Fundamental Group

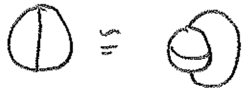
~~F17.8~~

~~(a) $\mathbb{R}^3 - A$ deformation retracts to T^2 so $\pi_1(\mathbb{R}^3 - A) \cong \pi_1(T^2) \cong \mathbb{Z}^2$~~

~~(b) $\mathbb{R}^3 - A - B$ deformation retracts to M_2. So $\pi_1(\mathbb{R}^3 - A - B) \cong \pi_1(M_2) \cong \mathbb{Z}^3$~~

~~(c)~~

F17.10

(a) $\mathbb{R}^3 - A$ deformation retracts to a sphere w/ a line through the middle  $\cong S^2 \vee S^1$. Therefore $\pi_1(\mathbb{R}^3 - A) \cong \pi_1(S^2) * \pi_1(S^1) \cong \mathbb{Z}$

(b) $\mathbb{R}^3 - A - B$ deformation retracts to the wedge of 2 spheres each w/ a line. Therefore $\pi_1(\mathbb{R}^3 - A - B) \cong \pi_1(S^2) * \pi_1(S^1) * \pi_1(S^2) * \pi_1(S^1) \cong \mathbb{Z} * \mathbb{Z}$

(c) Same reasoning, this deformation retracts to $S^2 \vee T^2$ so $\pi_1(\mathbb{R}^3 - A - B) \cong \mathbb{Z}^2$.

S18.7

(a) Van Kampen's, B^n n-ball

U ε-neighborhood of $M \setminus B$, V ε-neighborhood of $N \setminus B$. Then

$U \cap V$ deformation retracts to S^{n-1} simply connected

Van Kampen's yields $\pi_1(M \# N) \cong \pi_1(M) * \pi_1(N)$

(b) Mayer-Vietoris w/ U, V as above. Yields LES

$$\cdots \to H_k(S^{n-1}) \to H_k(M \setminus B) \oplus H_k(N \setminus B) \to H_k(M \# N) \to \cdots$$

By a similar Mayer-Vietoris, $H_k(M \setminus B) \cong H_k(M)$ for $k < n$ and $H_n(M \setminus B) \cong 0$. So

$$\cdots \to H_k(S^{n-1}) \to H_k(M) \oplus H_k(N) \to H_k(M \# N) \to \cdots$$ w/ $H_n(S^{n-1}) \to 0 \to H_n(M\#N)$

For $1 < k < n-2$, this gives $H_k(M \# N) \cong H_k(M) \oplus H_k(N)$.

For 0, $H_0(M \# N) \cong \mathbb{Z}$ since connected.

For 1,

$$0 \to H_1(M) \oplus H_1(N) \to H_1(M \# N) \xrightarrow{0} \mathbb{Z} \xrightarrow[\ker = 0]{\operatorname{im} = \mathbb{Z}} \mathbb{Z}^2 \xrightarrow[\ker = \mathbb{Z}]{\operatorname{im} = \mathbb{Z}} \mathbb{Z} \to 0$$

so $H_1(M) \oplus H_1(N) \cong H_1(M \# N)$.

$\longrightarrow$

S13.9

Recall S^2 is a 2-cover of $\mathbb{RP}^2$ w/ $\pi_1(S^2) \cong 0$ $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}$

and
$$H_k(S^2) \cong \begin{cases} \mathbb{Z} & k=0,2 \\ 0 & \text{else} \end{cases} \qquad H_k(\mathbb{RP}^2) \cong \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}/2\mathbb{Z} & k=1 \\ 0 & k=2, \text{else} \end{cases}$$

$$H^k_{dR}(S^2) \cong \begin{cases} \mathbb{R} & k=0,2 \\ 0 & \text{else} \end{cases} \qquad H^k_{dR}(\mathbb{RP}^2) \cong \begin{cases} \mathbb{R} & k=0 \\ 0 & \text{else} \end{cases}$$

Therefore (1), (2), (3) are false.

F13.9

By stereographic projection, $S^3 \setminus H$ is diffeomorphic to $\mathbb{R}^3$ excluding a line and circle around it (think removing z axis and $S^1 \subset xy$ plane). This deformation retracts to T^2. Therefore $\pi_1(S^3 \setminus H) \cong \pi_1(T^2) \cong \mathbb{Z}^2$,
and
$$H_k(S^3 \setminus H) \cong H_k(T^2) \cong \begin{cases} \mathbb{Z} & k=0,2 \\ \mathbb{Z}^2 & k=1 \\ 0 & \text{else} \end{cases}$$

S16.8

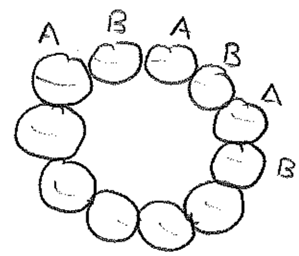
$$\mathbb{R}^3 - L_1 - L_2 - L_3 \cong S^2 \setminus 6 \text{ points} \cong \mathbb{R}^2 \setminus 5 \text{ points} \cong \bigvee_{k=1}^{5} S^1$$

Therefore $\pi_1(\mathbb{R}^3 - L_1 - L_2 - L_3) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.

F17.8

Fix $n \geq 1$. Wedge $2n$ copies of S^2 together in a circle, w/ alternating identifications as A, B (the 2 $\mathbb{RP}^2$s).

Then $p_* \pi_1(Y, y) \cong (ab)^n$.



Finally, ~~connected~~ closed + orientable gives $H_n(M\#N)\cong\mathbb{Z}$ so

$$0\to 0\to\mathbb{Z}\hookrightarrow\underset{0}{\mathbb{Z}}\xrightarrow{0} H_{n-1}(M)\oplus H_{n-1}(N)\to H_{n-1}(M\#N)\to 0$$

which gives $H_{n-1}(M\#N)\cong H_{n-1}(M)\oplus H_{n-1}(N)$.

Therefore

$$H_k(M\#N)\cong\begin{cases}\mathbb{Z} & k=0,n\\ H_k(M)\oplus H_k(N) & \text{else}\end{cases}$$

(c) when $n=2$, $M\cong M_g$ $N\cong M_h$ for g,h. Thus

$$M\#N\cong M_g\#M_h\cong M_{g+h}$$

which implies

$$H_k(M\#N)\cong\begin{cases}\mathbb{Z} & 0,2\\ \mathbb{Z}^{g+h} & 1\\ 0 & \text{else}\end{cases}$$

$$\pi_1(M\#N)\cong\mathbb{Z}^{g+h}$$

Corning Spares

F13.1

non-singular $\Rightarrow$ local diffeo

$\Rightarrow$ open map

(1) No.

$(0,1) \hookrightarrow (0,2)$ for injective

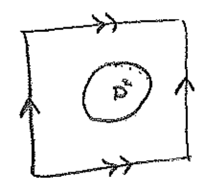
$[0,2) \to S^1$ for surjective

(2) No, $(0,1) \hookrightarrow [0,2]$ ~~cov~~

(3) yes, local diffeo

(4) No, $(0,1) \hookrightarrow (0,2)$ not closed

F13.7



M deformation retracts to $S^1 \vee S^1$. Therefore $\chi(M) = -1$.

Any ~~cover~~ ~~finite~~ 3-fold cover of M will be a closed orientable surface. ~~and wt~~ Therefore any 3-cover will be a genus g surface w/ n punctures for some n, denoted $\Sigma_{g,n}$.

We recall $\Sigma_{g,n}$ deformation retracts onto the wedge of $2g+n-1$ ~~cov~~ copies of S^1. ~~so~~ Then $\chi(\Sigma_{g,n}) = 1-2g-n+1 = 2-2g-n$.

~~So~~ If $\Sigma_{g,n}$ is a 3-fold cover, then $\chi(\Sigma_{g,n}) = 3\chi(M)$

$$\Rightarrow 2-2g-n = -3$$
$$5 = 2g+n$$

So $(g,n) = (0,5), (1,3), (2,1)$.

Boundary to boundary rules out $(0,5)$ and $(2,1)$.

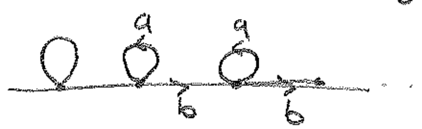
Therefore $\Sigma_{1,3}$ is the only option.

$\Sigma_{1,3}$ covers M via $(\theta,\phi) \mapsto (\theta, 3\phi)$.

S14.7

$S^1 \wedge S^1$ has ~~covers cover~~ fundamental group $\cong \langle a, b \rangle$.

$\langle a \rangle \subset \langle a, b \rangle$ is a non-normal subgroup.

Define $\tilde{X} =$ 

Then $p_* \pi_1(\tilde{X}) \cong \langle a \rangle$ which is not normal

S14.9

(a) $\mathbb{R}P^3 \# \mathbb{R}P^3 \cong \mathbb{R}P^2 \amalg S^2 \times [0,1] \amalg \mathbb{R}P^2 / \sim$ w/ $[x]$

$\mathbb{R}P^3 = D^3 \amalg \mathbb{R}P^2 / \sim$ w/ $x \sim [x]$ $\forall x \in S^2 = \partial D^3$.

so $\mathbb{R}P^3 \# \mathbb{R}P^3 \cong \mathbb{R}P^2 \amalg S^2 \times [0,1] \amalg \mathbb{R}P^2 / \sim$ w/ $(x,0) \sim [x]$, $(x,1) \sim [x]$ as before

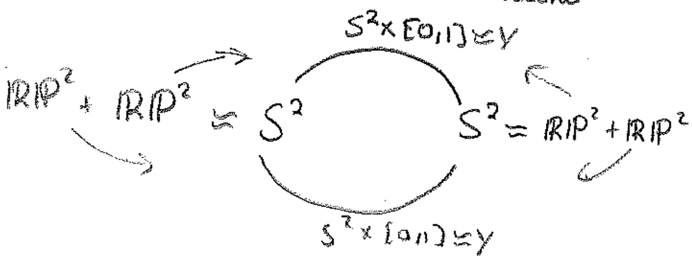
$\cong Y$

(b) $S^2 \times S^1$ is a double cover of Y via

$S^1 = [0,2] / 0 \sim 2$

w/ $S^1 \times \{0\}$ double covering first copy of $\mathbb{R}P^2$

$S^1 \times \{1\}$ " " second" "



F14.3

(a) Let f be the deck transformation on O. Then

$$\pi^*\omega = (\pi\circ f)^*\omega = f^*\pi^*\omega$$

and so $\int_O \pi^*\omega = \int_O f^*\pi^*\omega = -\int \pi^*\omega \Rightarrow \int \pi^*\omega = 0$

Since O is compact orientable, $\pi^*\omega$ is exact.

(b) $H^n_{dR}(O) \xleftarrow{\pi^*} H^n_{dR}(M)$ is an injection. So ω is exact.

Injection b/c it has inverse

$$(\pi^*)^{-1}(\eta)_p = \frac{1}{2}\sum_{k=1,2} \underbrace{\text{stuff}}$$

F14.9

$\mathbb{RP}^k$ has covers $\mathbb{RP}^k$ and S^k. Chain these together for cover of $\mathbb{RP}^{14} \vee \mathbb{RP}^{15}$

F15.8

Suppose $\mathbb{CP}^{2n}$ covers X. Then $\pi_1(X)$ acts on $\mathbb{CP}^{2n}$ ~~$\pi_1(\mathbb{CP}^{2n}) \cong 0$~~.

~~Supp So~~ To show $X \cong \mathbb{CP}^{2n}$, it suffices to show ~~$\pi_1(X) \cong 0$~~

$\pi_1(X) \cong \pi_1(\mathbb{CP}^{2n}) \cong 0$.

~~Any deck transfor.~~ Consider a deck transformation f. If f has a fixed point, then $f = id$. Calculating Lefschetz number

$$L(f) = \sum_{k=0}^{4n} (-1)^k \operatorname{tr}\left(f^* : H^k(\mathbb{CP}^{2n};\mathbb{Q}) \to H^k(\mathbb{CP}^{2n};\mathbb{Q})\right)$$

$$= \sum_{k=0}^{2n} \operatorname{tr}\left(f^* : H^{2k}(\mathbb{CP}^{2n};\mathbb{Q}) \to H^{2k}(\mathbb{CP}^{2n};\mathbb{Q})\right)$$

Recall ~~f~~ $H^{2k}(\mathbb{CP}^{2n};\mathbb{Q}) \cong \mathbb{Q}\ \forall\, k \le 2n$. Then if $f^* : H^2(\mathbb{CP}^{2n};\mathbb{Q}) \to H^2(\mathbb{CP}^{2n};\mathbb{Q})$ is multiplication by q,

$$L(f) = 1 + q + \dots + q^{2n} \neq 0$$

$\longrightarrow$

so f has a fixed point $\Rightarrow$ $f = id$.

Then $\pi_1(X) \cong 0 \Rightarrow X \cong \mathbb{CP}^{2n}$.

□

F16.7

$\mathbb{R}^n$ is the universal cover of $(S^1)^n$ so $p_* \pi_1(\mathbb{R}^n) \cong 0$.

Since $\pi_1(X)$ is finite, $f_* \pi_1(X)$ is finite.

Then $f_* \pi_1(X) \subset \pi_1((S^1)^n) \cong \mathbb{Z}^n$ is finite and hence $f_* \pi_1(X) \cong 0$.

Therefore f lifts to a map $\tilde{f}: X \to \mathbb{R}^n$ which is nullhomotopic.

This concludes.

□

S17.6

$f: Y \to X$, $p: \tilde{X} \to X$ covering

$f^*(\tilde{X}) = \{(y, \tilde{x}) : f(y) = p(\tilde{x})\} \subset Y \times \tilde{X}$

$f^*p : f^*(\tilde{X}) \to Y : (y, \tilde{x}) \to (f(y), p(\tilde{x}))$

S17.8

(a) G acts naturally on $\tilde{X}$ via deck transformation, diffeomorphisms $g: \tilde{X} \to \tilde{X}$ $\forall g \in G$. Then $\forall g \in G$, $g: \tilde{X} \to \tilde{X}$ induces a map $g_* : H_k(\tilde{X}) \to H_k$ $\forall k$.

(b)

F17.6

(a) orientable if $\exists$ an atlas $\{\varphi_\alpha, U_\alpha\}$ s.t. $\det(d(\varphi_\alpha \circ \varphi_\beta^{-1})) > 0$ $\forall$ $U_\alpha \cap U_\beta \neq \emptyset$.

(b) Define $\tilde{M} = \{(p, O_p) : p \in M,$ orientation O_p at $p\}$.
$\tilde{M}$ is given the topology $\{V_{U,O}\}$ where $U \subset M$ is open and can be given orientation O and $V_{U,O} = \{(p, O_p) : p \in U\}$.

Define $\pi : (p, O_p) \mapsto p$. Then since M is locally orientable, $\forall p \in M$ $\exists$ U s.t. U can be given $\pm O$ $\Rightarrow$ $\pi^{-1}(U) = V_{U,O} \amalg V_{U,-O}$ and $\pi|_{V_{U,O}} \cong id$ $\pi|_{V_{U,-O}} \cong id$.
On $V_{U,O}$, $T_p\tilde{M} \cong T_pM$ so can orient $\tilde{M}$ w/ O_p at (p, O_p).
Each connected component is a cover, so if $\tilde{M}$ is disconnected then M is orientable and connected $\Rightarrow$ non-orientable. $\square$

18S.1

(a) local diffeo $\Rightarrow$ open $\Rightarrow$ $f(M) \subset N$ is open
M compact $\Rightarrow$ $f(M) \subset N$ closed $\Rightarrow$ surjection
covering map is from stack of records theorem

(b) $\pi : (0,2) \to S^1 \cong \mathbb{R}/\mathbb{Z}$.

S18.10

(a) $\pi_1(\Delta^n) \cong 0$ so $\sigma_* \pi_1(\Delta^n) \cong 0 \subset p_* \pi_1(\tilde{X})$

(b) Fix $x_0 \in \Delta^n$. Then $p(f_1(x_0)) = p(f_2(x_0))$. Since $\tilde{X}$ is the universal cover, $\pi_1(X)$ acts transitively on $p^{-1}\{f_1(x_0)\}$ so $\exists$ $g \in \pi_1(X)$ s.t. $g \circ f_1(x_0) = f_2(x_0)$. $\therefore$ $g \circ f_1$ is a lift of σ which agrees w/ f_2 at a point, so $g \circ f_1 = f_2$.
Let $\Omega = \{x \in \Delta^n : g \circ f_1(x) = f_2(x)\}$. Then Ω is nonempty. Closed by continuity, and open b/c

F13.7

If n even then $H^n_{dR}(S^n) \cong \mathbb{R} \neq 0 = H^n_{dR}(\mathbb{RP}^n)$

~~If n odd~~

If n odd, $\pi^* : \underset{\cong\,\mathbb{R}}{H^n_{dR}(\mathbb{RP}^n)} \to \underset{\cong\,\mathbb{R}}{H^n_{dR}(S^n)}$ is injective and hence isomorphism.

If n even, orientation cover is S^n

If n odd, orientation cover is $\mathbb{RP}^n \amalg \mathbb{RP}^n$.

$\Rightarrow$ the deck transformation $x \mapsto -x$ is orientation reversing, so $S^n \to \mathbb{RP}^n$ is an orientation cover.

S19.6

$\forall p \in Y\ \exists\ U_p \ni p$ and $V_p^1, \dots, V_p^k \subset X$ w/ $f|_{V_p^j}$ a diffeo.
Then define $g : H^k(Y) \to H^k(X)$ ~~via~~ locally ~~via~~ ~~f~~ on U_p via

$$g(\eta) \doteq \frac{1}{k} \sum_{j=1}^{k} \left((f|_{V_p^j})^{-1} \right)^* \eta$$

g is well defined on cohomology and is an inverse for f^*.

F19.8

$(\mathbb{R}/2n\mathbb{Z})$ w/ ~~and~~ $(k, k+1)$ contracted to a point near $\forall k$. Then $p :$ $2k \mapsto N$

$2k+1 \mapsto S$

$(2k, 2k+1) \mapsto W$

$(2k+1, 2k) \mapsto E$

Universal cover is ~~to~~ ~~R~~ $\mathbb{R}/\sim$.

Schulitz #

S13.6

(a) Clear by linearity of $A \in GL_{n+1}(\mathbb{C})$

(b) If $[z_0 : \dots : z_n]$ is a fixed point of $\tilde{A} : \mathbb{C}P^n \to \mathbb{C}P^n$,

then $\tilde{A}[z_0 : \dots : z_n] = [z_0 : \dots : z_n]$

$\iff [A(z_0, \dots, z_n)] = [z_0 : \dots : z_n]$

$\iff A(z_0, \dots, z_n) = \lambda(z_0, \dots, z_n)$

so claimed result is derived.

(c) $\tilde{A}$ is a Lefschetz map iff $\forall$ fixed points p of $\tilde{A}$,
$d\tilde{A}_p$ does not have eigenvalue 1.

All eigenvalues have mult. $1 \Rightarrow A$ is diagonalizable.
Changing basis, WLOG $A = \mathrm{diag}(\lambda_0, \dots, \lambda_n)$.
Then $\tilde{A}$ has fixed point $[0 : \dots : 0 : \underset{i}{1} : 0 : \dots : 0] = z_i$
In local coordinates in U_i around z_i, $\mathbb{C}P^n \cong \mathbb{C}^n$ w/ $[z_0 : \dots : 1 : \dots : z_n] \mapsto (z_0, \dots, \hat{z_i}, \dots, z_n)$
Then $d\tilde{A}_{z_i} = \mathrm{diag}(\lambda_0/\lambda_i, \dots, \widehat{\lambda_i/\lambda_i}, \dots, \lambda_n/\lambda_i)$
since ~~this are all~~ eigenvalues are distinct, this concludes.

(d) A is homotopic to Id so

$$L(A) = L(Id) = \chi(\mathbb{C}P^n) = n+1$$

$\square$

S15.4

$f: \mathbb{RP}^n \to \mathbb{RP}^n$

(a) Suffices to show $L(f) \neq 0$. By definition
$$L(f) = \sum_{k=0}^{n} (-1)^k \operatorname{tr}(f^*: H^k(\mathbb{RP}^n;\mathbb{Q}) \to H^k(\mathbb{RP}^n;\mathbb{Q}))$$
Since n is even, UCT implies
$$H^k(\mathbb{RP}^n;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & k=0 \\ 0 & \text{else} \end{cases}$$

Then $L(f)=1$ as $f^*: H^0(\mathbb{RP}^n;\mathbb{Q}) \to H^0(\mathbb{RP}^n;\mathbb{Q})$ is the identity.

(b) $2n-1$ odd $\mathbb{RP}^{2n-1} \cong S^{2n-1}/x \sim -x$ and $S^{2n-1} \subset \mathbb{C}^n$. Then $f(z) = iz$ has no fixed point and descends to a map $f: \mathbb{RP}^{2n-1} \to \mathbb{RP}^{2n-1}$

since $iz \neq -z \ \forall z \in S^n$.

Distribution

F22.4

(1⇒3) Locally ∃ coordinates $x_1, \dots, x_n$ s.t. $\ker(\omega) = \mathbb{R}\langle \partial/\partial x_1, \dots, \partial/\partial x_{n-1} \rangle$

Then $\omega = f\,dx_n$ w/ f non-vanishing

$\kappa = \frac{df}{f} \Rightarrow \kappa \wedge \omega = df \wedge dx_n = d\omega$

(3⇒2)

$\omega \wedge d\omega = \omega \wedge \kappa \wedge \omega = -\omega \wedge \omega \wedge \kappa \Rightarrow \omega \wedge d\omega = 0$

(2⇒1) Frobenius theorem, $\ker(\omega)$ integrable iff $X, Y \in \ker \omega$ $\Rightarrow [X, Y] \in \ker \omega$.

If $X, Y \in \ker \omega$ then $\forall\, v \notin \ker \omega$

$0 = \omega \wedge d\omega(v, X, Y) = \omega(v)\, d\omega(X, Y) \Rightarrow d\omega(X, Y) = 0$

Then $d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega[X, Y]$

$0 = -\omega[X, Y]$

so $[X, Y] \in \ker \omega$. □

F14.5

if $\omega = \lambda\, df$ then $\omega \wedge d\omega = -\lambda\, df \wedge df \wedge d\lambda = 0$

if $\omega \wedge d\omega = 0$ then $\ker \omega$ is integrable so locally ∃ coordinates s.t. $\ker \omega = \mathbb{R}\langle \partial/\partial x_1, \dots, \partial/\partial x_{n-1} \rangle$

$\Rightarrow \omega = f\,dx_n$.

F18.4 $\ker \omega$ integrable

(1) locally, $\omega = f\,dx_n \Rightarrow \omega \wedge d\omega = 0$

(2) $\kappa = \frac{df}{f} \Rightarrow \kappa \wedge \omega = d\omega$

(3) $0 = d(d\omega) = d(\kappa \wedge \omega) = d\kappa \wedge \omega - \cancel{\kappa \wedge d\omega} = d\kappa \wedge \omega$ $\quad \kappa \wedge \kappa \wedge \omega = 0$

Poincaré stuff

Poincare Duality

$$H_{n-k}(X) \cong H^k_c(X)$$

F16.5

(1) $\int_N i^* \omega = \int_M \eta \wedge \omega \quad \forall\, \omega$

(2) Let $N = \{0\} \times S^1$. $\qquad a\,dx + b\,dy$

Then $\forall\, \omega = a\,dx + b\,dy$

$$\int_N i^* \omega = \int_{S^1} b\,dy = b = \int b\,dx \wedge dy = \int dx \wedge \omega$$

(3) $\pi\, dx$? ~~[illegible]~~ since $\deg \in \mathbb{Z}$

S19.10

this uses poincare duality

(1) n odd $\Rightarrow$ $\chi(\partial M) = 2\chi(M)$, $\chi(\partial M) = \chi(S^{n-1}) = 2$

(2) no

The Derivatives

F15.2

$H: [0,1] \times M \to N$ homotopy from g to f. Then

$$\begin{aligned} f^*\omega - g^*\omega &= H(1,\cdot)^*\omega - H(0,\cdot)^*\omega \\ &= \int_0^1 \frac{d}{dt} H(t,\cdot)^*\omega \, dt \\ &= \int_0^1 \frac{d}{dt} i_t H^*\omega \, dt \\ &= \int_0^1 \mathcal{L}_T H^*\omega \, dt \\ &= \int_0^1 d \circ i_T H^*\omega \, dt \\ &= d\left(\int_0^1 i_T H^*\omega \, dt \right) \end{aligned}$$

S17.4

(a) $\mathcal{L}_X \omega = (d \circ i_X + i_X \circ d)\omega$

(b) a flow $\Phi_t(p)$ for X preserves volume

$\iff \Phi_t^* dV = dV$

$\Rightarrow \mathcal{L}_X dV = 0$

$\iff d \circ i_X dV = 0$

$\iff \operatorname{div}(X) dV = 0$

If $\mathcal{L}_X dV = 0 \Rightarrow \Phi_{t_0}^* \mathcal{L}_X dV = 0$

$\Rightarrow \mathcal{L}_X \Phi_{t_0}^* dV = 0$

$\Rightarrow \frac{d}{dt} \Phi_t^* dV \Big|_{t_0} = 0 \quad \forall t_0$

$\Rightarrow \Phi_t^* dV = dV$

S19.4

$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$ recall: $[\mathcal{L}_X, i_Y] = i_{[X,Y]}$

$$[\mathcal{L}_X, d\circ i_Y + i_Y\circ d] = \mathcal{L}_X\circ d\circ i_Y - d\circ i_Y\circ\mathcal{L}_X + \mathcal{L}_X\circ i_Y\circ d - i_Y\circ d\circ\mathcal{L}_X$$

$$= d(\mathcal{L}_X\circ i_Y - i_Y\circ\mathcal{L}_X) + (\mathcal{L}_X\circ i_Y - i_Y\circ\mathcal{L}_X)\circ d$$

$$= d\circ[\mathcal{L}_X, i_Y] + [\mathcal{L}_X, i_Y]\circ d$$

$$= d\circ i_{[X,Y]} + i_{[X,Y]}\circ d$$

$$= \mathcal{L}_{[X,Y]}$$

S20.4 $[\mathcal{L}_X, i_Y] = i_{[X,Y]}$

ω $k+1$ form, $V_1, \dots, V_k$ vector fields

$([\mathcal{L}_X, i_Y]\omega)(V_1, \dots, V_k)$

$\cancel{(\mathcal{L}_X\omega)(V_1,\dots,V_k) = \mathcal{L}_X(\omega(V_1,\dots,V_k)) + \sum_{j=1}^{k}\omega(V_1,\dots,[X,V_j],\dots,V_k)}$

$$(i_Y\circ\mathcal{L}_X\omega)(V_1,\dots,V_k) = (\mathcal{L}_X\omega)(Y, V_1,\dots,V_k)$$

$$= \mathcal{L}_X(\omega(Y, V_1,\dots,V_k)) - \omega([X,Y], V_1,\dots,V_k) - \sum_{j=1}^{k}\omega(Y, V_1,\dots,[X,V_j],\dots,V_k)$$

$$(\mathcal{L}_X\circ i_Y\omega)(V_1,\dots,V_k) = \mathcal{L}_X(i_Y\omega(V_1,\dots,V_k)) - \sum_{j=1}^{k} i_Y\omega(V_1,\dots,[X,V_j],\dots,V_k)$$

$$= \mathcal{L}_X(\omega(Y,V_1,\dots,V_k)) - \sum_{j=1}^{k}\omega(Y,V_1,\dots,[X,V_j],\dots,V_k)$$

so $([\mathcal{L}_X, i_Y]\omega)(V_1,\dots,V_k) = \omega([X,Y], V_1,\dots,V_k)$

$$= (i_{[X,Y]}\omega)(V_1,\dots,V_k)$$

$\square$

F17.2

(b) $O(n)$ has trivializable tangent bundle.

$v_1, \ldots, v_n$ basis for $T_I O(n)$

for $p \in O(n)$, define $X_i(p) = p_*(v_i)$

F19.5

$$(L_g)_*[X,Y](f) = (L_g)_*\big(X(Y(f)) - Y(X(f))\big)$$

$$= X(Y(f \circ L_g)) - Y(X(f \circ L_g))$$

$$= X(L_{g*}Y(f)) - Y(L_{g*}X(f))$$

$$= X(Y(f)) - Y(X(f))$$

$$= [X,Y](f).$$

Stokes' Theorem

Divergence Theorem

$$\int_M \operatorname{div}(X)\,dV = \int_{\partial M} \langle X, N\rangle\, dA$$

N = normal VF to ∂M outwards

$\langle X, N\rangle$ = standard inner product

: $X = f\,\partial/\partial x + g\,\partial/\partial y + h\,\partial/\partial z$

$\operatorname{div}(X) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$

$dA = \iota^* i_N dV$ w/ $\iota : \partial M \hookrightarrow M$ inclusion

Define $T = X - \langle X, N\rangle N$, ~~tangent~~ ~~compond~~ component of X to ∂M

Then $\iota^* i_T dV = 0$ since T, X, Y are linearly dependent $\forall\, X, Y \in T\partial M$

so $\iota^* i_X dV = \langle X, N\rangle \iota^* i_N dV = \langle X, N\rangle dV$

Then by Stokes',

$$\begin{aligned}\int_{\partial M} \langle X, N\rangle\, dA &= \int_{\partial M} \iota^* i_X dV \\ &= \int_M d\, i_X dV \\ &= \int_M \operatorname{div}(X)\, dV\end{aligned}$$

$\square$

Parallelizable

p odd, q whatever,

$$
\begin{aligned}
T(S^p \times S^q) &\cong \pi_p^*(TS^p) \oplus \pi_q^*(TS^q) \\
&\cong \pi_p^*(\varepsilon \oplus \varepsilon^\perp) \oplus \pi_q^*(TS^q) \\
&\cong \pi_p^*(\varepsilon^\perp) \oplus \pi_q^*(TS^q) \oplus \varepsilon \\
&\cong \pi_p^*(\varepsilon^\perp) \oplus \pi_q^*(TS^q \oplus NS^q) \\
&\cong \pi_p^*(\varepsilon^\perp) \oplus \pi_q^*(T\mathbb{R}^{q+1}) \\
&\cong \pi_p^*(\varepsilon^\perp) \oplus \varepsilon^{q+1} \\
&\cong \pi_p^*(\varepsilon^\perp \oplus \varepsilon \oplus \varepsilon) \oplus \varepsilon^{q-1} \\
&\cong \pi_p^*(TS^p \oplus NS^p) \oplus \varepsilon^{q-1} \\
&\cong \pi_p^*(T\mathbb{R}^{p+1}) \oplus \varepsilon^{q-1} \\
&\cong \varepsilon^{p+q}
\end{aligned}
$$

as desired.

DIFF GEO PREP

from Yan's notes.

175.5

(a) We claim that this holds iff $\kappa = 1$.

Suppose $\kappa = 1$. Then

$$d\omega = \frac{(x^2+y^2)(2\,dx\wedge dy) - (2x\,dx + 2y\,dy)\wedge(-y\,dx + x\,dy)}{(x^2+y^2)^2}$$

$$= \frac{2\,dx\wedge dy}{x^2+y^2} + \frac{-2x^2\,dx\wedge dy - 2y^2\,dx\wedge dy}{(x^2+y^2)^2}$$

$$= 0$$

Now suppose $\exists$ a homotopy $H_t : S^1 \to \mathbb{R}^2\setminus\{0\}$, for $t\in[0,1]$, w/ $H_0 = \gamma_0$ and $H_1 = \gamma_1$. Then

$$\int_{\gamma_0}\omega - \int_{\gamma_1}\omega = \int_{S^1} H_0^*\omega - \int_{S^1} H_1^*\omega$$

$$= \int_{\partial(S^1\times[0,1])} H^*\omega$$

$$= \int_{S^1\times[0,1]} d(H^*\omega) = \int_{S^1\times[0,1]} H^*(d\omega) = 0$$

Therefore $\int_{\gamma_0}\omega = \int_{\gamma_1}\omega$ for all smoothly homotopic curves.

Suppose instead that $\int_{\gamma_0}\omega = \int_{\gamma_1}\omega$ for all closed loops γ_0, γ_1. Let γ_0 be the unit circle and γ_1 be the circle of radius r. We parametrize γ_0 by $(\cos t, \sin t)$ for $0\le t\le 2\pi$ and γ_1 by $r(\cos t, \sin t)$ for $0 \le t \le 2\pi$. Then

$$\int_{\gamma_0}\omega = \int_0^{2\pi} \frac{-\sin t(-\sin t\,dt) + \cos t(\cos t\,dt)}{1^\kappa} = \int_0^{2\pi} dt$$

$$= 2\pi$$

and

$$\int_{\gamma_1}\omega = \int_0^{2\pi} \frac{-r\sin t(-r\sin t\,dt) + r\cos t(r\cos t\,dt)}{r^{2\kappa}}$$

$$= \int_0^{2\pi} r^{2-2\kappa}\,dt = 2\pi r^{2-2\kappa}$$

Since $\int_{\gamma_0}\omega = \int_{\gamma_1}\omega$ for all r, this implies $\kappa = 1$.

(b)

We recall that $\mathbb{R}^2\setminus\{0\}$ deformation retracts onto S^1.
Therefore $\pi_1(\mathbb{R}^2\setminus\{0\}) \cong \pi_1 S^1 \cong \mathbb{Z}$.

Then any closed path γ in $\mathbb{R}^2\setminus\{0\}$ is homotopic to kS^1, for $k\in\mathbb{Z}$, where negative corresponds to clockwise loops and $0S^1$ is a point.

Then by the calculation in part a,

$$\int_\gamma \omega = \int_{kS^1} \omega = k\int_{S^1}\omega = 2\pi k$$

for $k\in\mathbb{Z}$ when $\kappa=1$.

$\square$

175.4 / 115.2

(a) Cartan's formula states
$$\mathcal{L}_X = d \circ i_X + i_X \circ d$$
for a vector field X.

Proof: We recall the definition of $\mathcal{L}_X$. If ϕ is a local ~~flo~~ flow of X at p, then
$$(\mathcal{L}_X \omega)_p = \lim_{h \to 0} \frac{1}{h}\left((\phi_h^* \omega)_p - \omega_p\right)$$
In particular, this formula implies that $\mathcal{L}_X \circ d = d \circ \mathcal{L}_X$ since pullbacks commute w/ the exterior derivative d.

We first show the claim in the case that ω is a 0-form.
$n = 0$. Then ω is a smooth function.
Then
$$\mathcal{L}_X \omega = X(f) = \lim_{h \to 0} \frac{1}{h}(\omega \circ \phi_h - \omega) = d\omega(X)$$
hence $i_X \omega = 0$, this implies the Cartan formula holds for $n = 0$.

We now show that the claim extends through the wedge product. Suppose that the claim holds for ω, η (w/ ω a k-form). Then by the Leibniz rule
$$\begin{aligned}
d \circ i_X(\omega \wedge \eta) + i_X \circ d(\omega \wedge \eta) &= d(i_X\omega \wedge \eta + (-1)^k \omega \wedge i_X \eta) + i_X(d\omega \wedge \eta + (-1)^k \omega \wedge d\eta) \\
&= d(i_X\omega) \wedge \eta + \cancel{(-1)^{k-1} i_X\omega \wedge d\eta} + \cancel{(-1)^k d\omega \wedge i_X\eta} + \omega \wedge d(i_X\eta) \\
&\quad + i_X(d\omega) \wedge \eta + \cancel{(-1)^{k+1} d\omega \wedge i_X \eta} + (-1)^k i_X\omega \wedge d\eta + \omega \wedge i_X(d\eta) \\
&= (d(i_X\omega) + i_X(d\omega)) \wedge \eta + \omega \wedge (d(i_X\eta) + i_X(d\eta)) \\
&= (\mathcal{L}_X \omega) \wedge \eta + \omega \wedge (\mathcal{L}_X \eta) \\
&= \mathcal{L}_X(\omega \wedge \eta)
\end{aligned}$$

Finally, we show the formula extends through the exterior derivative. Suppose it holds for ω. Then
$$\mathcal{L}_X(d\omega) = d \circ \mathcal{L}_X(\omega) = (d \circ d \circ i_X + d \circ i_X \circ d)\omega = d(i_X(d\omega)) = (d \circ i_X + i_X \circ d)(d\omega).$$
As any form can locally be written as the wedge product of differentials

of 0-forms (local coordinates), this implies that the formula holds locally for any form. Therefore Cartan's formula holds $\forall$ forms. $\square$

(b) Define $dV = dx \wedge dy \wedge dz$ to be the volume form.

Suppose that X has a flow Φ that preserves volume.
Then $\Phi^* dV = dV$ and so

$$\mathcal{L}_X dV = \lim_{h \to 0} \frac{1}{h}\left(\Phi_h^* dV - dV\right) = 0$$

Therefore by Cartan's formula, if $X = X_1 \partial/\partial x + X_2 \partial/\partial y + X_3 \partial/\partial z$,

$$\begin{aligned} 0 &= (d \circ i_X + i_X \circ d) dV \\ &= d(i_X dV) \\ &= d(X_1 dy \wedge dz - X_2 dx \wedge dz + X_3 dx \wedge dy) \\ &= \left(\frac{\partial X_1}{\partial x} + \frac{\partial X_2}{\partial y} + \frac{\partial X_3}{\partial z}\right) dx \wedge dy \wedge dz \\ &= \mathrm{Div}(X)\, dV \end{aligned}$$

Since dV is non-vanishing, this implies that $\mathrm{Div}(X) = 0$ as claimed.

Now suppose that $\mathrm{Div}(X) = 0$. Then by the above computations, $\mathcal{L}_X dV = 0$. Let Φ be a flow of X. Then $\forall h$.

$$\begin{aligned} \mathcal{L}_X(\Phi_h^* dV) &= (d \circ i_X + i_X \circ d)(\Phi_h^* dV) \\ &= \Phi_h^* (d \circ i_X + i_X \circ d) dV \\ &= 0 \end{aligned}$$

Therefore $\Phi_h^* dV$ is constant in h and hence $\Phi_h^* dV = \Phi_0^* dV = dV$.
Therefore Φ preserves volume. $\square$

Define

$$D = \ker(\underbrace{dx_3 - x_1 dx_2}_{\alpha}) \cap \ker(\underbrace{dx_1 - x_4 dx_2}_{\beta})$$

Since $T_p\mathbb{R}^4 \cong \mathbb{R}^4$, we can view α_p, β_p as ~~map~~ linear maps $\mathbb{R}^4 \to \mathbb{R}$

Then $\alpha_p = [0 \ -x_1 \ 1 \ 0]$ and $\beta_p = [1 \ -x_4 \ 0 \ 0]$ for $p = (x_1, x_2, x_3, x_4)$.

We can thus view

$$\ker \alpha_p \cap \ker \beta_p = \ker \begin{bmatrix} 1 & -x_4 & 0 & 0 \\ 0 & -x_1 & 1 & 0 \end{bmatrix} = \ker A_p$$

Analyzing the pivots, we find that A_p has rank 2 $\forall p$ and so has $\dim \ker A_p = 2$ $\forall p$. Therefore $D \equiv \{\ker A_p\}_p$ is a smooth distribution of rank 2 since A_p depends smoothly on p.

We claim that D is not an integrable distribution.

To show this, it suffices to ~~show~~ find $X, Y \in D$ s.t. $[X,Y] \notin D$.

Consider

$$X = \partial/\partial x_4$$

$$Y = x_4 \, \partial/\partial x_1 + \partial/\partial x_2 + x_1 \partial/\partial x_3$$

Then $\alpha(X) = \beta(X) = 0$ and

$$\alpha(Y) = x_1 - x_1 = 0$$

$$\beta(Y) = -x_4 + x_4 = 0$$

so $X, Y \in D$.

However,

$$\begin{aligned} [X,Y] &= XY - YX \\ &= \partial/\partial x_4 (x_4 \partial/\partial x_1 + \partial/\partial x_2 + x_1 \partial/\partial x_3) - Y(\partial/\partial x_4) \\ &= \partial/\partial x_1 - 0 \\ &= \partial/\partial x_1 \end{aligned}$$

Then $\alpha[X,Y] = 0$ but $\beta[X,Y] = 1$. and so $[X,Y] \notin D$.

Therefore D is not integrable. $\square$

175.1

Fix $x_1, \dots, x_n, y_1, \dots, y_n \in M$. Since M is connected, $\exists$ a smooth path γ_1 from x_1 to y_1. B/c M has dimension ≥ 2, and γ_1 is a smooth 1-dimensional submanifold, $M \setminus \gamma_1$ is connected. Therefore $\exists$ a smooth path γ_2 from x_2 to y_2 in $M \setminus \gamma_1$, so $\gamma_1 \cap \gamma_2 = \emptyset$. Iterating this process, we can construct $\gamma_i : [0,1] \to M$ from x_i to y_i s.t. $\gamma_i \cap \gamma_j = \emptyset \ \forall i \neq j$.

