive measure).

Problem 2: Let M_n be the space of $n \times n$ real matrices, viewed as the smooth manifold $\mathbb{R}^{n^2}$. Let M_n^k be the subset of rank k matrices. Show that M_n^k is a smooth submanifold of M_n.

> See Spring 2013 Problem 1b.

Problem 3: Let θ be the restriction of $(x_2dx_1 - x_1dx_2) + \ldots + (x_{2n}dx_{2n-1} - x_{2n-1}dx_{2n})$ to the unit sphere S^{2n-1}. Prove that $\ker\theta$ is a distribution on S^{2n-1}. Is it integrable?

For $\ker\theta$ to be a distribution, we either need $\theta \equiv 0$ or θ is nonvanishing, as otherwise, the pointwise dimension of $\ker(\theta)$ will not be constant. View $S^{2n-1} \subset \mathbb{R}^{2n} = \mathbb{C}^n$, and let $X_p = ip$ be a vector field on $\mathbb{R}^{2n}$. Since $p \perp ip$, X restricts to a nonvanishing vector field on the sphere. From the coordinate expression $X = \sum_{i=1}^{2n} x_{2i}\frac{\partial}{\partial x_{2i-1}} - x_{2i-1}\frac{\partial}{\partial x_{2i}}$, it is clear

$$\theta(X) = \sum_{i=1}^{2n} x_i^2 = 1$$

at all points on the sphere. Thus, θ is nonvanishing.

Meanwhile, $d\omega = \sum_{i=1}^{n} -2dx_{2i-1} \wedge dx_{2i}$, so that

$$\omega \wedge d\omega = \sum_{i=1}^{n}\sum_{j\neq i} -2x_{2i}dx_{2i-1} \wedge dx_{2j-1} \wedge dx_{2j} + 2x_{2i-1}dx_{2i} \wedge dx_{2j-1} \wedge dx_{2j}$$

Note that this gives us an expansion of the 3-form $\omega \wedge d\omega$ in coefficients of the basis, so that we see $\omega \wedge d\omega = 0$ if and only if all coefficients are zero. This amounts to saying $x_1 = x_2 = \ldots = x_{2n} = 0$. So we see on the sphere that $\omega \wedge d\omega \neq 0$. Thus, $\ker(\theta)$ is not integrable by Spring 2015 Problem 5d.

Problem 4: Let M be a compact smooth 3-manifold and let $\omega \in \Omega^1(M)$ be a nowhere zero 1-form, so that $\ker(\omega)$ is an integrable distribution. Prove the following:

a) $\omega \wedge d\omega = 0$.

See Fall 2013 Problem 5.

b) There is a 1-form α with $d\omega = \alpha \wedge \omega$.

First we show we can find such an α locally (i.e. on some open set of a given point). Then we may glue together such α via a partition of unity to get a globally defined form.

By linearity, since every form locally may be written as sums of terms of the form fdz (for local coordinates x, y, z), it suffices to check the case $\omega = fdz$ on this open set. Since ω is nonvanishing, f is never zero. Note on this set $d\omega = f_x dx \wedge dz + f_y dy \wedge dz$. Take $\alpha = \frac{f_x}{f}dx + \frac{f_y}{f}dy$ defined on this open set. Then $\alpha \wedge \omega = d\omega$ on this open set, as desired. By the above remarks, we may extend α to a globally defined form with this property.

c) $d\alpha \wedge \omega = 0$.

Note $0 = d(d\omega) = d(\alpha \wedge \omega) = d\alpha \wedge \omega - \alpha \wedge d\omega = d\alpha \wedge \omega - \alpha \wedge \alpha \wedge \omega = d\alpha \wedge \omega$, where $\alpha \wedge \alpha = 0$ since these are 1-forms. Thus $d\alpha \wedge \omega = 0$ as desired.

Problem 5: Let $M \subset \mathbb{R}^n$ be a compact $n-1$ submanifold, and let $D \subset \mathbb{R}^n$ with $\partial D = M$. Let $dV = dx_1 \wedge \ldots \wedge dx_n$ be the standard volume form on $\mathbb{R}^n$.

a) Define $dA \in \Omega^{n-1}(M)$, the standard volume form on M induced by the embedding i.

Let N be the outward pointing unit normal along M. Then $dA = i^*(i_N dV)$.

b) Prove that $i^*(i_X dV) = \langle X, N\rangle dA$ for any smooth vector field X on $\mathbb{R}^n$ (here N is the unit normal vector field along M, point outward from D).

Write $T = X - \langle X, N\rangle N$, so that T is tangent to M. Then notice $i^*(i_T dV)_p(Y_1, ..., Y_{n-1}) = (i_T dV)_p(di_p Y_1, ..., di_p Y_{n-1}) = dV_p(T, di_p Y_1, ..., di_p Y_{n-1}) = 0$, where we notice T and $di_p Y_1, ..., di_p Y_{n-1}$ are n vectors in $T_p M$ which is $(n-1)$-dimensional, so that they are linearly dependent. Hence

$$i^*(i_X dV) - i^*(i_{\langle X,N\rangle N} dV) = i^*(i_{X-\langle X,N\rangle N} dV) = i^*(i_T dV) = 0$$

Meanwhile, $i^*(i_{\langle X,N\rangle N} dV) = \langle X, N\rangle i^*(i_N dV) = \langle X, N\rangle dA$. Thus, we see $i^*(i_X dV) = \langle X, N\rangle dA$, as desired.

c) Prove that $\int_D L_X(dV) = \int_M \langle X, N\rangle dA$.

We have

$$\int_M \langle X, N\rangle dA = \int_{\partial D} i^*(i_X dV) = \int_D d(i_X dV) = \int_D (\mathcal{L}_X - i_X d) dV = \int_D \mathcal{L}_X dV$$

where we use $\mathcal{L}_X = i_X d + d i_X$, and the fact that $(i_X d)(dV) = i_X d^2 V = 0$.

d) Derive Gauss' Divergence Theorem from the $n = 3$ case.

Write $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$. Then

$$\mathcal{L}_X dV = \mathcal{L}_X(dx \wedge dy \wedge dz) = \mathcal{L}_X(dx) \wedge dy \wedge dz + dx \wedge \mathcal{L}_X(dy) \wedge dz + dx \wedge dy \wedge \mathcal{L}_X(dz)$$

$$= d(\mathcal{L}_X(x)) \wedge dy \wedge dz - d(\mathcal{L}_X(y)) \wedge dx \wedge dz + d(\mathcal{L}_X(z)) \wedge dx \wedge dy$$

Note $\mathcal{L}_X(f) = X(f)$ for a function f, so that $\mathcal{L}_X(x) = P$, $\mathcal{L}_X(y) = Q$, $\mathcal{L}_X(z) = R$. Then

$$\mathcal{L}_X(dV) = dP \wedge dy \wedge dz - dQ \wedge dx \wedge dz + dR \wedge dx \wedge dy$$

$$= P_x dx \wedge dy \wedge dz - Q_y dy \wedge dx \wedge dz + R_z dz \wedge dx \wedge dy = (P_x + Q_y + R_z)dV = div(X)dV$$

Hence by the previous part, we have

$$\int_D div(X) dV = \int_{\partial D} \langle X, N\rangle dA$$

as desired.

Problem 6: Can a finite rank free group have a finite index subgroup of smaller rank?

Solution: Let X be a wedge of n circles, whose corresponding oriented loops are labeled $a_1, ..., a_n$. Then $\pi_1(X) = \langle a_1, ..., a_n \rangle$ is a free group on n generators. An index k-subgroup of $\pi_1(X)$ corresponds to a connected k-fold covering space Y of X, with $\pi_1(Y)$ isomorphic to the corresponding subgroup. Note that k-fold covering spaces of X correspond to connected graphs on k vertices, such that at each vertex, we have one outgoing edge a_i and one incoming edge a_i for each i. Thus these are connected graphs with kn edges.

Hence we have Y is a connected graph on k vertices with kn edges. For a simple connected graph with v vertices and e edges, we may find a spanning tree with $e-1$ edges, so that when this contracts to a point, we are left with a wedge of $e-(v-1) = e-v+1$ circles. When the graph is not simple, we may make it simple: for each loop, add a vertex in the middle of the loop. Now the number of edges and the number of vertices has gone up by one, $e-v+1$ is invariant, there are no more loops and this is still homeomorphic to the original graph. Next, for every edge, add a vertex in between, so that there are no multiedges, $e-v+1$ is invariant and this is still homeomorphic to the original graph. From this we see Y deformation retracts to a wedge of $kn-k+1$ circles, so that $\pi_1(Y)$ is free on $kn-k+1$ generators.