Since $\gamma_1, \dots, \gamma_n$ are compact subsets of M, for each i we may choose an open neighborhood U_i of γ_i s.t. $U_i \cap U_j = \emptyset \ \forall i \neq j$.

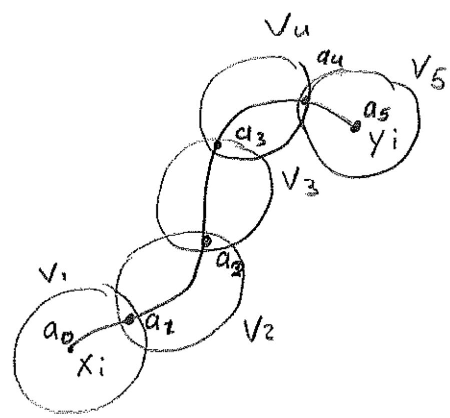
We claim that we may construct a diffeomorphism $f_i : M \to M$ s.t. $f_i = id$ on $M \setminus U_i$ and $f(x_i) = y_i$.

Assuming this claim, taking $f = f_1 \circ \dots \circ f_n$ ~~wot~~ would be a diffeomorphism $M \to M$ s.t. $f(x_i) = y_i \ \forall i$.

We now show the claim. Fix some i. Since M is smooth and U_i is open, for each $p \in \gamma_i$ $\exists$ a neighborhood V_p of p s.t. $V_p \subset U_i$ and V_p is diffeomorphic to an open ball in $\mathbb{R}^n$. Since γ_i is compact, we may take a finite partition $0 = t_1 < t_2 < \dots < t_k = 1$ of $[0,1]$ s.t. $V_{\gamma_i(t_1)}, \dots, V_{\gamma_i(t_k)}$ cover γ_i. By adding finitely many neighborhoods, we may assume that $V_{\gamma_i(t_j)} \cap V_{\gamma_i(t_{j+1})} \neq \emptyset \ \forall j$.

Define $V_j = V_{\gamma_i(t_j)}$ for ease of notation.

Let $a_0 = x_i$ and $a_k = y_i$. For each $j = 1, \dots, k-1$, choose $a_j \in V_j \cap V_{j+1} \cap \gamma_i$.



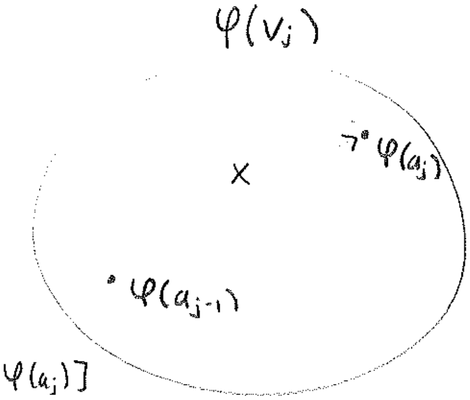
We claim that $\forall j = 1, \dots, k$ $\exists$ a diffeomorphism $f_i^j : M \to M$ s.t. $f_i^j = id$ on $M \setminus U_i$ and $f_i^j(a_{j-1}) = a_j$.

Assuming this claim, taking $f_i = f_i^k \circ \dots \circ f_i^1$ would then be a diffeomorphism $M \to M$ s.t. $f_i(x_i) = y_i$ and $f = id$ on $M \setminus U_i$.

To complete the proof, it thus suffices to construct f_i^j. $\longrightarrow$

Fix i, j. By construction, $a_{j-1}, a_j \in V_j$ and V_j is diffeomorphic to an open ball in $\mathbb{R}^n$, via a diffeomorphism φ.

Define X to be the vector field on $\varphi(V_j)$ given by the vector $\varphi(a_j) - \varphi(a_{j-1})$.



Let Ψ be a bump function s.t. $\Psi = 1$ on a neighborhood of the line segment $[\varphi(a_{j-1}), \varphi(a_j)]$ and $\Psi = 0$ outside of a compact neighborhood of $[\varphi(a_{j-1}), \varphi(a_j)]$ inside $\varphi(V_j)$. Then ΨX is a smooth vector field on $\varphi(V_j)$. Let Φ_t be the global flow of ΨX. ~~and assume that Φ~~ Then Φ_t is a diffeomorphism of $\varphi(V_j)$ that is the identity near $\partial \varphi(V_j)$. WLOG, assume that $\Phi_1(a_{j-1}) = a_j$.

We define $f_i^j = \varphi^{-1} \circ \Phi_t \circ \varphi$ on V_j and $f_i^j = id$ elsewhere.

Then f_i^j is a diffeomorphism $M \to M$ s.t. $f_i^j(a_{j-1}) = a_j$ and $f_i^j = id$ outside of U_i.

Letting $f_i = f_i^k \circ .. \circ f_i^1$ and $f = f_n \circ ... \circ f_1$ thus concludes. $\square$

175.2

Let K denote the space of skew-symmetric $2n \times 2n$ matrices.
We note that $\Omega^T = \begin{pmatrix} 0 & -I_n^T \\ I_n^T & 0 \end{pmatrix} = -\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = -\Omega$ and so $\Omega \in K$.
We also recall that K is a smooth submanifold of $M_{2n}(\mathbb{R})$ of dimension $(2n-1)+(2n-2)+\dots+1+0 = 2n-1 \cdot \frac{2n-1+1}{2} = n(2n-1)$.

Define $f: M_{2n} \to K$ by $f(A) = A^T \Omega A$. Then $f^{-1}(\Omega) = S$.
To show that S is a submanifold, it thus suffices to show that Ω is a regular value of f.

We recall that $T_A M_{2n} \cong M_{2n}$ and similarly that $T_A K \cong K$.
We can thus write $dF_A : M_{2n} \to K$ for all $A \in M_{2n}$.
Fix $A \in S$ and $M \in M_{2n}$. Then by definition,

$$\begin{aligned} dF_A(M) &= \lim_{t \to 0} \frac{1}{t}\left((A+Mt)^T \Omega (A+Mt) - A^T \Omega A \right) \\ &= \lim_{t \to 0} \frac{1}{t}\left(\Omega + tM^T \Omega A + tA^T \Omega M + t^2 M^T \Omega M - \Omega \right) \\ &= M^T \Omega A + A^T \Omega M \end{aligned}$$

Since Ω is skew-symmetric, we note that $dF_A(M) \in K \ \forall A, M$.
We claim that dF_A is surjective $\forall A \in S$.
Fix some $B \in K$. Then B can be written in block form as

$$B = \begin{bmatrix} C & D \\ D & C \end{bmatrix}$$

where C, D are skew-symmetric. Then

$$\begin{aligned} dF_A\left(\tfrac{1}{2} A \begin{bmatrix} -D & -C \\ C & D \end{bmatrix}\right) &= \begin{bmatrix} D & -C \\ C & -D \end{bmatrix} A^T \Omega A + A^T \Omega A \begin{bmatrix} -D & -C \\ C & D \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} D & -C \\ C & -D \end{bmatrix}\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}\begin{bmatrix} -D & -C \\ C & D \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} C & D \\ D & C \end{bmatrix} + \frac{1}{2}\begin{bmatrix} C & D \\ D & C \end{bmatrix} \\ &= B \end{aligned}$$

Therefore dF_A is surjective. As this holds $\forall A \in S$, this implies S is a smooth submanifold of dimension

$$4n^2 - n(2n-1) = 2n^2 + n. \qquad \square$$

175.3

We recall that S^n has a nonvanishing vector field iff n is odd.

Therefore by Poincaré-Hopf, $\chi(S^n)=0$ if n is odd.

~~and $\chi(S^n)$~~

We claim that $\chi(S^n)=2$ if n is even.

By Poincaré-Hopf, it suffices to construct a vector field ~~on~~ X on S^n s.t. X has 2 zeros, each w/ index 1.

For even $n=2k$, we can view

$$S^n \subset \mathbb{R}^{2k+1} \cong \mathbb{C}^k \times \mathbb{R}$$

We recall that $T_p S^n \cong \{v \in \mathbb{C}^k \times \mathbb{R} : \langle v, p\rangle = 0 \text{ (i.e. } p \perp v)\}$.

Therefore we construct ~~a~~ X on S^n via

$$X_{(p,x)} = (ip, 0)$$

~~Then~~ since $(ip,0)\cdot(p,x) = ip\cdot p = 0$, this is well-defined.

Moreover, X only has zeros at $(0,\pm 1)$.

We now compute the index of X at $(0,\pm1)$.

Let $U_\pm$ be the neighborhood $\{(p,x)\in S^n : \pm x > 0\}$.

Then $U_\pm$ is a neighborhood of $(0,\pm1)$ that does not contain $(0,\mp1)$.

Therefore $\operatorname{ind}_{(0,\pm1)} X = \deg X|_{\partial U_{\pm 1}}$

where $X|_{\partial U_\pm}$ can be realized as the map $\partial U_\pm \cong S^{n-1} \to S^{n-1}$. By construction, $X|_{\partial U_\pm} : p \longmapsto ip$

and so $\deg X|_{\partial U_\pm} = 1$. Therefore

$$\chi(S^n) = \operatorname{ind}_{(0,1)} X + \operatorname{ind}_{(0,-1)} X = 1+1 = 2$$

as claimed. $\square$

16F.1

Consider the open cover $M \setminus A, M \setminus B$ of M.

Let ψ, φ be a partition of unity subordinate to $\{M \setminus A, M \setminus B\}$.

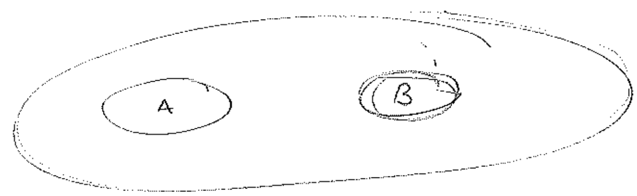
Then $\psi = 0$ on A and $\varphi = 0$ on B.

hence $\psi + \varphi = 1$, this implies

$\psi = 0$ on A and

$\psi = 1$ on B

as desired.



$\square$

17F. 2

Let $M_n(\mathbb{R})$ be the space of $n \times n$ real matrices

(a) Define $F: M_n(\mathbb{R}) \to S_n(\mathbb{R})$ where $S_n(\mathbb{R})$ is the space of $n \times n$ symmetric real matrices by

$$F: A \longmapsto AA^T$$

Then $F^{-1}(Id) = O(n)$. To show $O(n) \subset M_n(\mathbb{R})$ is a smooth submanifold, it thus suffices to show that Id is a regular value of F.

Fix some $A \in O(n)$. We claim that $dF_A(\cdot)$ is surjective. We recall that $T_M M_n(\mathbb{R}) \cong M_n(\mathbb{R})$ and similarly $T_M S_n(\mathbb{R}) \cong S_n(\mathbb{R})$. Fixing $M \in M_n(\mathbb{R})$, we compute

$$dF_A(M) = \lim_{t \to 0} \frac{(A+tM)(A+tM)^T - AA^T}{t}$$

$$= \lim_{t \to 0} \frac{t(MA^T + AM^T) + t^2 MM^T}{t}$$

$$= MA^T + AM^T$$

$MA^T + AM^T = C$
$M = \frac{1}{2} CA$
$\frac{1}{2} CAA^T + \frac{1}{2} AA^T C^T$

Now fix some $C \in S_n(\mathbb{R})$. Then

$$dF_A(\tfrac{1}{2}CA) = \frac{1}{2} CAA^T + \frac{1}{2} AA^T C^T$$

$$= \frac{1}{2} C + \frac{1}{2} C = C$$

Therefore $dF_A: T_A M_n(\mathbb{R}) \to T_{Id} S_n(\mathbb{R})$ is surjective $\forall A \in O(n)$. Thus ~~regular value~~ Id is a regular value of F and so $O(n) = F^{-1}(Id)$ is a smooth submanifold. □

(b) It suffices to construct vector fields

$$X_1, \dots, X_m \quad \text{w/ } m = \dim O(n)$$

s.t. $X_1, \dots, X_m$ is a basis for $T_p O(n)$ at all p.

Consider $T_{Id} O(n)$. Let $v_1, \dots, v_m$ be a basis for $T_{Id} O(n)$. ~~Define~~ We note that $O(n)$ is a Lie group and then define $X_1, \dots, X_m$ by

$$X_j(p) = dp_{Id} v_j$$

Since $O(n)$ is a ~~vect~~ Lie group, $dp_{Id} : T_{Id} O(n) \to T_p O(n)$ is an isomorphism. Therefore $X_1(p), \dots, X_m(p)$ is a basis for $T_p O(n)$ $\forall p$.

Therefore $O(n)$ has trivial tangent bundle. $\square$

17F.5

Since S^2 is compact and orientable, we recall that integration defines an isomorphism $H^2_{dR}(S^2) \to \mathbb{R}$ via

$$H^2_{dR} \to \mathbb{R} : \omega \longmapsto \int_{S^2} \omega$$

Therefore since all 2-forms on S^2 are closed,

$z^n dA$ is exact iff $\int_{S^2} z^n dA = 0$.

If n is odd, then $z^n dA$ is odd under reflections across the xy plane and hence $\int z^n dA = 0$.

If n is even, then z^n is non-negative and strictly positive outside of the xy plane. Therefore $\int z^n dA > 0$.

Thus conclude that $z^n dA$ is exact iff n is even.

17F.6 / 21F.7

(a) A manifold M is orientable if $\exists$ an atlas $\{U_\alpha, \phi_\alpha\}$ such that every ~~transition~~ map $\phi_\alpha \circ \phi_\beta^{-1}$ is orientation preserving. We define orientation preserving to be the requirement $\det(\phi_\alpha \circ \phi_\beta^{-1}) > 0$.

(b) Define

$$\tilde{M} = \{(p, O_p) : p \in M \text{ and } O_p \text{ is an orientation of } T_pM\}$$

We define a topology on $\tilde{M}$ as follows.
For each open set $U \subset M$ w/ ~~consistent~~ orientation O on U, we define $V_{(U,O)} = \{(p, O_p') : O_p' = O_p\}$. These sets define a basis for the topology on $\tilde{M}$.
Let π be the projection $\tilde{M} \to M : (p, O_p) \to p$.
Since any manifold is locally orientable, $\forall (p, O_p) \in \tilde{M}$ $\exists$ an open neighborhood U of p and orientation O' on U s.t. $O_p' = O_p$.
Then $\pi|_{V(U,O')}$ is a diffeomorphism onto U.
Therefore $\tilde{M}$ has a smooth structure.
Since any $p \in M$ has 2 orientations $O_p, -O_p$, this also implies that $\tilde{M}$ is a double cover for M.
Additionally, by construction of the topology on $\tilde{M}$, we can define an orientation on $\tilde{M}$ via orienting $T_{p,O_p}\tilde{M} \cong T_pM$ w/ O_p and $T_{p,-O_p}\tilde{M} \cong T_pM$ w/ $-O_p$.
Therefore $\tilde{M}$ is an orientable double cover.
Suppose that $\tilde{M}$ is disconnected. Then $\tilde{M} = U \amalg V$ where U, V are ~~???~~ open. Since $\tilde{M}$ is a double cover and M is connected, $\pi|_U : U \to M$ is a diffeomorphism. Then M is orientable $*$ $\square$

17F.1

It suffices to work locally. Therefore ~~it suffices to~~ by linearity it suffices to consider $\omega = f\,dx$.

By direct computation, $d\omega = df \wedge dx$ and so

$$d\omega(X,Y) = (df \wedge dx)(X,Y)$$
$$= df(X)dx(Y) - df(Y)dx(X)$$
$$= X(f)Y(x) - Y(f)X(x)$$

We also compute

$$X(\omega(Y)) = X(f\,dx(Y))$$
$$= X(fY(x))$$
$$= X(f)Y(x) + fXY(x)$$

and similarly $Y(\omega(X)) = Y(f)X(x) + fYX(x)$.

Then

$$d\omega(X,Y) - X(\omega(Y)) + Y(\omega(X)) = fYX(x) - fXY(x)$$
$$= -f[X,Y](x)$$
$$= -f\,dx[X,Y]$$
$$= -\omega([X,Y])$$

Which implies

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$$

as desired.

$\square$

015.2

$$\cancel{\int_M d\omega = \int_{\partial_0 M} \omega}$$

By Stokes' theorem, since $\kappa \wedge \beta$ is an $n\text{-}1$-form,

$$\int_M d(\kappa \wedge \beta) = \int_{\partial M} i^* \kappa \wedge \beta$$

$$= \int_{\partial_0 M} i_0^*(\kappa \wedge \beta) + \int_{\partial_1 M} i_1^*(\kappa \wedge \beta)$$

B/c pullbacks commute w/ wedge products,

$$\int_M d(\kappa \wedge \beta) = \int_{\partial_0 M} (i_0^* \kappa) \wedge (i_0^* \beta) + \int_{\partial_1 M} (i_1^* \kappa) \wedge (i_1^* \beta)$$

$$= 0$$

Then by the Leibniz rule, since κ is a p-form

$$0 = \int_M d\kappa \wedge \beta + (-1)^p \kappa \wedge d\beta$$

$$\Rightarrow \int_M d\kappa \wedge \beta = (-1)^{p+1} \int_M \kappa \wedge d\beta$$

As desired. $\square$

10F.2 / 16B.2

Suppose that X, Y are submanifolds of $\mathbb{R}^n$. We claim that $X+a \pitchfork Y$ for a.e $a \in \mathbb{R}^n$. To show this, we use the transversality theorem.

Define $F: X \times \mathbb{R}^n \to \mathbb{R}^n$ by $(x,a) \longmapsto x+a$.
We claim that F is transversal to Y. To show this, it suffices to show that $dF_{(x,a)}$ is surjective $\forall$ (x,a).
Working in local coordinates, we may write $dF_{(x,a)}$ as a $n \times (m+n)$ matrix where $m = \dim X$. Doing so,

$$dF_{(x,a)} = \begin{bmatrix} * & I \end{bmatrix}$$ where I is the $n \times n$ identity matrix.

Therefore $dF_{(x,a)}$ has rank n and so is surjective.
In particular, $\forall$ (x,a) s.t. $F(x,a) \in Y$,

$$\operatorname{Im}(dF_{(x,a)}) + T_{F(x,a)} Y \cong \mathbb{R}^n \cong T_{F(x,a)} \mathbb{R}^n$$

and so $F \pitchfork Y$.
B/c F is boundaryless, the transversality theorem then implies that $f_a = F(\cdot, a)$ is transversal to Y for a.e $a \in \mathbb{R}^n$.
Consider some a s.t. $f_a \pitchfork Y$. Then $\forall$ $x \in X$ s.t. $x+a \in Y$,

$$\operatorname{Im}(d_x f_a) + T_{x+a} Y = T_{x+a} \mathbb{R}^n$$

$$\Rightarrow T_{x+a}(X+a) + T_{x+a} Y = T_{x+a} \mathbb{R}^n$$

As this holds $\forall$ $x+a \in Y$, this implies that $T_z(X+a) + T_z Y = T_z \mathbb{R}^n$ $\forall$ $z \in X \cap Y$. Therefore $(X+a) \pitchfork Y$.
As this holds for a.e $a \in \mathbb{R}^n$, this concludes.

$\square$

10F.2 / 10S.2 - Alternate

Define $F: X \times Y \to \mathbb{R}^n$ by $(x,y) \mapsto y-x$. Then F is smooth and so Sard's theorem implies that for a.e $a \in \mathbb{R}^n$, a is a regular value of F.

Consider such an a. We claim $(X+a) \pitchfork Y$.
Suppose $\exists$ (x,y) s.t. $F(x,y) = a \iff x+a=y$.
Since a is a regular value of F,

$$
\begin{aligned}
n &= \dim\left(dF_{(x,y)}\left(T_{(x,y)}(X,Y)\right)\right) \\
&= \dim\left(dF_{(x,y)}\left(T_xX \oplus T_yY\right)\right) \\
&= \dim\left(T_xX + T_yY\right)
\end{aligned}
$$

Then since $x+a=y$,

$$T_{x+a}(X+a) + T_yY = T_y\mathbb{R}^n$$

As this holds $\forall$ x,y s.t. $x+a=y$, this implies that $(X+a) \pitchfork Y$.
Therefore $(X+a) \pitchfork Y$ for a.e $a \in \mathbb{R}^n$. $\square$

19F.1 / 18S.6 / 18F.5 /

Let M be a smooth compact 3-dimensional submanifold of $\mathbb{R}^3$ w/ smooth boundary ∂M. Let X be a smooth vector field on $\mathbb{R}^3$. The classical divergence theorem states that

$$\int_M \operatorname{div}(X)\, dV = \int_{\partial M} \langle X, N\rangle dA$$

where $\operatorname{div}(X) = \frac{\partial X_1}{\partial x} + \frac{\partial X_2}{\partial y} + \frac{\partial X_3}{\partial z}$ ~~there~~ for $X = X_1 \partial/\partial x + X_2 \partial/\partial y + X_3 \partial/\partial z$, N is the unit normal of ∂M w/ the boundary orientation, and $dA = \iota^* i_N dV$ where $\iota : \partial M \hookrightarrow M$ is the inclusion.

Define $T = X - \langle X, N\rangle N$ on ∂M. Then T is the component of X tangent to ∂M, and so can be viewed as a section of $T\partial M$.

We claim that $\iota^* i_T dV = 0$ on ∂M. Fix some $p \in \partial M$ and vectors $V_1, V_2 \in T_p(\partial M)$. Since $T_p(\partial M)$ is 2-dimensional, T, V_1, V_2 are linearly dependent. Then

$$\iota^* i_T dV(V_1, V_2) = \iota^* dV(T, V_1, V_2) = 0$$

Therefore $\iota^* i_T dV = 0$. By linearity, this implies that

$$0 = \iota^* i_T dV = \iota^* \left(i_X - \langle X, N\rangle i_N \right) dV$$

$$\Rightarrow \quad \iota^* i_X dV = \langle X, N\rangle \iota^* i_N dV = \langle X, N\rangle dA$$

By direct computation,

$$i_X dV = i_X (dx \wedge dy \wedge dz)$$

$$= X_1\, dy \wedge dz - X_2\, dx \wedge dz + X_3\, dy \wedge dz$$

and $$d(i_X dV) = \left(\frac{\partial X_1}{\partial x} + \frac{\partial X_2}{\partial y} + \frac{\partial X_3}{\partial z} \right) dx \wedge dy \wedge dz = \operatorname{div}(X)\, dV.$$

Then by Stokes' theorem

$$\int_M \operatorname{div}(X)\, dV = \int_M d(i_X dV) = \int_{\partial M} \iota^* i_X dV = \int_{\partial M} \langle X, N\rangle dA$$

as desired. $\square$

19F.3

Suppose first that $n = 2k-1$ is odd.

Consider $S^n \subset \mathbb{R}^{2k}$. For each $p \in S^n$, we can view $T_p S^n$ in the classical way as $T_p S^n \cong \{v \in \mathbb{R}^{2k} : v \cdot p = 0\}$.

We then define a non-vanishing vector field V on S^n by

$$V : (x_1, y_1, \ldots, x_k, y_k) \longmapsto (-y_1, x_1, -y_2, x_2, \ldots, -y_k, x_k).$$

Then V is well-defined and is non-vanishing $\overset{b/c}{\cancel{s.t.}}$ $p \neq 0 \ \forall \ p \in S^n$.

We recall that $\mathbb{RP}^n \cong S^n / x \sim -x$. We claim that V factors through the quotient map $\pi : S^n \to \mathbb{RP}^n$ to a nonvanishing vector field on $\mathbb{RP}^n$. To show this, it must be shown that $d\pi_p V_p = d\pi_{-p} V_{-p}$. Let f be the anti-podal map $p \mapsto -p$ on S^n.

Then $\pi \circ f = \pi$ and so

$$d\pi_p V_p = d(\pi \circ f)_p V_p = d\pi_{-p} df_p V_p = d\pi_{-p}(-V_p)$$

By construction, $-V_p = V_{-p}$ and so $d\pi_p V_p = d\pi_{-p} V_{-p}$.

Therefore V descends to a non-vanishing vector field on $\mathbb{RP}^n$ as desired.

Suppose instead that n is even. We recall that $\mathbb{RP}^n$ can be constructed w/ 1 k-cell for $k = 0, 1, \ldots, n$. Then since n is even,

$$\chi(\mathbb{RP}^n) = \sum_{k=0}^{n} (-1)^k = 1$$

By Poincare-Hopf, this implies that any vector field on $\mathbb{RP}^n$ must have a zero.

Therefore $\mathbb{RP}^n$ has a non-vanishing vector field iff n is odd.

19F.5

$$\cancel{(L_g)_* X = X \circ g}$$

$$\cancel{(L_g)_* [X,Y] = (L_g)_* (XY - YX)}$$

$$\cancel{= X}$$

Suppose that X, Y are left-invariant vector fields on G.
We recall that since G is a Lie group, $\forall g$ and $\forall$ vector fields Z on G,

$$(L_g)_* Z(f) = Z(f \circ L_g)$$

Then in particular, $\forall f$,

$$\begin{aligned}(L_g)_* [X,Y](f) &= (L_g)_* (X \circ Y(f) - Y \circ X(f)) \\ &= X \circ Y(f \circ L_g) - Y \circ X(f \circ L_g) \\ &= X \circ (L_g)_* Y(f) - Y \circ (L_g)_* X(f)\end{aligned}$$

Since X, Y are left-invariant,

$$\begin{aligned}(L_g)_* [X,Y](f) &= X \circ Y(f) - Y \circ X(f) \\ &= [X,Y](f)\end{aligned}$$

As this holds $\forall f$, this implies that $[X,Y]$ is left-invariant. $\square$

165.1

We recall that all straight lines in $\mathbb{R}^2$ can be written as $ax+by=c$ for some $a,b,c \in \mathbb{R}$ w/ $(a,b) \neq (0,0)$. Moreover, $\forall t \in \mathbb{R}\setminus\{0\}$ $tax+bty=ct$ defines the same line as $ax+by=c$.

Therefore we can view the space of all lines in $\mathbb{R}^2$ as a subset of $\mathbb{RP}^2$, specifically

$$U = \{[a:b:c] : (a,b) \neq (0,0)\} \subset \mathbb{RP}^2$$

Equivalently,

$$U = \mathbb{RP}^2 \setminus \{[0:0:1]\}$$

and so U is open subset of $\mathbb{RP}^2$. Therefore U can be given a smooth structure via the smooth structure on $\mathbb{RP}^2$.

We claim that U is not orientable.

We recall that $\mathbb{RP}^2$ is non-orientable and hence has a connected 2-sheeted orientation cover M. Let p_1, p_2 be the pre-images of $[0:0:1]$ in M. Then $M\setminus\{p_1,p_2\}$ is an orientation cover of U. Since M is 2-dimensional, $M\setminus\{p_1,p_2\}$ is connected and so U has a connected orientation cover. Therefore U is not orientable.

165.3

Let $a_1, \ldots, a_n$ denote the zeros of X.
For each i, let k_i denote the order of the zero at a_i.
Then we can write

$$X(z) = (z-a_i)^{k_i} q_i(z)$$

for each i where q_i is a polynomial w/ $q_i(a_i) \neq 0$
Then $X(z)$ ~~is of degree~~ has index k_i at a_i $\forall i$. Thus

$$\sum_{i=1}^{n} \operatorname{ind}_{a_i} X = \sum_{i=1}^{n} k_i = 2016$$

by the fundamental theorem of algebra. $\square$

165.10

(a) As given,

$$\kappa = x_1\, dx_2 \wedge dx_3 \wedge dx_4 - x_2\, dx_1 \wedge dx_3 \wedge dx_4 + x_3\, dx_1 \wedge dx_2 \wedge dx_4 - x_4\, dx_1 \wedge dx_2 \wedge dx_3$$

By Stokes' theorem,

$$\int_{S^3} i^*\kappa = \int_{B^4} d\kappa = 4\int_{B^4} dV = 4\mathrm{vol}(B^4)$$

where B^3 is the open unit ball in $\mathbb{R}^4$.

(b) By direct calculation,

$$d\gamma = \frac{-2k(x_1 dx_1 + x_2 dx_2 + x_3 dx_3 + x_4 dx_4)}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^{k+1}} \wedge \kappa + \frac{d\kappa}{(x_1^2 + \dots + x_4^2)^k}$$

$$= \frac{2}{(x_1^2 + \dots + x_4^2)}\left(dV - \frac{k(x_1 dx_1 + \dots + x_4 dx_4)\wedge\kappa}{(x_1^2 + \dots + x_4^2)}\right)$$

$$= \frac{2}{(x_1^2 + \dots + x_4^2)}\left(2dV - \frac{k(x_1^2 + \dots + x_4^2)dV}{(x_1^2 + \dots + x_4^2)}\right)$$

$$= \frac{2(2-k)dV}{(x_1^2 + \dots + x_4^2)}$$

Therefore $d\gamma = 0$ iff $k = 2$.

~~We recall that $\mathbb{R}^4 \setminus \{0\}$ deformation retracts onto $S^3 \subset \mathbb{R}^4 \setminus \{0\}$ via the homotopy~~

$$\cancel{H_t : x \longmapsto \frac{x}{|x|^{2t}}}$$

~~for $t \in [0,1]$.~~

Suppose that γ is exact for some k. Then $\gamma = d\eta$. Then since d commutes w/ pullbacks, $i^*\gamma$ is exact on S^3. Since $H^3_{dR}(S^3) \cong \mathbb{R}$ w/ isomorphism $\omega \mapsto \int_{S^3}\omega$, this implies that $\int_{S^3} i^*\gamma = 0$. However, by construction $i^*\gamma = i^*\kappa$. Therefore

$$0 = \int_{S^3} i^*\gamma = \int_{S^3} i^*\kappa = 4\mathrm{vol}(B^4) \neq 0$$

and so γ is never exact. $\square$

145.4 405 240 515

Suppose that $\omega_1, \dots, \omega_n$ are linearly dependent.
Then $\exists$ $f_1, \dots, f_{n-1}$ s.t. $\omega_n = f_1\omega_1 + \dots + f_{n-1}\omega_{n-1}$.
Therefore

$$\omega_1 \wedge \dots \wedge \omega_n = \sum_{i=1}^{n-1} f_i\, \omega_1 \wedge \dots \wedge \omega_{n-1} \wedge \omega_i$$

For all i, since ω_i is a 1-form, ~~the lab~~

$$\omega_i \wedge \omega_i \underset{\text{swap order}}{=} -\omega_i \wedge \omega_i$$

and so $\omega_i \wedge \omega_i = 0$. Then $\omega_1 \wedge \dots \wedge \omega_n = 0$ as desired.

Now suppose that $\omega_1, \dots, \omega_n$ are linearly independent.
Then $\forall$ $p \in M$, $(\omega_1)_p, \dots, (\omega_n)_p \in T_p^*M$ are linearly independent.
This implies $\exists$ dual vectors $v_1, \dots, v_n \in T_pM$ s.t. $(\omega_i)_p(v_j) = \delta_{ij}$.
Then $(\omega_1 \wedge \dots \wedge \omega_n)_p(v_1, \dots, v_n) = (\omega_1)_p(v_1) \cdot \dots \cdot (\omega_n)_p(v_n) = 1 \neq 0$.
Therefore $\omega_1 \wedge \dots \wedge \omega_n \neq 0$.

$\square$

19S.2

We recall that S^1 is ~~parallel~~ parallelizable since it admits a ~~nonze~~ nonvanishing vector field. Therefore

$$TS^1 \cong S^1 \times \mathbb{R}$$

Since $M = f^{-1}\{0\}$, M is a dimension n manifold (codimension 1).

~~w trivial normal bundle defined by~~ $\cancel{\nabla f}$.

Moreover, ∇f is a non-vanishing normal vector field on M and so M has a trivializable normal bundle. Then

$$NM \cong M \times \mathbb{R}$$

From this, it follows that $TM \times \mathbb{R} \cong M \times \mathbb{R}^{n+1}$ since $N_pM \oplus T_pM \cong \mathbb{R}^{n+1}$. Therefore, combining all these facts,

$$\begin{aligned} T(M \times S^1) &\cong TM \oplus TS^1 \\ &\cong TM \oplus (S^1 \times \mathbb{R}) \\ &\cong (TM \times \mathbb{R}) \oplus S^1 \\ &\cong M \times \mathbb{R}^{n+1} \times S^1 \\ &\cong M \times S^1 \times \mathbb{R}^{n+1} \end{aligned}$$

Therefore $T(M \times S^1)$ is trivializable and so $M \times S^1$ is parallelizable. □

19F.4

By Cartan's ~~ma~~ formula,

$$[\mathcal{L}_X, \mathcal{L}_Y] = [\mathcal{L}_X, d\circ i_Y + i_Y\circ d],$$
$$= [\mathcal{L}_X, d\circ i_Y] + [\mathcal{L}_X, i_Y\circ d]$$

By Cartan's formula, we also see that $\mathcal{L}_X$ commutes w/ d since

$$\mathcal{L}_X\circ d = (d\circ i_X + i_X\circ d) = d\circ i_X\circ d = d(i_X\circ d + d\circ i_X) = d\circ\mathcal{L}_X$$

Therefore

$$[\mathcal{L}_X, \mathcal{L}_Y] = d\circ[\mathcal{L}_X, i_Y] + [\mathcal{L}_X, i_Y]\circ d$$

We recall that $[\mathcal{L}_X, i_Y] = i_{[X,Y]}$. (Shown in lemma if time permits.) Therefore

$$[\mathcal{L}_X, \mathcal{L}_Y] = d\circ i_{[X,Y]} + i_{[X,Y]}\circ d = \mathcal{L}_{[X,Y]}$$

as desired.

$\square$

195.5

Suppose w is exact. Then $w = d\eta$ and so $\forall f : S^1 \to M$,

$$\int_{S^1} f^* w = \int_{S^1} d(f^*\eta)$$

We recall that since S^1 is compact and orientable, $H^1_{dR}(S^1) \cong \mathbb{R}$ w/ isomorphism $\theta \mapsto \int_{S^1}\theta$. Therefore

$$\int_{S^1} f^* w = \int_{S^1} d(f^*\eta) = 0$$

as desired, since w is closed.

Now suppose $\int_{S^1} f^* w = 0 \quad \forall\, f : S^1 \to M$.

~~Restrict~~ to path-connected case.

Fix a base point p_0. For each $p \in M$, $\exists$ a path $\gamma(p)$ from p_0 to p. ~~Since $\int_{S^1} f^* w = 0$~~,

Define

$$g(p) = \int_{\gamma(p)} w .$$

Since $\int_{S^1} f^* w = 0 \ \forall f$, g is independent of the choice of $\gamma(p)$. In particular, g is smooth by working locally.

Then by the fundamental theorem of calculus,

$$dg_p = \frac{d}{dt}\Big|_{t=1} \int_{\gamma(p)} w = w_p$$

~~so~~ and so $dg = w$ as desired.

$\square$

195.6

We construct an inverse map to show injectivity.

Since f is a covering map, for each p $\exists$ a neighborhood U of p s.t. $f^{-1}(U) = \coprod_{j=1}^{k} U_j$ for k independent of p, and $f : U_j \to U$ is a diffeomorphism.

Define g locally on U via

$$g|_U(\omega) = \frac{1}{k}\sum_{j=1}^{k} ((f|_{U_j})^{-1})^*(\omega|_U)$$

Since f is a smooth covering map, this is well-defined and extends globally.

We claim that g is well-defined on the level of cohomology.

Consider $\omega + d\kappa$. Then $\forall$ p, U as above

$$\begin{aligned} g|_U(\omega + d\kappa) &= \frac{1}{k}\sum_{j=1}^{k}((f|_{U_j})^{-1})^*(\omega|_U + d\kappa|_U) \\ &= g|_U(\omega) + \frac{1}{k}\sum_{j=1}^{k}((f|_{U_j})^{-1})^*(d\kappa|_U) \\ &= g|_U(\omega) + d\left(\frac{1}{k}\sum_{j=1}^{k}((f|_{U_j})^{-1})^*(\kappa|_U)\right) \end{aligned}$$

Therefore g is well-defined on cohomology.

Finally, for any $\omega \in H^k_{dR}(X)$, $\forall$ p, U,

$$\begin{aligned} g \circ f^*|_U(\omega) &= \frac{1}{k}\sum_{j=1}^{k}((f|_{U_j})^{-1})^*(f^*\omega|_U) \\ &= \frac{1}{k}\sum_{j=1}^{k}((f|_{U_j})^{-1} f|_U)^*(\omega|_U) \\ &= \frac{1}{k}\sum_{j=1}^{k}\omega|_U \\ &= \omega|_U \end{aligned}$$

and so $g \circ f^* = id$. Therefore f^* is injective. $\square$

18F.3

To show that $\ker(\Theta)$ is a distribution, it must be shown that $\ker\Theta$ is of consistent dimension. To show this, it suffices to show that $\operatorname{rank}\Theta_p = 1 \ \forall p \in S^{2n-1}$.

Since Θ maps onto $\mathbb{R}$, it suffices to find $v \in T_pS^{2n-1}$ $\forall p$ s.t. $\Theta_p(v) \neq 0$.

In fact, we may find a smooth vector field X s.t. $\Theta(X) = 1 \neq 0$.

Define $X = x^2\, \partial/\partial x^1 - x^1\, \partial/\partial x^2 + \dots + x^{2n}\, \partial/\partial x^{2n-1} - x^{2n-1}\, \partial/\partial x^{2n}$.

Then $X \in TS^{2n-1}$ since $X_p \perp p \ \forall p$. Moreover, by direct calculation,

$$\Theta(X) = (x^2)^2 + (x^1)^2 + \dots + (x^{2n})^2 + (x^{2n-1})^2 = 1 \neq 0$$

and so $\ker\Theta$ is of consistent dimension.

By direct computation,

$$d\Theta = -2\,dx^1\wedge dx^2 - \dots - 2\,dx^{2n-1}\wedge dx^{2n}$$

$$\Rightarrow \Theta\wedge d\Theta = -2\sum_{j=1}^{n}\sum_{\substack{i=1\\ i\neq j}}^{n}\left(x^{2j}dx^{2j-1}\wedge dx^i\wedge dx^{i+1} - x^{2j-1}dx^{2j}\wedge dx^i\wedge dx^{i+1}\right)$$

We note that there are no repeated terms $dx^a\wedge dx^b\wedge dx^c$ in this sum and so $(\Theta\wedge d\Theta)_p = 0$ iff $p = 0$. Since $p \neq 0 \ \forall p \in S^{2n-1}$, this implies $\Theta\wedge d\Theta \neq 0$. Frobenius' theorem then implies $\ker\Theta$ is not integrable.

$\square$

155.6

On S^2, $x^2+y^2+z^2=1$. Therefore by Stokes' theorem,

$$\int_{S^2} \iota^*\omega = \int_{S^2} \iota^*\left(\frac{x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= \int_{S^2} \iota^*(x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy)$$

$$= \int_{B^3} d(x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy)$$

$$= \int_{B^3} 3\,dV$$

$$= 3\,\mathrm{vol}(B^3) = 4\pi$$

depending on chosen normalization.

By direct computation $\forall\ (x,y,z) \neq 0$,

$$d\omega = \frac{3r^3\,dx\wedge dy\wedge dz - (x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy)\wedge \frac{3}{2}r(2x\,dx + 2y\,dy + 2z\,dz)}{r^6}$$

$$= \frac{3r^3 dV - \frac{3}{2}r(2x^2\,dx\wedge dy\wedge dz + 2y^2\,dx\wedge dy\wedge dz + 2z^2\,dV)}{r^6}$$

$$= \frac{(3r^3 - 3r^3)dV}{r^6} = 0$$

where $r = \sqrt{x^2+y^2+z^2}$ for ease of ~~nota~~ computations.

By definition, $j:(x,y,z) = (3x, 2y, 8z)$ pulls back S^2 diffeomorphically onto the ellipse $E = \{\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{64} = 1\}$.
We note that E contains S^2 and so we can consider the annular region A between S^2 and E.

Then by Stokes' theorem,

$$0 = \int_A d\omega = \int_{S^2} \iota^*\omega - \int_E k^*\omega = 4\pi - \int_{S^2} j^*\omega$$

and so $\int_{S^2} j^*\omega = 4\pi$ where $k: E \hookrightarrow \mathbb{R}^3\setminus\{0\}$ is the inclusion. $\square$

155.7

We say that a differential form w is closed if its exterior derivative $dw = 0$ and exact if $\exists$ some form η s.t. $d\eta = w$. Since $d^2 = 0$, if w is exact then it is closed. Therefore we may consider the quotient

$$H^k_{dR} = \frac{\{\text{closed } k\text{-forms}\}}{\{\text{exact } k\text{-forms}\}} = \frac{\ker d_k}{\operatorname{im} d_{k-1}}$$

where $d_k : \Omega^k \to \Omega^{k+1}$ is the exterior derivative on k-forms. By convention we take $d_{-1} = 0$.

We claim that $H^k_{dR}(S^1) \cong \mathbb{R}$ if $k = 0, 1$ and 0 ~~else~~ otherwise. Since S^1 has no non-trivial k-forms for $k \geq 2$, it follows that $H^k_{dR}(S^1) \cong 0 \quad \forall \quad k \geq 2$.

We first consider $k = 0$. Suppose that f is a 0-form i.e. a smooth function, on S^1, such that $df = 0$. Then, since S^1 has a global coordinate θ,

$$0 = df = \frac{\partial f}{\partial \theta} d\theta$$

and so $\frac{\partial f}{\partial \theta} = 0$. Therefore f is a constant. Since any constant is closed, this implies that $H^0_{dR}(S^1) \cong 0$.

Now consider $k = 1$. We claim that $[w] \mapsto \int_{S^1} w$ for $[w] \in H^1_{dR}(S^1)$ is a well-defined isomorphism $H^1_{dR}(S^1) \to \mathbb{R}$.

To show it is well-defined on cohomology, it suffices to show that $\int_{S^1} w + d\eta = \int_{S^1} w$. By Stokes' theorem, $\int_{S^1} d\eta = 0$ and so $\int_{S^1} w + d\eta = \int_{S^1} w$.

Therefore the map is well-defined on cohomology.

We now show injectivity. Suppose that $\int_{S^1} w = \int_{S^1} \varphi$.
Then $\int_{S^1} (w-\varphi) = 0$. Let $w - \varphi = g\,d\theta$. Then $\int_{S^1} g\,d\theta = 0$
which implies that $g = f'$ for some f. Then $g\,d\theta = df$ and
so $w = \varphi + df$. Therefore if $\int_{S^1} w = \int_{S^1} \varphi$ then $w = \varphi$ on the level
of cohomology. Thus the map is ~~surject~~ injective.
Finally, for surjectivity, we note that $\int_{S^1} d\theta = 2\pi$ and so
$\int \frac{a}{2\pi} d\theta = a$. $\forall a \in \mathbb{R}$. Therefore the map is surjective and hence
$H^1_{dR}(S^1) \cong \mathbb{R}$.

$\square$

15S.10/ 12F.7/ 21F.5

Suppose that $H^{2n+1}_{dR}(M) \cong \mathbb{R}^k$. Since M is compact and orientable, $H^{4n+2}_{dR}(M) \cong \mathbb{R}$.

We recall that the wedge product $\wedge : H^{2n+1}_{dR}(M) \times H^{2n+1}_{dR}(M) \to H^{4n+2}_{dR}(M)$ acts as an alternating bilinear map on cohomology. Therefore $\exists$ an equivalent alternating bilinear map $\hat{\wedge} : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$.

Since $\hat{\wedge}$ is bilinear, $\exists$ ~~an anti-symmetric~~ a matrix A s.t.

$$\hat{\wedge}(v,u) = v^T A u$$

Since $\hat{\wedge}$ is alternating, A is anti-symmetric.

We claim that A is invertible. Suppose $\exists$ u s.t. $Au=0$. Then $v^T A u = 0 \quad \forall \; v \in \mathbb{R}^k$. In particular, letting w be the $2n+1$-form equivalent to u, this implies that

$$\eta \wedge w = 0$$

$\forall$ $2n+1$-forms η. Then w, η are linearly dependent $\forall \eta$. or local coordinates

This is only possible if $w=0$. Then $u=0$ and so $\ker A = 0$. Since A is a $k \times k$ matrix, this implies A is invertible.

Then by anti-symmetry,

$$0 \neq \det A = \det(A^T) = (-1)^k \det A$$

and so k is even.

$\square$

18F.7

We recall that $\mathbb{RP}^n$ is orientable iff n is odd.
In particular, this implies that $H^n_{dR}(\mathbb{RP}^n) \cong \mathbb{R}$ iff n is odd. Since $H^n_{dR}(S^n) \cong \mathbb{R}$ $\forall n$, this implies that if $\pi : S^n \to \mathbb{RP}^n$ induces an isomorphism on cohomology, then n must be odd.

Now suppose that n is odd. We aim to show that π induces an isomorphism on cohomology.
We recall that covering maps induce injections on ^{dR} cohomology. This can be shown in a standard way by constructing an inverse. Since all deRham cohomology groups of S^n and $\mathbb{RP}^n$ are finite dimensional, this implies that π^* is an isomorphism.

H

If n is odd then $\mathbb{RP}^n$ is orientable and so has orientation cover $\mathbb{RP}^n \sqcup \mathbb{RP}^n$.

If n is even, then ~~S^n is a connec~~ $\mathbb{RP}^n$ is not orientable. Since S^n is a connected 2-fold cover of $\mathbb{RP}^n$, this implies that ~~$\mathbb{RP}^n$~~ S^n is the orientation double cover.

By definition, the orientation cover is the 2-fold cover such that the non-trivial deck transformation reverses orientation. If n is even then ~~$x \mapsto$~~ the deck transformation $x \mapsto -x$ is of deg $(-1)^{n+1} = -1$ and so reverses orientation. Therefore S^n is the orientation cover iff n ~~is~~ is even. $\square$

155.3

It suffices to show the result locally since we may then extend f_{ij} globally via a partition of unity.

Since $\phi_1 \wedge \dots \wedge \phi_K$ is non-vanishing, $\{\phi_1, \dots, \phi_K\}$ are linearly independent. Locally, we may extend $\phi_1, \dots, \phi_K$ to a basis $\phi_1, \dots, \phi_n$. Then $\forall\, i\ \exists$ smooth functions f_{ij} s.t.

$$\omega_i = \sum_{j=1}^{n} f_{ij}\, \phi_j$$

Taking the wedge w/ $\phi_1 \wedge \dots \wedge \phi_K = \omega_1 \wedge \dots \wedge \omega_K$, we find that

$$\sum_{j=1}^{n} f_{ij}\, \phi_j \wedge \phi_1 \wedge \dots \wedge \phi_K = \omega_i \wedge \omega_1 \wedge \dots \wedge \omega_K$$

$$\Rightarrow \quad \sum_{j=k+1}^{n} f_{ij}\, \phi_j \wedge \phi_1 \wedge \dots \wedge \phi_K = 0$$

Since $\phi_1 \wedge \dots \wedge \phi_K$ is non-vanishing and $\phi_1, \dots, \phi_n$ are linearly independent, this implies $f_{ij} = 0\ \forall\, j \geq k+1$. Therefore $\forall\, i$

$$\omega_i = \sum_{j=1}^{k} f_{ij}\, \phi_j$$

as desired. $\square$

15F.1 / 10F.3

(a) Define $F: M_n \to \mathbb{R} : A \mapsto \det A$. Then F is smooth and $F^{-1}(1) = SL_n$. To show $SL_n \subset M_n$ is a smooth submanifold, it then suffices to show that 1 is a regular value of F.

Consider $A \in SL_n$. We aim to show that $dF_A : TM_n \to T\mathbb{R}$ is surjective. We recall $TM_n \cong M_n$ and $T\mathbb{R} \cong \mathbb{R}$. Then $dF_A : M_n \to \mathbb{R}$.
Take some $k \in \mathbb{R}$. Then $\forall A \in SL_n$

$$dF_A\left(\tfrac{k}{n}A\right) = \lim_{t \to 0} \frac{\det(A + t\frac{k}{n}A) - \det A}{t}$$

$$= \lim_{t \to 0} \frac{\det A \det(I + t\frac{k}{n}I) - 1}{t}$$

$$= \lim_{t \to 0} \frac{(1 + kt/n)^n - 1}{t}$$

$$= \frac{d}{dt}\left(1 + \frac{kt}{n}\right)^n \Big|_{t=0} = \frac{k}{n} n \left(1 + \frac{kt}{n}\right)^{n-1} \Big|_{t=0} = k$$

so dF_A is surjective. Therefore dF_A is surjective $\forall A \in SL_n$.
Then 1 is a regular value of F and so SL_n is a smooth submanifold of M_n.

(b) Identify the tangent space of SL_n at I

Since $SL_n = F^{-1}(1)$, it follows that $T_I SL_n \cong \ker dF_I$.
We recall that $\det(I + tB)$ can be expanded as

$$\det(I + tB) = 1 + t \operatorname{Tr} B + O(t^2)$$

Therefore

$$dF_I(B) = \lim_{t \to 0} \frac{1 + t\operatorname{Tr}B + O(t^2) - 1}{t} = \operatorname{Tr} B.$$

Then $B \in \ker dF_I$ iff $\operatorname{Tr}(B) = 0$.
Therefore $T_I SL_n \cong \{B \in M_n : \operatorname{Tr} B = 0\}$.

$\square$

(c) We recall Poincare-Hopf, which states that the Euler Characteristic of a compact ~~orientable m~~ oriented manifold is the sum of indices of a finitely vanishing vector field.

In order to apply Poincare-Hopf, we show that SL_n is homotopic to SO_n, the space of orthogonal matrices. We recall that SO_n is a compact Lie group.

Since SO_n is a Lie group, SO_n admits a non-vanishing vector field V. Let $v \in T_I SO_n$ be nonzero and let $m_g(h) = gh$ $\forall g \in SO_n$. Then $dm_g|_I$ is an isomorphism $T_I SO_n \to T_g SO_n$. We ~~then def~~ then define V on SO_n via

$$V_g = d(m_g)|_I v$$

which is non-vanishing. Therefore $\chi\, SO_n = 0$.

We now show SL_n is homotopic to SO_n.

For $A \in SL_n$, we recall the polar decomposition $A = OP$ where O is ~~and~~ orthogonal and P is positive definite. Note O, P depend smoothly on A.

Define H_t by

$$H_t : A \mapsto (1-t)A + t\,\frac{O}{\det O}$$

Then $H_0 = id$ and $H_1 : A \mapsto \frac{O}{\det O} \in SO_n$. Therefore H_t is a homotopy $SL_n \to SO_n$.

Alternate: $H_t : A \mapsto \dfrac{(1-t)A + tO}{\det((1-t)A + tO)}$. This is well defined since $\det((1-t)A + tO) = \det((1-t)P + tI)\det U$ which has positive eigenvalues b/c P is positive and $\det U \neq 0$.

Therefore $\chi\, SL_n = \chi(SO_n) = 0$. $\square$

15F.3 Suppose ω is a $k+1$-form.

Let $Z_1, \dots, Z_k$ be vector fields on M. Then by Cartan's formula,

$$([\mathcal{L}_X, i_Y]\omega)(Z_1, \dots, Z_k) =$$

$$(\mathcal{L}_X \circ i_Y \omega - i_Y \mathcal{L}_X \omega)(Z_1, \dots, Z_k)$$

We recall that $\forall$ m-forms η and vector fields $V_1, \dots, V_m$

$$(\mathcal{L}_X \eta)(V_1, \dots, V_m) = \mathcal{L}_X(\eta(V_1, \dots, V_m)) - \sum_{i=1}^{m} \eta(V_1, \dots, \mathcal{L}_X V_i, \dots, V_m)$$

Therefore

$$(\mathcal{L}_X \circ i_Y \omega)(Z_1, \dots, Z_k) = \mathcal{L}_X(i_Y \omega(Z_1, \dots, Z_k)) - \sum_{j=1}^{k} i_Y \omega(Z_1, \dots, [X, Z_j], \dots, Z_k)$$

$$= \mathcal{L}_X(\omega(Y, Z_1, \dots, Z_k)) - \sum_{j=1}^{k} \omega(Y, Z_1, \dots, [X, Z_j], \dots, Z_k)$$

and similarly

$$(i_Y \circ \mathcal{L}_X \omega)(Z_1, \dots, Z_k) = (\mathcal{L}_X \omega)(Y, Z_1, \dots, Z_k)$$

$$= \mathcal{L}_X(\omega(Y, Z_1, \dots, Z_k)) - \omega([X, Y], Z_1, \dots, Z_k) - \sum_{j=1}^{k} \omega(Y, Z_1, \dots [X, Z_j], \dots, Z_k)$$

Therefore

$$([\mathcal{L}_X, i_Y]\omega)(Z_1, \dots, Z_k) = \omega([X, Y], Z_1, \dots, Z_k) = i_{[X,Y]}\omega\,(Z_1, \dots, Z_k)$$

and so $[\mathcal{L}_X, i_Y]\omega = i_{[X,Y]}\omega$ as desired.

$\square$

15F. 4

We recall that the Poincaré dual of C is $\eta \in H^2_{dR}(M)$ on M s.t. $\forall$ $\theta \in H^1_{dR}(M)$

$$\int_C \iota^* \theta = \int_M \theta \wedge \eta$$

where $\iota : C \hookrightarrow M$ is the inclusion.

We recall that $H^1_{dR}(M) \cong \mathbb{R}^3$ and that all 1-forms on M can be written as $a\,dx + b\,dy + c\,dz$ on the level of cohomology.

Moreover, since L goes from $(0,1,1)$ to $(1,3,5)$, we see that $\pi(L)$ wraps 1 time in the x direction, 2 times in the y direction and 4 times in the z direction. Therefore, $\forall$ $\theta = a\,dx + b\,dy + c\,dz$,

$$\int_C \iota^* w = a\int_C \iota^* dx + b\int_C \iota^* dy + c\int_C \iota^* dz$$
$$= a + 2b + 4c$$

Taking $\eta = dy \wedge dz + 2\,dz \wedge dx + 4\,dx \wedge dy$ then implies that $\forall$ $w = a\,dx + b\,dy + c\,dz$,

$$\int_M w \wedge \eta = (a+2b+4c)\int_M dx \wedge dy \wedge dz = a + 2b + 4c = \int_C \iota^* w$$

~~and so~~ Therefore η is the Poincaré dual of C, as claimed

$\square$

15F.6 / 12F.3

We first show connected. To do so, it suffices to show path connected.

Suppose $\exists\ a, b \in \mathbb{R}^n \setminus M$. Then since $\mathbb{R}^n$ is path connected, $\exists$ a path $\tilde{\gamma} : [0,1] \to \mathbb{R}^n$ s.t. $\tilde{\gamma}(0) = a$, $\tilde{\gamma}(1) = b$.

By the extension theorem, since $\{0,1\} \subset [0,1]$ is closed, there exists a homotopic path $\gamma : [0,1] \to \mathbb{R}^n$ s.t. $\gamma(0) = \tilde{\gamma}(0) = a$, $\gamma(1) = \tilde{\gamma}(1) = b$ and $\gamma \pitchfork M$.

Suppose $\exists\ p \in \gamma \cap M$. Then $T_pM + \operatorname{im}(d\gamma_p) = T_p\mathbb{R}^n$ and so

$$n = \dim T_p\mathbb{R}^n = \dim(T_pM + \operatorname{im}(d\gamma_p)) \le m + 1 < n - 1$$

which is a contradiction. Therefore $\gamma \cap M = \emptyset$ and so $\mathbb{R}^n \setminus M$ is path connected.

To show simply connected, we proceed similarly.

Suppose $\exists$ a closed loop $\gamma : S^1 \to \mathbb{R}^n \setminus M$. Since $\mathbb{R}^n$ is simply connected, $\exists$ a homotopy $\tilde{\gamma}_t : S^1 \to \mathbb{R}^n$ s.t. $\tilde{\gamma}_0 = \gamma$ and $\tilde{\gamma}_1 = \tilde{p} \in \mathbb{R}^n$. Since $\mathbb{R}^n \setminus M$ is path connected, we may extend $\tilde{\gamma}_t$ continuously so that $\tilde{\gamma}_2 = p \in \mathbb{R}^n \setminus M$.

B/c $S^1 \times \{0\} \amalg S^1 \times \{2\}$ is closed in $S^1 \times [0,2]$, the extension theorem implies $\exists$ a homotopic map γ_t s.t. $\gamma_0 = \tilde{\gamma}_0 = \gamma$ and $\gamma_2 = \tilde{\gamma}_2 = p$, and s.t. $\gamma_t \pitchfork M$.

Suppose $\exists\ q \in \operatorname{im}\gamma_t \cap M$. Then by transversality,

$$n = \dim T_q\mathbb{R}^n = \dim(T_qM + \operatorname{im} d_q(\gamma_t)) \le m + 2 < n$$

which is a contradiction. Therefore $\gamma_t \cap M = \emptyset$ and so $\gamma_t : S^1 \to \mathbb{R}^n \setminus M\ \forall\ t$. Therefore γ is homotopic to p in $M \setminus \mathbb{R}^n$ and so $\mathbb{R}^n \setminus M$ is simply connected. $\square$

21F.1

Let M denote the space of $n \times k$ matrices and let S denote the space of $k \times k$ symmetric matrices. We note that $\dim M = nk$ and $\dim S = k + (k-1) + \cdots + 1 = \frac{k(k+1)}{2}$.

We view $V_k \subset M$ via the ~~diffeomorphism~~ inclusion $V_k \to M : (v_1, \ldots, v_k) \mapsto [v_1, \ldots v_k]$.

Define $F : M \to S : A \mapsto A^T A$. Then by definition of orthonormal,

$$V_k \cong F^{-1}(I).$$

To show that V_k is a smooth manifold, it then suffices to show I is a regular value of F. We recall that $T_A M \cong M$ and $T_B S \cong S$, so it suffices to show that $dF_A : M \to S$ is surjective $\forall\, A \in V_k$.

Fix some $A \in V_k$ and $B \in S$. Define $C = \frac{1}{2} AB$. Then

$$\begin{aligned}
dF_A(C) &= \lim_{t \to 0} \frac{(A+tC)^T(A+tC) - A^T A}{t} \\
&= \lim_{t \to 0} \frac{A^T A + tC^T A + tA^T C + t^2 C^T C - A^T A}{t} \\
&= C^T A + A^T C \\
&= \tfrac{1}{2} B^T + \tfrac{1}{2} B = B \quad (\text{since } B \in S)
\end{aligned}$$

Therefore dF_A is surjective $\forall\, A \in V_k$ and so I is a regular value of F. Then $V_k \cong F^{-1}(I)$ is a smooth submanifold of dimension

$$\dim V_k = nk - \frac{k(k+1)}{2} = n\left(k - \frac{k+1}{2}\right)$$

as desired.

$\square$

21F:2

WLOG suppose that p is odd.

Let π_p, π_q denote the projections from $S^p \times S^q$ to the respective factors.

Then

$$T(S^p \times S^q) \cong \pi_p^*(TS^p) \oplus \pi_q^*(TS^q).$$

Since p is odd, S^p admits a non-vanishing vector field.

Therefore $TS^p \cong \varepsilon \oplus \varepsilon^\perp$ where ε is a trivial line bundle.

Then

$$T(S^p \times S^q) \cong \pi_p^*(\varepsilon^\perp) \oplus \varepsilon \oplus \pi_q^*(TS^q).$$

~~Since~~ Viewing $S^q \subset \mathbb{R}^{q+1}$, we note that NS^q is trivializable.

Therefore $\varepsilon \cong \pi_q^*(NS^q)$ and so

$$\begin{aligned} T(S^p \times S^q) &\cong \pi_p^*(\varepsilon^\perp) \oplus \pi_q^*(NS^q \oplus TS^q) \\ &\cong \pi_p^*(\varepsilon^\perp) \oplus \pi_q^*(T\mathbb{R}^{q+1}) \end{aligned}$$

Since $\mathbb{R}^{q+1}$ is parallelizable, $T\mathbb{R}^{q+1} \cong \varepsilon^{q+1}$. Therefore

$$\begin{aligned} T(S^p \times S^q) &\cong \pi_p^*(\varepsilon^\perp) \oplus \varepsilon^{q+1} \\ &\cong \pi_p^*(\varepsilon^\perp \oplus \varepsilon) \oplus \varepsilon \oplus \varepsilon^{q-1} \\ &\cong \pi_p^*(TS^p) \oplus \varepsilon \oplus \varepsilon^{q-1} \end{aligned}$$

Repeating the same steps for $\pi_p^*(TS^p) \oplus \varepsilon$, we see that

$$\pi_p^*(TS^p) \oplus \varepsilon \cong \varepsilon^{p+1}$$

Therefore

$$T(S^p \times S^q) \cong \varepsilon^{p+q}$$

and so $S^p \times S^q$ is parallelizable. $\square$

21F.4

(⇒) Suppose first that $\omega = d\eta$ for some $\eta \in \Omega_c^{n-1}(\mathbb{R}^n)$.

Let B be a ball in $\mathbb{R}^n$ containing the support of η.

Then by Stokes' theorem,

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} d\eta = \int_B d\eta = \int_{\partial B} \iota^* \eta = \int_{\partial B} 0 = 0$$

as desired.

~~(⇐) Now suppose instead that~~ $\cancel{\int_{\mathbb{R}^n} \omega = 0}$.

~~Let B be a ball containing the support of ω.~~

Then $\int_B \omega = \int_{\mathbb{R}^n} \omega = 0$

(⇐) Define $f : H_c^n(\mathbb{R}^n) \to \mathbb{R}$ by $\omega \mapsto \int_{\mathbb{R}^n} \omega$. By the previous direction, this is well-defined on cohomology.

By the linearity of integration, f is linear.

We claim that f is an isomorphism.

First we show surjective. Let $\varphi \geq 0$ be a smooth bump function, s.t. $\varphi = 1$ on $B(0,1)$ and $\varphi = 0$ on $\mathbb{R} \setminus B(0,2)$.

Then $f(\varphi \, dV) = \int_{\mathbb{R}^n} \varphi \, dV \geq \int_{B(0,1)} \varphi \, dV = \mathrm{vol}(B(0,1)) > 0$.

By linearity, this then implies surjectiveness.

We recall by Poincaré duality that $H_c^n(\mathbb{R}^n) \cong H^0_{dR}(\mathbb{R}^n) \cong \mathbb{R}$.

Therefore f is a linear ~~iso~~ surjective map between same dimension spaces and hence is an isomorphism.

Then ω is exact iff $f(\omega) = \int_{\mathbb{R}^n} \omega = 0$, as desired. □

GeoTop

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We show (b) only as (a) is a ~~specific~~ specific instance of (b).

Fix $k \geq 1$ and let T be the set of rank k $m \times n$ matrices.
We recall that a matrix M is rank k iff M contains an invertible $k \times k$ submatrix (in the sense of removing rows/columns from M) and no $j \times j$ invertible submatrices for $j > k$.

Let M_1 denote the matrices of the block form

$$M = \begin{array}{cc} \overbrace{}^{k} & \overbrace{}^{n-k} \\ \end{array}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{array}{l} \}k \\ \}m-k \end{array}$$

where A is a $k \times k$ invertible matrix. We first aim to show that M is of rank k iff $D - CA^{-1}B = 0$. Consider the map

$$\varphi : \begin{bmatrix} A & B \\ C & D \end{bmatrix} \longmapsto \begin{bmatrix} A^{-1} & 0 \\ -CA^{-1} & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{bmatrix}$$

By direct computation, $\det \begin{bmatrix} A^{-1} & 0 \\ -CA^{-1} & I \end{bmatrix} = \det A^{-1} \neq 0$ and so $\begin{bmatrix} A^{-1} & 0 \\ -CA^{-1} & I \end{bmatrix}$ is invertible w/ inverse $\begin{bmatrix} A & 0 \\ C & I \end{bmatrix}$. Therefore φ is invertible and hence a ~~df~~ diffeomorphism since it is matrix multiplication/rearrangement.

In particular, $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is rank k iff $\begin{bmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{bmatrix}$ is rank k. Since I is invertible, if $D - CA^{-1}B \neq 0$ then we can find a larger invertible submatrix of $\begin{bmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{bmatrix}$. Therefore $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is ~~so~~ rank k iff $D - CA^{-1}B = 0$.

Define $\tilde{f} : M_1 \to \underset{\cong\, M_{(m-k)\times(n-k)}}{\mathbb{R}^{(m-k)(n-k)}}$ by $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto D - CA^{-1}B$. We claim that 0 is a regular value of $\tilde{f}$. Fix some $E \in M_{(m-k)\times(n-k)}$ then $\forall \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \tilde{f}^{-1}$

$$d\tilde{f}_{\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]}\left(\begin{bmatrix} 0 & 0 \\ 0 & E \end{bmatrix}\right) = \lim_{t \to 0} \frac{D + Et - CA^{-1}B - D + CA^{-1}B}{t} = E$$

and so $d\tilde{f}_M$ is ~~so~~ surjective $\forall M \in \tilde{f}^{-1}\{0\}$. Therefore 0 is a regular value of $\tilde{f}$.

We now ~~soon~~ aim to extend $\tilde{f}$. M...n.

We note that the ~~determinant~~ of a submatrix is a smooth map and so we can partition $M_{m\times n}$ into the union

$$M_{m\times n} = \underbrace{\{\text{no } k\times k \text{ invertible submatrices}\}}_{\mathcal{F}^c} \cup \underbrace{\{\exists \text{ a } k\times k \text{ invertible submatrix}\}}_{\mathcal{F}}$$

where $\mathcal{F}$ is open and $\mathcal{F}^c$ is closed. B/c $\mathcal{F}$ is open, $\mathcal{F}$ is a submanifold of $M_{m\times n}$ of dimension mn. We now aim to extend $\tilde{f}$ to all of $\mathcal{F}$.

We note that each $m\times n$ matrix has $(m-k)(n-k)$ $k\times k$ submatrices. Let the positions of these submatrices be enumerated $1, \ldots, (m-k)(n-k)$. Since the ~~determinate~~ determinant of a $k\times k$ submatrix is smooth, we can write $\mathcal{F}$ as the union of open sets

$$\mathcal{F} = \bigcup_{i=1}^{(m-k)(n-k)} \underbrace{\{k\times k \text{ submatrix in position } i \text{ is invertible}\}}_{\mathcal{F}_i}$$

On $\mathcal{F}_i$, by permuting rows and columns, $\exists$ a diffeomorphism $\varphi_i : \mathcal{F}_i \to M$.
Let $\{\psi_i\}$ be a partition of unity subordinate to $\{\mathcal{F}_i\}$. Then define $f : \mathcal{F} \to \mathbb{R}^{(m-k)(n-k)}$ by

$$f = \sum_{i=1}^{(m-k)(n-k)} \psi_i \tilde{f} \circ \varphi_i$$

By the earlier reasoning for $\tilde{f}$, 0 is a regular value of f w/ $f^{-1}\{0\} = T$. Therefore T is a submanifold of dimension

$$mn - (m-k)(n-k) = mk + nk - k^2 = k(m+n-k)$$

$\square$

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(a) Let $\{t_n\}$ be a partition of $[0,1]$ s.t. $c|_{[t_n, t_{n+1}]}$ is smooth.
We define
$$\int_c \omega = \sum_n \int_{t_n}^{t_{n+1}} c^* \omega$$
Since integration ~~of a smooth~~ in $\mathbb{R}$ is well-defined, this is as well.

(b)

($\Rightarrow$) Suppose that $\omega = df$ for some smooth f. Then by direct computation, $\forall$ ~~dr paths~~ piecewise smooth c,
$$\int_c \omega = \int_c df = \sum_n \int_{t_n}^{t_{n+1}} c^* df = \sum_n \int_{t_n}^{t_{n+1}} d(f \circ c) = \sum_n \int_{t_n}^{t_{n+1}} (f \circ c)'(t)\,dt$$
(FTC) $$= \sum_n \big(f(c(t_{n+1})) - f(c(t_n)) \big) = f(c(1)) - f(c(0))$$
In particular, if c is closed then $\int_c \omega = 0$.

($\Leftarrow$) Suppose instead that $\int_c \omega = 0$ $\forall$ piecewise smooth closed curves c. We aim to show that ~~M is~~ ω is exact. WLOG we may assume that M is path-connected as otherwise we may repeat this argument on all path-connected components of M.

Fix some $p_0 \in M$. Since M is path-connected, $\forall p \in M$ $\exists$ a smooth path $\gamma(p)$ from p_0 to p. Define $f: M \to \mathbb{R}$ by
$$f(p) = \int_{\gamma(p)} \omega$$
We first claim that f is well-defined, i.e that $f(p)$ is independent of the choice of path $\gamma(p)$. Fix $p \in M$ and suppose $\exists$ two smooth paths γ_1, γ_2 from p_0 to p.

$\longrightarrow$

Then $\gamma_1 - \gamma_2 : [0,2] \to M$ is a piecewise smooth 'closed' path.
Assumption and definition then implies

$$0 = \int_{\gamma_1 - \gamma_2} w = \int_0^1 \gamma_1^* w + \int_0^1 (-\gamma_1^*) w = \int_{\gamma_1} w - \int_{\gamma_2} w$$

and so $\int_{\gamma_1} w = \int_{\gamma_2} w$. Therefore f is independent of the choice of path $\gamma(p)$ and hence is well-defined.

B/c w is smooth and $\gamma(p)$ can be locally chosen to be vary smoothly w/ p, it then follows that $f(p)$ is a smooth function.
~~Moreover, by definition, since $\gamma(p)$ is smooth~~

$$\cancel{df_p = d\left(\int_{\gamma(p)} w\right) = d\left(\int_0^1 \gamma(p)^* w\right)}$$

Fix some $p \in M$ and local coordinates $\{x_1, \dots, x_n\}$ on a neighborhood U of p.
Then on U, w may be written $w = \sum_i g_i \, dx_i$.
By definition,

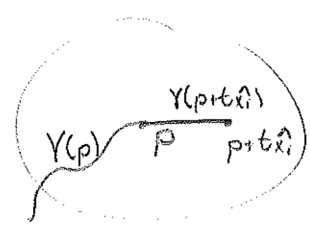
$$df_p = \sum_i \frac{\partial f}{\partial x_i}\Big|_p (dx_i)_p$$

and

$$\frac{\partial f}{\partial x_i}\Big|_p = \lim_{t \to 0} \frac{f(p + t\hat{x}_i) - f(p)}{t}$$

hence f is independent of the path chosen, we may choose $\gamma(p)$, $\gamma(p+t\hat{x}_i)$ s.t. they agree up to p and $\gamma(p+t\hat{x}_i)$ travels in the $\hat{x}_i$ direction at unit speed.

$$\frac{\partial f}{\partial x_i}\Big|_p = \lim_{t \to 0} \frac{1}{t} \int_p^{p+t\hat{x}_i} g_i \, dx_i$$



$$= \lim_{t \to 0} \frac{1}{t} \int_0^t g_i \, ds$$

$$\text{(FTC)} \quad = g_i(p)$$

Therefore $df_p = \sum_i g_i(p) \, dx_i = w_p$ and so $df = w$ as desired. $\square$

(3) $\underline{405\ 846\ 515}$

(a) Two smooth submanifolds $\overset{S_1, S_2}{\vee}$ are transversal, denoted $S_1 \pitchfork S_2$, if $\forall\, p \in S_1 \cap S_2$, $T_pS_1 + T_pS_2 = T_pM$.

(b) Suppose that $S_1 \pitchfork S_2$.

If $S_1 \cap S_2 = \emptyset$, then this claim fails. This can be seen by taking two disjoint open sets $U_1, U_2 \subset M$. Then $U_1 \cap U_2 = \emptyset$ but $\dim U_1 + \dim U_2 - \dim M = \dim M$. Therefore we assume $S_1 \cap S_2 \neq \emptyset$.

Let $\iota_1 : S_1 \hookrightarrow M$ and $\iota_2 : S_2 \hookrightarrow M$ be ~~the~~ ~~smo~~ the given smooth embeddings. Fix $p \in S_1 \cap S_2$. Then by the implicit function theorem, $\exists$ a neighborhood $U_1 \subset S_1$ of p that is the zero set of functions $f = (f_1, \ldots, f_{n-n_1})$ where $n_1 = \dim S_1$.* Similarly, $\exists$ a neighborhood $U_2 \subset S_2$ of p that is the zero set of functions $g = (g_1, \ldots, g_{n-n_2})$ where $n_2 = \dim S_2$.*

Let $U = U_1 \cap U_2 \subset S_1 \cap S_2$. By construction, $U = \{ f_1 = 0, \ldots, f_{n-n_1} = 0, g_1 = 0, \ldots, g_{n-n_2} = 0 \}$. We claim that 0 is a regular value of $\varphi = (f_1, \ldots, f_{n-n_1}, g_1, \ldots, g_{n-n_2})$. By definition, $\varphi : U \subset M \to \mathbb{R}^{2n-n_1-n_2}$. Therefore it must be shown that $d\varphi_x$ is of rank $2n - n_1 - n_2$ or equivalently that ~~$\dim\ker \varphi_x$~~ $\dim\ker d\varphi_x = n - (2n - n_1 - n_2) = n_1 + n_2 - n$. ~~By dire~~ $\forall\, x \in U$. By direct computation, $d\varphi_x = df_x \oplus dg_x$ and so

$$\begin{aligned}
\dim\ker d\varphi_x &= \dim(\ker df_x \cap \ker dg_x) \\
&= \dim\ker df_x + \dim\ker dg_x - \dim(\ker df_x + \ker dg_x) \\
&= n_1 + n_2 - \dim(T_xS_1 + T_xS_2) \\
&= n_1 + n_2 - n
\end{aligned}$$

b/c $S_1 \pitchfork S_2$. Therefore 0 is a regular value of φ and so $U = \varphi^{-1}(0)$ is a submanifold of M of dimension $n - (n - n_1) - (n - n_2) = n_1 + n_2 - n$. As such a U can be found $\forall\, p \in S_1 \cap S_2$, this implies $S_1 \cap S_2$ is a submanifold of dimension $n_1 + n_2 - n$. $\square$

* and zero is a regular value.

4) 405 846 515   Fix some $p \in S$.

($\Leftarrow$) Suppose $\exists$ constants $\lambda_1, \ldots, \lambda_k$ s.t.

$$df_p = \sum \lambda_i dF^i_p$$

Let $\iota : S \hookrightarrow M$ be the inclusion of S in M.
Then $f|_S = f \circ \iota$ and so

$$\begin{aligned} d(f\circ\iota)_p &= df_p \circ d\iota_{\iota^{-1}(p)} \\ &= \sum \lambda_i dF^i_p \circ d\iota_{\iota^{-1}(p)} \\ &= \sum \lambda_i d(F^i \circ \iota)_p \end{aligned}$$

Since F^i is constant on S, $d(F^i \circ \iota)_p = 0 \Rightarrow d(f|_S)_p = d(f\circ\iota)_p = 0$
as desired.

($\Rightarrow$) Suppose that $f|_S = f\circ\iota$ has a critical point at p.
Then $df_p \circ d\iota_{\iota^{-1}(p)} = 0$ and so $\ker df_p \supset T_pS$. ~~In particular~~ Heuristically,
$\ker df_p$ is only nonzero in directions ~~normal~~ perpendicular to S.
Since $S = F^{-1}(c)$, we know that $\operatorname{span}\{dF^1_p, \ldots, dF^k_p\} = T_pS^{\perp} \subset T_pM$.
Therefore $\exists$ constants $\lambda_1, \ldots, \lambda_k$ s.t.

$$df_p = \sum \lambda_i dF^i_p$$

as desired.

$\square$

(5) 405 846 515

Let M be a smooth orientable compact n-manifold w/ boundary. ~~of dimension n~~.

Suppose for the sake of contradiction that $\exists$ a smooth retract $r: M \to \partial M$.

By Sard's theorem, $\exists\ p \in \partial M$ s.t. p is a regular value of r. Since $\{p\}$ is of codimension $n-1$ in ∂M, the regular value theorem implies that $r^{-1}\{p\}$ is a submanifold of dimension $n-(n-1)=1$ in M. B/c M is compact and $r^{-1}\{p\}$ is closed since $\{p\}$ is closed, it follows that $r^{-1}\{p\}$ is a compact 1-dimensional manifold.
Therefore by the classification of compact 1-dimensional manifolds, $r^{-1}\{p\}$ is the disjoint union of line segments, and copies of S^1.
In particular, $\#(\partial(r^{-1}\{p\}))$ is even if it is finite.
However, since $r = id$ on ∂M,

$$\partial(r^{-1}\{p\}) = (\partial r)^{-1}\{p\} = p$$

which is odd. Therefore no such retract r exists. $\square$

⑥ 405 846 515

Fix $A \in GL_{n+1}(\mathbb{C})$.

(a) We note that A is a smooth linear map $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$. Since A is linear and invertible, $\ker A = \{0\}$. Therefore $A: \mathbb{C}^{n+1}\setminus\{0\} \to \mathbb{C}^{n+1}\setminus\{0\}$ is smooth. To show that this descends to a smooth map $\mathbb{CP}^n \to \mathbb{CP}^n$, it suffices to show that A factors through the quotient $\mathbb{C}^{n+1}\setminus\{0\} \to \mathbb{C}^{n+1}\setminus\{0\}/\sim = \mathbb{CP}^n$ where $\kappa z = z$ $\forall \kappa \in \mathbb{C}\setminus\{0\}$.

We define $A: \mathbb{CP}^n \to \mathbb{CP}^n$ by $A[z_0 : \dots : z_n] = [A(z_0, \dots, z_n)]$. By linearity, $\forall \kappa \in \mathbb{C}\setminus\{0\}$,

$$A[\kappa z_0 : \dots : \kappa z_n] = [A(\kappa z_0, \dots, \kappa z_n)] = [\kappa A(z_0, \dots, z_n)] = [A(z_0, \dots, z_n)] = [z_0 : \dots : z_n].$$

Therefore $A: \mathbb{CP}^n \to \mathbb{CP}^n$ is well-defined and smooth.

(b) Suppose that $[z_0 : \dots : z_n]$ is a fixed point of A. Then

$$[z_0 : \dots : z_n] = A[z_0 : \dots : z_n] = [A(z_0, \dots, z_n)]$$

and so $A(z_0, \dots, z_n) = \kappa(z_0, \dots, z_n)$ for some $\kappa \neq 0$. By definition, this implies that $\forall$ fixed points $[z_0 : \dots : z_n]$, $(z_0, \dots, z_n)$ is an eigenvector of A.

(c) We recall that a map φ is Lefschetz if $\forall$ fixed points x, $\det(d\varphi_x - I) \neq 0$.

Suppose that all eigenvalues of A are distinct. Then by changing bases, we may assume that $\operatorname{diag}(\lambda_0, \dots, \lambda_n) = A$.

By the previous part, the only possible fixed points of A are of the form $[0 : \dots : \underset{i^{th}}{1} : \dots : 0]$. Moreover, we note that $\forall i$,

$$A[0 : \dots : \underset{i^{th}}{i} : \dots : 0] = [0 : \dots : \lambda_i : \dots : 0] = [0 : \dots : 1 : \dots : 0]$$

and so the fixed points of A are precisely $\{[0 : \dots : \underset{i^{th}}{1} : \dots : 0] : i\}$.

Fix i and consider the neighborhood

$$U_i = \{[z_0 : \dots : z_i : \dots : z_n] : z_i \neq 0\}$$

We note that U_i is diffeomorphic to $\mathbb{C}^n$, via the map $\varphi : [z_0 : \dots : z_i : \dots : z_n] = [\frac{z_0}{z_i} : \dots : 1 : \dots : \frac{z_n}{z_i}] \longleftrightarrow (\frac{z_0}{z_i}, \dots, \hat{1}, \dots, \frac{z_n}{z_i})$.

W/ this, we may represent A as a map $A : \mathbb{C}^n \to \mathbb{C}^n$ w/

$$A(z_0, \dots, \hat{1}, \dots, z_n) = \varphi(A[z_0 : \dots : 1 : \dots : z_n]) = \varphi([\lambda_0 z_0 : \dots : \lambda_i : \dots : \lambda_n z_n])$$

$$= (\frac{\lambda_0}{\lambda_i} z_0, \dots, \hat{1}, \dots, \frac{\lambda_n}{\lambda_i} z_n)$$

Therefore $A = \mathrm{diag}(\frac{\lambda_0}{\lambda_i}, \dots, \hat{1}, \dots, \frac{\lambda_n}{\lambda_i})$ on $\mathbb{C}^n \cong U_i$.

In particular, $dA = \mathrm{diag}(\frac{\lambda_0}{\lambda_i}, \dots, \hat{1}, \dots, \frac{\lambda_n}{\lambda_i})$ since $TU_i \cong T\mathbb{C}^n \cong \mathbb{C}^n$

$$\Rightarrow \quad dA - I = \mathrm{diag}(\frac{\lambda_0}{\lambda_i} - 1, \dots, \hat{0}, \dots, \frac{\lambda_n}{\lambda_i} - 1)$$

$$\Rightarrow \quad \det(dA - I) \neq 0 \quad \text{since} \quad \frac{\lambda_j}{\lambda_i} \neq 1 \ \forall j \neq i.$$

Therefore $[0 : \dots : \underbrace{1}_{i\text{th}} : \dots : 0]$ is a Lefschetz fixed point $\forall i$ and so A is a Lefschetz map if all eigenvalues are distinct.

(d) We assume that A has distinct eigenvalues again.

We recall that the Lefschetz number of A is the sum of local Lefschetz numbers which are $\pm$ depending on ~~the~~ ~~sign of~~ whether $dA_x - I$ is orientation preserving or reversing.