Thus we conclude an index k subgroup of a rank n free group $kn-k+1$. However, we have $n \geq 1 \Rightarrow (k-1)n \geq k-1 \Rightarrow kn-n \geq k-1 \Rightarrow kn-k+1 \geq n$. Thus a finite index subgroup will always have smaller rank. *Alternative Solution*: An alternative way to compute the rank of $\pi_1(Y)$ is to notice $\chi(Y) = k \cdot \chi(X) = k \cdot (1-n)$ since Y is a k-fold cover of X. Meanwhile $\chi(Y) = \text{rank}(H_0(Y)) - \text{rank}(H_1(Y)) = 1 - \text{rank}(H_1(Y))$. Since $\pi_1(Y)$ is free of finite rank, $H_1(Y)$ is free abelian of the same rank, so that we have $\chi(Y) = 1 - \text{rank}(\pi_1(Y))$. From this we see $\pi_1(Y)$ has rank $1-k(1-n) = 1-k+kn = kn-k+1$. The above argument then follows.

Problem 7: Prove that the covering map $\pi : S^n \to \mathbb{RP}^n$ induces an isomorphism on de Rham cohomology if and only if n is odd. What is the orientation double cover of $\mathbb{RP}^n$?

Recall from Spring 2011 Problem 8 the homology of $\mathbb{RP}^n$. Applying universal coefficient and de Rham's Theorem, we can obtain the de Rham cohomology of $\mathbb{RP}^n$ as

$$H^k_{dR}(\mathbb{RP}^n) = \begin{cases} \mathbb{R} & k = 0 \\ \mathbb{R} & k = n \text{ and } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

Of course,

$$H^k_{dR}(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

Meanwhile, by the proposition in Spring 2012 Problem 9c, π^* is injective. By dimensionality, we see $\pi^* : H^k_{dR}(\mathbb{RP}^n) \to H^k_{dR}(S^n)$ is an isomorphism for all k with the possible exception of $k = n$. In this case, we see it is an isomorphism if and only if n is odd. Hence, π^* is an isomorphism if and only if n is odd.

An oriented connected double cover of a nonorientable manifold which has an orientation reversing deck transformation must be the orientation double cover. For n even, Since $\pi : S^n \twoheadrightarrow \mathbb{RP}^n$ has the deck transformation $x \mapsto -x$ which has degree $(-1)^{n+1} = -1$ which is orientation reversing, we see S^n is the orientation double cover of $\mathbb{RP}^n$ in this case. Otherwise, if n is odd, $\mathbb{RP}^n$ is orientable, so that its orientation double cover is $\mathbb{RP}^n \sqcup \mathbb{RP}^n$.

Problem 8: Assume that the integral homology of a space is $\mathbb{Z}$ in grading 0, $\mathbb{Z}$ in grading 2, $\mathbb{Z}/2\mathbb{Z}$ in grading 3, and 0 otherwise.

a) What are the integral cohomology groups?

By Universal Coefficients, we have

$$H^i(X) = \mathrm{Hom}_{\mathbb{Z}}(H_i(X), \mathbb{Z}) \oplus Ext(H_{i-1}(X), \mathbb{Z})$$

where we set $H_{-1} = 0$. Recall $Ext(A, B) = 0$ if A is free or projective. Otherwise, $Ext(\mathbb{Z}/n\mathbb{Z}, B) = \mathrm{coker}(B \xrightarrow{n} B)$. From this we see that the Ext term is only crucial for $i = 4$. Otherwise we see

$$H^0(X) = \mathrm{Hom}_{\mathbb{Z}}(H_0(X), \mathbb{Z}) \cong \mathbb{Z}$$

$$H^1(X) = \mathrm{Hom}_{\mathbb{Z}}(H_1(X), \mathbb{Z}) = 0$$

$$H^2(X) = \mathrm{Hom}_{\mathbb{Z}}(H_2(X), \mathbb{Z}) = \mathbb{Z}$$

$$H^3(X) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$$

$$H^4(X) = \mathrm{Hom}_{\mathbb{Z}}(H_4(X), \mathbb{Z}) \oplus Ext(H_3(X), \mathbb{Z}) = Ext(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = \mathrm{coker}(Z \xrightarrow{2} \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

and the higher cohomology groups are zero.

b) Construct a simply connected CW complex X with the given homology.