As calculated in (c), $\forall$ fixed points $z^i = [0 : \dots : \underbrace{1}_{i\text{th}} : \dots : 0]$,

$$\det(dA_x - I) = \prod_{j \neq i} (\frac{\lambda_j}{\lambda_i} - 1)$$

Therefore the local Lefschetz number of A at z^i is given by

$$L_{z^i}(A) = \mathrm{sign} \prod_{j \neq i} (\frac{\lambda_j}{\lambda_i} - 1)$$

$$= \mathrm{sign}(\lambda_i^{-n} \prod_{j \neq i} (\lambda_j - \lambda_i))$$

(d) We recall that Lefschetz number is homotopy invariant and that $L(id) = \chi(M)$.
Because $GL_{n+1}(\mathbb{C})$ is connected, $\exists$ a path and hence a homotopy, from A to the identity matrix. This ~~so~~ descends to a homotopy from $A : \mathbb{CP}^n \to \mathbb{CP}^n$ to $id : \mathbb{CP}^n \to \mathbb{CP}^n$.

Therefore

$$L(A) = L(id) = \chi(\mathbb{CP}^n).$$

We recall that the CW structure of $\mathbb{CP}^n$ has ~~a cell~~ a single cell in every even dimension. Therefore

$$L(A) = \sum_{i=0}^{n} 1 = n+1$$

as desired.

$\square$

(7) 405 846 515

Let $F: S^n \to S^n$ be a continuous map. Then F induces a map $F_*: H_n(S^n) \to H_n(S^n)$ on top homology. We recall that $H_n(S^n) \cong \mathbb{Z}$ and so F_* gives a homomorphism $\mathbb{Z} \to \mathbb{Z}$. As the only homomorphisms $\mathbb{Z} \to \mathbb{Z}$ are multiplication by an integer, $\exists!\ k \in \mathbb{Z}$ s.t. $F_*: \mathbb{Z} \to \mathbb{Z}$ is multiplication by k. We define $\deg F = k$.

~~Additionally, we recall recall that the isomorphism $H_n(S^n) \to \mathbb{Z}$~~

Therefore, for all n-forms ω on S^n

$$\int_{S^n} F^* \omega = \int_{F_* S^n} \omega = \int_{(\deg F) S^n} \omega = \deg F \int_{S^n} \omega$$

where S^n is viewed as an n-cycle. ~~we define integration for~~ ~~Depending on whether~~

(b). Suppose F has no fixed point. We claim that F is homotopic to $-\mathrm{id}$. Consider the straight line homotopy

$$H_t(x) = \frac{(1-t)F(x) + t(-x)}{|(1-t)F(x) + t(-x)|} = \frac{(1-t)F(x) - tx}{|(1-t)F(x) - tx|}$$

To show H_t is well-defined, it must be shown that $(1-t)F(x) - tx \neq 0\ \forall\, t, x$. Suppose $\exists\, t, x$ s.t. $(1-t)F(x) - tx = 0$. We note that since $F(x) \neq 0$ and $-x \neq 0$, $t \neq 0, 1$. Therefore

$$F(x) = \frac{t}{1-t} x \implies$$

$$\implies \quad 1 = |F(x)| = \left|\frac{t}{1-t}\right| |x| = \left|\frac{t}{t-1}\right|$$

and so $t = 1/2$. However, then $F(x) = \frac{1/2}{1 - 1/2} x = x$ which contradicts the fact that F has no fixed points. Therefore $(1-t)F(x) - tx \neq 0\ \forall\, x, t$ and so H_t is a homotopy from $H_0 = F$ to $H_1 = -\mathrm{id}$.

Then

$$\deg F = \deg(-\mathrm{id}) = (-1)^{n+1}$$

as desired. $\square$

⑧ 405 846 515 Let $X = D^n \cup_f D^n$.

(a) We define

$$D^n \cup_f D^n = D^n \times \{0,1\} / \sim$$

where $(x,0) \sim (f(x),1)$ $\forall x \in \partial D^n \cong S^{n-1}$.
This is to say that $D^n \cup_f D^n$ is two copies of D^n w/ boundaries attached via f.

(b) Equivalently, for $n \geq 2$ and $\deg f = k$, we construct $D^n \cup_f D^n$ via

$$\left.\begin{array}{l} 1 \text{ 0-cell}: p \\ 1 \text{ (n-1)-cell}: e \text{ w/ } \partial_{n-1} e = p = 0 \end{array}\right\} S^{n-1}$$

$$2 \text{ n-cells}: A, B \text{ w/ } \partial_n A = e, \quad \partial_n B = ke$$

Direct computation then gives the homology groups

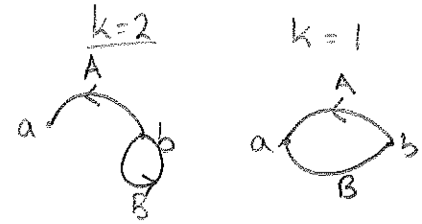
$$H_0(X) = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \frac{\mathbb{Z}\langle p \rangle}{0} \cong \mathbb{Z}$$

$$H_k(X) \cong 0 \quad \text{for } 1 \leq k \leq n-2$$

$$H_{n-1}(X) = \frac{\ker \partial_{n-1}}{\operatorname{Im} \partial_n} = \frac{\mathbb{Z}\langle e \rangle}{\mathbb{Z}\langle e, ke \rangle} \cong 0$$

$$H_n(X) = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}} = \mathbb{Z}\langle B - kA \rangle \cong \mathbb{Z}$$

and 0 for all higher order homologies.



For $n=1$, we instead have the CW complex

$$2 \text{ 0-cells}: a, b$$

$$2 \text{ 1-cells}: A, B \text{ w/ } \partial A = a - b$$

$$\partial B = f(a) - f(b) = \begin{cases} 0 & k = \pm 2 \\ \pm(a-b) & k = \pm 1 \end{cases}$$

For both $k = \pm 1, 2$, this deformation retracts onto S^1 and so

$$H_k(D^n \cup_f D^n) = \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & \text{else} \end{cases}$$

(c) If f is a homeomorphism, then $\deg f = \pm 1$ and so we have the CW complex

$\underline{n \geq 2}$

1 0-cell : p

1 $(n-1)$-cell : e w/ $\partial e = p = 0$

2 n-cells : A, B w/ $\partial A = e$
$\partial B = \pm e$

$\underline{n = 1}$

2 0-cells : a, b

2 1-cells : A, B w/ $\partial A = a - b$
$\partial B = \pm (a-b)$

By reversing the orientation of B if $\deg f = 1$, this yields

$\underline{n \geq 2}$

1 0-cell : p

1 $(n-1)$-cell : e w/ $\partial e = p = 0$

2 n-cells : A, B w/ $\partial A = \partial B = e$

$\underline{n = 1}$

2 0-cells : a, b

2 1-cells : A, B w/ $\partial A = \partial B = a - b$.

In both cases, this is precisely the CW complex for S^n. Therefore $D^n \cup_f D^n$ is homeomorphic to S^n if f is a homeomorphism since all ~~these~~ CW complexes for a ~~all~~ space are homeomorphic. $\square$

(9) 405 846 515

(a) No, M and N need not have the same fundamental group.
We recall that S^2 is a 2-sheeted cover of $\mathbb{RP}^2$.
By considering their CW complexes,

S^2	$\mathbb{RP}^2$
1 0-cell: p	1 0-cell: q
1 2-cell: A w/ $\partial_2 A = p = 0$	1 1-cell: a w/ $\partial_1 a = q - q$
	1 2-cell: B w/ $\partial_2 B = 2a$

we can compute their fundamental groups as being generated by their 1-cells w/ relations given by the boundaries of their 2-cells.
Then $\pi_1(S^2) \cong \langle 0 | 0 \rangle \cong 0$ and $\pi_1(\mathbb{RP}^2) \cong \langle a | 2a \rangle \cong \mathbb{Z}/2\mathbb{Z}$.
Then $\pi_1(S^2) \not\cong \pi_1(\mathbb{RP}^2)$.

(c) No, M and N need not have the same singular homology.
Consider $S^2 \to \mathbb{RP}^2$ as done in (a). It suffices to compute cellular homology over $\mathbb{Z}$ as that is equivalent. By direct computation w/ the CW complexes above,

$H_1(S^2) = 0$

$H_2(S^2) \cong \mathbb{Z}$

$$H_1(\mathbb{RP}^2) = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\mathbb{Z}\langle a \rangle}{\mathbb{Z}\langle 2a \rangle} \cong \mathbb{Z}/2\mathbb{Z}$$

$$H_2(\mathbb{RP}^2) = \frac{\ker \partial_2}{\operatorname{im} \partial_1} = \frac{\{0\}}{\{0\}} \cong 0$$

and so their homologies do not agree.

(b) No, their deRham cohomologies need not agree.
Consider $S^2, \mathbb{RP}^2$ as above. By universal coefficients,

$$H^2_{dR}(S^2; \mathbb{R}) \cong H_2(S^2; \mathbb{R}) \cong \mathbb{R}$$

$$H^2_{dR}(\mathbb{RP}^2; \mathbb{R}) \cong H_2(\mathbb{RP}^2; \mathbb{R}) \cong 0$$

and so their deRham cohomologies need not agree.

$\square$

⑩ 405 846 515

~~Let A, X be endowed w/ a simplicial (or cellular structure)~~
~~and let $C_n(X), C_n(A)$ denote~~

Let $C_n(X), C_n(A)$ denote the free abelian groups over the singular n-simplices in X, A respectively. Let $\iota: A \hookrightarrow X$ denote the inclusion of A in X. Then ~~it~~ ι induces an inclusion $\iota_*: C_n(A) \hookrightarrow C_n(X)$ and so we can consider the quotient $C_n(X,A) = C_n(X)/C_n(A)$.
Since the boundary map $\partial: C_n(X) \to C_{n-1}(X)$ agrees w/ the boundary map $\partial: C_n(A) \to C_{n-1}(A)$ on their overlap, we ~~have a~~ have a well-defined boundary map $\partial_n: C_n(X,A) \to C_{n-1}(X,A)$.
Since $\partial_n(C_n(X)) \subset C_{n-1}(X)$, and $\partial_n(C_n(A)) \subset C_{n-1}(A)$, it follows that we have a well-defined chain complex

$$\cdots \to C_n(X,A) \xrightarrow{\partial_n} C_{n-1}(X,A) \to \cdots$$

We then define the relative homology $H_n(X,A) = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$ on this sequence. ~~Letting~~

Letting π_n be the quotient map $C_n(X) \to C_n(X,A)$, we get a SES

$$0 \to C_n(A) \xrightarrow{\iota_*} C_n(X) \xrightarrow{\pi_n} C_n(X,A) \to 0$$

The standard snake lemma proof then yields a LES on homology.

GeoTop

Fall 2012

① 405846515

(a) Let $\exists\, A \in SL_2(\mathbb{R})$. We recall the polar decomposition

$$A = UP$$

where U is an orthogonal matrix and P is a symmetric positive semi-definite matrix. Then $\det A = \det U \cdot \det P$. B/c $\det U = \pm 1$ and $\det P \geq 0$, it follows: $\det P, \det U = 1$. Therefore $U \in SO(2)$ and hence is a rotation $U = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ for $\theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$.

Then

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \text{w/} \quad ac - b^2 = 1 \Rightarrow c = \sqrt{\frac{b^2+1}{a}}$$

(note $a, c \neq 0$ since $b^2 \geq 0$)

and so

$$A = \begin{bmatrix} a & b \\ b & \sqrt{\frac{b^2+1}{a}} \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Since A is invertible, this decomposition is unique and smooth in A. Therefore $SL_2(\mathbb{R}) \to S^1 \times \mathbb{R}^2 : A \mapsto (\theta, a, b)$ is a diffeomorphism.

(b) Now suppose $A \in SL_2(\mathbb{C})$. Again, by ~~Po~~ polar decomposition $\exists$ unitary U and positive semi-definite Hermitian P s.t.

$$A = UP$$

By the same reasoning as above, $\det U = \det P = 1$ and so $U \in SU_2(\mathbb{C})$. Then $U = \begin{bmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix}$ w $\underbrace{|\alpha|^2 + |\beta|^2 = 1}_{(\alpha,\beta) \in S^2}$. Therefore

$$A = \begin{bmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix}\begin{bmatrix} a & \bar{b} \\ b & c \end{bmatrix}$$

We note that $a, c \in \mathbb{R}$ w/ $c = \frac{|b|^2+1}{a}$ since $\det P = 1$. As before,
↳ Sylvester's criterion

U, P are unique and vary smoothly w/ A, so
$SL_2(\mathbb{C}) \to S^2 \times \mathbb{R}^3 \cong S^1 \times \mathbb{R} \times \mathbb{C} : A \mapsto (\alpha, \beta, a, c)$ is a diffeomorphism. □

② 405 846 515

We recall that $\mathbb{RP}^{2n-1}$ can be viewed as

$$\mathbb{RP}^{2n-1} = S^{2n-1}/\sim$$

where $S^{2n-1} \subset \mathbb{R}^{2n}$ and $\sim$ is the anti-podal identification $x \sim -x$.
We aim to construct a non-vanishing smooth vector field on S^{2n} and then transfer that to $\mathbb{RP}^{2n-1}$.

Consider some $x = (x_1, y_1, \dots, x_n, y_n) \in S^{2n-1} \subset \mathbb{R}^{2n}$. We recall that $T_x S^{2n-1} \cong \{ z \in \mathbb{R}^{2n} : z \perp x \}$. In particular,

$$(y_1, -x_1, \dots, y_n, -x_n) \in T_x S^{2n-1}$$

since $x \cdot (y_1, -x_1, \dots, y_n, -x_n) = \sum_i x_i y_i - x_i y_i = 0$.
The map $x \longmapsto (y_1, -x_1, \dots, y_n, -x_n)$ is smooth b/c is only a reordering and scaling of coordinates. Therefore

$$V : S^{2n-1} \to TS^{2n-1} : x \longmapsto (y_1, -x_1, \dots, y_n, -x_n) \in T_x S^{2n-1}$$

is a smooth vector field on S^{2n-1}. Moreover, since $0 \notin S^{2n-1}$, V is everywhere non-vanishing.

~~We now show that V factors through $\sim$ to a non-vanishing VF on $\mathbb{RP}^{2n-1}$. To show this it suffices to show that~~

$$[V_x] = [V_{-x}]$$

~~By direct computation, for $x = (x_1, y_1, \dots, x_n, y_n)$~~

$$V_{-x} = ((-y_1), -(-x_1), \dots, (-y_n), -(-x_n)) = -(y_1, -x_1, \dots, y_n, -x_n) = -V_x$$

~~and so $[V_x] =$~~

We now show that V factors through the quotient map $\pi : S^{2n-1} \to \mathbb{RP}^{2n-1}$.

We recall that π induces a map $d\pi : TS^{2n-1} \to T\mathbb{RP}^{2n-1}$. It then suffices to show that

$$d\pi_x V_x = d\pi_{-x} V_{-x}$$

Let f be the antipodal map $x \mapsto -x$. Then $\pi \circ f = \pi$. Direct computation then implies, w/ $x = (x_1, y_1, \dots, x_n, y_n)$

$$\cancel{d\pi_{-x} V_{-x} = d(\pi \circ f)_x ((-y_1), -(-x_1), \dots, (-y_n), -(-x_n))}$$

$$\cancel{= d\pi_x(-(y_1, -x_1, \dots, y_n, -x_n))}$$

$$\begin{aligned}
d\pi_x V_x &= d(\pi \circ f)_x V_x \\
&= d\pi_{f(x)} df_x V_x \\
&= d\pi_{-x}(-V_x) \\
&= d\pi_{-x}(-(y_1, -x_1, \dots, y_n, -x_n)) \\
&= d\pi_{-x}((-y_1), -(-x_1), \dots, (-y_n), -(-x_n)) \\
&= d\pi_{-x} V_{-x}
\end{aligned}$$

as desired. Therefore V defines ~~as~~ a well-defined vector field on $\mathbb{RP}^{2n-1}$. Since V is non-vanishing, the vector field on $\mathbb{RP}^{2n-1}$ is non-vanishing.

$\square$

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We recall the extension theorem.

<u>theorem</u> (extension theorem): Let $Z \subset Y$ be a closed manifold and $C \subset X$ a closed set. Let $f: X \to Y$ satisfy $f|_C \pitchfork Z$ and $\partial f \pitchfork Z$ on $C \cap \partial X$. Then $\exists$ a homotopic map $g: X \to Y$ s.t. $g|_C = f|_C$ and $g \pitchfork Z$.

Suppose $\exists$ $x, y \in \mathbb{R}^n \setminus M$. Let $f: [0,1] \to \mathbb{R}^n$ be the straight line

$$f(t) = (1-t)x + ty$$

Since M is a manifold of dimension $<n$, M is closed.

Let $C = \{0,1\}$. Then $C \subset [0,1]$ is closed. ~~and~~ Moreover, $C \cap f^{-1}(M) = \emptyset$, and $C \cap (\partial f)^{-1}(M) = \emptyset$, so $f|_C \pitchfork M$ and $\partial f \pitchfork Z$ on $C \cap \partial X = \partial X$. The extension theorem then implies that $\exists$ $g: [0,1] \to \mathbb{R}^n$ s.t. $g(0) = x$, $g(1) = y$, ~~and~~ g is homotopic to f and $g \pitchfork M$. In particular, g is a path from x to y.

Since $[0,1]$ is one-dimensional, $\forall p \in [0,1]$, $\dim(\operatorname{im} dg_p) = 1$.

Therefore, if $\exists$ p s.t. $g(p) \in M$ then

$$n = \dim(\operatorname{im} dg_p + T_{g(p)}M) \leq 1 + n - 2 = n - 1 < n$$

which is a contradiction. Therefore $g[0,1] \subset \mathbb{R}^n \setminus M$. and so $\mathbb{R}^n \setminus M$ is path connected and likewise connected.

We now claim that $\mathbb{R}^n \setminus M$ is simply connected.

Let $\gamma: S^1 \to \mathbb{R}^n \setminus M$ be a closed path. Let $h: [0,1] \times S^1 \to \mathbb{R}^n$ be the straight line homotopy in $\mathbb{R}^n$,

$$h_s(t) = h(s,t) = (1-s)\gamma(t) + s\gamma(0)$$

Let $C = \{0,1\} \times S^1 = \partial([0,1] \times S^1)$. Then C is closed, ~~and~~ $C \cap h^{-1}(M) = \emptyset$ and $C \cap (\partial h)^{-1}(M) = \emptyset$ so $h|_C \pitchfork M$ and $\partial h|_C \pitchfork M$.

$\longrightarrow$

The extension theorem then implies that $\exists$ a homotopy

$$k : [0,1] \times S^1 \to \mathbb{R}^n$$

s.t. $k \pitchfork M$ and $k|_{\{0,1\} \times S^1} = h|_{\{0,1\} \times S^1}$. Since h is a homotopy from γ to $\gamma(0)$ and $k|_{\{0,1\}\times S^1} = h|_{\{0,1\}\times S^1}$, it follows that k is a homotopy from γ to $\gamma(0)$. Moreover, since $[0,1]\times S^1$ is 2-dimensional,

$$\dim(\operatorname{im} dk_p) \leq 2 \quad \forall\, p \in [0,1]\times S^1$$

Therefore, if $\exists\, p \in [0,1]\times S^1$ s.t. $k(p) \in M$, then

$$n = \dim\left(\operatorname{im} dk_p + T_{k(p)}M\right) < 2 + n - 2 = n$$

which is a contradiction. Then $\operatorname{im}(k) \in \mathbb{R}^n \setminus M$ and so k is a homotopy in $\mathbb{R}^n\setminus M$ from γ to $\gamma(0)$. Therefore γ is contractible in $\mathbb{R}^n \setminus M$. As this holds $\forall$ closed ~~loop~~ paths in $\mathbb{R}^n\setminus M$, this implies that $\mathbb{R}^n \setminus M$ is simply connected. $\square$

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(a) Fix $n \geq 1$ and $k \in \mathbb{Z}$. We aim to construct a continuous map $f: S^n \to S^n$ of degree k.

If $k=0$ then a constant map satisfies.

Now suppose $k>0$. Let $B_1, \dots, B_k$ be disjoint open n-disks in S^n.

Consider the quotient map $\pi: S^n \to S^n/(S^n \setminus (B_1 \cup \dots \cup B_k))$.

Since the one-point compactification of B^n is S^n, we note that $\overline{B_i}/\partial B_i \cong S^n$. Therefore $S^n/(S^n \setminus (B_1 \cup \dots \cup B_k)) \cong \bigvee_{i=1}^{k} S^n$ where the wedge sum is at some point $p \in S^n$.

Then π ~~induces~~ defines a continuous map $\tilde{\pi}: S^n \to \bigvee_{i=1}^{k} S^n$.

Finally, define $f: S^n \to S^n$ by $S^n \xrightarrow{\tilde{\pi}} \bigvee_{i=1}^{k} S^n \xrightarrow{g} S^n$ where g is the identity or a reflection on each S^n and maps into a single copy of S^n, where the identity or reflection on each S^n is chosen such that f is orientation-preserving.

To compute the degree of f, it suffices to consider local degree.

Choose some $q \in S^n$ s.t. $f^{-1}(q)$ is finite (i.e. $q \neq p$). By construction $|f^{-1}(q)| = k$ and $\forall \hat{q} \in f^{-1}(q)$, f is locally an orientation-preserving diffeomorphism. Therefore $\deg_{\hat{q}} f = 1$ and so $\deg f = k$, as desired.

Finally consider $k<0$. Repeating the ~~above~~ above argument w/ $|k|$ disks and g chosen to reverse orientation, we find a suitable f.

(b) The same construction as above holds, since $\forall k$

$$X/(X \setminus (B_1 \cup \dots \cup B_{|k|})) \cong \bigvee_{i=1}^{k} S^n$$

□

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($\Rightarrow$) Suppose that Δ is integrable. Fix some $p \in \Omega$. Then by definition $\exists$ local coordinates $(x_1, \dots, x_n)$ defined on a neighborhood $V \subset \Omega$ of x s.t. $\Delta = \mathbb{R}\langle \partial/\partial x_1, \dots, \partial/\partial x_k \rangle$ on V.

Define $u_i = x_{k+i}$. Since $\Delta = \mathbb{R}\langle X_1, \dots, X_k \rangle = \mathbb{R}\langle \partial/\partial x_1, \dots, \partial/\partial x_k \rangle$ on V, we can ~~write~~ locally write

$$X_j = \sum_{i=1}^{k} f_{ij}\, \partial/\partial x_i$$

Then
$$X_j(u_\ell) = \sum_{i=1}^{k} f_{ij}\, \partial/\partial x_i (u_\ell) = \sum f_{ij}\, \partial/\partial x_i (x_{k+\ell}) = 0$$

on V. In particular, $u_\ell \in Z_V$ $\forall\, \ell = 1, \dots, n-k$. Moreover, since $\{x_i\}$ are coordinates for V, it follows that $\{du_\ell = dx_{k+\ell}\}_\ell$ are linearly ~~de~~ independent on V.

($\Leftarrow$) Suppose ~~statement~~ (b) holds. Fix some $p \in \Omega$. Then $\exists$ a neighborhood $V \subset \Omega$ of ~~x~~ and $n-k$ functions $u_1, \dots, u_{n-k} \in Z_V$ s.t. $du_1, \dots, du_{n-k}$ are linearly independent on V.

~~Let $\phi_i(t,q)$ be a local flow of X_i defined on a neighborhood U_i of~~ p.
~~$\phi_i : (-\varepsilon_i, \varepsilon_i) \times U_i \to \Omega$~~ ~~Define $U = V \cap U_1 \cap \dots \cap U_k$. Then U is a neighborhood~~ of p.

We recall that a distribution is integrable iff $\forall$ 1-forms ω that annihilate Δ, $d\omega$ also annihilates Δ. Let $I(\Delta) = \{$1-form ω : ω annihilates $\Delta\}$. Since $\dim \Delta = k$, it follows that $\dim I(\Delta) = n-k$. Since $du_1, \dots, du_{n-k} \in I(\Delta)$ ~~and~~ are linearly dependent, it follows that they span $I(\Delta)$ near p. In particular, any 1-form $\omega \in I(\Delta)$ can be locally written ~~by~~ near p as
$$\omega = \sum_{i=1}^{n-k} f_i\, du_i \Rightarrow d\omega = \sum_{i=1}^{n-k} df_i \wedge du_i$$

~~* that is $d\eta$ annihilates Δ $\forall$ 1-forms η~~

Then $\forall X, Y \in \Delta$,

$$\begin{aligned} d\omega(X, Y) &= \sum_i df_i \wedge du_i(X, Y) \\ &= \sum_i \big(df_i(X) du_i(Y) - df_i(Y) du_i(X) \big) \\ &= \sum_i 0 \\ &= 0 \end{aligned}$$

and so $d\omega$ annihilates Δ. Therefore Δ is an integrable distribution

7) 405 846 515 It suffices to work w/ deRham cohomology.

Suppose that $H^{2n+1}(M;\mathbb{R}) \cong \mathbb{R}^k$.

B/c M is compact and orientable, we know that $H^{4n+2}(M;\mathbb{R}) \cong \mathbb{R}$.

Therefore, the wedge product $\wedge : H^{2n+1}(M;\mathbb{R}) \times H^{2n+1}(M;\mathbb{R}) \Rightarrow H^{4n+2}(M;\mathbb{R})$

defines a map $A : \mathbb{R}^k \times \mathbb{R}^k \Rightarrow \mathbb{R}$. Since the wedge product is bilinear and alternating, it follows that A is bilinear and alternating. In particular, A can be represented by a $k \times k$ skew-symmetric matrix.

We claim that A is an invertible matrix. To show this, it suffices to show that $\ker A = \{0\}$. In particular, it suffices to show that if $\exists u$ s.t. $A(v,u) = v^T A u = 0 \ \forall v$, then $u = 0$.

Suppose that such a u exists. Then $\exists$ a corresponding form ω s.t. $\eta \wedge \omega = 0 \ \forall \eta \in H^{2n+1}(M;\mathbb{R})$.

Let $x_1, \dots, x_{2n+2}$ be local coordinates for M. Then locally,

$$\omega = \sum_{0 \le i_1 < \dots < i_{2n+1} \le 4n+2} f_{i_1,\dots,i_{2n+1}} \, dx_{i_1} \wedge \dots \wedge dx_{i_{2n+1}}$$

By the given property of ω, this implies that locally, $\forall \ 0 \le i_1 < \dots < i_{2n+1} \le 4n+2$

$$0 = dx_{i_1} \wedge \dots \wedge dx_{i_{2n+1}} \wedge \omega = f_{i_1,\dots,i_{2n+1}} \, dV \Rightarrow f_{i_1,\dots,i_{2n+1}} = 0$$

and so $\omega = 0$ locally and hence identically. Therefore A is an invertible matrix. Since A is skew-symmetric,

$$\det(A) = \det(A^T) = \det(-A) = (-1)^k \det(A)$$

B/c $\det A \neq 0$, this implies k is ~~constant~~, as desired. $\square$

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Suppose on the contrary that $\exists$ a compact 3-dimensional M s.t. $\partial M = \mathbb{RP}^2$. Since $\mathbb{RP}^2$ is connected, $\exists$ a connected component M_1 of M s.t. $\partial M_1 = \mathbb{RP}^2$ and $\partial M_i = \emptyset$ $\forall$ other connected components. Therefore we may assume M is connected WLOG by restricting to M_1 if necessary.

We first establish 2 ~~4~~ lemmas.

Lemma 1: Suppose X is an odd-dimensional closed manifold, ~~$\mathbb{R}$-oriented~~ w/o boundary.

Then $\chi(X) = 0$.

proof: We calculate $\chi(X)$ via homology ~~deRham cohomology~~ over $\mathbb{Z}/2\mathbb{Z}$ since X is $\mathbb{Z}/2\mathbb{Z}$-orientable. ~~Let $H^n(X) = H^n_{dR}(X)$.~~ since $\mathbb{R}$ is a field,

By the universal coefficients theorem,

$$\begin{aligned} H^k(X;\mathbb{R}) &\cong \mathrm{Hom}(H_k(X;\mathbb{R}),\mathbb{R}) \oplus \mathrm{Ext}^1_{\mathbb{R}}(H_{k-1}(X;\mathbb{R}),\mathbb{R}) \\ &\cong \mathrm{Hom}(H_k(X;\mathbb{R}),\mathbb{R}) \\ &\cong H_k(X;\mathbb{R}). \end{aligned}$$

Moreover, since X is compact, Poincaré duality implies

$$H^k(X;\mathbb{R}) \cong H_{n-k}(X;\mathbb{R}). \implies H_k(X;\mathbb{R}) \cong H_{n-k}(X;\mathbb{R})$$

Therefore since X is $\mathbb{R}$-oriented, since n is odd

$$\begin{aligned} \chi(X) &= \sum_{i=0}^{n} (-1)^i \operatorname{rank}(H_i(X;\mathbb{R})) \\ &= \sum_{i=0}^{\frac{n-1}{2}} (-1)^i \operatorname{rank}(H_i(X;\mathbb{R})) + (-1)^{n-i} \operatorname{rank}(H_{n-i}(X;\mathbb{R})) \\ &= \sum_{i=0}^{\frac{n-1}{2}} \underbrace{\operatorname{rank}(H_i(X;\mathbb{R}))\left((-1)^i + (-1)^{n-i}\right)}_{=0} \\ &= 0 \end{aligned}$$

$\square$

which is what was to be shown.

replace all $\mathbb{R}$'s w/ $\mathbb{Z}/2\mathbb{Z}$.

Lemma 2: Suppose X is a compact odd-dimensional manifold w/ boundary.

Then $2\chi(M) = \chi(\partial M)$.

Proof: Let M_1, M_2 be 2 copies of M. Let $\tilde{M}$ be the doubling of M defined by $\tilde{M} = M_1 \amalg (\partial M \times [0,1]) \amalg M_2 / \sim$ where

$m_1 \sim (m_1, 0)$ $\forall m_1 \in \partial M_1$, $(m_1, 0) \in \partial M \times [0,1]$ and

$m_2 \sim (m_2, 1)$ $\forall m_2 \in \partial M_2$, $(m_2, 1) \in \partial M \times [0,1]$. Which is to say that $\tilde{M}$ is two copies of M attached via a thickening of their boundary. We now proceed via Mayer-Vietoris.

Let $U = M_1 \amalg \partial M \times [0,1) / \sim$ and $V = \partial M \times (0,1] \amalg M_2 / \sim$.

Then U, V ~~deformation retract onto~~ are open subsets of $\tilde{M}$,

U, V deformation retract onto $M_1, M_2 \cong M$ respectively,

$U \cap V$ deformation retracts onto ∂M, and

$U \cup V = \tilde{M}$.

We then acquire the LES

$$\cdots \longrightarrow H_k(U \cap V) \longrightarrow H_k(U) \oplus H_k(V) \to H_k(U \cup V) \to \cdots$$

$$H_k(U\cap V) \cong H_k(\partial M), \quad H_k(U)\oplus H_k(V) \cong H_k(M)^2, \quad H_k(U\cup V) \cong H_k(\tilde{M})$$

Taking an alternating sum and rearranging, we find that

$$0 = \sum_{k=0}^{n} \left((-1)^k \operatorname{rank}(H_k(\tilde{M})) - (-1)^k \operatorname{rank}(H_k(M)^2) + (-1)^k \operatorname{rank}(H_k(\partial M)) \right)$$

$$= \sum_{k=0}^{n} (-1)^k \operatorname{rank}(H_k(\tilde{M})) - 2 \sum_{k=1}^{n} (-1)^k \operatorname{rank}(H_k(M)) + \sum_{k=1}^{n} (-1)^k \operatorname{rank}(H_k(\partial M))$$

$$= \chi(\tilde{M}) - 2\chi(M) + \chi(\partial M)$$

Since $\tilde{M}$ is odd-dimensional, w/o boundary, and compact, the previous lemma implies $\chi(\tilde{M}) = 0$. Then $\chi(\partial M) = 2\chi(M)$. $\square$

Based on lemma 2, $2\chi(M) = \chi(\partial M) = \chi(\mathbb{RP}^2)$. However we recall that $\chi(\mathbb{RP}^2) = 1$ $\divideontimes$. $\square$

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~~We note that~~

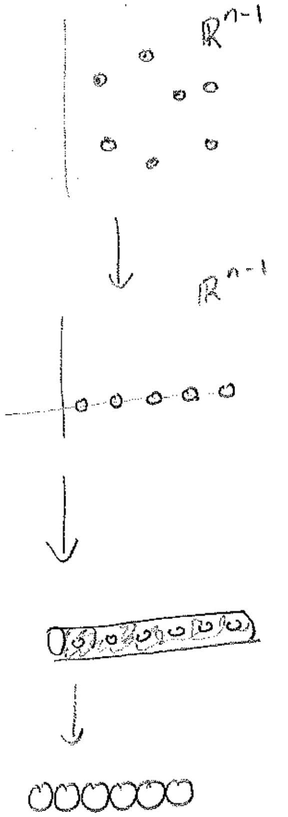
Let $X = \mathbb{R}^n \setminus (L_1 \cup \cdots \cup L_n)$. If $n = 0,1$, then $X = \phi$ and the homology is trivial. We first take $n > 2$.

We note that X deformation retracts onto $S^{n} \setminus \{2n \text{ points}\}$ via the ~~hom~~ straight line homotopy equivalence

$$x \longmapsto (1-t)x + t\frac{x}{|x|}$$

Therefore it suffices to compute the ~~2n-p~~ homology of a 2n-punctured sphere.

We note that the 2n-punctured sphere is diffeomorphic to ~~$\mathbb{R}^{n-1}$~~ the 2n-1-punctured $\mathbb{R}^{n-1}$ via a stereographic projection.



Via a diffeomorphism, we may arrange all the punctures in a straight line. From here, the space deformation retracts onto a 2n-1-punctured cylinder^{n-1}. Finally, since a 1-punctured cylinder deformation retracts onto S^{n-2}, our space deformation retracts onto the wedge of 2n-1 copies of S^{n-2}.

Therefore

$$H_k(X) \cong H_k\Big(\bigvee_{j=1}^{2n-1} S^{2n-2}\Big) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}^{2n-1} & k = n-2 \\ 0 & \text{else} \end{cases}$$

which is what was to be found.

For $n=2$, X is the disjoint union of 4 contractible spaces. Therefore

$$H_k(X) \cong \begin{cases} \mathbb{Z}^4 & k=0 \\ 0 & \text{else} \end{cases}.$$

$\square$

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~~(a) Method 1: Kunneth formula~~

~~We recall the Kunneth formula, which states~~

~~$H_k(A\times B) = \bigoplus_{j=0}^{k} H_j(A)\otimes H_{k-j}(B)$~~

(a) We construct S^1 w/ one 0-cell p and 1 1-cell e w/ $\partial e = p - p$. Let X have k-cells $e_1^k, \ldots, e_{n_k}^k$ for $k = 0, \ldots, n$.

Then $X\times S^1$ has k-cells $e_i^k \times p$ and $e_j^{k-1}\times e$. By the Leibniz rule, $\forall k, i, j$

$$\partial_k(e_i^k \times p) = \partial_k e_i^k \times p + (-1)^k e_i^k \times \partial_0 p = \partial_k e_i^k \times p$$

$$\partial_k(e_j^{k-1}\times e) = \partial_{k-1} e_j^{k-1}\times e + (-1)^{k-1} e_j^{k-1}\times \partial_1 e = \partial_{k-1} e_j^{k-1}\times e$$

Then
$$\operatorname{im}\partial_k = (\operatorname{im}\partial_k)\times p \oplus (\operatorname{im}\partial_{k-1})\times e$$
$$\ker\partial_k = (\ker\partial_k)\times p \oplus (\ker\partial_{k-1})\times e$$

In particular,

$$H_k(X\times S^1) = \frac{\ker\partial_k}{\operatorname{im}\partial_{k+1}} = \frac{(\ker\partial_k)\times p \oplus (\ker\partial_{k-1})\times e}{(\operatorname{im}\partial_{k+1})\times p \oplus (\operatorname{im}\partial_k)\times e}$$

Since $(\operatorname{im}\partial_{k+1})\times p \subset (\ker\partial_k)\times p$ and $(\operatorname{im}\partial_k)\times e \subset (\ker\partial_{k-1})\times e$ w/ $(\ker\partial_k)\times p \cap (\ker\partial_{k-1})\times e = \emptyset$, it follows that

$$H_k(X\times S^1) = \frac{(\ker\partial_k)\times p}{(\operatorname{im}\partial_{k+1})\times p} \oplus \frac{(\ker\partial_{k-1})\times e}{(\operatorname{im}\partial_k)\times e}$$

$$\cong \frac{\ker\partial_k}{\operatorname{im}\partial_{k+1}} \oplus \frac{\ker\partial_{k-1}}{\operatorname{im}\partial_k} \cong H_k(X)\oplus H_{k-1}(X).$$

(b) Fix $n\geq 0$. We recall that $\mathbb{CP}^n$ is constructed w/ 1 cell in every even dimension. Therefore

$$H_k(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & k \text{ even} \\ 0 & \text{else} \end{cases}$$

By part a, this implies $H_k(\mathbb{CP}^n\times S^1) \cong H_k(\mathbb{CP}^n)\oplus H_{k-1}(\mathbb{CP}^n) \cong \mathbb{Z}$ $\forall k = 0, \ldots, 2n+1$, as desired.

□

GeoTop

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MATH 3215

(1) 405 846 515

Let M be a compact smooth manifold. We claim $\exists$ a smooth embedding $\Psi : M \to \mathbb{R}^k$ for some $k \geq 1$.

By definition of a smooth manifold, $\exists$ an atlas (U_i, φ_i) of M. Since M is compact, $\exists$ a finite subcover $\{U_1, \dots, U_m\}$ of M.

Let $\psi_1, \dots, \psi_m$ be a partition of unity subordinate to $\{U_1, \dots, U_m\}$.

Define $\Psi : M \to \mathbb{R}^{mn+m}$

$$\Psi = (\psi_1\varphi_1, \dots, \psi_m\varphi_m, \psi_1, \dots, \psi_m)$$

Since ψ_i, φ_j are smooth, Ψ is smooth. It then suffices to show Ψ is an injective immersion.

We first show injection. Suppose $\Psi(x) = \Psi(y)$ for some $x, y \in M$. Then $\psi_i(x) = \psi_i(y)$ $\forall i$ and $\psi_i(x)\varphi_i(x) = \psi_i(y)\varphi_i(y)$ $\forall i$. Since ψ_i is supported on U_i and $\{U_i\}$ covers M, this implies that $\exists\, i$ s.t. $x, y \in U_i$. Since $\psi_i(x) = \psi_i(y)$ and $\psi_i(x)\varphi_i(x) = \psi_i(y)\varphi_i(y)$, it follows that $\varphi_i(x) = \varphi_i(y)$. B/c $\varphi_i : U_i \to \varphi_i(U_i)$ is a diffeomorphism, ~~this~~ and hence injective, this implies $x = y$. Therefore Ψ is injective.

It remains to show $d\Psi_p$ is injective $\forall p$.

Suppose $\exists\, v, w \in T_xM$ s.t. $d\Psi_x(v) = d\Psi_x(w)$ for some $x \in U_i \subset M$. Then b/c

$$d\Psi_x = \begin{bmatrix} d_x(\psi_1\varphi_1) \\ \vdots \\ d_x(\psi_m\varphi_m) \\ \hline (d\psi_1)_x \\ \vdots \\ (d\psi_m)_x \end{bmatrix} \quad \text{in block form}$$

it follows that $d\psi_i(v) = d\psi_i(w)$, and $d_x(\underbrace{\psi_i\varphi_i}_{M \to \mathbb{R}^n})(v) = d_x(\psi_i\varphi_i)(w)$.

Then by the product rule

$$\cancel{(d\psi_i)_x(v)\cdot\varphi_i(x)} + \cancel{\psi_i(x)}\cdot d(\varphi_i)_x(v) = \cancel{(d\psi_i)_x(w)\cdot\varphi_i(x)} + \cancel{\psi_i(x)}\cdot d(\varphi_i)_x(w)$$

$$d(\varphi_i)_x(v) = d(\varphi_i)_x(w)$$

Since φ_i is a diffeomorphism, $d\varphi_i$ is ~~[illegible]~~ injective. Hence $v = w$ and so $d\Psi_x$ is injective. Therefore Ψ is an immersion. $\square$

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we recall that $\mathbb{R}P^n = (\mathbb{R}^{n+1}\setminus\{0\})/\sim$ where $\sim$ is the identification $x \sim cx$ $\forall\, c \in \mathbb{R}\setminus\{0\}$. We denote elements of $\mathbb{R}P^n$ via

$$[x_0 : \dots : x_n] = [(x_0, \dots, x_n)]$$

for any $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}\setminus\{0\}$ To show $\mathbb{R}P^n$ is a smooth manifold of dimension n, we ~~construct~~ coordinate charts on $\mathbb{R}P^n$ and check that the transitions are smooth.