Attach a 4-cell to S^3 via a degree 2 map $S^3 \to S^3$. This gives us a CW complex Y with a 4-cell F, a 3-cell α and a 0-cell v, with $\partial F = 2\alpha$. Thus we have a cell complex

$$0 \to C_4 = \mathbb{Z} \xrightarrow{2} C_3 = \mathbb{Z} \to 0 \to 0 \to C_0 = \mathbb{Z} \to 0$$

from which it is clear $H_4(Y) = 0, H_3(Y) = \mathbb{Z}/2\mathbb{Z}, H_2(Y) = 0, H_1(Y) = 0, H_0(Y) = \mathbb{Z}$, and all other homology groups are zero.

Note Y is simply connected, as its 2-skeleton is a point. Finally, $Y \vee S^2$ has the desired homology (we have $H_i(Y \vee S^2) = H_i(Y)$ for $i \neq 2$, and $H_2(Y \vee S^2) = H_2(Y) \oplus H_2(S^2) = \mathbb{Z}$). It remains simply connected as Y, S^2 both are. Thus $Y \vee S^2$ is the desired space.

c) Construct another simply connected CW complex Z with the same homology, which is not homotopy equivalent to X.

Attach two 4-cells F_1, F_2 to $S^3 \vee S^3 = A \vee B$ via maps $S^3 \to A = S^3 \hookrightarrow A \vee B$ of degree 1 and $S^3 \to B = S^3 \hookrightarrow A \vee B$ of degree 2. Note then by cellular boundary formula that $\partial F_1 = A$ and $\partial F_2 = B$. We have a chain complex

$$0 \to C_4 = \mathbb{Z}^2 \xrightarrow{\partial_4} C_3 = \mathbb{Z}^2 \to 0 \to 0 \to C_0 = \mathbb{Z} \to 0$$

where $\partial_4((1,0)) = (1,0)$ and $p_4((0,1)) = (0,2)$. Then $H_4(Z) = \ker(\partial_4) = 0$, $H_3(Z) = \mathbb{Z}^2/\langle(1,0),(0,2)\rangle \cong \mathbb{Z}/2\mathbb{Z}$, $H_2(Z) = H_1(Z) = 0, H_0(Z) = \mathbb{Z}$, and all other homology groups are zero.

With a similar argument as before, Z is simply connected, as is $Z \vee S^2$, and the latter is the desired space.

There is one more construction that works here: it suffices to add a 2-cell to $\mathbb{RP}^4$ via the attaching word a, where a is the 1-cell, and call this space W. Then we get chain complex

$$0 \to C_4 = \mathbb{Z} \xrightarrow{2} C_3 = \mathbb{Z} \xrightarrow{0} C_2 = \mathbb{Z}^2 \xrightarrow{0} C_1 = \mathbb{Z}^2 \xrightarrow{0} C_0 = \mathbb{Z} \to 0$$

with $\partial_2((1,0)) = 2a$ the usual boundary, and $\partial_2((0,1)) = 1a$ for the new 2-cell. Now $H_4(W) = 0, H_3(W) = \mathbb{Z}, H_2(W) = \ker(\partial_2) = \text{span}((1,-2)) \cong \mathbb{Z}$, $H_1(W) = 0$, $H_0(W) = \mathbb{Z}$. Moreover, by Hatcher Proposition 1.26, $\pi_1(W) = \langle a | 2a, a\rangle = 0$. So W is yet another space with the desired properties.

It is hard to formally determine why $Y \vee S^2, Z \vee S^2, W$ are not homotopy equivalent. SKIP!

Problem 9: Let X be a connected CW-complex. Show that there is a natural isomorphism $\tilde{H}_k(\Sigma X; M) \cong \tilde{H}_{k-1}(X; M)$ for all k and all abelian groups M.

See Spring 2016 Problem 9.

Problem 10: Let Y be a connected and simply connected CW-complex.

a) Compute the fundamental group of $Y \vee S^1$.

We have $\pi_1(Y \vee S^1) = \pi_1(Y) * \pi_1(S^1) = 0 * \mathbb{Z} = \mathbb{Z}$.

b) Describe the universal covering $Y \vee S^1$, together with the action of the deck transformations.

In general, the universal cover of a wedge $A \vee B$ is an infinite bipartite tree whose vertices are copies of $\tilde{A}$ or $\tilde{B}$ the universal covers of A, B, and these are glued along lifts of the base point. In this case, Y is its own universal cover (with the base point only lifting to one point p) and S^1 has universal cover $\mathbb{R}$ with the base point lifting to $\mathbb{Z} \subset \mathbb{R}$. Thus, the universal cover of $Y \vee S^1$ is $\mathbb{R}$ with a copy of Y glued via p at $x \in \mathbb{Z}$ for every $x \in \mathbb{Z}$. The deck transformations are entirely coming from the deck transformations $\mathbb{R} \to S^1$ by the calculation in part a, so that they must all be the translations by k for $k \in \mathbb{Z}$.

c) Describe all finite covers of $Y \vee S^1$, again with the action of the deck transformations.

Finite covers may be obtained from the answer in part c via quotienting by the action of a finite index subgroups of $\pi_1(Y \vee S^1) \cong \mathbb{Z}$. These are $k\mathbb{Z} \subset \mathbb{Z}$ for $k > 0$. Quotienting out by this action, we see 0 and k are identified in $\mathbb{R}$, as are the corresponding copies of Y. We are left with a circle with k base points, with a copy of Y glued via p at each base point. Note that this corresponds to the k-fold cover of S^1, $S^1 \to S^1$ via $z \mapsto z^k$.

d) Describe what changes in the first two parts for $Y = \mathbb{RP}^2$.

For $Y = \mathbb{RP}^2$, we have $\pi_1(Y) = \langle a | a^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}$. Writing $\pi_1(S^1) = \langle b \rangle$, we see $\pi_1(Y \times S^1) = \pi_1(Y) * \pi_1(S^1) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z} = \langle a, b | a^2 \rangle$.

Thus we see the universal cover of $\mathbb{RP}^2 \wedge S^1$ is an infinite bipartite tree of copies of $\mathbb{R}$ and S^2 glued at lifts of the base point. Here is an alternative description following a similar argument as the end of Fall 2010 Problem 7c. Note that $\mathbb{RP}^2$ is the Cayley complex for $\mathbb{Z}/2\mathbb{Z}$, and S^1 is the Cayley complex for $\mathbb{Z}$. In fact, $X = \mathbb{RP}^2 \vee S^1$ is the Cayley complex for $G = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}$. Thus, its universal cover can be constructed as follows: the vertices of $\tilde{X}$ are the elements of G; we have directed edges from g to ga and g to gb for each $g \in G$. At each vertex, follow the relation a^2 (along the edges) to give the attaching word for a 2-cell. In fact, since we attach a 2-cell via a^2 at g as well as at ga, we see we actually get a copy of S^2 between g and ga for each $g \in G$. This gives us a bunch of copies of S^2 connected by edges at the base points, as desired.

19 Spring 2019

Problem 1: Let M be a smooth manifold. Show that there exists a smooth proper map $f : M \to \mathbb{R}$.

We follow the argument of Lee Proposition 2.28. Take a countable open cover of M, $M = \cup_{j=1}^{\infty} V_j$ with each $\overline{V_j}$ compact. Take $\{\psi_j\}_{j=1}^{\infty}$ a partition of unity suboordinate to this cover. Write $f(p) = \sum_{j=1}^{\infty} j \cdot \psi_j(p)$. Notice that for $p \in M$, since $\sum_{j=1}^{\infty} \psi_j(p) = 1$, we have $\psi_j(p) = 0$ for all $j \geq N_p$ for our partition of unity. Thus, $0 \leq f(p) \leq N_p$ is also finite at p.

Notice if $p \notin \cup_{j=1}^{N} \overline{V_j}$, then $\psi_j(p) = 0$ for $j \leq N$. Hence $\sum_{j>N} \psi_j(p) = 1$, and hence $f(p) = \sum_{j>N} j\psi_j(p) \geq N + 1 > N$. We conclude if $f(p) \leq N$, then $p \in \cup_{j=1}^{N} \overline{V_j}$, which is compact by construction. Hence each $f^{-1}[0, N]$ is compact. Hence we see if $b \leq N \in \mathbb{Z}$, then $f^{-1}([a, b]) \subset f^{-1}([0, N])$ is a closed subset of a compact set and hence also compact. Thus, f is proper.

Problem 2: A smooth manifold Y of dimension n is called parallelizable if there exist n linearly independent vector fields v_i on Y. Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function with 0 a regular value, and let $M = f^{-1}(\{0\})$. Show that $M \times S^1$ is parallelizable.