Let $U_i = \{[x_0 : \dots : x_n] \in \mathbb{R}P^n : x_i \neq 0\}$. Then since $x_i \neq 0$,

$U_i = \{[\frac{x_0}{x_i} : \dots : \underbrace{1}_{i^{th}\text{ position}} : \dots : \frac{x_n}{x_i}] \in \mathbb{R}P^n\}$. This gives a natural homeomorphism w/ $\mathbb{R}^n$ via the map $\varphi_i : U_i \to \mathbb{R}^n : [\frac{x_0}{x_i} : \dots : 1 : \dots : \frac{x_n}{x_i}] \to (\frac{x_0}{x_i}, \dots, \hat{1}, \dots, \frac{x_n}{x_i})$.

By construction, since $\mathbb{R}P^n$ is the quotient of $\mathbb{R}^{n+1}\setminus\{0\}$, $\forall\, [x_0 : \dots : x_n] \in \mathbb{R}P^n$ $\exists$ some j s.t. $x_j \neq 0$. Therefore $\mathbb{R}P^n = \bigcup_i U_i$

Moreover, since $\mathbb{R}^n$ is open and the above map is a homeomorphism, U_i is open $\forall i$. Therefore $(U_i, \varphi_i)_i$ is an atlas for $\mathbb{R}P^n$.

It remains to check that the transition functions are smooth.

~~By construction, $\forall i,j$ $\varphi_j(U_i \cap U_j) = \mathbb{R}^n \setminus (i^{th}$ axis).~~

Up to reordering coordinates, we may assume that $i = n-1$, $j = n$ and check the transition function $\varphi_{n-1} \circ \varphi_n^{-1}$.

By construction, $\varphi_n(U_n \cap U_{n-1}) = \mathbb{R}^n \setminus ((n-1)^{th}$ axis)

$\varphi_{n-1}(U_n \cap U_{n-1}) = \mathbb{R}^n \setminus (n^{th}$ axis)

Then

$\varphi_n \circ \varphi_{n-1}^{-1} : \mathbb{R}^n \setminus (n^{th} \text{ axis}) \to \mathbb{R}^n \setminus ((n-1)^{th} \text{ axis})$

$$: (x_0, \ldots, x_{n-2}, x_n \neq 0) = (x_0, \ldots, x_{n-2}, \hat{1}, x_n \neq 0)$$

$$\xrightarrow{\varphi_{n-1}^{-1}} [x_0 : \ldots : x_{n-2} : 1 : x_n \neq 0]$$

$$= \left[\frac{x_0}{x_n} : \ldots : \frac{1}{x_n} : 1 \right]$$

$$\xrightarrow{\varphi_n} \left(\frac{x_0}{x_n}, \ldots, \frac{x_{n-2}}{x_n}, \frac{1}{x_n} \right)$$

Re-indexing to standard coordinates,

$$\varphi_n \circ \varphi_{n-1}^{-1} : (x_1, \ldots, x_n \neq 0) \to \left(\frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}, \frac{1}{x_n} \neq 0 \right)$$

which is ~~smooth~~ a diffeomorphism since $x_n \neq 0$. $\square$

③ 405 846 515

Suppose for the sake of contradiction that an immersion

$$f : M \to T^n$$

exists. We ~~aim to~~ recall that $\mathbb{R}^n$ is the universal cover of T^n via the ~~covering~~ quotient map $p : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n \cong T^n$.

Since $\pi_1(M) = 0$, $f^*(\pi_1(M)) = 0 \subset p^*(\pi_1(\mathbb{R}^n))$. Therefore $\exists$ a lift $\tilde{f} : M \to \mathbb{R}^n$ s.t. $p \circ \tilde{f} = f$. Since f is an immersion, $\tilde{f}$ is an immersion. Since $M, \mathbb{R}^n$ are both dimension n, $d\tilde{f}_q$ is invertible $\forall q \in M$. The inverse function theorem then implies that $\tilde{f}$ is a local diffeomorphism and hence an open map. Therefore $\tilde{f}(M) \subset \mathbb{R}^n$ is open. Since M is compact, $\tilde{f}(M)$ is compact b/c $\tilde{f}$ is continuous. Therefore $\tilde{f}(M)$ is an open compact subset of $\mathbb{R}^n$ which is a contradiction. $\divideontimes$

$\square$

<u>alternate</u>

Since M, T^n are compact and of the same dimension, the stack of records theorem implies that M is a covering space of T^n w/ covering map $f : M \to T^n$. Since M is simply connected, M is ~~the~~ a universal cover of T^n. We recall that $\mathbb{R}^n$ is a universal cover of T^n and so $M \cong \mathbb{R}^n$. However $\mathbb{R}^n$ is not compact $\divideontimes$.

(4) 405 846 515

Let $p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0$, $n>0$, be a non-constant complex polynomial.
To show that $p(z)$ has a zero, we ~~argue via~~ consider the ~~image~~ closed path

$$\gamma_R : S^1 \to \mathbb{C} : \theta \mapsto p(Re^{i\theta})$$

for $R \geq 0$ and argue via the winding number about 0.

Consider $R=0$. Then $\gamma_0(\theta) = a_0 \ \forall \theta$. If $a_0 = 0$ then p has a zero and we are done. Otherwise suppose $a_0 \neq 0$, then γ_0 winds around 0 0 times.

As $R \to \infty$, we know that $|z|^n$ dominates $|a_{n-1}z^{n-1} + \dots + a_0|$. Choose R sufficiently large so that $|a_{n-1}z^{n-1} + \dots + a_0| < |z|^n$ $\forall$ $|z| = R$. Then by the standard "dog on a leash" theorem,

γ_R and $\gamma : S^1 \to \mathbb{C} : \theta \mapsto R^n e^{in\theta}$ have the same winding number about zero. Since γ winds around zero $n>0$ times, this implies that γ_R winds around 0 $n>0$ times.

We note that γ_R is homotopic to γ_0 via the homotopy

$$\gamma_t : \theta \mapsto p(te^{i\theta})$$

~~However~~ We likewise recall that the winding number around 0 is invariant under homotopies that avoid 0. Therefore, b/c $n \neq 0$, $\exists\, t, \theta$ s.t.

$$p(te^{i\theta}) = \gamma_t(\theta) = 0$$

and so p has a zero at $te^{i\theta}$.

$\square$

(5) 405 646 515

~~It suffices to work locally since d is defined locally.~~

~~Given local coordinates~~ $x_1,\dots,x_n$ ~~around~~ $q\in N$, ~~we can express~~ α ~~as~~

$$\alpha = \sum_{1\le i_1<\dots<i_p\le n} g_{i_1,\dots,i_p}\, dx_{i_1}\wedge\dots\wedge dx_{i_p}$$

~~By linearity, it then suffices to consider p-forms~~ $w = g\, dx_{i_1}\wedge\dots\wedge dx_{i_p}$.

~~By definition of the pullback,~~

$$(f^*w)_q = (g\circ f)(q)(d_{f(q)}x_{i_1}\circ d_q f)\wedge\dots\wedge(d_{f(q)}x_{i_p}\circ d_q f)$$

~~Then~~

$$d_{f(q)}(f^*w) = d_{f(q)}(g\circ f)(d_{f(q)}x_{i_1}\circ d_q f)\wedge\dots\wedge(d_{f(q)}x_{i_p}\circ d_q f)$$

We proceed by induction on p.

Suppose that κ is a 0-form. Then the chain rule implies that $\forall q\in M$

$$d_q(f^*\kappa) = d(\kappa\circ f) = d_{f(q)}\kappa\circ d_q f = f^*(d\kappa)_q$$

so $d(f^*\kappa) = f^*(d\kappa)$.

Now suppose the claim holds $\forall\, k<p$ for $p\geq 1$. Since d is defined locally, it suffices to work locally. Moreover, by linearity of f^*, d, it suffices to consider forms of the form $\kappa = g\, dx_{i_1}\wedge\dots\wedge dx_{i_p}$.

Let $w = g\, dx_{i_2}\wedge\dots\wedge dx_{i_p}$. Then w is a $p-1$-form and $\kappa = dx_{i_1}\wedge w$.

Direct computation and the inductive assumption imply

$$\begin{aligned}
d(f^*\kappa) &= d(f^*(dx_{i_1}\wedge w)) = d((f^*dx_{i_1})\wedge(f^*w)) \\
&= d(f^*dx_{i_1})\wedge(f^*w) \quad (-1)^1 (f^*dx_{i_1})\wedge d(f^*w) \\
\text{(inductive assumption)} &= d(d(f^*x_{i_1}))\wedge(f^*w) - (f^*dx_{i_1})\wedge f^*(dw) \\
&= -f^*(dx_{i_1}\wedge dw) \\
&= f^*(d\kappa)
\end{aligned}$$

As desired. $\square$

⑦ 405 846 515

(a) Let B^3 be the closed unit ball in $\mathbb{R}^3$. Then $\partial B^3 = S^2 \subset \mathbb{R}^3$ and so Stokes's theorem implies that

$$\int_{S^2} i^*w = \int_{\partial B^3} i^* w = \int_{B^3} d(w|_{B^3}) = \int_{B^3} (2x+1)\, dx \wedge dy \wedge dz$$

$$= 2\int_{B^3} x\, dV + \int_{B^3} dV$$

By symmetry, $\int_{B^3} x\, dV = 0$. So

$$\int_{S^2} w = \int_{B^3} dV = \mathrm{vol}(B^3)$$

the value of $\mathrm{vol}(B^3)$ differs depending on normalization but is usually $\mathrm{vol}(B^3) = \frac{4}{3}\pi$.

(b) Suppose that such an κ exists. Then by Stokes's,

$$\int_{S^2} w = \int_{S^2} i^* w = \int_{S^2} i^* \kappa = \int_{B^3} d\kappa = \int_{B^3} 0 = 0$$

which contradicts part (a). Therefore no such κ exists.

---

⑥ 405 846 515

(a) Let M be a smooth manifold and let $\Omega_k(M)$ be the collection of its k-forms. ~~we then~~ ~~the zero~~ W/ the exterior derivative $d_k : \Omega_k(M) \to \Omega_{k+1}(M)$, this gives a cochain complex

$$\cdots \leftarrow \Omega_{k+1} \xleftarrow{d_k} \Omega_k \xleftarrow{d_{k-1}} \Omega_{k-1} \leftarrow \cdots \xleftarrow{d_0} \Omega_0 \leftarrow 0$$

and the de Rham cohomology groups are then given by

$$H^k_{dR}(M) = \frac{\ker d_k}{\operatorname{im} d_{k-1}}$$

(b) <u>de Rham's theorem</u>

Suppose M is a smooth manifold. Let $H^k(M;\mathbb{R})$ be the singular cohomology of M w/ $\mathbb{R}$ coefficients, given by

$$H^k(M;\mathbb{R}) \cong \mathrm{Hom}(H_k(M;\mathbb{R}); \mathbb{R})$$

by the UCT. Then $\exists$ an isomorphism $H^k_{dR}(M) \to H^k(M;\mathbb{R})$ given by $H^k_{dR}(M) \to \mathrm{Hom}(H_k(M;\mathbb{R}), \mathbb{R}) : w \to I(w)$

w/ $I(w) : H_k(M;\mathbb{R}) \to \mathbb{R} : [\alpha] \to \int_\alpha w$.

$\square$

(a) 405 846 515

Based on the figure, we can express X via the CW complex

$$\begin{aligned} &1 \text{ 0-cell}: \ p \\ &1 \text{ 1-cell}: \ e \quad \text{w/} \ \partial_1 e = p - p \\ &1 \text{ 2-cell}: \ f \quad \text{w/} \ \partial_2 f = 5e \end{aligned}$$

Then $\operatorname{im}\partial_1 = 0$, $\ker \partial_1 = \mathbb{Z}(e)$, $\operatorname{im}\partial_2 = \mathbb{Z}(5e) = 5\mathbb{Z}(e)$ and $\ker\partial_2 = 0$.
This gives the chain complex

$$0 \xrightarrow{\partial_3} C_2(X) = \mathbb{Z}(f) \xrightarrow{\partial_2} C_1(X) = \mathbb{Z}(e) \xrightarrow{\partial_1} C_0(X) = \mathbb{Z}(p) \xrightarrow{\partial_0} 0$$

and the corresponding homology groups

$$H_0(X) = \frac{\ker \partial_0}{\operatorname{im}\partial_1} = \frac{\mathbb{Z}(p)}{0} \cong \mathbb{Z}$$

$$H_1(X) = \frac{\ker\partial_1}{\operatorname{im}\partial_2} = \frac{\mathbb{Z}(e)}{5\mathbb{Z}(e)} \cong \mathbb{Z}/5\mathbb{Z}$$

$$H_2(X) = \frac{\ker\partial_2}{\operatorname{im}\partial_3} = \ker\partial_2 = 0$$

w/ $H_k(X) = 0 \ \forall \ k > 2$.

This can be shortened to:
This gives the chain complex

$$0 \to C_2 \cong \mathbb{Z} \xrightarrow{5} C_1 \cong \mathbb{Z} \xrightarrow{0} C_0 \cong \mathbb{Z} \to 0$$

and therefore gives homology groups

$$H_2(X) \cong 0, \quad H_1(X) \cong \mathbb{Z}/5\mathbb{Z}, \quad H_0(X) \cong \mathbb{Z}$$

and $H_k(X) = 0 \ \forall k > 2$.

To obtain the cohomology, we dualize the chain complex to get the cochain complex. Dualizing, we find

$$C_i^* = \operatorname{Hom}(C_i, \mathbb{Z}) \cong \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$$

$$\delta_2 = 5, \quad \delta_1 = 0, \quad \delta_0 = 0$$

which gives the cochain complex

$$0 \xleftarrow{0} C_2^* \cong \mathbb{Z} \xleftarrow{5} C_1^* \cong \mathbb{Z} \xleftarrow{0} C_0^* \cong \mathbb{Z} \xleftarrow{0} 0$$

and cohomology groups

$$H^2(X) = \frac{\ker\delta_3}{\operatorname{im}\delta_2} \cong \mathbb{Z}/5\mathbb{Z}, \quad H^1(X) = \frac{\ker\delta_2}{\operatorname{im}\delta_1} \cong 0, \quad H^0(X) \cong 0$$

and $H^k(X) = 0 \ \forall \ k > 2$.

alternatively, the universal coefficients theorem implies that

$$0 \to \mathrm{Ext}(H_{n-1}(X), \mathbb{Z}) \to H^n(X;\mathbb{Z}) \to \mathrm{Hom}(H_n(X);\mathbb{Z}) \to 0$$

is a split exact sequence. We then compute

$$\mathrm{Ext}(H_0(X), \mathbb{Z}) \cong \mathrm{Ext}(\mathbb{Z}, \mathbb{Z}) = 0$$
$$\mathrm{Ext}(H_1(X), \mathbb{Z}) \cong \mathrm{Ext}(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/5\mathbb{Z}$$
$$\mathrm{Ext}(H_2(X), \mathbb{Z}) \cong \mathrm{Ext}(0, \mathbb{Z}) \cong 0$$

$$\mathrm{Hom}(H_0(X), \mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$$
$$\mathrm{Hom}(H_1(X), \mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}/5\mathbb{Z}, \mathbb{Z}) \cong 0$$
$$\mathrm{Hom}(H_2(X), \mathbb{Z}) \cong \mathrm{Hom}(0, \mathbb{Z}) \cong 0$$

with all other indices being 0. Then

$$H^0(X) \cong \mathrm{Ext}(H_{-1}(X), \mathbb{Z}) \oplus \mathrm{Hom}(H_0(X), \mathbb{Z}) \cong \mathbb{Z}$$
$$H^1(X) \cong \mathrm{Ext}(H_0(X), \mathbb{Z}) \oplus \mathrm{Hom}(H_1(X), \mathbb{Z}) \cong 0$$
$$H^2(X) \cong \mathrm{Ext}(H_1(X), \mathbb{Z}) \oplus \mathrm{Hom}(H_2(X), \mathbb{Z}) \cong \mathbb{Z}/5\mathbb{Z}$$

w/ all other cohomologies being 0. $\square$

GEOTOP

SPRING 2011

(1) 405 346 515

Suppose V is a smooth vector field on a smooth manifold of dimension n. Suppose $\exists\, p$ s.t. $V(p) \neq 0$. We claim that there exists a coordinate system $(x_1, \dots, x_n)$ s.t. $V = \partial/\partial x_1$ in a neighborhood of p.

As we are working locally at p, we may assume WLOG that we are working on $\mathbb{R}^n$ at $p = 0$. Further, by rotating and scaling, we may assume that $V_0 = \partial/\partial x_1$.

Let ϕ_t be ~~the~~ a local flow of V near 0, defined on some neighborhood ~~U~~ of 0. Define $\chi : U \to \mathbb{R}^n$ by

$$\chi(a_1, \dots, a_n) = \phi_{a_1}(0, a_2, \dots, a_n)$$

where U is a sufficiently small neighborhood of the origin.

We claim that χ^{-1} is our desired coordinate system.

Express V as $V = f_1 \partial/\partial x_1 + \dots + f_n \partial/\partial x_n$ w/ $f = (f_1, \dots, f_n)$.

Then

$$\begin{aligned} \frac{\partial}{\partial x_1} \chi(a_1, \dots, a_n) &= \partial/\partial x_1\, \phi_{a_1}(0, a_2, \dots, a_n) \\ &= V_{\phi_{a_1}(0, a_2, \dots, a_n)} \\ &= V_{\chi(a_1, \dots, a_n)} \\ &= f(\chi(a_1, \dots, a_n)) \end{aligned}$$

We claim that $(d\chi)_0$ is non-singular. Calculating $\partial/\partial x_1\, \chi|_0$ in general yields

and evaluating at 0 then gives $\frac{\partial}{\partial x_1}\chi = (1, 0, \dots, 0)$.

For $i = 2, \dots, n$, we find that

$$\begin{aligned} \frac{\partial}{\partial x_i} \chi\Big|_0 &= \frac{\partial}{\partial x_i} \phi_{x_1}(0, x_2, \dots, x_n) = \frac{\partial}{\partial x_i} \phi_0(0, 0, \dots, x_i, \dots 0) \\ &= \frac{\partial}{\partial x_i} (0, \dots, x_i, \dots, 0) = (0, \dots, \underbrace{1}_{i^{th}\text{ position}}, \dots, 0) \end{aligned}$$

Therefore $(d\chi)_0$ is non-singular. The inverse function theorem

(2) 405 846 515

a) We aim to demonstrate Cartan's formula

$$\mathcal{L}_X = d \circ i_X + i_X \circ d$$

To do so, we work locally. By linearity, it suffices to consider k-forms of the form $f\,dx_1 \wedge \ldots \wedge dx_k$. We proceed by induction.

First, we consider a 0-form f. ~~Since~~ By definition,

$$(\mathcal{L}_X(f))_p = \lim_{h \to 0} \frac{f \circ \phi_h(p) - f(p)}{h} = df(X)_p$$

where ϕ_h is the flow of X (local to p if needed).

Additionally,

$$(d \circ i_X(f) + i_X \circ d(f))_p = (i_X \circ df)_p = df(X)_p$$

~~Since ϕ_h is a curve through p w/ $\frac{d}{dt}\phi_t(p)\big|_{t=0} = X_p$.~~

$\therefore \quad (Xf)_p = (f \circ \phi_t(p))'(0) = (\mathcal{L}_X(f))_p$ as desired.

Now suppose the result holds $\forall$ m-forms for $m<k$. Consider a k-form $f\,dx_1 \wedge \ldots \wedge dx_k = dx_1 \wedge \underbrace{(f\,dx_2 \wedge \ldots \wedge dx_k)}_{\eta}$.

Note η is a $(k-1)$-form. Then by the inductive hypothesis

$$\begin{aligned}
\mathcal{L}_X(dx_1 \wedge \eta) &= \mathcal{L}_X(dx_1) \wedge \eta + dx_1 \wedge \mathcal{L}_X(\eta) \\
&= (\underbrace{\cancel{i_X \circ d(dx_1)}}_{\to 0} + d \circ i_X(dx_1)) \wedge \eta + dx_1 \wedge (i_X \circ d(\eta) + d \circ i_X(\eta)) \\
&= d(X(x_1)) \wedge \eta + dx_1 \wedge (i_X \circ d(\eta) + d \circ i_X(\eta))
\end{aligned}$$

Additionally, by the Leibniz rule

$$\begin{aligned}
(d \circ i_X + i_X \circ d)(dx_1 \wedge \eta) &= d(i_X(dx_1 \wedge \eta)) - i_X(dx_1 \wedge d\eta) \\
&= d(X(x_1) \wedge \eta - dx_1 \wedge i_X(\eta)) - i_X(dx_1 \wedge d\eta) \\
&= d(X(x_1)) \wedge \eta + \cancel{X(x_1) \wedge d\eta} + dx_1 \wedge d\,i_X(\eta) \\
&\quad - \cancel{X(x_1) \wedge d\eta} + dx_1 \wedge i_X(d\eta) \\
&= \mathcal{L}_X(dx_1 \wedge \eta)
\end{aligned}$$

as desired

③ 405 840 515

a) We note that $\mathbb{R}^3 \setminus \{0\}$ deformation retracts onto S^2 via the map $x \mapsto \frac{x}{\|x\|}$. Therefore
$$H^2(\mathbb{R}^3\setminus\{0\}) \cong H^2(S^2) \cong \mathbb{R}$$
In particular, this implies that $\exists$ a closed 2-form φ on $\mathbb{R}^3\setminus\{0\}$ s.t. φ is not exact, as desired.

b) Let φ be such a form. We recall that since S^2 is compact, the map $H^2(S^2) \to \mathbb{R} : \omega \mapsto \int_{S^2}\omega$ is an isomorphism. Therefore since $[\varphi] \neq [0] \in H^2(S^2)$, it follows that $\int_{S^2}\varphi \neq 0$.

Let $f: S^2 \to S^2$ be smooth. We recall that ~~the~~ $\deg f$ is defined as follows. The map $f: S^2 \to S^2$ induces a ~~map~~ homomorphism $f_*: H_2(S^2) \to H_2(S^2)$. ~~via $\omega \mapsto \omega \circ f$ As~~ B/c ~~$H^2(S$~~ $H_2(S^2) \cong \mathbb{Z}$, it follows that f induces a homomorphism $f_*: \mathbb{Z} \to \mathbb{Z}$ which must be ~~ent~~ multiplication by an integer, which we call $\deg f$.
Then
$$\int_{S^2} f^*\varphi = \int_{f_* S^2}\varphi$$

The space $f_* S^2$ is a $\deg f$-fold cover of S^2, and so
$$\int_{S^2} f^*\varphi = \int_{f_*S^2}\varphi = \deg f \underbrace{\int_{S^2}\varphi}_{\neq 0}$$

$$\Rightarrow \quad \deg f = \frac{\int_{S^2} f^*\varphi}{\int_{S^2}\varphi}$$

□

(4) 405 BUG 515

Suppose that φ is a 2-form on S^2 s.t. $\int_{S^2}\varphi = 0$.
We claim that φ is exact.

Let $N = S^2 \setminus \{\text{north pole}\}$ and $S = S^2\setminus\{\text{south pole}\}$.
Then $N \cong S \cong \mathbb{R}^2$. Since $\mathbb{R}^2$ is contractible, the Poincaré lemma then implies that $\varphi|_N, \varphi|_S$ are exact and $\exists\ \theta_1 \in \Omega^2(N)$ and $\theta_2 \in \Omega^2(S)$ s.t. $d\theta_1 = \varphi|_N$ and $d\theta_2 = \varphi|_S$.
We now aim to glue θ_1 and θ_2 together to find η s.t. $d\eta = \varphi$.
Let U be the upper hemisphere of S^2 and L the lower hemisphere. Since $U\cup L = S^2$ and $U\cap L$ has measure 0, it follows that

$$0 = \int_{S^2}\varphi = \int_U \varphi|_U + \int_L \varphi|_L = \int_U d\theta_1|_U + \int_L d\theta_2|_L$$

Taking the standard orientation on S^2, Stokes's then implies

$$0 = \int_U d\theta_1|_U + \int_L d\theta_2|_L = \int_{\partial U}\theta_1|_{\partial U} - \int_{\partial L}\theta_2|_{\partial L}$$

$$\Rightarrow \quad 0 = \int_{S^1}(\theta_1 - \theta_2)|_{S^1}$$

In particular, by the S^1 version of this result, $(\theta_1-\theta_2)|_{S^1}$ is exact on S^1.

B/c $N\cap S$ deformation retracts onto $U\cap L \cong S^1$, via i^*, it follows that i^* is an isomorphism on cohomology. Therefore
$\theta_1|_{N\cap S} - \theta_2|_{N\cap S}$ is exact b/c $i^*(\theta_1|_{N\cap S} - \theta_2|_{N\cap S}) = (\theta_1-\theta_2)|_{S^1}$ is exact. Then write $\theta_1|_{N\cap S} - \theta_2|_{N\cap S} = df$.
Let ψ_1, ψ_2 be a partition of unity subordinate to N, S.
Define

$$\eta = \begin{cases} \theta_1 - d(f\psi_1) & \text{on } N \\ \theta_2 + d(f\psi_2) & \text{on } S\end{cases}$$

(5) 405 346 515

Suppose that $V: U \to S^2$ is a smooth map, considered as a VF of unit vectors w/ $U = \mathbb{R}^3 \setminus \{p_1, \dots, p_n\}$ where $p_1, \dots, p_n \in B^3$

We aim to explain why $\deg V|_{S^2} = \sum_{i=1}^{n} \mathrm{ind}_{p_i} V$.

For each p_i choose an open neighborhood U_i of p_i s.t. $U_i \subset B^3$ and $U_i \cap U_j = \phi$ for $i \neq j$. This is possible b/c $p_1, \dots, p_n$ are ~~distinct~~ and strictly inside B^3 which is open.

Consider $X = B^3 \setminus (\bigcup_i U_i)$. Then $\partial X = S^2 \amalg (\bigcup_i U_i)$

Since $V: \partial X \to S^2$ is a map between 2-manifolds, S^2 is connected, ~~the extension theorem implies that~~ and V extends to all of X, the extension theorem then implies that $\deg V|_{\partial X} = 0$. Since $\partial X = S^2 - (\bigcup_i U_i)$, when orientation is taken into account, this implies that

$$0 = \deg V|_{S^2} - \deg V|_{\cup U_i}$$

$$\Rightarrow \deg V|_{S^2} = \sum_i \deg V|_{U_i}$$

$$= \sum_i \mathrm{ind}_{p_i} V$$

As desired

$\square$

⑥ 405 846 518

a) Let A_*, B_*, C_* be chain complexes. Then $0 \to A_* \xrightarrow{f_*} B_* \xrightarrow{g_*} C_* \to 0$ is a short exact sequence of chain complexes if $\forall n$,

$$0 \to A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \to 0$$

is a short exact sequence. To be clear, $0 \to A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \to 0$ is a short exact sequence if $\operatorname{im} f_n = \ker g_n$, $\ker f_n = 0$, $\operatorname{im} g_n = C_n$.

(7) 405 846 515

a) We define $\mathbb{CP}^n$ as $S^{2n+1}/\sim$ where $S^{2n+1} \subset \mathbb{C}^{n+1}$ in the usual way and $z \sim w$ if $z = \lambda w$ for some $\lambda \in \mathbb{C}^\times$.
We denote the equivalence class of $(z_0, \dots, z_n) \in \mathbb{C}^{n+1}$ as $[z_0 : \dots : z_n]$.

b) We define the cell structure of $\mathbb{CP}^n$ recursively as follows.

For $\mathbb{CP}^0$, we note that $S^1/\sim \cong \{pt\}$ since $z/w \in \mathbb{C}$ $\forall z, w \in S^1$.
Therefore $\mathbb{CP}^0$ has the structure of a single 0-cell.

Consider $\mathbb{CP}^n$ for $n \geq 1$. We claim that $\mathbb{CP}^n$ can be expressed as an $2n$-cell $e^{2n} = D^{2n}$ attached to $\mathbb{CP}^{n-1}$ via the map

$$\partial D^{2n} \cong S^{2n-1} \to \mathbb{CP}^{n-1} : (z_0, \dots, z_{n-1}) \longmapsto [z_0 : \dots : z_{n-1}]$$

To see this, we decompose $\mathbb{CP}^n$ into $U \amalg V$ where $U \cong \mathbb{CP}^{n-1}$, $V \cong D^{2n}$ and $U \amalg V$ are attached as above.

Define $U = \{[z_0 : \dots : z_{n-1} : 0] \in \mathbb{CP}^n\}$. Then $U \cong \mathbb{CP}^{n-1}$ by the map $[z_0 : \dots : z_{n-1} : 0] \mapsto [z_0 : \dots : z_{n-1}]$.

Consider $\mathbb{CP}^n \setminus U = V$. Then $V = \{[z_0 : \dots : z_n] : z_n \neq 0\}$.
B/c $|z_n| = \frac{\bar{z}_n}{|z_n|} z_n$ for $z_n \neq 0$ where $\left|\frac{\bar{z}_n}{|z_n|}\right| = 1$, it follows that $V = \{[z_0 : \dots : z_{n-1} : x] : x > 0\}$. Since $(z_0, \dots, z_{n-1}, x) \in S^{2n+1}$, it follows that $V = \{[z_0 : \dots : z_{n-1} : \sqrt{1 - \sum_{i<n} |z_i|^2}]\}$. Then $V \cong D^n$ via the map $V \to D : [z_0 : \dots : z_{n-1} : \sqrt{1 - \sum_{i<n}|z_i|^2}] \to (z_0, \dots, z_{n-1})$.
We note this is well-defined since $|z_0|^2 + \dots + |z_{n-1}|^2 < 1$.
Since $\mathbb{CP}^n = U \amalg V$ w/ $\partial V = U \cong \mathbb{CP}^{n-1}$ via the above attaching map, this completes the claim.

Therefore, by the above recursion, $\mathbb{CP}^n$ can be expressed as a CW complex w/ one cell in each even dimension.

$\longrightarrow$

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(a) By the same reasoning as the previous problem, we can express $\mathbb{RP}^2$ w/ the CW complex

1 0-cell : p.

1 1-cell : e w/ $\partial_1 e = p - p = 0$

1 2-cell : f w/ $\partial_2(f) = 2e$ via the map $\partial f \cong S^1 \to e \cong \mathbb{RP}^1 : (x,y) \longmapsto [x:y]$.

Then we acquire the chain complex

$$0 \to \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \to 0$$

Therefore

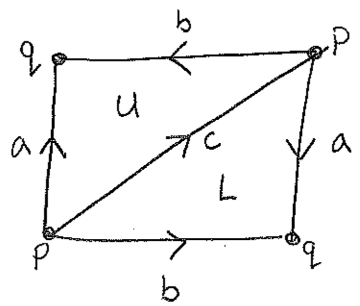
$$H_2(\mathbb{RP}^2) \cong \ker \partial_2 \cong 0$$

$$H_1(\mathbb{RP}^2) \cong \frac{\ker \partial_1}{\operatorname{Im} \partial_2} \cong \mathbb{Z}/2\mathbb{Z}$$

$$H_0(\mathbb{RP}^2) \cong \frac{\ker 0}{\operatorname{Im} \partial_1} \cong \mathbb{Z}/0 \cong \mathbb{Z}$$

$$H_k(\mathbb{RP}^2) \cong 0 \quad \forall k > 2.$$

Alternate: We recall that $\mathbb{RP}^2$ can be represented simplicially by



$c + (b-a) = 0$

$c - (b+a) = 0$

$c = b - a$

Then $\partial_2 U = c + (b-a)$, $\partial_2 L = (b-a) - c$

$\partial_1 a = q - p \qquad \partial_1 b = q - p \qquad \partial_1 c = p - p = 0$

Then $H_0(\mathbb{RP}^2) \cong \operatorname{im} \partial_1 = \mathbb{Z}(p-q) \cong \mathbb{Z}$

$$H_1(\mathbb{RP}^2) \cong \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\mathbb{Z}(b-a, c)}{\mathbb{Z}(c+(b-a), (b-a)-c)}$$

$$\cong \mathbb{Z}(b-a, c)/(c = b-a,\ c + b - a = 0)$$

$$\cong \mathbb{Z}/2\mathbb{Z}$$

$$H_2(\mathbb{RP}^2) \cong \ker \partial_2 = 0.$$

As desired. $\square$

⑨ 405 846 515

a) <u>theorem</u> (Lefschetz fixed point)
If $f: X \to X$ is a smooth map on a compact orientable manifold w/ $L(f) \neq 0$, then f has a fixed point.
Here $L(f)$ is the Lefschetz number and is defined as

$$L(f) = \sum_{k \geq 0} (-1)^k \, tr\left(f_*|_{H_k(X) \to H_k(X)}\right)$$

(b) Suppose that $\exists$ a smooth map $f: \mathbb{C}P^{2n} \to \mathbb{C}P^{2n}$.
We recall that $H^{2k}(\mathbb{C}P^{2n}) \cong \mathbb{Z}$ (this can be obtained nearly immediately from earlier via the universal coeffs theorem)
Let w denote the generator of $H^2(\mathbb{C}P^{2n})$. Then $\underbrace{w \wedge \dots \wedge w}_{k \text{ copies}}$ generates $H^{2k}(\mathbb{C}P^{2n})$ $\forall k \geq 0$.

Consider $f_*: H^2(\mathbb{C}P^{2n}) \cong \mathbb{Z} \to H^2(\mathbb{C}P^{2n}) \cong \mathbb{Z}$. Then f_* is a homomorphism $\mathbb{Z} \to \mathbb{Z}$ and so is multiplication by some $m \in \mathbb{Z}$. Then $f_*(w) = m \cdot w$ and so $f_*(\underbrace{w \wedge \dots \wedge w}_{k \text{ times}}) = m^k \cdot w \wedge \dots \wedge$
Therefore $\forall k \geq 0$, $f_*|_{H^{2k}(\mathbb{C}P^{2n})}$ is multiplication by m^k.

We now compute $L(f)$. By definition and Poincaré duality, since $\mathbb{C}P^{2n}$ is closed and orientable,

$$L(f) = \sum_{k \geq 0} (-1)^k \, tr\left(f_*|_{H^k(\mathbb{C}P^{2n}) \to H^k(\mathbb{C}P^{2n})}\right)$$

$$= \sum_{k \geq 0} (-1)^k \, tr\left(f_*|_{H_k(\mathbb{C}P^{2n}) \to H_k(\mathbb{C}P^{2n})}\right)$$

$\left(\begin{matrix} H_k(\mathbb{C}P^{2n}) \cong 0 \\ \text{for odd } k \end{matrix}\right)$ $$= \sum_{j \geq 0} tr\left(f_*|_{H_{2j}(\mathbb{C}P^{2n})}\right)$$

$$= \sum_{j \geq 0}^{2n} tr(m^j) = \sum_{j=0}^{2n} m^j = \begin{cases} \dfrac{m^{2n+1} - 1}{m-1} & (m \neq 1) \\ n+1 & m = 1 \end{cases}$$

In either case, $L(f) \neq 0$, which concludes. $\square$

(b) We can decompose $\mathbb{RP}^2 \times \mathbb{RP}^2$ into cells according to the cell decomposition of $\mathbb{RP}^2$. Let p_i, q_i, a_i, b_i, A_i for $i=1,2$ be the cells of two copies of $\mathbb{RP}^2$, as before. Then $\mathbb{RP}^2 \times \mathbb{RP}^2$ has the cellular structure

4 0-cells : $(p_1,p_2), (p_1,q_2), (q_1,p_2), (q_1,q_2)$

8 1-cells : $(a_1,p_2), (a_1,q_2), (b_1,p_2), (b_1,q_2)$
$(p_1,a_2), (p_1,b_2), (q_1,a_2), (q_1,b_2)$

8 2-cells : $(A_1,p_2), (A_1,q_2), (p_1,A_2), (q_1,A_2)$
$(a_1,a_2), (a_1,b_2), (b_1,a_2), (b_1,b_2)$

4 3-cells : $(A_1,a_2), (A_1,b_2), (a_1,A_2), (b_1,A_2)$

1 4-cell : (A_1,A_2)

Moreover,
$$\begin{aligned}\partial_4(A_1,A_2) &= (A_1, 2(b_2-a_2)) + (2(b_1-a_1), A_2) \\ &= 2\big((A_1,b_2) - (A_1,a_2) + (b_1,A_2) - (a_1,A_2)\big)\end{aligned}$$

Decomposing $\mathbb{RP}^2$ as done in the first solution, we give the same decomposition to 2 copies of $\mathbb{RP}^2$ w/ cells p_i, e_i, f_i $i=1,2$. This yields a corresponding decomposition of $\mathbb{RP}^2 \times \mathbb{RP}^2$ w/

1 0-cells : (p_1,p_2)

2 1-cells : $(e_1,p_2), (p_1,e_2)$

3 2-cells : $(f_1,p_2), (p_1,f_2), (e_1,e_2)$

2 3-cells : $(f_1,e_2), (e_1,f_2)$

1 4-cell : (f_1,f_2)

Moreover,

$$\partial_4(f_1,f_2) = (\partial_2 f_1, f_2) + (f_1, \partial_2 f_2)$$
$$= (2e_1, f_2) + (f_1, 2e_2) = 2((e_1,f_2)+(f_1,e_2))$$

$$\partial_3(f_1,e_2) = (\partial_2 f_1, e_2) + (f_1, \partial_1 e_2)$$
$$= (2e_1, e_2) = 2(e_1,e_2)$$

(ignore X)

$$\partial_3(e_1,f_2) = (\partial_1 e_1, f_2) - (e_1, \partial_2 f_2)$$
$$= -2(e_1,e_2)$$

Then

$$H_3(\mathbb{RP}^2 \times \mathbb{RP}^2) = \frac{\ker \partial_3}{\operatorname{Im} \partial_4}$$
$$= \frac{\mathbb{Z}((f_1,e_2)+(e_1,f_2))}{\mathbb{Z}(2(e_1,f_2)+2(f_1,e_2))}$$
$$\cong \mathbb{Z}/2\mathbb{Z}$$

which in particular has a nonzero element, $(f_1,e_2)+(e_1,f_2)$.

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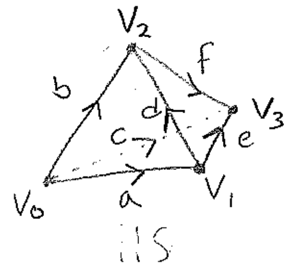
We decompose the tetrahedron X into simplicial complexes as follows.



$$\Rightarrow \quad X = \underset{A}{[v_0, v_1, v_2]} \cup \underset{D}{[v_1, v_2, v_3]} \cup \underset{C}{[v_0, v_2, v_3]} \cup \underset{B}{[v_0, v_1, v_3]}$$

Then we assume the chain complex

$$0 \to \mathbb{Z}(A,B,C,D) \xrightarrow{\partial_2} \mathbb{Z}(a,b,c,d,e,f) \xrightarrow{\partial_1} \mathbb{Z}(v_0,v_1,v_2,v_3) \to 0$$

w/ $\partial_2 A = a+d-b$, $\partial_2 B = a+e-c$, $\partial_2 C = b+f-c$, $\partial_2 D = d+f-e$

$\partial_1 a = v_1 - v_0$, $\partial_1 b = v_2 - v_0$, $\partial_2 c = v_3 - v_0$, $\partial_2 d = v_2 - v_1$, $\partial_2 e = v_3 - v_1$

$\partial_2 f = v_3 - v_2$

Taking a,b,c,d,e,f and A,B,C,D and v_0,v_1,v_2,v_3 as bases, we can thus represent ∂_2 and ∂_1 as the matrices,

$$\partial_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad \partial_1 = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Consider $[x,y,z,w]^T \in \ker \partial_2$. By the above matrix, we have

$$\begin{aligned} x+y=0 &\Rightarrow x=-y \\ -x+z=0 &\Rightarrow x=z \\ x+w=0 &\Rightarrow x=-w \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

hence $[1,-1,1,-1]^T \in \ker \partial_2$, this implies that $\ker \partial_2 = \mathbb{Z}(A-B+C-D) \cong \mathbb{Z}$.

Spring 2010 GeoTop

1 a ≈ ~~b √~~

2 ?

3 √

~~4 a ≈ b √~~

5 a √ b ≈

6 a √ b c

7 √

8 ≈

9 a √ b ≈

10 √

11 a √ b √

12 ?