Recall a similar problem Spring 2010 Problem 2. Here, we have NM is trivial if and only if we have a nonvanishing normal vector field, since M has codimension 1. Taking $X = \nabla f$, we have f is nonvanishing on M since 0 is a regular value of f, and since M is a level set fo f, we have X is everywhere normal to M. Thus, NM is trivial, and we have as before

$$T(S^1 \times M) = \pi_{S^1}^*(TS^1) \oplus \pi_M^*(TM) = (S^1 \times M \times \mathbb{R}) \oplus \pi_M^*(TM)$$

$$= \pi_M^*(NM) \oplus \pi_M^*(TM) = \pi_M^*(NM \oplus TM) = \pi_M^*(M \times \mathbb{R}^{n+1}) = S^1 \times M \times \mathbb{R}^{n+1}$$

so that $S^1 \times M$ is parallelizable, as desired.

Problem 3: Show that the antipodal map $A : S^n \to S^n$ is homotopic to the identity if and only if n is odd.

See Spring 2014 Problem 3.

Problem 4: Prove that $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$.

First, see from Fall 2015 Problem 3 that

$$[\mathcal{L}_X, i_Y] = i_{[X,Y]}$$

Also recall Cartan's magic formula $\mathcal{L}_Y = i_Y d + d i_Y$. Finally, note that $\mathcal{L}_X$ commutes with d. From these observations, we get

$$[\mathcal{L}_X, \mathcal{L}_Y] = [\mathcal{L}_X, i_Y d + d i_Y] = [\mathcal{L}_X, i_Y d] + [\mathcal{L}_X, d i_Y]$$

$$= \mathcal{L}_X i_Y d - i_Y d \mathcal{L}_X + \mathcal{L}_X d i_Y - d i_Y \mathcal{L}_X$$

$$= \mathcal{L}_X i_Y d - i_Y \mathcal{L}_X d + d \mathcal{L}_X i_Y - d i_Y \mathcal{L}_X$$

$$= [\mathcal{L}_X, i_Y] d + d[\mathcal{L}_X, i_Y] = i_{[X,Y]} d + d i_{[X,Y]} = \mathcal{L}_{[X,Y]}$$

as desired.

Problem 5: Show that a closed 1-form ω on a manifold M is exact if and only if $\int_{S^1} f^*\omega = 0$ for every smooth map $f : S^1 \to M$.

See Spring 2013 Problem 2b.

Problem 6: Let $f : X \to Y$ be a smooth, finite covering map between smooth manifolds. Show that the induced map on de Rham cohomology $f^* : H^k(Y) \to H^k(X)$ is injective.

See the proposition stated in Spring 2012 Problem 9c.

Problem 7: Let $X = [0,1]$ and $A = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}, n \geq 1\}$. Show that $H_1(X, A)$ is not isomorphic to $H_1(X/A)$.

See Hatcher Example 1.25. Note X/A is actually the Hawaiian earring. Hatcher shows $\pi_1(X/A)$ surjects onto the uncountable group $\Pi_{i=1}^{\infty}\mathbb{Z}$. Since the latter is abelian, this homomorphism factors through to give a surjective map $H_1(X/A)$ onto the uncountable group $\Pi_{i=1}^{\infty}\mathbb{Z}$. In particular, $H_1(X/A)$ is uncountable.

Meanwhile, by the LES for relative homology, we have

$$\widetilde{H}_1(X) \to H_1(X, A) \to \widetilde{H}_0(A) \to \widetilde{H}_0(X)$$

Notice $\widetilde{H}_0(X) = 0$ since X is connected. Meanwhile, $\widetilde{H}_1(X) = 0$ since X is contractible. Hence $H_1(X, A)$ injects into $\widetilde{H}_0(A)$. Note $H_0(A)$ is the direct sum of one copy of $\mathbb{Z}$ for each path component of A. Since there are countably many components, and the direct sum only consists of finite sums, we have $H_0(A)$, and hence $\widetilde{H}_0(A)$, is countable. Thus $H_1(X, A)$ is countable and not equal to $H_1(X/A)$, as desired.

We construct the aforementioned surjection below. Write $I_n = [\frac{1}{n+1}, 1/n] \subset X = [0,1]$. Write $C_n = I_n/A$ as the image of $X \mapsto X/A$. Then note $C_n \cong S^1$ since only the endpoints are identified. Note $X = \{0\} \cup \bigcup_{n=1}^{\infty}[\frac{1}{n+1}, \frac{1}{n}]$, so that $X/A = \cup_{n=1}^{\infty} C_n$.

Write $B_n = [0, \frac{1}{n+1}] \cup [\frac{1}{n}, 1]$. Note $A \subset B_n$, so that the projection $X \to X/B_n = C_n$ factors through to $r_n : X/A \to C_n$, which is a retract onto C_n. Hence $(r_n)_* : \pi_1(X/A) \twoheadrightarrow \pi_1(C_n) = \pi_1(S^1) \cong \mathbb{Z}$ is surjective (since $r_n \circ i_n = id_{B_n}$).

Thus we have a map $r_* : \pi_1(X/A) \to \prod_{i=1}^{\infty} \pi_1(C_n) = \prod_{i=1}^{\infty} \mathbb{Z}$ given by $r_*\gamma = ((r_1)_*\gamma, (r_2)_*\gamma, ...)$. In fact, r_* is surjective. To see this, let $(a_1, a_2, ...) \in \prod_{i=1}^{\infty} \mathbb{Z}$ be arbitrary. Pick a loop $\gamma_n : [0,1] \to C_n$ corresponding to $a_n \in \mathbb{Z} \cong \pi_1(C_n)$. If $i_n : C_n \to X/A$ is the inclusion, note $(i_n)_*[\gamma_n] = [i_n \circ \gamma_n] \in \pi_1(C_n)$, and $(r_n)_*[i_n \circ \gamma_n] = [\gamma_n] \in \pi_1(\mathbb{C}_n)$.

Write $\tau_n : I_n = [\frac{1}{n+1}, \frac{1}{n}] \to C_n \subset X/A$ as $\tau_n(t) = i_n \circ \gamma_n\left(\frac{t - \frac{1}{n+1}}{\frac{1}{n} - \frac{1}{n+1}}\right)$. Write $\tau : [0,1] \to X/A$ via $\tau(0) = 0 \in X/A$, and $\tau(x) = \tau_n(x)$ for $x \in I_n$. Since $X = \{0\} \cup_{n=1}^{\infty} I_n$ and the τ_n agree on $\{0\} \cup_{n=1}^{\infty} \partial I_n = A$, we see τ is a continuous loop in X/A. Meanwhile, $(r_n) * \tau = \gamma_n \in \pi_1(C_n)$, which corresponds to $a_n \in \mathbb{Z}$. Thus, $r_*\tau = (a_1, a_2, ...)$. Hence r_* is surjective, as desired.

Problem 8:

a) Show that any continuous map $\mathbb{RP}^2 \to S^1 \times S^1$ is nullhomotopic.

> See Fall 2016 Problem 7.

b) Find, with proof, a continuous map $f : S^1 \times S^1 \to \mathbb{RP}^2$ that is not nullhomotopic.

> See Spring 2010 Problem 10b. Write $T = S^1 \times S^1$. We have a map $g : T \to S^2$ which is not nullhomotopic. Consider the projection map $\pi : S^2 \to \mathbb{RP}^2$. Suppose $f = \pi \circ g$ is nullhomotopic. Write
>
> $$H : T \times [0,1] \to \mathbb{RP}^2$$
>
> with $H(-,0) = f(-)$ and $H(-,1) = c$ the constant map for fixed $c \in \mathbb{RP}^2$.
>
> Clearly f lifts to a map g to the universal cover of S^2, by construction. Thus $f_*\pi_1(T) \subset \pi_*\pi_1(S^2) = 0$, so that $f_*\pi_1(T) = 0$.
>
> Meanwhile, let $i : T \to T \times [0,1]$ for the inclusion $i(x) = (x,0)$. Then $H \circ i = f$. Notice since $T \times [0,1]$ deformation retracts to $T = T \times \{0\}$, i induces an isomorphism on π_1. Hence, we have $H_* i_* = f_* = 0$, and i_* is an isomorphism, so that $H_* = 0$. Picking base point $p \in T$, $0 \in [0,1]$, $x = g(p) \in S^2$ and $[x] = f(p) \in \mathbb{RP}^2$, we have the covering space $\pi : (S^2, x) \to (\mathbb{RP}^2, [x])$ and a map $H : (T \times [0,1], (p,0)) \to (\mathbb{RP}^2, [x])$, which by the lifting criterion (Hatcher Proposition 1.33) lifts to a map $K : (T \times [0,1], (p,0)) \to (S^2, x)$ with $\pi \circ K = H$.
>
> Note $K \circ i$ is a lift of $H \circ i = f$, and $(K \circ i)(p) = K(p,0) = x = g(p)$. Thus the lefts $K \circ i$ and g of f agree at a point, so by Proposition 1.34, $K \circ i = g$. Meanwhile, $K(-,1)$ is a lift of the constant map and hence is also a constant map (its image must be connected so cannot be two antipodal points). Thus, K gives a homotopy between g and a constant map. Since g is not null-homotopic, we get a contradiction. Thus f is not null-homotopic, so that it is the desired map.
>
> *Alternative solution*: See Hatcher Proposition 1.30 - any homotopy of f with a constant map would lift to a homotopy of g with a lift of the constant map, which must also be constant by connectedness.