13 √

GeoTop

Spring 2010

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a) To show 0 is a regular value of F, it ~~suffices to~~ must be shown that dF_A is surjective $\forall A \in F^{-1}(0)$.
Fix some $A \in F^{-1}(0)$. We recall that $T_A M_n \cong M_n$ and $T_{F(A)} S_n \cong S_n$. Consider some $C \in S_n$. Then by direct computation,

$$dF_A(\tfrac{1}{2}CA) = \lim_{t\to 0} \frac{(A+\frac{t}{2}CA)(A+\frac{t}{2}CA)^T - I - AA^T + I}{t}$$

$$= \lim_{t\to 0} \frac{\cancel{AA^T} + \frac{t}{2}CAA^T + \frac{t}{2}A(CA)^T + \frac{t^2}{4}(CA)(CA)^T - \cancel{AA^T}}{t}$$

$$= \lim_{t\to 0} \tfrac{1}{2}(CAA^T + AA^TC^T) + \tfrac{t}{4}(CA)(CA)^T$$

$$= \tfrac{1}{2}(CAA^T + AA^TC^T) = C$$

where the final inequality follows b/c $F(A)=0 \Rightarrow AA^T = I$ and $C \in S_n$.
Therefore $dF_A : T_A M_n \to T_{F(A)} S_n$ is surjective $\forall A \in F^{-1}(0)$ and so F has 0 as a regular value.

b) By construction $F(A)=0$ iff $AA^T = I$ and so $F(A)=0$ iff $A^{-1} = A^T$. Then $F^{-1}(0) = O(n)$. The regular value theorem and part a then imply that $O(n)$ is a smooth submanifold of M_n.

c) B/c $F: M_n \to S_n$ and $\dim M_n = n^2$, $\dim S_n = n + (n-1) + \dots + 1 = \frac{n(n+1)}{2}$, the regular value theorem implies $\dim O(n) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

Since $O(n) = F^{-1}(0)$, it follows that

$$T_I O(n) = \ker(dF_I) \quad \text{w/ } dF_I : T_I M_n \cong M_n \to T_{F(I)=0} S_n \cong S_n$$

By direct calculation, $\forall B \in M_n$

$$dF_I(B) = \lim_{t\to 0} \frac{(I+tB)(I+tB)^T - I}{t}$$

$$= \lim_{t\to 0} \frac{I - I + tB + tB^T + t^2 BB^T}{t}$$

$$= B + B^T$$

So $\ker(dF_I) = \{B \in M_n : B + B^T = 0\} = \{B \in M_n : B^T = -B\}$ which is the skew-symmetric matrices $\square$

(2) 405 846 515

<u>Lemma</u>: the product of parallelizable manifolds is parallelizable.

<u>proof</u>: take M, N parallelizable w/ $\dim M = m$, $\dim N = n$.
Then $\exists$ vector fields $V_1, \dots, V_m$ on M and $U_1, \dots, U_n$ on N s.t. $V_1(p), \dots, V_m(p)$ are linearly independent $\forall p \in M$ and $U_1(q), \dots, U_n(q)$ are lin. ind. $\forall q \in N$.
We can extend V_i, U_j to vector fields $\hat{V}_i, \hat{U}_j$ on $M \times N$, $\forall i, j$ by taking

$$\hat{V}_i(p,q) = (V_i(p), 0) \in T_{(p,q)}(M \times N) \cong \pi_M^* T_pM \oplus \pi_N^* T_qN$$

and

$$\hat{U}_j(p,q) = (0, U_j(q)) \in T_{(p,q)}(M \times N) \cong \pi_M^* T_pM \oplus \pi_N^* T_qN$$

where π_M, π_N are the canonical projections $M \times N \to M$ and $M \times N \to N$.
Then $\forall (p,q) \in M \times N$, $\hat{V}_1(p,q), \dots, \hat{V}_m(p,q), \hat{U}_1(p,q), \dots, \hat{U}_n(p,q)$ are linearly independent. Then $M \times N$ is parallelizable. ▷

We recall that parallelizability is equivalent to the trivializability of the tangent bundle. Additionally, we recall that

$$TS^1 \cong S^1 \times \mathbb{R} \quad , \quad \cancel{\mathbb{R} \oplus TS^n \cong \mathbb{R}^{n+1}}$$

since ~~the tangent bundle~~ S^1 has a nowhere vanishing vector field.
Moreover, ~~since we~~ it follows that $TS^n \oplus \mathbb{R} \cong S^n \times \mathbb{R}^{n+1}$ since the additional $\mathbb{R}$ term can be realized as the normal bundle to $S^n \subset \mathbb{R}^{n+1}$! Then

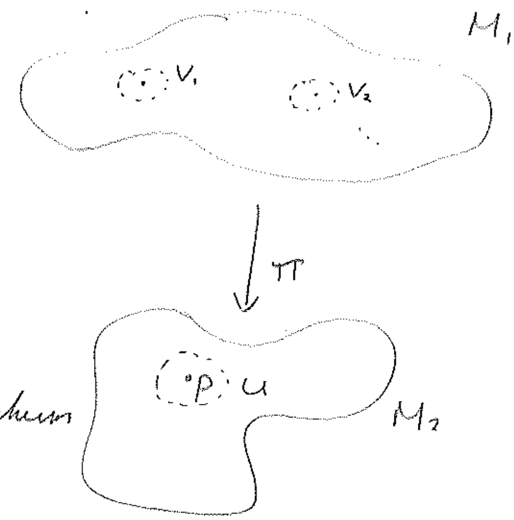
$$\begin{aligned}
T(T^2 \times S^n) &\cong \pi_{S^1}^{1*} TS^1 \oplus \pi_{S^1}^{2*} TS^1 \oplus \pi_{S^n}^* TS^n \\
&\cong \pi_{S^1}^{1*} TS^1 \oplus S^1 \times \mathbb{R} \oplus \pi_{S^n}^* TS^n \\
&\cong \pi_{S^1}^{1*} TS^1 \oplus S^1 \times S^n \times \mathbb{R}^{n+1} \\
&\cong S^1 \times S^1 \times S^n \times \mathbb{R}^{n+2} \\
&\cong (T^2 \times S^n) \times \mathbb{R}^{n+2}
\end{aligned}$$

so $T(T^2 \times S^n)$ is trivializable and hence parallelizable.

③ 405 846 515

Suppose M_1 is compact (*)

a) ~~Since π is C^∞,~~ to show it is a covering map, it suffices to show we first aim that $\forall\, p \in M_2$, $\exists$ an open neighborhood U of p s.t. $\pi^{-1}(U) = \coprod_{i=1}^{n} V_i$ for open V_i s.t. $\pi|_{V_i} : V_i \to U$ is a homeomorphism.



Consider $\pi^{-1}(p)$: $\forall\, q \in \pi^{-1}(p)$, the differential $d\pi_q : T_qM_1 \to T_pM_2$ is an isomorphism and hence surjective. Therefore p is a regular value of π and so $\pi^{-1}(p)$ is a 0-dimensional submanifold of M_1. In particular, this implies that the pre-image $\pi^{-1}(p)$ is discrete and countable. Since M_1 is compact, any discrete set is finite. Therefore $\pi^{-1}(p) = q_1, \dots, q_n$.

Since $d\pi_{p_i}$ is an isomorphism $\forall i$, the inverse function theorem implies that π is a local diffeomorphism. Therefore $\forall i$ $\exists$ an open neighborhood $\hat{V}_i$ of q_i s.t. $\pi|_{\hat{V}_i}$ is a diffeomorphism. Since $\{q_i\}$ is discrete, we may restrict $\hat{V}_i$ s.t. $\{\hat{V}_i\}_i$ is pairwise disjoint. Since $\pi|_{\hat{V}_i}$ is a diffeomorphism, it is open. Therefore $\hat{U}_i := \pi(\hat{V}_i)$ is an open neighborhood of p. *

Let $U' = \hat{U}_1 \cap \dots \cap \hat{U}_n$. Then U' is an open neighborhood of p.

Let $V_i' = \hat{V}_i \cap \pi^{-1}(U)$. Then V_i' is an open neighborhood of q_i and $\pi^{-1}(U) \supset V_1' \cup \dots \cup V_n'$.

Let $V' = V_1' \cup \dots \cup V_n'$ and $Z = M_1 \setminus V'$. Then Z is closed and hence compact since M_1 is compact. Then $\pi(Z)$ is closed.

By construction, $\pi(Z) \not\ni p$ so $U' \setminus \pi(Z)$ is an open neighborhood of p. Take $U = U' \setminus \pi(Z)$ and $V_i = V_i' \cap \pi^{-1}(U)$. Then since

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We claim that

$$H_j(\mathbb{R}^n \setminus \{x_1, \ldots, x_k\}) \cong \begin{cases} \mathbb{Z} & j=0 \\ \mathbb{Z}^k & j=n-1 \\ 0 & \text{else} \end{cases}$$

To show this, we proceed by induction on k, the # of points removed. Suppose $k=0$. Then we recall

$$H_j(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z} & j=0 \\ 0 & \text{else} \end{cases}$$

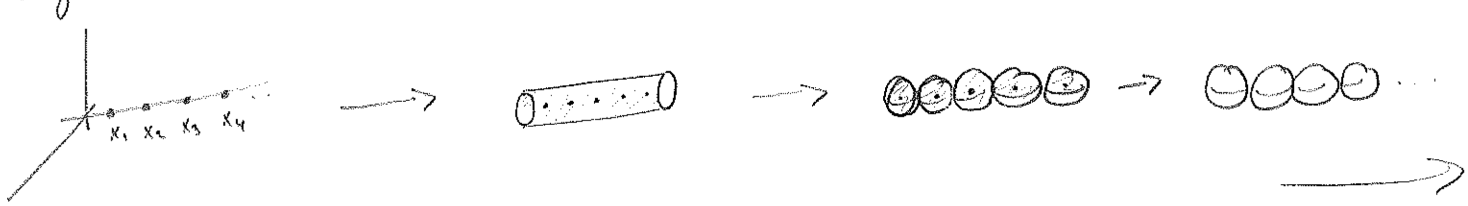
which is consistent w/ our claim.

Suppose $k=1$. Then $\mathbb{R}^n \setminus \{x_1\}$ deformation retracts onto S^{n-1}, and so

$$H_j(\mathbb{R}^n \setminus \{x_1\}) \cong \begin{cases} \mathbb{Z} & j=0,1 \\ 0 & \text{else} \end{cases}$$

which aligns w/ our claim.

We claim that $\mathbb{R}^n \setminus \{x_1, \ldots, x_k\}$ deformation retracts onto the wedge of k copies of S^{n-1}. There exists a diffeomorphism $\mathbb{R}^n \to \mathbb{R}^n$ s.t. $x_i \to (i, 0, \ldots, 0)$. In the ~~standard~~ obvious way, we can then deformation retract $\mathbb{R}^n \setminus \{x_1, \ldots, x_k\}$ onto the solid cylinder of length $k+1$ and radius 1 centered on the 1st axis, excluding the points $x_1, \ldots, x_k$. From there we deformation retract radially inwards towards the wedge k solid spheres, excluding their centers. Finally, we deformation retract radially outwards the inside of each solid sphere onto its boundary. Composing these, we find that $\mathbb{R}^n \setminus \{x_1, \ldots, x_k\}$ deformation retracts onto $\bigvee_{j=1}^k S^{n-1}$.



We recall that the singular ~~holomo~~ homology of S^{n-1} is given by

$$H_j(S^{n-1}) = \begin{cases} \mathbb{Z} & j=0,n-1 \\ 0 & \text{else} \end{cases}$$

We additionally recall that the homology of a wedge sum is the direct sum of homologies, excluding the 0^{th} homology. Therefore

$$H_j(\mathbb{R}^n \setminus \{x_1,\dots x_k\}) \cong H_j\left(\bigvee_{i=1}^k S^{n-1}\right) \cong \begin{cases} \mathbb{Z} & j=0 \\ \mathbb{Z}^k & j=n-1 \\ 0 & \text{else} \end{cases}$$

where the 0^{th} homology follows from the fact that ~~$\mathbb{R}$~~ $\bigvee_{i=1}^k S^{n-1}$ is connected. $\square$

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a) Let U, V be two disjoint open ~~subsets of~~ disks in M s.t. $\bar{U} \cap \bar{V} = \emptyset$ and $U \cong \mathbb{R}^n$, $V \cong \mathbb{R}^n$. Since M is orientable, there exists an orientation on $M \setminus U \cup V$ and an induced orientation on $\partial U, \partial V$. We then glue one end of a cylinder C of fixed length to ∂U and the other to ∂V such that orientation is preserved.

b) Consider S^2 w/ g handles ~~conn~~ attached. Denote this by S^2_g.

If $g=1$, then it is clear that $S^2_g \cong M_g$, the orientable genus g surface. Repeating this w/ increasing g, we find that $S^2_g \cong M_g$ $\forall g$.

We claim $\pi_1(M_g) \neq 0$ $\forall$ $g>0$. We recall that M_g can be constructed via.

1 0-cell : p

$2g$ 1-cells : $a_1\, b_1\, a_2\, b_2, \ldots, a_g\, b_g$ w/ $\partial a_i = \partial b_i = p - p$

1 2-cell : f w/ $\partial f = a_1 + b_1 - a_1 - b_1 + \ldots + a_g + b_g - a_g - b_g$

Then $H^1(M_g) = \dfrac{\ker \partial_1}{\operatorname{Im} \partial_2} \cong \mathbb{Z}^{2g}$ since $\partial_1 = 0$ and $\partial_2 = 0$.

Since H^1 is the abelianization of $\pi_1(M_g)$, this implies that $\pi_1(M_g) \neq 0$ and so S^2_g is not simply connected. □

(8) 405 846 515

a) Let $\exists$ $F: S^2 \to S^2$. Then F induces a ~~map~~ homomorphism $F_*: H_2(S^2) \to H_2(S^2)$.
Since $H_2(S^2) \cong \mathbb{Z}$, F_* can be viewed as a homomorphism $F^*: \mathbb{Z} \to \mathbb{Z}$. As the only such homomorphisms are multiplication by an ~~integ~~ integer, it follows that $F_*(a) = ka$ for some $k \in \mathbb{Z}$. We let $k = \deg F$. No choices were necessary in this construction and the above ~~den~~ description guarantees well-defined.

b.) ~~Let $\overline{B_1}, \dots, \overline{B_K}$ be disjoint disks on S^2.~~
For each B_i, $\exists$ a C^∞ map $B_i \to S^n$ that extends to a ~~map~~ $\varphi_i: \overline{B_i} \to S^n$ w/ $\partial B_i \to N$.
Define $F: S^2 \to S^2$ by $F|_{\overline{B_i}} = \varphi_i$ and $F|_{S^n \setminus \cup \overline{B_i}} = N$.
Then F is C^∞ and

9

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Let $V = P\,\partial/\partial x + Q\,\partial/\partial y + R\,\partial/\partial z$. Then as usual,

$$\mathrm{div}(V) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

This implies that

$$\mathrm{div}(V)\,d(\mathrm{vol}) = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\wedge dy\wedge dz$$

Take $\omega = P\,dy\wedge dz - Q\,dx\wedge dz + R\,dx\wedge dy$. Then

$$d\omega = \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right)\wedge dy\wedge dz + dQ\wedge dx\wedge dz + dR\wedge dx\wedge dy$$

$$= \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\wedge dy\wedge dz$$

Stoke's theorem for manifolds w/ boundaries then gives

$$\int_M \mathrm{div}(V)\,d(\mathrm{vol}) = \int_M d\omega = \int_{\partial M}\omega$$

$$= \int_{\partial M}\left(P\,dy\wedge dz - Q\,dx\wedge dz + R\,dx\wedge dy\right)$$

$$= \int_{\partial M} V$$

Let N be the unit normal vector field to M. We ~~recall~~ recall that the divergence theorem states

$$\int_M \mathrm{div}(V)\,d(\mathrm{vol}) = \int_{\partial M}\langle V,N\rangle dS$$

where $dS = \iota^* i_N d(\mathrm{vol})$ w/ ι the ~~t~~ embedding of ∂M into $\mathbb{R}^n$.

We first claim $\iota^* i_X d(\mathrm{vol}) = \langle V,N\rangle dS$. Let $T = V - \langle V,N\rangle N$. Then T is precisely the portion of V tangent to ∂M and so T is a vector field on ∂M. We claim $\iota^* i_T d(\mathrm{vol}) = 0$. For any $p\in\partial M$ and for any vector fields $V_1,\dots,V_{n-1}\in T_p(\partial M)$, we see that

$$\left(\iota^* i_T d(\mathrm{vol})\right)_p(V_1,\dots,V_{n-1}) = d(\mathrm{vol})(T, d\iota_p V_1,\dots d\iota_p V_{n-1})$$

Since ∂M is $n-1$-dimensional and $T, d\iota_p V_1,\dots,d\iota_p V_{n-1}$ are n elements of $T_p\partial M$, it follows they are linearly dependent and so

$$\left(\iota^* i_T d(\mathrm{vol})_p\right)(V_1,\dots,V_{n-1}) = 0$$

As this holds $\forall$ vector fields, it follows that $\iota^* i_T d(\mathrm{vol}) = 0$ and $\Rightarrow \iota^* i_V d(\mathrm{vol}) = \langle V,N\rangle dS$. $\longrightarrow$

Let $V = P\,\partial/\partial x + Q\,\partial/\partial y + R\,\partial/\partial z$. ~~Then~~

$$\cancel{\iota^* i_V d(\text{vol}) = P\,dy\wedge dz - Q\,dx\wedge dz + R\,dy\wedge dz.}$$

Stokes theorem then implies

$$\int_{\partial M} \langle V, N\rangle dS = \int_{\partial M} \iota^* i_V d(\text{vol})$$

$$= \int_M d(i_V d(\text{vol}))$$

By construction

$$d(i_V d(\text{vol})) = d(P\,dy\wedge dz - Q\,dx\wedge dz + R\,dy\wedge dz)$$

$$= \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\wedge dy\wedge dz$$

$$= \text{div}(V)\, d(\text{vol})$$

And so

$$\int_{\partial M} \langle V, N\rangle dS = \int_M \text{div}(V)\, d(\text{vol})$$

as desired. $\square$

(10) 405 846 515   We assume $n>1$.

a) We recall that the universal cover of $S^1\times\cdots\times S^1 \cong T^k$ is $\mathbb{R}^k$. Let $\pi:\mathbb{R}^k\to T^k$ be the canonical projection.

$$\begin{tikzcd} & \mathbb{R}^k \arrow[d, "\pi"] \\ S^n \arrow[ur, dotted, "\tilde{F}"] \arrow[r, "F"'] & T^k \end{tikzcd}$$

Since $n>1$, $\pi_1(S^n)=0$. Then since $\pi_1(\mathbb{R}^k)=0$, it follows that $F^*(\pi_1(S^n))\subset\pi^*(\pi_1(\mathbb{R}^k))$, and so $\exists$ a lift $\tilde{F}:S^n\to\mathbb{R}^k$ of F. B/c $\mathbb{R}^k$ is contractible, $\exists$ a homotopy $\tilde{h}_t:S^n\to\mathbb{R}^k$ from $\tilde{F}$ to a constant map. This descends to a homotopy $h_t:S^n\to T^k$ from F to a constant map. Therefore F is nullhomotopic.

b) Pick some open $U\subset T^n$ s.t. $U\cong\mathbb{R}^n\cong S^n\setminus\{N\}$. Since T^n is a smooth manifold, this is possible. Let $\varphi:U\to S^n\setminus\{N\}$ be this diffeomorphism and define

$F:T^n\to S^n$ by $F|_U=\varphi$ and $F|_{T^n\setminus U}=N$. Then F is cont. by construction and can be made smooth if necessary.

Consider a regular value x of F. Since a regular value must have finite pre-images, $x\neq N$. Since $F(p)\neq N$ iff $p\in U$, and $F|_U=\varphi$, a diffeomorphism, it follows that $F^{-1}(x)$ consists of one point in U. Since $F|_U$ is a homeomorphism, ~~its degree~~ it then follows that $\deg F=\pm1\neq0$. Since degree is homotopy invariant, this implies that F is not nullhomotopic.

c) We now use degree defined via differential forms. For each i, let ω_i be a nowhere vanishing volume form on S^{n_i}. Then $\omega_1 \wedge \dots \wedge \omega_k$ is a nowhere vanishing volume form on $S^{n_1} \times \dots \times S^{n_k}$. Normalizing, we may assume

$$\int_{S^{n_1} \times \dots \times S^{n_k}} \omega_1 \wedge \dots \wedge \omega_k = 1$$

so that

$$\deg F = \int_{S^n} F^*(\omega_1 \wedge \dots \wedge \omega_k) = \int_{S^n} F^*\omega_1 \wedge \dots \wedge F^*\omega_k$$

Consider $F^*\omega_1$. Since ω_1 is a closed form on S^{n_1}, it follows that ω_1 extended to $S^{n_1} \times \dots \times S^{n_k}$ is also closed. Then $F^*\omega_1$ is a closed n_1-form on S^n. Since $0 < n_1 < n$, it then follows that $F^*\omega_1$ is exact and so $\exists\, \eta$ s.t. $d\eta = F^*\omega_1$. Then $d(\eta \wedge F^*\omega_2 \wedge \dots \wedge F^*\omega_k) = F^*(\omega_1 \wedge \dots \wedge \omega_k)$. Stokes theorem then implies

$$\deg F = \int_{\partial S^n} \eta \wedge F^*\omega_2 \wedge \dots \wedge F^*\omega_k = \int_{\emptyset} \sim = 0$$

As desired.

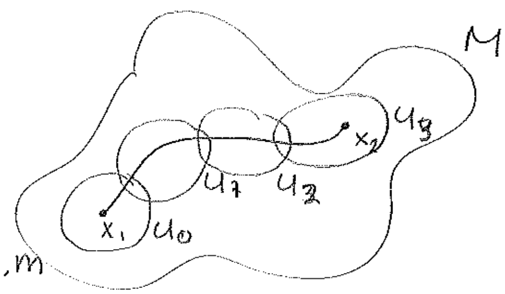
$\square$

FALL, 2010 - Geotop

~~1 ✓~~ ✓
~~2 ✓~~ ✓
~~3 a ✓ b ✓ c ✓~~ ✓
4 a ≈ b ≈ ✓
5 ≈ ?
~~6 ✓~~ ✓
~~7 ✓~~ ✓
8 ✗ ?
~~9 ✓~~ ✓
10 ≈ ✓

① 405 846 515

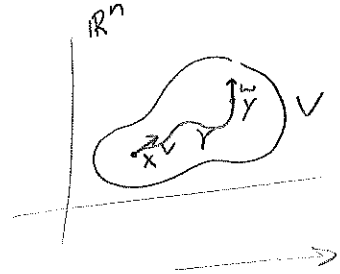
Since M is smooth and connected, $\exists$ a path $\gamma:[0,1]\to M$ from x_1 to x_2. Here, we conflate γ w/ its image $\gamma([0,1])$. Since $[0,1]$ is compact, $\gamma\subset M$ is compact. Since M is a smooth manifold, $\forall x\in\gamma$ $\exists$ a neighborhood U_x of x s.t. U_x is diffeomorphic to a subset of $\mathbb{R}^n$ w/ $n=\dim(M)$. Since γ is compact, $\exists$ a finite subcover $U_0,\dots,U_m$ of γ. By choosing sufficiently small initial sets U_x and rearranging our sets, we may ~~assume~~ assume that $x_1\in U_0$, $x_2\in U_m$, the index of U_i increases as t goes from 0 to 1, ~~that~~ and that $U_i\cap U_j\neq\phi$ iff $|i-j|\le 1$. This is illustrated to the right. For each $i=1,\dots,m$ choose $z_i\in U_i\cap U_{i+1}\cap\gamma$, and let $z_0=x_1$ and $z_{m+1}=x_2$. *



To construct ϕ, we aim to construct diffeomorphisms ϕ_i for $i=0,\dots,m$ s.t. $\phi_i(z_i)=z_{i+1}$ and $d(\phi_i)_{z_i}(u_i)=u_{i+1}$. Then $\phi=\phi_m\circ\dots\circ\phi_0$ will satisfy $\phi(x_1)=x_2$ w/ $d\phi_{x_1}(v_1)=v_2$.

By construction, $\forall i$ $z_i,z_{i+1}\in U_i$. Therefore, since $U_i\cong V_i\subset\mathbb{R}^n$ we have reduced to the case of connected $M=V\subset\mathbb{R}^n$ and we desire a diffeomorphism ϕ which sends $x\mapsto y$ and $d\phi_x(v)=w$ for specified x,y,v,w and is equal to the identity outside of a compact ~~neigh~~ neighborhood of x and y.

Since we are working in $V\subset\mathbb{R}^n$, $\exists$ a smooth path $\gamma:[0,1]\to V$ s.t. $\gamma(0)=x$, $\gamma(1)=y$, ~~and~~ $\gamma'(0)=v$ and $\gamma'(1)=w$.



* Additionally choose nonzero $u_i\in T_{z_i}M$ w/ $u_0=v_1$ and $u_{m+1}=v_2$.

(2) 405 846 515 Let $m = \dim(X)$

We aim to use the transversality theorem.

Define $F: X \times \mathbb{R}^n \to \mathbb{R}^n$ by $(x,a) \mapsto x + a$. ~~w/ $\iota: X \to \mathbb{R}^n$ the inclusion,~~

We aim to show $F \pitchfork Y$.

Suppose $\exists$ (x,a) s.t. $F(x,a) \in Y$. Then,

$$dF_{(x,a)} : T_{(x,a)}(X \times \mathbb{R}^n) \cong (T_x X) \oplus (T_a \mathbb{R}^n) \to T_{F(x,a)} \mathbb{R}^n \cong \mathbb{R}^n$$

In local coordinates $(x_1, \dots, x_m, a_1, \dots, a_n)$ of $X \times \mathbb{R}^n$ we may write $dF_{(x,a)}$ as a $(n+m) \times n$ matrix. By construction we see that $dF_{(x,a)}$ has an $n \times n$ identity matrix in the furthest right position. Then $\operatorname{rank}(dF_{(x,a)}) \geq n \Rightarrow \operatorname{rank}(dF_{(x,a)}) = n$ and so $dF_{(x,a)} : T_{(x,a)}(X \times \mathbb{R}^n) \to T_{F(x,a)} \mathbb{R}^n$ is surjective.

In particular, this implies

$$dF_{(x,a)}\left(T_{(x,a)}(X \times \mathbb{R})\right) + T_{F(x,a)} Y = T_{F(x,a)} \mathbb{R}^n$$

As this holds $\forall$ $(x,a) \in F^{-1}(Y)$, this implies that $F \pitchfork Y$.

The transversality theorem then implies that

$$f_a = F(\cdot, a) : X \to \mathbb{R}^n$$

is transversal to Y for a.e. a. By construction, since f_a is translation by a, $d(f_a) = \mathrm{id}$ and so $\forall$ $x \in f_a^{-1}(Y)$,

$$d(f_a)_x(T_x X) + T_{f_a(x)} Y = T_{f_a(x)} \mathbb{R}^n$$

$$\Rightarrow \quad T_{f_a(x)}(X + a) + T_{f_a(x)} Y = T_{f_a(x)} \mathbb{R}^n$$

and so $X + a \pitchfork Y$ for a.e. a.

3) 405 846 515

a) We use the regular value theorem.
Consider $F: M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ defined by $F(M) = \det M$.
Then $F^{-1}(1) = SL(n,\mathbb{R})$. To show $SL(n,\mathbb{R})$ is a smooth submanifold, it then suffices to show 1 is a regular value of F. Let $\exists$ some $M \in SL(n,\mathbb{R})$. Then $\forall\, T \in M_{n\times n}(\mathbb{R}) \cong T_M M_{n\times n}(\mathbb{R})$

$$dF_M(T) = \lim_{t\to 0} \frac{\det(M+tT) - \det(M)}{t}$$
$$= \lim_{t\to 0} \frac{\det(M)(\det(I+tM^{-1}T) - 1)}{t}$$
$$= \lim_{t\to 0} \frac{\det(I+tM^{-1}T) - 1}{t}$$

Consider some $k \in \mathbb{R}^1 \cong T_{F(M)}\mathbb{R}$. Then w/ $T = \frac{k}{n}M$, T

$$dF_M(kM) = \lim_{t\to 0} \frac{\det((1+\frac{tk}{n})I) - 1}{t}$$
$$= \lim_{t\to 0} \frac{(1+\frac{tk}{n})^n - 1}{t}$$
$$= \lim_{t\to 0} k + \binom{n}{2} t (\tfrac{k}{n})^2 + \dots + t^{n-1}(k/n)^n$$
$$= k.$$

and so ~~1 is a regular value of F~~ dF_M is surjective $\forall M \in F^{-1}(1)$
$\Rightarrow$ 1 is a regular value of F. Then $F^{-1}(1) = SL(n,\mathbb{R})$ is a smooth submanifold of $M_{n\times n}(\mathbb{R})$.

b) By the previous part, it follows that

$$T_I SL(n,\mathbb{R}) = \ker(dF_I)$$

By direct calculation,

$$dF_I(A) = \lim_{t\to 0} \frac{\det(I+tA) - 1}{t}$$

since $\det(I+tA) \approx 1 + Tr(A)t$ for sufficiently small t, it follows that

$$dF_I(A) = Tr(A)$$

so

$$T_I SL(n,\mathbb{R}) = \{A : Tr(A) = 0\}.$$

(4) 405 846 515

a) A cochain homotopy between f_0^* and f_1^* is a collection of linear maps $h_n : \Omega^n(N) \to \Omega^{n-1}(M)$ s.t.

$$f_1^* - f_0^* = d\circ h + h\circ d$$

[If such a cochain homotopy exists then $f_0^* = f_1^*$ on the level of cohomology since ω closed implies $(f_1^* - f_0^*)\omega = d(h(\omega)) =$ exact].

b) Let $\exists$ some $\omega \in \Omega^*(N)$. Then by definition,

$$\phi_1^*\omega - \phi_0^*\omega = \int_0^1 \left(\frac{d}{dt}\phi_t^*\omega\Big|_{t=s}\right) ds$$

By definition of the Lie derivative and Cartan's formula,

$$\phi_1^*\omega - \phi_0^*\omega = \int_0^1 \mathcal{L}_X(\omega)\, ds$$

$$= \int_0^1 (d\circ i_X + i_X\circ d)(\omega)\, ds$$

$$= d\circ\left(\int_0^1 i_X(\omega)\, ds\right) + \int_0^1 i_X(d\omega)\, ds$$

Define $h : \Omega^n(N) \to \Omega^{n-1}(M)$ by $h(\omega) = \int_0^1 i_X(\omega)\, ds$. Then $\phi_1^* - \phi_0^* = d\circ h + h\circ d$ as desired. $\square$

⑤ 405 846 515

a) By definition, ~~for $p=(x_1,\dots,x_{2n})$~~

$$\cancel{X_p(x_{2j-1}) = \frac{d}{dt}g_t(p)\Big|_{t=0}(x_j) = \frac{d}{dt}\left(\cos(t)x_{2j-1}(p) - \sin(t)x_{2j}(p)\right)\Big|_{t=0}}$$

$$= -x_{2j}(p)$$

$$\cancel{X_p(x_{2j}) = \frac{d}{dt}g_t(p)\Big|_{t=0}(x_{2j}) = \frac{d}{dt}\left(\sin(t)x_{2j-1}(p) - \cos(t)x_{2j}(p)\right)\Big|_{t=0}}$$

$$\cancel{= x_{2j-1}(p)}$$

so

$$X = \sum_{j=1}^{n}\left(-x_{2j}\,\partial/\partial x_{2j-1} + x_{2j-1}\,\partial/\partial x_{2j}\right)$$

Additionally, $d\omega = 0$ by definition. Then by Cartan's,

$$\mathcal{L}_X(\omega) = d\circ i_X(\omega)$$

$$= d\left(\sum_{j=1}^{n} -x_{2j}\,dx_{2j} - x_{2j-1}\,dx_{2j-1}\right) = d\left(\sum_{k=1}^{2n} -x_k\,dx_k\right)$$

$$= \sum_{k=1}^{2n}\left(-dx_k\wedge dx_k\right)$$

$$= 0$$

Finally, we find f s.t. $df = i_X(\omega) = \sum_{k=1}^{2n} -x_k\,dx_k$.
It then follows that $f = \sum_{k=1}^{2n} \frac{-1}{2}x_k^2$ satisfies this. □

b)

(6) 405 846 515

Suppose on the contrary. Define

$$H:[0,1]\times S^n \to S^n : (t,x) \mapsto \frac{(1-t)f(x)+t(-x)}{|(1-t)f(x)+t(-x)|}$$

To show H is smooth it suffices to show the denominator is non-vanishing. Suppose on the contrary that $(1-t)f(x)+t(-x)=0$ for some t,x. Then $|(1-t)f(x)|=|tx| \Rightarrow (1-t)|f(x)|=t|x|$ $\Rightarrow (1-t)=t \Rightarrow t=1/2$ and so $\frac{1}{2}f(x)-\frac{1}{2}x=0 \Rightarrow f(x)=x$ $\divideontimes$. Therefore the denominator is non-vanishing and so H is smooth.

By construction, H is a homotopy from f to $-\mathrm{id}$. Since degree is homotopy invariant, this implies $\deg f = (-1)^{n+1}$ b/c $\deg(-\mathrm{id}) = (-1)^{n+1}$. However, this contradicts the given fact that $\deg f \neq (-1)^{n+1}$ so f must have a fixed point. $\square$

(7) 405846515

a) Let G be a finitely presented group given by

$$G = \langle a_1, \dots, a_n \mid f_1, \dots, f_m \rangle$$

Define a CW complex X given by

1 0-cell: p

n 1-cells: $a_1, \dots, a_n$ w/ $\partial a_i = p - p$

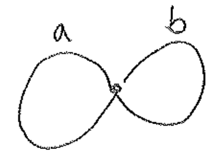
m 2-cells: $A_1, \dots, A_m$ w/ $\partial A_i = f_i$

Since the fundamental group of a CW complex is equivalent to the fundamental group of its 1-skeleton modulo the attaching of its 2-cells, we find that since $X^{(1)} \cong \underbrace{S^1 \wedge \dots \wedge S^1}_{n \text{ times}}$

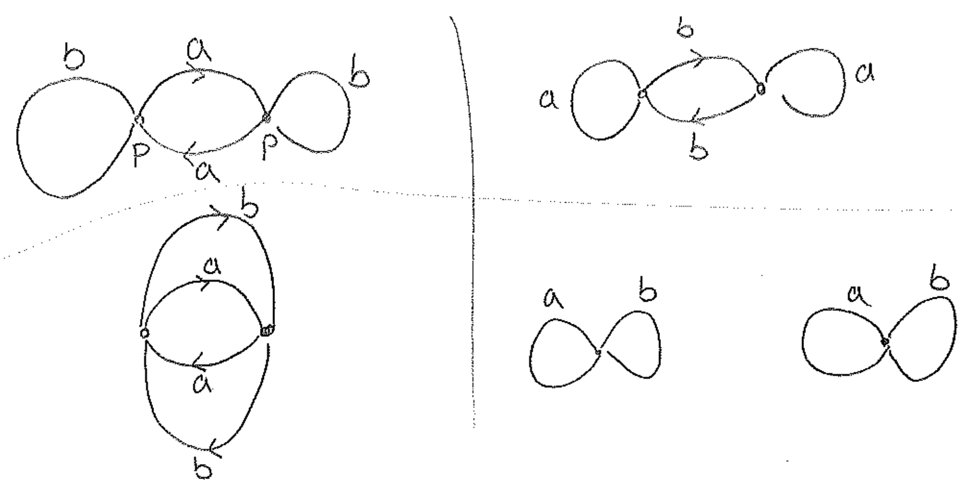
$$\pi_1(X) = \langle a_1, \dots, a_n \mid f_1, \dots, f_m \rangle$$

b) Consider $X = S^1 \wedge S^1$. Then $\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$ as desired.

c) Since X is realized as a connected graph w/ 1 vertex and 2 edges, ~~we~~ it follows that any 2-sheeted covering spaces of X can be viewed as a ~~connected~~ graph w/ 2 vertices and 4 edges. Given

$X =$ 

we then see all 2-sheeted ~~cover~~ ~~connected~~ coverings are



so $\exists$ 4 2-sheeted covering spaces of X, w/ 3 connected.

8) 405 846 515

To show that $\pi_1($

⑨ 405 846 515

Suppose $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic. WLOG suppose $n \leq m$. Then $\exists$ a homeomorphism $f: \mathbb{R}^n \to \mathbb{R}^m$. ~~Then $f^*: H^n(\mathbb{R}^n) \to H^m(\mathbb{R}^m)$ is an isomorphism.~~
~~Since $H^n(\mathbb{R}^n) \cong$~~ Then $f^*: H_c^n(\mathbb{R}^n) \to H_c^n(\mathbb{R}^m)$ is an isomorphism.
We recall that by Poincaré duality,

$$H_c^k(\mathbb{R}^l) \cong \begin{cases} \mathbb{R} & k = 0, l \\ 0 & \text{otherwise} \end{cases}$$

and so $H_c^n(\mathbb{R}^n) \cong H_c^n(\mathbb{R}^m) \Rightarrow \mathbb{R} \cong H_c^n(\mathbb{R}^m) \Rightarrow n = m$ as desired. □

~~If $n=0$ or $m=0$, the result is immediate as $\mathbb{R}^n$ is compact~~

Or ~~defo~~ remove a point and deformation retract to a sphere.

Need to deal w/ $n=0$ case.

(10) ~~We recall~~ 405 846 515 . . .

We recall that N_g can be constructed via the cell complex

$$\begin{aligned}
&1 \text{ 0-cell}: \ p \\
&g \text{ 1-cells}: \ a_1, \dots, a_g \ \text{ w/ } \ \partial a_i = p - p \\
&1 \text{ 2-cell}: \ f \ \text{ w/ } \ \partial f = 2a_1 + 2a_2 + \dots + 2a_g.
\end{aligned}$$

This implies the fundamental group is given by

$$\pi_1(N_g) = \langle a_1, \dots a_g \mid a_1^2 a_2^2 \dots a_g^2 \rangle$$

To compute the homology, we get the following chain complex

$$0 \to \underset{2}{\mathbb{Z}} \xrightarrow{\partial_2} \underset{1}{\mathbb{Z}^g} \xrightarrow{\partial_1} \underset{0}{\mathbb{Z}} \xrightarrow{\partial_0} 0$$

where $\partial_1 = 0$ since $\partial_1 a_i = p - p = 0 \ \forall i$ and $\partial_2 f = 2a_1 + \dots + 2a_g$. So, $\partial_2 : \mathbb{Z} \to \mathbb{Z}^g : a \mapsto (2a, 2a, \dots, 2a)$.

Then

$$H_2(N_g) = \frac{\ker(\partial_2)}{\operatorname{Im}(\partial_3)} = \ker(\partial_2) = \{a : 2a = 0\} = 0$$

$$H_1(N_g) = \frac{\ker(\partial_1)}{\operatorname{Im}(\partial_2)} = \mathbb{Z}^g / \{2x : x \in \mathbb{Z}^g\} \cong \mathbb{Z}^{g-1} \times \mathbb{Z}/2\mathbb{Z}$$

$$H_0(N_g) = \frac{\ker \partial_0}{\operatorname{Im}(\partial_1)} = \mathbb{Z}/0 = \mathbb{Z}$$

all other homologies are trivial. $\square$

GeoTop

Spring 2020

(1) 405 846 515

Suppose that g is a locally constant function s.t. f is homotopic to g. Let $M_1, \ldots$ denote the connected components of M. We claim that g is constant on M_i for each i.

Fix i and some $p \in M_i$. Let $\Omega = \{ q \in M_i : g(q) = g(p) \}$.
Since $p \in \Omega$, Ω is non-empty.
B/c g is locally constant, $\forall\, q \in \Omega$ $\exists$ an open neighborhood U of q s.t. $g(x) = g(q) = g(p)$ $\forall x \in U$. Then $U \subset \Omega$ and so Ω is open.
Finally, by the continuity of g, Ω is closed.
Therefore since M_i is connected, Ω is non-empty open and closed, $\Omega = M_i$. Therefore $g(q) = g(p)$ $\forall\, q \in M_i$ and so g is constant on each M_i. Let $g = c_i$ on M_i.
Therefore, for a k-form ω, $\forall i$, $\forall p \in M_i$

$$\left((g|_{M_i})^*(\omega|_{M_i})\right)_p = (\omega|_{M_i})_{c_i} \circ \cancelto{0}{d_p(g|_{M_i})} = 0$$

and so $g^* = 0$ on M, b/c this holds $\forall\, i$.
In particular, on the level of ~~cohot~~ cohomology $g^* = 0$.
Then since f is homotopic to g, $f^* = 0$ on the level of cohomology. In particular, if ω is closed then $f^*\omega$ is exact.

$\square$

② 405 846 515

(a) Define

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$$

Then $d\omega = 0$ so ω is closed and

$$\omega \wedge \omega = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + dx_3 \wedge dx_4 \wedge dx_1 \wedge dx_2$$
$$= dV + dV = 2dV$$

which is non-vanishing.

(b) Suppose that such an ω ~~exists~~ on S^4 exists. We recall that

$$H^k_{dR}(S^4) \cong \begin{cases} \mathbb{R} & k = 0, 4 \\ 0 & \text{else} \end{cases}$$

Therefore since ω is closed, $\omega = d\eta$ for some 1-form η.