Problem 9: Let W be the space obtained by attaching two 2-cells to S^1, one by the map $z \to z^4$ and the other by $z \to z^7$.

a) Compute the homology groups of W with $\mathbb{Z}$ coefficients.

> By the standard argument, we have a cell complex
>
> $$0 \to C_2 = \mathbb{Z}^2 \xrightarrow{\partial_2} C_1 = \mathbb{Z} \xrightarrow{0} C_0 = \mathbb{Z} \to 0$$
>
> where by the cellular boundary formula, we have $\partial F_1 = 4e$ and $\partial F_2 = 7e$ as the boundaries of the two faces. Hence, ∂_2 is surjective, and we get $H_1(W) = 0$. Meanwhile, $H_0(W) = \mathbb{Z}$, and $H_2(W) = \ker(\partial_2)$. It is clear the kernel is generated by $7F_1 - 4F_2$. Thus, $H_2(W) \cong \langle (7, -4) \rangle \cong \mathbb{Z}$.

b) Is W homotopy equivalent to S^2?

> One might be tempted to compute fundamental groups, but note that by Hatcher Proposition 1.26, $\pi_1(W) = \langle e | 4e, 7e \rangle = \langle e | e \rangle = 0$.
>
> Skip!

Problem 10: Suppose that M is a compact, connected, orientable topological n-manifold with boundary a rational homology sphere, i.e. $H_*(\partial M; \mathbb{Q}) \cong H_*(S^{n-1}; \mathbb{Q})$.

a) Assuming that n is odd, use Poincare duality (with $\mathbb{Q}$ coefficients) to show that M has Euler characteristic $\chi(M) = 1$.

> Recall from the third proposition here that $\chi(\partial M) = 2\chi(M)$. Since ∂M has the same $\mathbb{Q}$ homology as the sphere, we have $\chi(\partial M) = \chi(S^{n-1}) = (-1)^{(n-1)} + 1 = 2$ for n odd. Thus, $\chi(M) = 1$ as desired.

b) Assuming that $n \equiv 2 \bmod 4$, show that the Euler characteristic of M is odd.

> Write $n = 4k+2$. Lefshetz duality gives $H_i(M, \partial M; \mathbb{Q}) = H^{n-i}(M; \mathbb{Q}) = H_{n-i}(M; \mathbb{Q})$ by universal coefficient. Meanwhile, the LES for relative homology gives, by $\widetilde{H}_i(\partial M; \mathbb{Q}) = \widetilde{H}_{i-1}(\partial M; \mathbb{Q}) = 0$ for $i \neq 4k+2, 4k+1$, that $H_i(M, \partial M; \mathbb{Q}) \cong \widetilde{H}_i(M; \mathbb{Q})$ for $i \neq 4k+2, 4k+1$. Thus $\widetilde{H}_i(M; \mathbb{Q}) = H_{n-i}(M; \mathbb{Q})$. It suffices to notice this holds for $0 \leq i \leq 2k+1$. From this we see (since top homology of M must be zero) that
>
> $$\chi(M) = 1 + \sum_{i=1}^{2k} (-1)^i 2 \dim_{\mathbb{Q}}(H_i(M; \mathbb{Q})) - \dim_{\mathbb{Q}} H_{2k+1}(M; \mathbb{Q})$$
>
> So it suffices to show $H_{2k+1}(M; \mathbb{Q}) = H^{2k+1}(M, \partial M; \mathbb{Q})$ is even dimensional. However, we have a nondegenerate alternating bilinear form $H^{2k+1}(M, \partial M; \mathbb{Q}) \times H^{2k+1}(M, \partial M; \mathbb{Q}) \to H^{4k+2}(M, \partial M; \mathbb{Q}) = H_0(M; \mathbb{Q}) = \mathbb{Q}$ given by the cup product. By a similar argument as Fall 2012 Problem 7, we conclude $H^{2k+1}(M, \partial M; \mathbb{Q})$ is even dimensional. Thus $\chi(M)$ is odd, as desired.