Then $\omega \wedge \omega = d(\eta \wedge \omega)$ and so $\omega \wedge \omega$ is exact.

Since $H^4_{dR}(S^4) \to \mathbb{R} : \theta \longmapsto \int_{S^4} \theta$ is an isomorphism, this implies that $\int_{S^4} \omega \wedge \omega = 0$.

Alternatively, Stokes's gives $\int_{S^4} \omega \wedge \omega = \int_{\partial S^4} \eta \wedge \omega = \int_{\emptyset} \eta \wedge \omega = 0.$

Since $\omega \wedge \omega$ is non-vanishing volume form and S^4 is connected, $\omega \wedge \omega$ is strictly ~~posited~~ positive or strictly negative.

In either case, $\int_{S^4} \omega \wedge \omega \neq 0$ which is a contradiction.

Therefore no such ω exists. $\square$

③ 405 846 515

Suppose that $\exists f$ s.t. $f\omega$ is closed. Then

$$\begin{aligned} 0 &= d(f\omega) \\ &= df \wedge \omega + f \wedge d\omega \\ &= \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)\wedge\omega + f\left(2\,dx\wedge dy\right) \\ &= x\frac{\partial f}{\partial x}dx\wedge dy - \frac{\partial f}{\partial x}dx\wedge dz + y\frac{\partial f}{\partial y}dx\wedge dy - \frac{\partial f}{\partial y}dy\wedge dz \\ &\quad + -x\frac{\partial f}{\partial z}dy\wedge dz + y\frac{\partial f}{\partial z}dx\wedge dz + 2f\,dx\wedge dy \end{aligned}$$

and so

$$\left(2f + x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)dx\wedge dy = 0$$

$$\left(y\frac{\partial f}{\partial z} - \frac{\partial f}{\partial x}\right)dx\wedge dz = 0$$

$$-\left(\frac{\partial f}{\partial y} + x\frac{\partial f}{\partial z}\right)dy\wedge dz = 0$$

which implies $\frac{\partial f}{\partial x} = y\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial y} = -x\frac{\partial f}{\partial z}$.

Then

$$2f + x\left(y\frac{\partial f}{\partial z}\right) + y\left(-x\frac{\partial f}{\partial z}\right) = 0$$

$$\Rightarrow \quad 2f = 0$$

$$\Rightarrow \quad f = 0$$

Therefore if $f\omega$ is closed then $f=0$, which is not non-vanishing. $\square$

④ 405 846 515

Fix vector fields X, Y. We aim to show that

$$[\mathcal{L}_X, i_Y] = i_{[X,Y]}.$$

Let ω be a k-form and let $Z_1, \ldots, Z_{k-1}$ be vector fields. Then by direct computation,

$$\mathcal{L}_X(i_Y(\omega(Z_1,\ldots,Z_{k-1}))) = \mathcal{L}_X(\omega(Y, Z_1,\ldots,Z_{k-1}))$$

$$\cancel{= (\mathcal{L}_X)\omega}$$

$$= (\mathcal{L}_X\omega)(Y, Z_1,\ldots,Z_{k-1}) - \omega([X,Y], Z_1,\ldots,Z_{k-1})$$

$$- \sum_j \omega(Y, Z_1,\ldots,[X,Z_j],\ldots,Z_{k-1})$$

swap these

Similarly,

$$i_Y \circ \mathcal{L}_X(\omega(Z_1,\ldots,Z_{k-1})) = i_Y\Big((\mathcal{L}_X\omega)(Z_1,\ldots,Z_{k-1}) - \sum_j \omega(Z_1,\ldots,[X,Z_j],\ldots,Z_{k-1})\Big)$$

$$= (\mathcal{L}_X\omega)(Y, Z_1,\ldots,Z_{k-1}) - \sum_j \omega(Y, Z_1,\ldots,[X,Z_j],\ldots,Z_{k-1})$$

swap

Therefore

$$\cancel{[\mathcal{L}_X, i_Y](\omega(Z_1,\ldots,Z_{k-1}))}$$

$$\Big[[\mathcal{L}_X, i_Y]\omega\Big](Z_1,\ldots,Z_{k-1}) = \omega([X,Y], Z_1,\ldots,Z_{k-1})$$

$$= (i_{[X,Y]}\omega)(Z_1,\ldots,Z_{k-1})$$

As this holds $\forall\ \omega$ and $\forall\ Z_1,\ldots,Z_{k-1}$, this concludes

$$[\mathcal{L}_X, i_Y] = i_{[X,Y]}.$$

as desired. □

5 405 846 515

Suppose that $\mathbb{C}P^{2n}$ covers X.

Then $\pi_1(X)$ acts on $\mathbb{C}P^{2n}$ via deck transformations.

Suppose that $g \in \pi_1(X)$ is one such deck transformation

We aim to show that $g = id$. To do so, it suffices to show that g has a fixed point and hence that $L(g) \neq 0$.

We recall that, by the Lefschetz trace formula,

$$L(g) = \sum_{i=0}^{2n} (-1)^i \, \cancel{\text{rank}}\ \text{tr} \left(g_*^* : H_i^*(\mathbb{C}P^{2n};\mathbb{Q}) \to H_i^*(\mathbb{C}P^{2n};\mathbb{Q}) \right)$$

We recall that

$$H_*^i(\mathbb{C}P^{2n}) = \begin{cases} \cancel{\mathbb{Z}}\ \mathbb{Q} & k \text{ } \cancel{\text{odd}} \text{ even} \\ 0 & \text{else} \end{cases}$$

Therefore

$$L(g) = \sum_{k=0}^{n} \text{tr}\left(g_*^* : H_k^{2k}(\mathbb{C}P^{2n}) \to H_k^{2k}(\mathbb{C}P^{2n}) \right)$$

$$= \sum_{k=0}^{n} \text{tr}\left(g^* : \mathbb{Q} \to \mathbb{Q} \right)$$

~~Consider~~ $\cancel{g^* : H^2(\mathbb{C}P^{2n};\mathbb{Q}) \to H^2(\mathbb{C}P^{2n};\mathbb{Q})}$.

Since $H^{2k}(\mathbb{C}P^{4n}) \cong \mathbb{Q}$ $\forall k$, it follows that if ω generates $H^2(\mathbb{C}P^{2n})$ then ω^k generates $H^{2k}(\mathbb{C}P^{2n})$ $\forall k$.

Therefore if $g^* : H^2(\mathbb{C}P^{2n}) \to H^2(\mathbb{C}P^{2n})$ is multiplication by α, then $g^* : H^{2k}(\mathbb{C}P^{2n}) \to H^{2k}(\mathbb{C}P^{2n})$ is multiplication by α^k.

In ~~part~~ particular,

$$L(g) = \sum_{k=0}^{n} \alpha^{2k}$$

B/c $\alpha^{2k} \geq 0$ $\forall k > 0$ and $\alpha^0 = 1$, this implies that

$L(g) \neq 0$.

Therefore g has a fixed point and hence is ~~a~~ the identity b/c it's deck transformation.

Therefore $\pi_1(X)$ is the identity on $\mathbb{CP}^{2n}$, and hence

~~$X \cong \oplus 1$~~ X is $\mathbb{CP}^{2n}$. $\square$

⑥ 405 846 515

Suppose that $f: S^2 \times S^2 \to \mathbb{C}P^2$ is continuous.
Then f induces a map on the cohomology rings

$$f^*: H^*\mathbb{C}P^2 \to H^*(S^2 \times S^2)$$

We recall that

$$H^k \mathbb{C}P^2 \cong \begin{cases} \mathbb{Z} & k \overset{=0,2,4}{\cancel{\text{even}}} \\ 0 & \text{else} \end{cases} \quad H^k(S^2) = \begin{cases} \mathbb{Z} & k=0,2 \\ 0 & \text{else} \end{cases} \Rightarrow H^k(S^2 \times S^2) = \begin{cases} \mathbb{Z} & k=0,4 \\ \mathbb{Z}^2 & k=2 \\ 0 & \text{else} \end{cases}$$

Then

$$H^*(\mathbb{C}P^2) \cong \mathbb{Z}[x^2]/x^6 \quad \text{and} \quad H^*(S^2 \times S^2) \cong \mathbb{Z}[a^2, b^2]/(a^4, b^4)$$

hence f^* must preserve degree,

$$f^*(x^2) = \alpha a^2 + \beta b^2$$

for some $\alpha, \beta \in \mathbb{Z}$. Then

$$f^*(x^4) = f^*(x^2) * f^*(x^2)$$
$$= \alpha^2 a^4 + \beta^2 b^4 + 2\alpha\beta a^2 b^2$$
$$= 2\alpha\beta a^2 b^2$$

Therefore on top cohomology, f^* is multiplication by $2\alpha\beta$, ~~which is even~~ since a^2b^2 generates $H^4(S^2 \times S^2)$.
hence $2\alpha\beta$ is even, this concludes that $\deg f = 2\alpha\beta$ is even. □

(7) 405 846 515

We recall the LES on ~~relative~~ reduced homology.

$$\cdots \longrightarrow \tilde{H}_k(\{x\}) \to \tilde{H}_k(X) \to H_k(X,x) \to \cdots$$

For all k, we have $\tilde{H}_k(\{x\})=0$ and so we acquire the SES

$$0 \to \tilde{H}_k(X) \to H_k(X,x) \to 0$$

which implies that $\forall\, k$: $\forall x \in X$ $\exists$ an isomorphism $\phi_x^X : \tilde{H}_k(X) \to H_k(X,x)$

Define $\eta_{xy}^X = \phi_y^X \circ (\phi_x^X)^{-1} : H_k(X,x) \to \tilde{H}_k(X) \to H_k(X,y)$.

Then η_{xy}^X is an isomorphism, $\eta_{xx}^X = id$, and

$$\begin{aligned}\eta_{yz}^X \circ \eta_{xy}^X &= \phi_z^X \circ \cancel{(\phi_y^X)^{-1} \circ \phi_y^X} \circ (\phi_x^X)^{-1} \\ &= \phi_z^X \circ (\phi_x^X)^{-1} \\ &= \eta_{xz}^X\end{aligned}$$

Finally, let $f: X \to Y$ be continuous. Then f induces maps $f_*: H_k(X,y) \to H_k(Y,f(y))$ and $f_*: H_k(X,x) \to H_k(Y,f(x))$. Moreover, since f induces a map on the LES,

$$f_* \circ \phi_x^X = \phi_{f(x)}^Y \circ f_* \implies f_* \circ (\phi_x^X)^{-1} = (\phi_{f(x)}^Y)^{-1} \circ f_*$$

Then

$$f_* \circ \eta_{xy}^X = f_* \circ \phi_y^X \circ (\phi_x^X)^{-1} = \phi_{f(y)}^Y \circ \phi_{f(x)}^Y \circ f_* = \eta_{f(x)f(y)}^Y \circ f_*$$

as desired. $\square$

(8) 405 846 515

We recall the universal coefficient theorem, which states

$$0 \to \operatorname{Ext}(H_{n-1}(C), G) \to H^n(C;G) \to \operatorname{Hom}(H_n(C), G) \to 0$$

is a split exact sequence. Then

$$H^n(C;G) \cong \operatorname{Hom}(H_n(C),G) \oplus \operatorname{Ext}(H_{n-1}(C),G)$$

As given,

$$H_k(X) = \begin{cases} \mathbb{Z} & k=0 \\ 0 & k=1 \\ \mathbb{Z}/2\mathbb{Z} & k=2 \\ \mathbb{Z}/3\mathbb{Z} & k=3 \\ 0 & \text{else} \end{cases}$$

To compute $H_k(X;\mathbb{Z}/6\mathbb{Z})$, we compute

$$\operatorname{Hom}(H_0(X), \mathbb{Z}/6\mathbb{Z}) \cong \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}) \cong \mathbb{Z}/6\mathbb{Z}$$
$$\operatorname{Hom}(H_1(X), \mathbb{Z}/6\mathbb{Z}) \cong \operatorname{Hom}(0, \mathbb{Z}/6\mathbb{Z}) \cong 0$$
$$\operatorname{Hom}(H_2(X); \mathbb{Z}/6\mathbb{Z}) \cong \operatorname{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$
$$\operatorname{Hom}(H_3(X); \mathbb{Z}/6\mathbb{Z}) \cong \operatorname{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$$

$$\operatorname{Hom}(H_4(X), \mathbb{Z}/6\mathbb{Z}) \cong 0$$

$$\operatorname{Ext}(H_0(X), \mathbb{Z}/6\mathbb{Z}) \cong \operatorname{Ext}(\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}) \cong 0$$
$$\operatorname{Ext}(H_1(X), \mathbb{Z}/6\mathbb{Z}) \cong \operatorname{Ext}(0, \mathbb{Z}/6\mathbb{Z}) \cong 0$$
$$\operatorname{Ext}(H_2(X), \mathbb{Z}/6\mathbb{Z}) \cong \operatorname{Ext}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$
$$\operatorname{Ext}(H_3(X), \mathbb{Z}/6\mathbb{Z}) \cong \operatorname{Ext}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$$

Then

$$H_k(X;\mathbb{Z}/6\mathbb{Z}) = \begin{cases} \mathbb{Z}/6\mathbb{Z} & k=0 \\ 0 & k=1 \\ \mathbb{Z}/2\mathbb{Z} & k=2 \\ \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \quad k=3 \cong \mathbb{Z}/6\mathbb{Z} & k=3 \\ \mathbb{Z}/3\mathbb{Z} & k=4 \end{cases}$$

To construct such a space, define X as

1 0-cell : p

0 1-cells :

1 2-cell : f w/ $\partial f = p = 0$

2 3-cell : A, B w/ ∂A attached to f via a deg 2 map
$\partial B = p = 0$

1 4-cell : C w/ ∂C attached to B via a deg 3 map

Then

$$H_0(X) = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} \cong \mathbb{Z}\langle p \rangle \cong \mathbb{Z}$$

$$H_1(X) \cong 0$$

$$H_2(X) \cong \frac{\ker \partial_2}{\operatorname{Im} \partial_3} \cong \frac{\mathbb{Z}\langle f \rangle}{\mathbb{Z}\langle 2f \rangle} \cong \mathbb{Z}/2\mathbb{Z}$$

$$H_3(X) \cong \frac{\ker \partial_3}{\operatorname{Im} \partial_4} \cong \frac{\mathbb{Z}\langle B \rangle}{\mathbb{Z}\langle 3B \rangle} \cong \mathbb{Z}/3\mathbb{Z}$$

$$H_4(X) \cong \frac{\ker \partial_4}{\operatorname{Im} \partial_5} \cong 0$$

$\square$

as desired.

⑨ 405

$\chi(M_n) = 2-2n$

⑨ 405 846 515

We recall that M_n, the compact orientable genus n surface is constructed via

$$1 \text{ 0-cell}: p$$
$$2n \text{ 1-cells}: a_1, b_1, \ldots, a_n, b_n \text{ w/ } \partial a_i = \partial b_i = p - p = 0$$
$$1 \text{ 2-cell}: f \text{ w/ } \partial f = a_1 + b_1 - a_1 - b_1 + \ldots + a_n + b_n - a_n - b_n$$

Therefore $\pi_1(M_n) \cong \langle a_1, b_1, \ldots, a_n, b_n \mid a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_n b_n a_n^{-1} b_n^{-1} \rangle = G_n$.

Additionally, we recall that the finite index subgroups of $\pi_1(M_n)$ correspond to the finite-covering spaces $\overset{(X,p)}{\vee}$ of M_n via

$$p_* \pi_1(X) \subset \pi_1(M_n)$$

where the # of sheets is the index of $p_* \pi_1(X)$. I.e,

$$(\pi_1(M_n) : p_* \pi_1(X)) = \# p^{-1}(x_0) \quad \forall x_0 \in M_n$$

Therefore, to look for finite index subgroups isomorphic to G_m, we are looking for finite sheeted covers M_m of M_n.

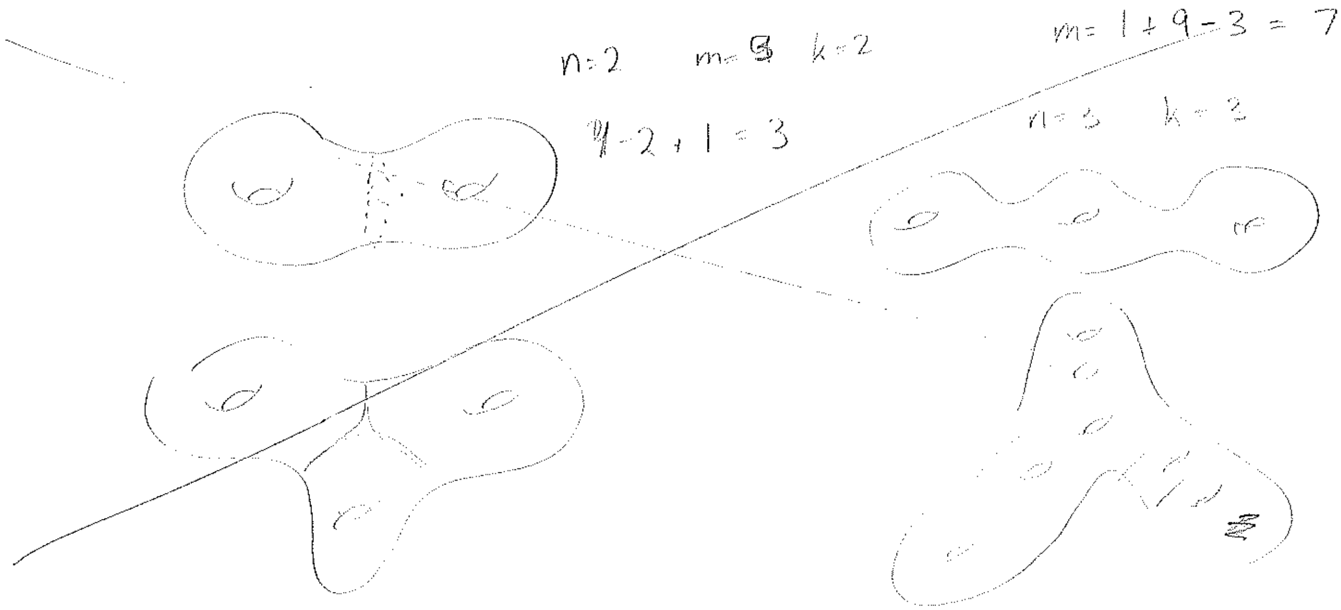
By the CW complex above, we note that $\chi(M_n) = 2-2n$.

Therefore, if M_m is a k-sheeted covering of M_n, then

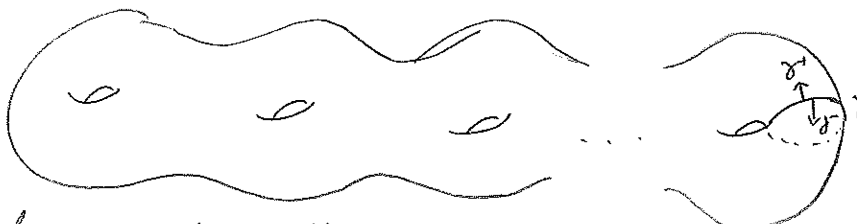
$$\chi(M_m) = k\chi(M_n) \iff 2-2m = k(2-2n)$$
$$\iff 1-m = k-kn$$
$$\iff m = kn-k+1$$

We claim that $m = k(n-1)+1$ is a sufficient condition.

$\longrightarrow$



Suppose $m = k(n-1)+1$. To cover M_n by M_m, we first draw M_n as



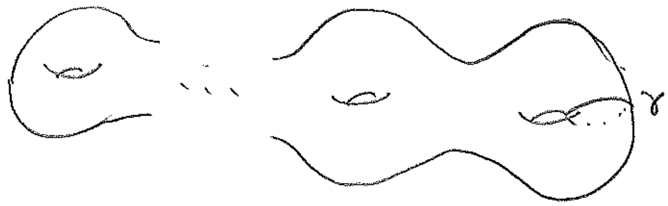
and identifying a loop through a hole of M_n as γ, and note the two sides of γ, denoted γ^+ and γ^- above.
We then take k copies of M_n, $C_1, \dots, C_k$ w/ $\gamma_1, \dots, \gamma_k$
and attach C_1 to C_2 by identifying γ_1^+ w/ γ_2^-,
C_2 to C_3 by identifying γ_2^+ w/ γ_3^-

$\vdots$

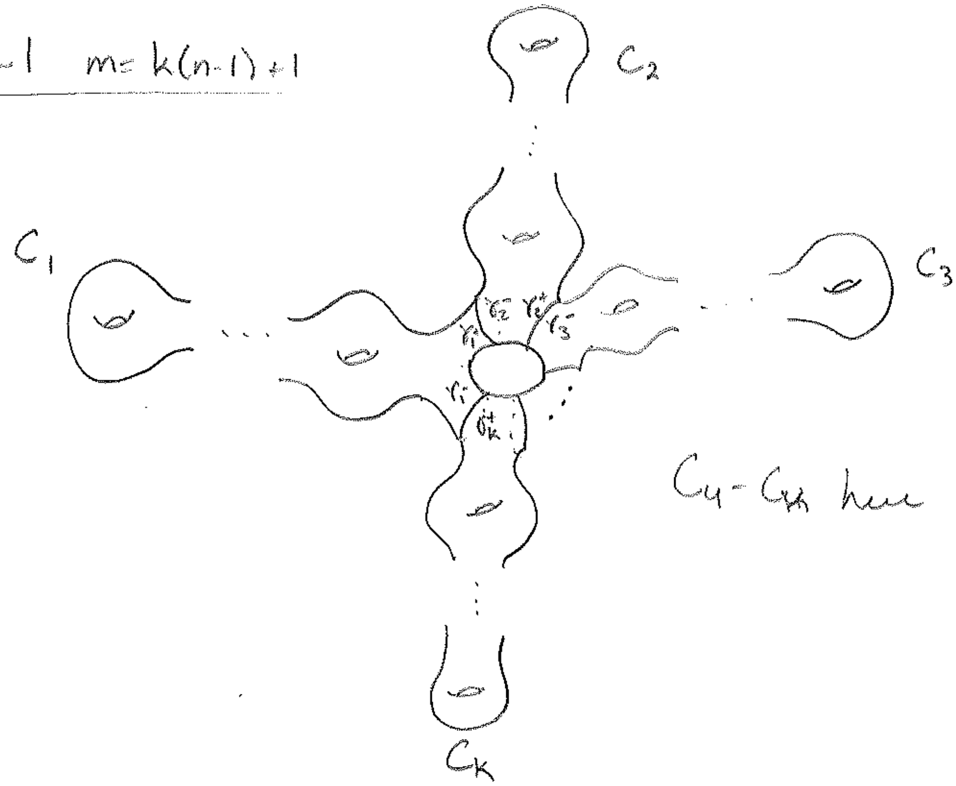
C_k to C_1 by identifying γ_k^+ w/ γ_1^-

When split at γ, each C_i has $n-1$ holes. Attaching all copies of γ then creates a hole, so the ~~resulting~~ resulting k-cover has $k(n-1)+1 = m$ holes and hence is isomorphic to M_m. This is pictured on the following page.

$\underline{M_n}$

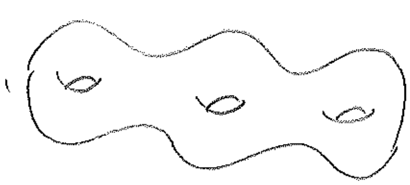


$\underline{M_m \text{ w/ } m = k(n-1)+1}$

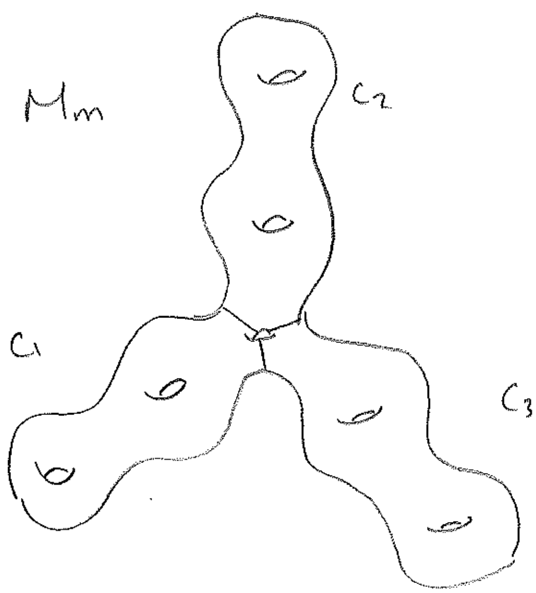


example: $n = 3$, $k = 3$, $m = 7$

M_n



M_m



(10) 405 846 5'5

Let U be an ε-neighborhood of $D^2 \times S^1 \times \{0\}/\sim$ that deformation retracts onto $D^2 \times S^1 \times \{0\}/\sim$ and similarly for V w/ $D^2 \times S^1 \times \{1\}/\sim$. Then $U \cap V$ deformation retracts onto $\partial D^2 \times S^1 \times \{0,1\}/\sim \cong T^2$ and $U \cup V = X$. We note that $D^2 \times S^1 \times \{0\}/\sim$ deformation retracts onto S^1. Mayer-Vietoris then gives the LES

$$\cdots \longrightarrow H_k(T^2) \to H_k(S^1)^{\oplus 2} \to H_k(X) \to \cdots \qquad (1)$$

We recall

$$H_k(T^2) = \begin{cases} \mathbb{Z} & k=0,2 \\ \mathbb{Z}^2 & k=1 \\ 0 & \text{else} \end{cases} \qquad H_k(S^1) = \begin{cases} \mathbb{Z} & k=0,1 \\ 0 & \text{else} \end{cases}$$

From (1), we obtain the SES.

$$0 \to H_3(X) \to \mathbb{Z} \to 0$$

so $H_3(X) \cong \mathbb{Z}$.

From (1) we also have

$$0 \to H_2(X) \hookrightarrow \underset{\mathbb{Z}^2}{\underset{\cong}{H_1(T^2)}} \to \underset{\mathbb{Z}^2}{\underset{\cong}{H_1(S^1)^{\oplus 2}}} \to H_1(X) \to \underset{\mathbb{Z}}{\underset{\cong}{H_0(T^2)}} \to \underset{\mathbb{Z}^2}{\underset{\cong}{H_0(S^1)^{\oplus 2}}} \to \underset{\mathbb{Z}}{\underset{\cong}{H_0(X)}} \to 0$$

where $H_0(X) \cong \mathbb{Z}$ since X is path connected.

Hence $H_0(S^1)^{\oplus 2} \to H_0(X)$ is a surjection, $H_0(T^2) \to H_0(S^1)^{\oplus 2}$ has kernel $\cong \mathbb{Z}$ and so $H_1(X) \to H_0(T^2)$ is the 0 map. Then we have the SES

$$0 \to H_2(X) \hookrightarrow \underset{\mathbb{Z}^2}{\underset{\cong}{H_1(T^2)}} \to \underset{\mathbb{Z}^2}{\underset{\cong}{H_1(S^1)^{\oplus 2}}} \twoheadrightarrow H_1(X) \to 0 \qquad (2)$$

~~we~~ The map $H_1(T^2) \to H_1(S^1) \oplus H_1(S^1)$ is given by the inclusions

$$T^2 \hookrightarrow D^2 \times S^1 \times \{0\}/\sim$$

$$T^2 \hookrightarrow D^2 \times S^1 \times \{1\}/\sim$$

Where T^2 is the shared boundary in X.

By construction, these are

$$T^2 \to D^2 \times S^1 \times \{0\}/\sim \; : \quad (x,y) \longmapsto (x,y,0)$$

$$T^2 \to D^2 \times S^1 \times \{1\}/\sim \; : \quad (x,y) \longmapsto (xy^5, y, 1)$$

When D^2 is contracted to a point these both become

$$T^2 \to S^1 \; : \; (x,y) \longmapsto y$$

and so

$$H_1(T^2) \to H_1(S^1) \oplus H_1(S^1) : (a,b) \longmapsto (b,b)$$

In particular, the map has image $\cong \mathbb{Z}$ and kernel $\cong \mathbb{Z}$.
By exactness of (2), this gives $H_2(X) \cong \mathbb{Z}$ and $H_1(X) \cong \mathbb{Z}$.
To conclude,

$$H_k(X) \cong \begin{cases} \mathbb{Z} & k = 0,1,2,3 \\ 0 & \text{else} \end{cases}$$

which is what was to be found. $\square$

GeoTop

Fall 2019

① 405 346 515

Let M be a compact submanifold of $\mathbb{R}^3$ of dimension 3 w/ smooth boundary ∂M. The classical divergence theorem states that for a smooth vector field X,

$$\int_{\partial M} \langle X, N\rangle dA = \int_M \operatorname{div}(X)\, dV$$

where dV is the volume form on M given by dV on $\mathbb{R}^3$, $\operatorname{div}(X) = \frac{\partial X}{\partial x} + \frac{\partial X}{\partial y} + \frac{\partial X}{\partial z}$, $\langle X, N\rangle$ is the inner product of X w/ N, the unit normal of ∂M, and dA is the induced ~~surface~~ area form on ∂M given by $dA = \iota^* i_N dV$ w/ $\iota: \partial M \to M$ is the inclusion.

Define $T = X - \langle X, N\rangle X$ on ∂M to be the ~~tangent~~ component of X to ∂M. We claim that $\iota^* i_T dV = 0$. Let Y_1, Y_2 be vector fields in ∂M. Then $\{T, Y_1, Y_2\}$ is linearly dependent ~~and so~~ b/c ∂M is 2-dimensional and so

$$\iota^* i_T dV(Y_1, Y_2) = \iota^* dV(T, Y_1, Y_2) = 0 \Rightarrow \iota^* i_T dV = 0.$$

Linearity then implies that $\iota^* i_X dV = \langle X, N\rangle \iota^* i_N dV = \langle X, N\rangle dA$. Therefore, by Stokes's theorem,

$$\int_{\partial M} \langle X, N\rangle dA = \int_{\partial M} \iota^* i_X dV$$

$$= \int_M d \circ i_X dV$$

Let $X = f\,\partial/\partial x + g\,\partial/\partial y + h\,\partial/\partial z$. Then

$$\begin{aligned} d\, i_X dV &= d(i_X dx dy dz) \\ &= d(f dy dz - g dx dz + h dx dy) \\ &= \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) dx dy dz \\ &= \operatorname{Div}(X) dV \end{aligned}$$

and so

$$\int_{\partial M} \langle X, N\rangle dA = \int_M \operatorname{Div}(X) dV$$

as desired. ◻

② 405 846 515   Let $X = \mathbb{RP}^{n+m}/\mathbb{RP}^n$

Consider $\mathbb{RP}^n \subset \mathbb{RP}^{n+m}$. We recall that $\mathbb{RP}^k$ can be constructed from $\mathbb{RP}^{k-1}$ by attaching a k-disk B^k to $\mathbb{RP}^{k-1}$ via the map

$$(x_0, \dots, x_k) \longmapsto [x_0 : \dots : x_k]$$

Therefore, $\mathbb{RP}^{n+m}/\mathbb{RP}^n$ has the structure

$$\begin{array}{llll} 1 & 0\text{-cell} & p & \\ 1 & n+1\text{-cell} & e_{n+1} & \text{w/ } \partial e_{n+1} = p = 0 \\ 1 & n+2\text{-cell} & e_{n+2} & \text{w/ } \partial e_{n+2} \text{ attached via the quotient map } \xrightarrow{\text{anti-podal identification}} \text{(degree } 1+(-1)^n) \\ & \vdots & & \\ 1 & n+m\text{-cell} & e_{n+m} & \text{w/ } \partial e_{n+m} \text{ attached via a deg } 1+(-1)^{n+m-1} \text{ map} \end{array}$$

We note that the quotient only collapses the $0, 1, \dots, n$-cells of $\mathbb{RP}^{n+m}$ to a point. Therefore, it only affects the boundary maps of the $k \le n+1$ cells and leaves all higher cells unaffected. Consider $H_{n+1}(\mathbb{RP}^{n+m}/\mathbb{RP}^n)$. By definition

$$H_{n+1}(X) = \frac{\ker \partial_{n+1}}{\operatorname{im} \partial_{n+2}} \cong \frac{\mathbb{Z}}{(1+(-1)^n)\mathbb{Z}} = \begin{cases} \mathbb{Z}/2\mathbb{Z} & n \text{ even} \\ \mathbb{Z} & n \text{ odd} \end{cases}$$

As all cells and boundary maps are the same for $k > n+1$ and all k-skeletons are trivial for $k \le n$, it follows that

$$H_k(X) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k = 1, \dots, n \\ \mathbb{Z} & k = n+1 \text{ w/ } n \text{ even} \\ \mathbb{Z}/2\mathbb{Z} & k = n+1 \text{ w/ } n \text{ odd} \\ H_k(\mathbb{RP}^{n+m}) & k > n+1 \end{cases}$$

which is what was to be computed. □

(2) 405 846 515 ALTERNATE, NOT FINISHED

~~We proved by the LES on relative~~ If $m=0$ then this is trivial. We assume $m \geq 1$.

We note that $\forall\, m \geq 0$, $\mathbb{R}P^n \subset \mathbb{R}^{n+m}$ has a neighborhood which deformation retracts onto $\mathbb{R}P^n$. Therefore $(\mathbb{R}P^{n+m}, \mathbb{R}P^n)$ is a good pair and

$$\tilde{H}_k(\mathbb{R}P^{n+m}/\mathbb{R}P^n) \cong H_k(\mathbb{R}P^{n+m}, \mathbb{R}P^n)$$

For $k>0$, this implies that it suffices to find $H_k(\mathbb{R}P^{n+m}, \mathbb{R}P^n)$.
For $k=0$, we note that $\mathbb{R}P^{n+m}/\mathbb{R}P^n$ is the quotient of a connected space and hence connected, so $H_0(\mathbb{R}P^{n+m}/\mathbb{R}P^n) \cong \mathbb{Z}$.

We recall the LES for relative homology, which is

$$\cdots \longrightarrow H_k(\mathbb{R}P^n) \to H_k(\mathbb{R}P^{n+m}) \longrightarrow H_k(\mathbb{R}P^{n+m}, \mathbb{R}P^n) \to \cdots \qquad (*)$$

We recall that the homology of $\mathbb{R}P^l$ in general is given by

$$H_k(\mathbb{R}P^l) = \begin{cases} \mathbb{Z} & k=0 \text{ or } k=l \text{ is odd} \\ \mathbb{Z}/2\mathbb{Z} & 0<k<l \text{ is odd} \\ 0 & \text{else} \end{cases}$$

Suppose first that n is even. Then $\forall\, k>n$, we acquire the SES

$$H_k(\mathbb{R}P^n) = 0 \to H_k(\mathbb{R}P^{n+m}) \to H_k(\mathbb{R}P^{n+m}, \mathbb{R}P^n) \to 0 = H_{k-1}(\mathbb{R}P^n)$$

$$\Rightarrow H_k(\mathbb{R}P^{n+m}, \mathbb{R}P^n) \cong H_k(\mathbb{R}P^{n+m})$$

Now consider $1 \leq k < n$. Since $\mathbb{R}P^{n+m}$ and $\mathbb{R}P^n$ have the same l-skeleton $\forall\, l = 0, \ldots, n$, it follows that the map $H_k(\mathbb{R}P^n) \to H_k(\mathbb{R}P^{n+m})$ in $(*)$ is an isomorphism.* Therefore, $\forall\, 1 \leq k < n$ by properties of an exact sequence

$$H_k(\mathbb{R}P^n) \leftrightarrow H_k(\mathbb{R}P^{n+m}) \xrightarrow{0} \underset{\overset{\|}{0}}{H_k(\mathbb{R}P^{n+m}, \mathbb{R}P^n)} \xrightarrow[\text{im}=0]{\ker=0} H_{k-1}(\mathbb{R}P^n) \leftrightarrow H_{k-1}(\mathbb{R}P$$

* In the $k=n$ case, since n is even, $H_n(\mathbb{R}P^n) = H_n(\mathbb{R}P^{n+m}) = 0$ so the map is still an isomorphism.

and so for $1 \le k \le n$,

$$H_k(\mathbb{RP}^{n+m}, \mathbb{RP}^n) = 0$$

This yields, for n even,

$$H_k(\mathbb{RP}^{n+m}/\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & k=0 \\ 0 & 1 \le k \le n \\ H_k(\mathbb{RP}^{n+m}) & k > n \end{cases}$$

Now consider n odd. For $k < n$ and $k > n+1$, the same reasoning holds. To find the $k = n, n+1$ cases, we have the exact sequence

$$0 \to 0 \hookrightarrow H_{n+1}(\mathbb{RP}^{n+m}, \mathbb{RP}^n) \hookrightarrow H_n(\mathbb{RP}^n) \to H_n(\mathbb{RP}^{n+m}) \twoheadrightarrow H_n(\mathbb{RP}^{n+m}, \mathbb{RP}^n) \to 0$$

$H_n(\mathbb{RP}^n) \cong \mathbb{Z}$, $H_n(\mathbb{RP}^{n+m}) \cong \mathbb{Z}/2\mathbb{Z}$

Bk $n \pm 1$ is even.

$\mathbb{RP}^n \subset \mathbb{RP}^m$

$\mathbb{RP}^n \hookrightarrow \mathbb{RP}^{n+1}$ $[x_0 : \cdots : x_n] \mapsto [x_0 : \cdots : x_n : 0]$

$\mathbb{RP}^n \hookrightarrow \mathbb{RP}^{n+m}$ $[x_0 : \cdots : x_n] \mapsto [x_0 : \cdots : x_n : \underbrace{0 : \cdots : 0}_{m}]$

$\mathbb{RP}^{n+m}/\mathbb{RP}^n$

1 ... cell ...

1 ... cell ... attached ...

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We recall that $\mathbb{RP}^n$ is compact as the quotient of S^n which is compact. Additionally, since $\mathbb{RP}^n$ can be constructed w/ 1 k-cell $\forall$ $k = 0, 1, \ldots, n$, it follows that ~~$\mathbb{RP}$~~

$$\chi(\mathbb{RP}^n) = \begin{cases} 0 & k \text{ odd} \\ 1 & k \text{ even} \end{cases}$$

Therefore if n is even then $\mathbb{RP}^n$ does not admit a non-vanishing vector field.

Suppose that n is odd. Express $\mathbb{RP}^n$ as $S^n/\sim$ where $\sim$ is the anti-podal identification and $S^n \subset \mathbb{R}^{n+1}$.

We recall that $\forall$ $p \in S^n$, $T_pS^n \cong \{p\}^\perp$ smoothly in p.

We can therefore construct a nonvanishing VF V on S^n via the map $V: (x_1, y_1, x_2, y_2, \ldots, x_{\frac{n+1}{2}}, y_{\frac{n+1}{2}}) \longmapsto (-y_1, x_1, -y_2, x_2, \ldots, -y_{\frac{n+1}{2}}, x_{\frac{n+1}{2}})$ since

$$p \cdot V_p = 0 \Rightarrow p \perp V_p.$$

Let $\pi: S^n \to \mathbb{RP}^n$ be the anti-podal quotient map. To show that $(\pi_* V)_{\pi(p)} = d\pi_p V_p$ is a well-defined VF on $\mathbb{RP}^n$, it must be shown that $d\pi_p V_p = d\pi_{-p} V_{-p}$. Let $f: p \mapsto -p$ be the anti-podal map. Then b/c $\pi \circ f = \pi$

$$d\pi_p = d(\pi \circ f)_p = d\pi_{f(p)} df_p = -d\pi_{-p}$$

Moreover, by construction,

$$V_p = (y_1, -x_1, \ldots) = -(-y_1, x_1, \ldots) = -V_p$$

and so $d\pi_p V_p = d\pi_{-p} V_{-p}$. Then V factors through π to a VF $\pi_* V$ on $\mathbb{RP}^n$. Since V is non-vanishing, so is $\pi_* V$, as desired. $\square$

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Let $X = S^1 \times S^1 \cong T^2$ and $Y = X \times [0,1] / ((x,y),0) \sim ((y,x),1)$. The standard mapping cone LES then yields a LES

$$\cdots \to H_k(X) \xrightarrow{1-f_*} H_k(X) \to H_k(Y) \to H_{k-1}(X) \to \cdots$$

where $f: X \to X : (x,y) \mapsto (y,x)$.
We recall that by the Kunneth formula,

$$H_k(X) = \begin{cases} \mathbb{Z} & k = 0,2 \\ \mathbb{Z}^2 & k = 1 \\ 0 & \text{else} \end{cases}$$

which yields the exact sequence

$$0 \to H_3(Y) \hookrightarrow \underset{2}{\mathbb{Z}} \xrightarrow{1-f_*} \underset{2}{\mathbb{Z}} \to H_2(Y) \to \underset{2}{\mathbb{Z}^2} \xrightarrow{1-f_*} \underset{1}{\mathbb{Z}^2} \to H_1(Y) \to \underset{0}{\mathbb{Z}} \to \underset{0}{\mathbb{Z}} \to H_0(Y) \to 0$$

Since Y is connected, $H_0(Y) \cong \mathbb{Z}$.

$H_1(\mathbb{R}^2) - \frac{\partial}{\partial}\, dx\, dy$

~~$\omega = \frac{\partial}{\partial_1}(x,y)\,dx + \frac{\partial}{\partial_2}(x,y)\,dy$~~

$f_* \omega =$

$S^1 \times S^1$ — 1 0-cell p

2 1-cells a, b

$\partial a = \partial b = p - p = 0$

1 2-cell A

$\partial A = a + b - a - b = 0$

B/c f swaps entries, we note that $f_* : H_2(X) \cong \mathbb{Z} \to H_2(X) \cong \mathbb{Z}$ is multiplication by -1, and $f_* : H_1(X) \cong \mathbb{Z}^2 \to H_1(X) \cong \mathbb{Z}^2 : (a,b) \mapsto (b,a)$.
Then

$$0 \to H_3(Y) \hookrightarrow \mathbb{Z} \xrightarrow{2} \underset{\ker = 2\mathbb{Z}}{\mathbb{Z}} \to H_2(Y) \to \mathbb{Z}^2 \xrightarrow{(a,b) \mapsto (a-b,\, b-a)} \mathbb{Z}^2 \to H_1(Y) \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to \mathbb{Z} \to 0$$

Since $H_3(Y) \to \mathbb{Z}$ is injective and $\ker 2 = 0$, it follows that $H_3(Y) \cong 0$.

B/c $\mathbb{Z} \to H_2(Y)$ has kernel $2\mathbb{Z}$ and $\mathbb{Z}^2 \to \mathbb{Z}^2$ has kernel $\cong \mathbb{Z}$, it follows that

$$0 \to \mathbb{Z}/2\mathbb{Z} \to H_2(Y) \to \mathbb{Z} \to 0$$

is a short exact sequence and hence $H_2(Y) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

B/c $\mathbb{Z}^2 \to H_1(Y)$ has kernel $\{(a,-a) : a \in \mathbb{Z}\} \cong \mathbb{Z}$ and $H_1(Y) \to \mathbb{Z}$ has image $\mathbb{Z}$,

$$0 \to \mathbb{Z} \to H_1(Y) \to \mathbb{Z} \to 0$$

is a short exact sequence so $H_1(Y) \cong \mathbb{Z}^2$. Therefore

$$H_k(Y) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}^2 & k=1 \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & k=2 \\ 0 & \text{else} \end{cases}$$

as desired.

$\square$

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Since multiplication by g is a diffeomorphism,

$$(L_g)_*(X)(f) = X(f \circ L_g)$$

$\forall$ g, f. Therefore, by direct computation, $\forall$ left-invariant X, Y

$$\begin{aligned}(L_g)_*[X,Y](f) &= (L_g)_*(X \circ Y - Y \circ X)(f) \\ &= (X \circ Y \circ L_g(f) - Y \circ X \circ L_g(f)) \\ &= X(L_{g*}Y(f)) - Y(L_{g*}X(f)) \\ &= X(Y(f)) - Y(X(f)) \\ &= [X,Y](f)\end{aligned}$$

As this holds $\forall$ f, $[X,Y]$ is left-invariant.

$\square$

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We use the convention $E_0 = E_n$ and $N_{n+1} = N_1$ for ease of notation.

Define $X_n = \{N_i, S_i, E_i, W_i : i = 1, 2, \dots, n\}$ w/ topology generated by
$\{\{E_i\}, \{W_i\}, \{E_i, S_i, W_i\}, \{E_{i-1}, N_i, W_i\}, \{E_i, W_i\}, \{E_{i-1}, W_i\} \mid i\}$.

Define $f_n : X_n \to X$ by $N_i \mapsto N$, $E_i \mapsto E$, $S_i \mapsto S$, $W_i \mapsto W$ $\forall i$.
Then f_n is continuous since

$$f_n^{-1}\{E\} = \bigcup_{i=1}^{n} \{E_i\} = \text{open}$$
$$f_n^{-1}\{W\} = \bigcup_{i=1}^{n} \{W_i\} = \text{open}$$
$$\vdots$$
$$f_n^{-1}\{S, E, W\} = \bigcup_{i=1}^{n} \{S_i, E_i, W_i\} = \text{open}$$

We first claim that X_n is path connected. B/c X_n is countable, it suffices to construct paths $N_i \to W_i \to S_i \to E_i \to N_{i+1}$.
~~Define~~

$$\gamma_{N_i \to W_i} : [0,1] \to X_n : t \mapsto \begin{cases} N_i & t = 0 \\ W_i & \text{else} \end{cases}$$

$$\gamma_{W_i \to S_i} : [0,1] \to X_n : t \mapsto \begin{cases} W_i & t < 1 \\ S_i & t = 1 \end{cases}$$

$$\gamma_{S_i \to E_i} : [0,1] \to X_n : t \mapsto \begin{cases} S_i & t = 0 \\ E_i & \text{else} \end{cases}$$

$$\gamma_{E_i \to N_{i+1}} : [0,1] \to X_n : t \mapsto \begin{cases} E_i & t < 1 \\ N_{i+1} & t = 1 \end{cases}$$

Each of these are continuous paths since

$$\gamma_{N_i \to W_i}^{-1} : \begin{cases} \{W_i\} \mapsto (0,1] & \text{open } \checkmark \\ \{E_{i-1}, N_i, W_i\} \mapsto [0,1] & \text{open } \checkmark \\ \{E_i, W_i\} \mapsto (0,1] & \text{open} \\ \{E_{i-1}, W_i\} \mapsto (0,1] & \text{open } \checkmark \\ \{S_i, E_i, W_i\} \mapsto (0,1] & \text{open} \\ \text{else} \mapsto \emptyset & \text{open } \checkmark \end{cases}$$

similar computations hold $\forall$ other paths. Therefore X_n is path connected.

~~Consider N. By definition, $\{N,E,W\}$ is an open neighborhood of N and~~
~~$f_n^{-1}\{N;E,W\} = [0,1/2)\cup(1/2,3/2)\cup\dots\cup(n-1/2,n]$~~
~~$= \coprod$~~

It remains to show that f_n is an n-fold cover.

Consider E. We have that $\{E\}$ is an open neighborhood of E w/ $f_n^{-1}\{E\} = \coprod_{i=1}^{n} \{E_i\}$, where $\{E_i\}$ open, $f_n|_{E_i} = \mathrm{id}$. Therefore f_n satisfies the covering map definition at E. A symmetric argument holds for W.

Consider N. We have that $\{E, N, W\}$ is an open neighborhood of N w/ $f_n^{-1}\{E,N,W\} = \coprod_{i=1}^{n}\{E_{i-1}, N_i, W_i\}$, open. By construction, $f_n|_{\{E_{i-1},N_i,W_i\}} = \mathrm{id}$ and so f_n satisfies the definition at N.
The same reasoning holds for S.
Therefore f_n is an n-fold cover.

W/ this construction, we find that the universal cover will be

$$X_\infty = \{N_i, W_i, S_i, E_i : i \in \mathbb{Z}\}$$

w/ the topology generated by the same kinds of sets as before.
The covering map is then given by

$$f : \begin{cases} E_i \longmapsto E \\ N_i \longmapsto N \\ W_i \longmapsto W \\ S_i \longmapsto S \end{cases}$$

as before.

$\square$

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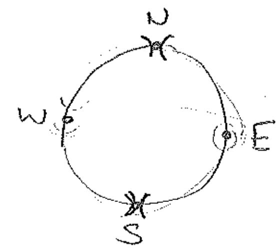
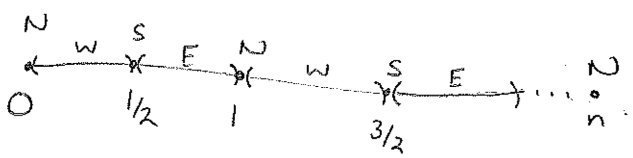
Fix some $n \geq 1$. Consider the segment $[0,n]/0 \sim n$. Define $f_n : [0,n]/\sim \to X$ by



$$f_n(x) = \begin{cases} N & x = 0, 1, \dots, n-1 \\ S & x = 1/2, 3/2, \dots, n - 1/2 \\ W & x \in (0, 1/2) \cup (1, 3/2) \cup \dots \cup (n-1, n-1/2) \\ E & x \in (1/2, 1) \cup (3/2, 2) \cup \dots \cup (n-1/2, n) \end{cases}$$

This can be visualized by



here we have given $[0,n]/\sim$ the usual topology of S^1.

First we claim that f_n is continuous. By construction

$$f_n^{-1}\{E\} = (1/2, 1) \cup \dots \cup (n-1/2, n) = \text{open} \checkmark$$

$$f_n^{-1}\{W\} = (0, 1/2) \cup \dots \cup (n-1, n-1/2) = \text{open} \checkmark$$

$$f_n^{-1}\{E, W\} = [0,n] \setminus \{0, 1/2, 1, \dots, n-1/2\} = \text{open} \checkmark$$

$$f_n^{-1}\{N, E, W\} = [0,n] \setminus \{1/2, 3/2, \dots, n-1/2\} = \text{open} \checkmark$$

$$f_n^{-1}\{S, E, W\} = [0,n] \setminus \{0, 1, \dots, n-1\} = \text{open} \checkmark$$

and so f_n is continuous.

We now show that f_n is an n-fold cover. We only consider N, W as the remaining points will follow symmetry arguments.

9) 405 846 515

(a) Let $0 \to A \to B \to C \to 0$ be a SES of chain complexes. To get the boundary map in the associated LES, we wish to define the map $H_n(C) \to H_{n-1}(A)$.

Let $i: A \to B$ and $j: B \to C$ be the maps between the chain complexes.

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, hook, "i_n"] \arrow[d, "\partial_A"] & B_n \arrow[r, two heads, "j_n"] \arrow[d, "\partial_B"] & C_n \arrow[r] \arrow[d, "\partial_C"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, hook, "i_{n-1}"'] \arrow[d] & B_{n-1} \arrow[r, two heads, "j_{n-1}"'] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Consider some $[c_n] \in H_n(C)$. Since $j_n: B_n \to C_n$ is surjective, $\exists$ some $b_n \in B_n$ s.t. $j_n(b_n) = c_n$. Then since j is a chain map, $0 = \partial_C(c_n) = \partial_C(j_n(b_n))$
$= j_{n-1}(\partial_B b_n)$
so $\partial_B b_n \in \ker(j_{n-1})$. Then ~~since~~ by exactness $\exists$ $a_{n-1} \in A_{n-1}$ s.t. $i_{n-1}(a_{n-1}) = \partial_B b_n$. Define $\delta_n\ H_n(C) \to H_{n-1}(A) : [c_n] \mapsto [a_{n-1}]$.
To show that δ_n is well-defined, it must be shown that δ_n is independent of the choice of representative of $[c_n]$ and the choice of pre-image b_n.

Suppose $\exists$ $\partial_C c_{n+1} \in C_n$, and consider $c_n + \partial_C c_{n+1} \in [c_n]$. (*)
Then $\exists$ $b_{n+1} \in B_{n+1}$ s.t. $j_{n+1}(b_{n+1}) = c_{n+1}$, and so
$j_n(b_n + \partial b_{n+1}) = c_n + \partial c_{n+1}$
Applying ∂_B to $b_n + \partial b_{n+1}$, we get ∂b_n. Therefore, this alternative representation of $[c_n]$ will yield the same $[a_{n-1}]$ since it yields the same $\partial b_n \in B_{n-1}$.

We first show that δ is independent of the choice of $b_n \in B_n$ $\longrightarrow$

Suppose $\exists\, b_n, b_n' \in B_n$ s.t. $j_n(b_n) = j_n(b_n') = c_n$.

Then $j_n(b_n - b_n') = 0 \Rightarrow b_n - b_n' \in \ker j_n$. By exactness, $\exists\, \check{a}_n \in A_n$ s.t. $i_n(\check{a}_n) = b_n - b_n'$.

As before, choose $a_{n-1}, a_{n-1}' \in A_{n-1}$ s.t. $i_{n-1}(a_{n-1}) = \partial_B b_n$, and $i_{n-1}(a_{n-1}') = \partial_B b_n'$. We claim that $[a_{n-1}] = [a_{n-1}']$.

By construction,

$$\begin{aligned} i_{n-1}(a_{n-1} - a_{n-1}') &= \partial_B b_n - \partial_B b_n' = \partial_B(b_n - b_n') \\ &= \partial_B(i_n(\check{a}_n)) \\ &= i_{n-1}(\partial_A(\check{a}_n)) \end{aligned}$$

Since i_{n-1} is injective, this implies $a_{n-1} - a_{n-1}' = \partial_A(\check{a}_n)$ and so $[a_{n-1}] = [a_{n-1}']$. Therefore the choice of b_n does not affect the map δ_n.

We now show that the choice of representative of $[c_n]$ does not affect the map.

(insert $*$ here)

Therefore δ_n is well-defined on homology.

Finally, ~~we note that~~

$$\begin{aligned} i_{n-1*}(\delta_n[c_n]) &= i_{n-1*}([a_{n-1}]) \\ &= [i_{n-1} a_{n-1}] \\ &= [\partial_B b_n] = 0 \end{aligned}$$

~~so $\mathrm{im}(\delta_n) \subset \ker(i_{n-1*})$.~~

Moreover, $\forall\, b_n \in B_n$,

$$\delta_n j_{n*}[b_n] = \delta_n[j_n(b_n)]$$

$$=$$

We do not show exactness of this LES.

(b) Tensoring the chain complex w/ the long exact sequence yields the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathbb{Z}/5\mathbb{Z} \arrow[r, "5"] \arrow[d, "5=0"] & \mathbb{Z}/25\mathbb{Z} \arrow[r] \arrow[d, "5"] & \mathbb{Z}/5\mathbb{Z} \arrow[r] \arrow[d, "5=0"] & 0 & (1) \\
0 \arrow[r] & \mathbb{Z}/5\mathbb{Z} \arrow[r, "5"] \arrow[d] & \mathbb{Z}/25\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/5\mathbb{Z} \arrow[r] \arrow[d] & 0 & (0) \\
 & 0 & 0 & 0 & &
\end{tikzcd}$$

In all the instances of $\mathbb{Z}/5\mathbb{Z}$ above, the homology is $\mathbb{Z}/5\mathbb{Z}$ since the maps are trivial. Let $A_1, B_1, C_1, A_0, B_0, C_0$ denote the spaces, reading from left to right. We then drop $[\cdot]$ when considering homology. Following the construction from part a, consider some $[c_1] \in H_1(C) \cong \mathbb{Z}/5\mathbb{Z}$. Then $\exists\ b_1 \in B_1 \cong \mathbb{Z}/25\mathbb{Z}$ s.t. $b_1 \mapsto c_1$. By construction of the map $\mathbb{Z}/25\mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}$ we may choose b_1 "=" c_1 in the sense of $\mathbb{Z}$. Then $\partial b_1 = 5b_1 \neq 0$ and we may choose $a_0 \in A_0 \cong \mathbb{Z}/5\mathbb{Z}$ s.t. $a_0 = b_1 \Rightarrow 5a_0 = \partial b_1 = 5b_1$. Then $\delta: c_1 \mapsto a_0 = b_1 = c_1$ is the identity map on $\mathbb{Z}/5\mathbb{Z}$. $\square$

GeoTop

Spring 2019

① 405 846 515

For each $p \in M$ $\exists$ a neighborhood U_p of p s.t. $\overline{U_p}$ is compact. ~~Consider~~ This can be constructed explicitly for any $p \in M$ by taking a neighborhood ~~that~~ diffeomorphic to a subset of $\mathbb{R}^n$ and then taking the pre-image of a ball.

~~Consider the open cover $\{U_p\}_{p \in M}$ of M and let $\{\psi_p\}$ be a partition of unity subordinate to $\{U_p\}$.~~

Since M is second-countable, we may restrict each U_p to a small neighborhood of p s.t. $\overline{U_p}$ remains compact and $\{U_p\}_p$ is a countable open cover of M.* Enumerate these sets $\{U_1, U_2, \ldots\}$ and let $\{\psi_i\}$ be a partition of unity subordinate to $\{U_i\}$.

Define $f: M \to \mathbb{R}$ by $f(p) = \sum_i i\psi_i(p)$. Since $\{U_i\}$ is locally finite, f is a finite sum at all p and so is well-defined. Since ψ_i is smooth, this similarly implies that f is smooth.

We aim to show f is proper. Let K be a compact subset of $\mathbb{R}$. Since f is continuous, $f^{-1}(K)$ is closed. To show $f^{-1}(K)$ is compact, it then suffices to show $f^{-1}(K)$ is contained in a compact set.

Since $K \subset \mathbb{R}$ is compact, $\exists$ n s.t. $K \subset [-n, n]$. Consider $f^{-1}[-n, n]$. Since f is non-negative, $f^{-1}[-n, n] = f^{-1}[0, n]$.

~~By construction, we note that~~

$\longrightarrow$

---

* additionally, by paracompactness, we may restrict $\{U_p\}$ to a locally finite cover.

By construction, $\forall\, p \in M$,

$$|f(p)| = \left|\sum_{i=1}^{\infty} i\,\psi_i(p)\right| \geq \min\{i : \psi_i(p) > 0\} \left|\cancelto{1}{\textstyle\sum_i \psi_i(p)}\right|$$
$$\geq \min\{i : \psi_i(p) > 0\}$$

Therefore $f^{-1}[0,n] \subset U_1 \cup \dots \cup U_n \subset \overline{U_1} \cup \dots \cup \overline{U_n}$ which is compact by construction. Therefore $f^{-1}(K)$ is contained in a compact subset of M and hence is compact. Then f is proper as desired. $\square$

$$\left|\sum_i i\,\psi_i(p)\right| \cancel{\leq \max_{i:\psi_i \neq 0}} \; i$$

③ 405 846 515

Consider the anti-podal map $f: S^n \to S^n : x \mapsto -x$. Since $0 \notin S^n$, f does not have any fixed points and hence is a Lefschetz map. We aim to calculate the Lefschetz number of f.

By definition,

$$L(f) = \sum_{j=0}^{n} (-1)^j \, tr(f_* : H_j(S^n) \to H_j(S^n)).$$

We recall that $H_j(S^n) = \begin{cases} \mathbb{Z} & j = 0, n \\ 0 & \text{else} \end{cases}$. Therefore

$$L(f) = tr(f_* : H_0(S^n) \to H_0(S^n)) + (-1)^n tr(f_* : H_n(S^n) \to H_n(S^n))$$

Recalling the CW structure of S^n, we know that f_* fixes the 0-cell, inverts the n-cell if n is odd, and fixes the n-cell if n is even. Then:

$$L(f) = 1 + (-1)^n (-1)^n = 2$$

For the identity, we recall the Lefschetz number is

$$L(id) = \chi(S^n) = 1 + (-1)^n$$

Moreover, since f does not have any fixed points, $L(f) = 0$.
For the identity, we recall that $L(id) = \chi(S^n) = 1 + (-1)^n$.
Therefore if n is even, $L(f) = 0 \neq 2 = L(id)$ and so f and id are not homotopic. It then remains to show that they are in the case of n odd.

Suppose n is odd. Then we view $S^n \subset \mathbb{R}^{n+1} = (\mathbb{R}^2)^{\frac{n+1}{2}}$.
Let R_θ be the rotation by θ on $\mathbb{R}^2$. Then

$$f_\theta = R_\theta \oplus \dots \oplus R_\theta : S^n \to S^n$$

is a homotopy from $f_0 = id$ to $f_\pi = -id$ as desired. □

(4) 405 846 515

We recall Cartan's formula which states

$$\mathcal{L}_X = d \circ i_X + i_X \circ d$$

From this, it follows that $\mathcal{L}_X$ commutes w/ d since

$$\mathcal{L}_X \circ d = (d \circ i_X + i_X \circ d) \circ d = d \circ i_X \circ d = d \circ (d \circ i_X + i_X \circ d) = d \circ \mathcal{L}_X$$

Additionally, as will be shown in a lemma if time permits,

$$[\mathcal{L}_X, i_Y] = i_{[X,Y]}$$

Combining these facts, direct computation yields

$$\begin{aligned}
[\mathcal{L}_X, \mathcal{L}_Y] &= [\mathcal{L}_X, d \circ i_Y] + [\mathcal{L}_X, i_Y \circ d] \\
&= (\mathcal{L}_X \circ d \circ i_Y - d \circ i_Y \circ \mathcal{L}_X) + (\mathcal{L}_X \circ i_Y \circ d - i_Y \circ d \circ \mathcal{L}_X) \\
(d \circ \mathcal{L}_X = \mathcal{L}_X \circ d) \quad &= (d \circ \mathcal{L}_X \circ i_Y - d \circ i_Y \circ \mathcal{L}_X) + (\mathcal{L}_X \circ i_Y \circ d - i_Y \circ \mathcal{L}_X \circ d) \\
&= d \circ [\mathcal{L}_X, i_Y] + [\mathcal{L}_X, i_Y] \circ d \\
([\mathcal{L}_X, i_Y] = i_{[X,Y]}) \quad &= d \circ i_{[X,Y]} + i_{[X,Y]} \circ d \\
&= \mathcal{L}_{[X,Y]}
\end{aligned}$$

as desired.

$\square$

lemma $\rightarrow$

Lemma $[\mathcal{L}_X, i_Y] = i_{[X,Y]}$

proof: Fix a $k+1$ form ω and k vector fields $X_1, \dots, X_k$.

Then by direct computation

$$\mathcal{L}_X \circ i_Y (\omega(X_1,\dots,X_k)) = \mathcal{L}_X(\omega(Y, X_1, \dots, X_k))$$
$$= (\mathcal{L}_X \omega)(Y, X_1, \dots, X_k) + \omega(\mathcal{L}_X Y, X_1, \dots, X_k) + \sum_j \omega(Y, X_1, \dots, \mathcal{L}_X X_j, \dots X_k)$$

and

$$i_Y \circ \mathcal{L}_X(\omega(X_1,\dots,X_k)) = i_Y\Big((\mathcal{L}_X\omega)(X_1,\dots,X_k) + \sum_j \omega(X_1,\dots,\mathcal{L}_X X_j,\dots,X_k)\Big)$$
$$= (\mathcal{L}_X \omega)(Y, X_1, \dots, X_k) + \sum_j \omega(Y, X_1, \dots, \mathcal{L}_X X_j, \dots, X_k)$$

Therefore

$$[\mathcal{L}_X, i_Y](\omega(X_1,\dots,X_k)) = \omega(\mathcal{L}_X Y, X_1, \dots, X_k)$$
$$= \omega([X,Y], X_1, \dots, X_k)$$
$$= i_{[X,Y]}\, \omega(X_1, \dots, X_k)$$

and so $[\mathcal{L}_X, i_Y] = i_{[X,Y]}$.

$\square$

(5) 405 846 515

~~Suppose that~~ w is ~~exact~~. Then

($\Rightarrow$) Suppose first that w is exact, i.e $w = dg$ for some smooth g. Consider a smooth map $f: S^1 \to M$. B/c f^* commutes w/ the exterior derivative, Stokes' law implies

$$\int_{S^1} f^* w = \int_{S^1} d(f^* g) = \int_{\phi} f^* g = 0$$

as desired. Alternatively, we may note that $H^1_{dR}(S^1) \cong \mathbb{R}$ w/ isomorphism $\eta \mapsto \int_{S^1} \eta$. Therefore since $f^* w = d(f^* g)$ is 0 on the level of cohomology, $\int_{S^1} f^* w = 0$.

($\Leftarrow$) Suppose instead that $\int_{S^1} f^* w = 0$ $\forall$ smooth $f: S^1 \to M$. Let $M_1, M_2, \ldots$ denote the connected components of M and fix $p_i \in M_i$. Since M is a smooth manifold, M_i is path connected $\forall i$. Therefore $\forall x \in M_i$ $\exists$ a smooth path $\gamma(x)$ from p_i to x.

Define $g: M \to \mathbb{R}$ by

$$g(x) = \int_{\gamma(x)} w$$

To show that g is well-defined, it must be shown that g is independent of the choice of path $\gamma(x)$. Suppose $\exists$ 2 smooth paths γ_1, γ_2 from p_i to $x \in M_i$. Consider the path $\gamma_1 - \gamma_2$. By construction, $\gamma_1 - \gamma_2$ is smooth except potentially at p_i, x. Working locally at p_i, x we can construct a homotopic smooth closed path $\tilde{\gamma}$. Let $f: S_1 \to M$ be the smooth map defining $\tilde{\gamma}$. Then by homotopy invariance and assumption

$$\int_{\gamma_1} w - \int_{\gamma_2} w = \int_{\gamma_1 - \gamma_2} w = \int_{\tilde{\gamma}} w = \int_{S^1} f^* w = 0$$

$\longrightarrow$

Therefore g is independent of the choice of path $\gamma(x)$.
Since w is smooth, this implies that $g: M \to \mathbb{R}$ is smooth
$\gamma(x)$ can be chosen smoothly in x.

We finally claim that $dg = w$. To show this, it suffices to show it pointwise.
By the FTC, $\forall x \in M$,

$$dg_x = \frac{d}{dt}\Big|_{t=1} \int_{\gamma(x)} w = w_{\gamma(x)(1)} = w_x$$

and so w is exact w/ $dg = w$. $\square$

⑥ 405 846 515

Suppose $\exists\ \omega, \eta \in \Omega^n(Y)$ s.t. $f^*\omega - f^*\eta = d\varphi$ for some $n-1$-form φ on X.

~~$\omega-\eta$ is exact.~~ We claim that $\omega - \eta$ is exact

---

$\cancel{(f^*\omega)_{f(p)} = df_p\,\omega_p}$ $\quad (f^*\omega)_{f(p)} = \omega_p \circ df_p$

$df : TX \to TY$

$\cancel{\omega - \eta}$ $\quad f^*\omega - f^*\eta = d$

$\omega - \eta = d\theta \Rightarrow f^*(\omega-\eta) = d f^*\theta$

f local diffeo

of $\omega - \eta$

To show that $\omega-\eta$ is exact, it suffices to construct a $(n-1)$-form Ψ locally such that $d\Psi = \omega - \eta$. Fix some $p \in Y$. ~~Since~~
~~f is a finite covering map $\exists$ a neighborhood of y) and disjoint~~
Let $p_1, \dots, p_k$ be the pre-images of p under f, which are finite since f is a finite covering map. There ~~exists~~ exists a neighborhood U of p and disjoint neighbourhoods U_i of p_i s.t. $f|_{U_i}$ is a diffeomorphism onto U. Define Ψ on U by

$$\Psi_q = \frac{1}{k}\sum_{i=1}^{k} (f|_{U_i})_* \,\varphi_{(f|_{U_i})^{-1}(q)}$$

Since $f|_{U_i}$ is a diffeomorphism $\forall i$, this is well-defined and smooth on U. Moreover, on U,

$$\begin{aligned} d\Psi_q &= \frac{1}{k}\sum_{i=1}^{k} (f|_{U_i})_* d\varphi_{(f|_{U_i})^{-1}(q)} \\ &= \frac{1}{k}\sum_{i=1}^{k} (f|_{U_i})_* (f^*\omega - f^*\eta)_{(f|_{U_i})^{-1}(q)} \\ &= \frac{1}{k}\sum_{i=1}^{k} (\omega - \eta)_q \\ &= (\omega-\eta)_q \end{aligned}$$

Therefore $\omega - \eta$ is exact. This implies that f^* is injective on cohomology. $\square$

7) 405 846 515

~~we first calculate $H_1(X,A)$. Let $C_k(X)$ denote the k-chains of X~~ and ~~$C_k(A)$ denote the k chains of A.~~ Since A is a union of points, we have

$$C_2(X,A) \cong C_2(X) \cong 0, \quad C_1(X,A) = C_1(X) \cong \mathbb{Z}$$

We first calculate $H_1(X,A)$. We recall the LES

$$0 \to H_1(A) \to H_1(X) \to H_1(X,A) \to H_0(A) \to H_0(X) \to H_0(A) \to H_0(X,A) \to 0$$

Since $X = [0,1]$, X is contractible and hence $H_1(X) = 0$. Therefore $H_1(X,A) \hookrightarrow H_0(A)$. Since A is the countable union of 0-cells, $H_0(A)$ is countably generated. Therefore $H_1(X,A)$ is countably generated.

We claim that $H_1(X/A)$ is not countably generated. We recall that $H_1(X/A)$ is the abelianization of $\pi_1(X/A)$ and so it suffices to show $\pi_1(X/A)$ is not countably generated. To do so, it suffices to construct a surjection $\pi_1(X/A) \to \bigotimes_{i=1}^{\infty} \mathbb{Z}$ *

Fix some $x \in \bigotimes_{i=1}^{\infty} \mathbb{Z}$ w/ binary expansion $x = (x_1, x_2, \dots)$.

Define $\gamma : [0,1] \to X/A$ s.t. $\gamma(0) = [1]$ and s.t. γ traverses $[\frac{1}{n+1}, \frac{1}{n}]/A$ x_n times during time $[1/n+1, 1/n]$.

Therefore $\pi_1(X/A) \to \bigotimes_{i=1}^{\infty} \mathbb{Z}$ is a surjection and $H_1(X/A)$ is uncountably generated.

---

* Define $\pi_1(X/A) \to \bigotimes_{i=1}^{\infty} \mathbb{Z}$ by $\gamma \mapsto (x_1, x_2, \dots)$ where x_n is the # of times γ traverses $[1/n+1, 1/n]/A$. Since each loop of $[1/n+1, 1/n]/A$ is non-contractible, this is well-defined on $\pi_1(X/A)$.

8) 405 046 515

(a) Let $f : \mathbb{R}P^2 \to T^2 = S^1 \times S^1$ be continuous.

w/ covering map $q : \mathbb{R}^2 \to S^1 \times S^1$

We recall that $\mathbb{R}^2$ is the universal cover of T^2 and claim that f lifts to a continuous map $\tilde{f} : \mathbb{R}P^2 \to \mathbb{R}^2$.

We recall that $\mathbb{R}P^2$ has CW structure

1 0-cell : p

1 1-cell : e w/ $\partial e = p - p = 0$

1 2-cell : f w/ $\partial f = 2e$

Therefore $\pi_1(\mathbb{R}P^2) \cong \langle e \mid 2e \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

Similarly, since $S^1 \times S^1 \cong T^2$ has ~~cell~~ CW structure

1 0-cell : p

2 1-cells : a, b w/ $\partial a = \partial b = p - p = 0$

1 2-cell : f w/ $\partial f = a + b - a - b = 0$

it follows that $\pi_1(S^1 \times S^1) \cong \langle a, b \mid a + b - a - b \rangle = \langle a, b \rangle \cong \mathbb{Z}^2$.

Consider $f_*(\pi_1(\mathbb{R}P^2)) \subset \pi_1(S^1 \times S^1)$. Since the only finite subgroup of $\mathbb{Z}^2$ is 0, it follows that $f_*(\pi_1(\mathbb{R}P^2)) = 0$. Therefore

$$f_*(\pi_1(\mathbb{R}P^2)) \subset q^* \pi_1(\mathbb{R}^2)$$

and so $\exists$ a lift $\tilde{f} : \mathbb{R}P^2 \to \mathbb{R}^2$ of f. By the straight line homotopy $\tilde{f}_t = (1-t)\tilde{f}$, we note that $\tilde{f}$ is nullhomotopic.

Then $f_t = q \circ \tilde{f}_t$ is a homotopy from $q(\tilde{f}_0) = q \circ \tilde{f} = f$ to $q \circ \tilde{f}_1 = q(0)$ and so f is nullhomotopic as desired.

(b) As calculated above, $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$. Therefore $\exists$ a path $\gamma : S^1 \to \mathbb{R}P^2$ that is not nullhomotopic.

Define $f : S^1 \times S^1 \to \mathbb{R}P^2 : (x, y) \longmapsto \gamma(x)$.

Then f is continuous since $(x, y) \longmapsto x$ and γ are continuous and f is not null homotopic since γ is not nullhomotopic. $\square$

(9) 405 846 515

(a) As given, W is constructed as

$$1 \text{ 0-cell}: p$$
$$1 \text{ 1-cell}: e \quad w/\ \partial_1 e = p - p = 0$$
$$2 \text{ 2-cells}: A, B \quad w/\ \partial_2 A = 4e,\ \partial_2 B = 7e$$

This gives the chain complex

$$0 \to \underset{(2)}{\mathbb{Z}\langle A, B\rangle} \xrightarrow{\partial_2} \underset{(1)}{\mathbb{Z}\langle e\rangle} \xrightarrow{\partial_1} \underset{(0)}{\mathbb{Z}\langle p\rangle} \to 0$$

and hence the homology groups

$$H_0(W) = \frac{\ker(0)}{\operatorname{im}\partial_1} = \frac{\mathbb{Z}\langle p\rangle}{0} \cong \mathbb{Z}$$

$$H_1(W) = \frac{\ker\partial_1}{\operatorname{im}\partial_2} = \frac{\mathbb{Z}\langle e\rangle}{\mathbb{Z}\langle 4e, 7e\rangle} = \frac{\mathbb{Z}\langle e\rangle}{\mathbb{Z}\langle e\rangle} = 0 \quad \text{since } 4,7 \text{ are coprime}$$

$$H_2(W) = \frac{\ker\partial_2}{\operatorname{im}\partial_3} = \frac{\mathbb{Z}\langle 7A - 4B\rangle}{0} \cong \mathbb{Z}$$

Which is what was to be found.

(b) We claim that W is not homotopy equivalent to S^2.

To show this, it suffices to show that $\pi_1(S^2) \not\cong \pi_1(W)$.

We recall that $\pi_1(S^2) = 0$. This can be computed explicitly by noting that for any closed loop $\gamma \subset S^2$, we may find $p \in S^2$ s.t. $p \notin \gamma$. A stereographic projection gives $\pi: S^2 \setminus \{p\} \leftrightarrow \mathbb{R}^2$. Considering $\pi(\gamma)$, $\exists$ a straight line homotopy of $\pi(\gamma)$ to $0 \in \mathbb{R}^2$, which pulls back to a homotopy from γ to a point. Therefore every closed loop on S^2 is homotopic to a point and so $\pi_1(S^2) \cong 0$.

To compute $\pi_1(W)$, we recall that for any CW complex, the fundamental group has a presentation where the generators are the 1-cells and the relations are the boundaries of the 2-cells. This yields

$$\pi_1(W) \cong \langle e \mid 4e, 7e\rangle \cong 0 \quad \text{which doesn't help.}$$

GeoTop

SPRING 2014

① 405 846 515

(a) Let φ be a smooth bump function s.t.
$\varphi = 1$ on $(-1,1)$ and $\varphi = 0$ outside $(-2,2)$, and $\varphi : \mathbb{R} \to [0,1]$
Define $f : \mathbb{R} \to \mathbb{R}^2$ by

$$f(x) = \big(x(1-\varphi(x)),\ |x|(1-\varphi(x))\big)$$

Then $\forall x \in \mathbb{R}$, $f(x) \in T'$ since $|x(1-\varphi(x))| = |x|(1-\varphi(x))$.
Additionally, since $1-\varphi(x)$ is 0 near 0 and 1 away from 0,

$x(1-\varphi(x)) \to 0$ as $x \to 0$

$x(1-\varphi(x)) \to \infty$ as $x \to \infty$

$x(1-\varphi(x)) \to -\infty$ as $x \to -\infty$

The intermediate value theorem then implies that $x(1-\varphi(x))$ is surjective onto $\mathbb{R}$. Therefore $\operatorname{im} f = T'$.
Finally, since x, $\varphi(x)$ are smooth and $|x|$ is smooth except at 0, to show that f is smooth it suffices to show that f is smooth at 0. By construction, $1-\varphi(x) = 0$ on $(-1,1)$. Therefore f is identically 0 on a neighborhood of 0 and hence is smooth.

(b) No, f cannot be an immersion.
Heuristically, if f were an immersion then T' would be a smooth submanifold of $\mathbb{R}^2$. However, because of the corner at $(0,0)$, T' is not smooth.

~~Suppose on the contrary that f is an immersion. Then T' is a smooth submanifold of $\mathbb{R}^2$ w/ the usual smooth structure. Then $\exists$ a neighborhood hence the smooth structure on $\mathbb{R}^2$ is the identity, this implies that $\exists$ a neighborhood of 0 s.t.~~

Option 1:

Suppose that such an immersion exists. Then locally it is the canonical immersion, i.e. locally $\exists$ coordinates x_1, x_2 s.t. $f(t) = (t, 0)$. ~~Then~~ Looking locally at the origin, this would imply that $\exists$ a diffeomorphism from ~~the~~ a corner to a line, which is impossible.

Option 2:

By definition, any immersion f ~~would~~ is of the form

$$f(t) = (g(t), |g(t)|)$$

WLOG suppose that $f(0) = (0,0)$.

To be an ~~immersion,~~ immersion,

$$df_0 = g'(0)\,dx + |g|'(0)\,dy$$

must be injective

IDK man,

num

② 405 346 515

By definition of a manifold w/ boundary, $\forall\, p \in \partial W$ $\exists$ ~~an~~ open neighborhood U_p of p and a diffeomorphism $\varphi_p : U_p \to V_p \subset H$ where $H = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$ w/ $n = \dim W$, s.t. φ_p takes ∂W to $\partial H = \{(x_1, \dots, x_{n-1}, 0) \in \mathbb{R}^n\}$.

Let $U = W \setminus \partial W$. Then $\{U, U_p\}$ is an open cover of W.

By paracompactness, we may extend $\{U, U_p\}$ to a countable locally finite cover $\{U, U_1, U_2, \dots\}$. Let $\{\Psi, \Psi_i\}$ be a partition of unity subordinate to $\{U, U_i\}$.

Define $\pi : \mathbb{R}^n \to \mathbb{R}^n : (x_1, \dots, x_n) \longmapsto (x_1, \dots, x_{n-1}, 0)$. W/ this, define F_i on U_i by

$$F_i = f \circ \varphi_i^{-1} \circ \pi \circ \varphi_i$$

Then for $p \in U_i \cap \partial W$, $\pi(\varphi_i(p)) = \varphi_i(p)$ so $F_i = f$ on $\partial W \cap U_i$.

Additionally, since f, φ_i, π are smooth, F_i is smooth on U_i.

Define $F : W \to \mathbb{R}^n$ by

$$F = \sum_i \Psi_i F_i = \sum_i \Psi_i\, f \circ \varphi_i^{-1} \circ \pi \circ \varphi_i$$

By local finiteness, this sum is locally finite and so F is smooth.

Additionally, by construction $\sum_i \Psi_i = 1$ on ∂W since $U \cap \partial W = \emptyset$.

Therefore on ∂W, $F = \sum_i \Psi_i f = f$, as desired.

$\square$

Alternatively, this can be done via the tubular neighborhood theorem.

3) 405 846 515

Suppose that n is even. Then id has degree 1 and $-$id has degree -1. This can be seen either by remembering that $\deg(-id) = (-1)^{n+1}$ on S^n or by noticing that by viewing $S^n \subset \mathbb{R}^{n+1}$, $-$id flips $n+1$ coordinates and hence has degree $(-1)^{n+1}$. Since degree is homotopy invariant, this implies that $-$id is not homotopic to id ~~if~~ if n is even.

Now suppose that n is odd. Then $n = 2k-1$ and we can view $S^n \subset \mathbb{C}^k$. Consider the map $H_t : S^n \to S^n : z \mapsto e^{\pi i t} z$.
Then $H_0 = id$ and $H_1 = -id$ and H_t is continuous.
Therefore H_t is a homotopy from id to $-$id. $\square$

(4) 405 846 515

($\Rightarrow$). Suppose $\omega_1, \ldots, \omega_k$ are linearly independent 1-forms on M.
Then $\forall p \in M$, $(\omega_1)_p, \ldots, (\omega_k)_p \in T_p^*M$ are linearly independent.
Therefore $\exists$ ~~$v_1, \ldots, v_k \in T_pM$ s.t.~~ dual vectors $v_1, \ldots, v_k \in T_pM$ s.t. $(\omega_i)_p(v_j) = \delta_{ij}$. Then $(\omega_1 \wedge \ldots \wedge \omega_k)_p(v_1, \ldots, v_k) = 1 \neq 0$.
Then $\omega_1 \wedge \ldots \wedge \omega_k \neq 0$.

($\Leftarrow$) Suppose $\omega_1, \ldots, \omega_k$ are linearly dependent. Then $\exists\, a_1, \ldots, a_{k-1}$ s.t.

$$\omega_k = a_1\omega_1 + \ldots + a_{k-1}\omega_{k-1}$$

Then

$$\begin{aligned} \omega_1 \wedge \ldots \wedge \omega_{k-1} \wedge \omega_k &= \omega_1 \wedge \ldots \wedge \omega_{k-1} \wedge \Big(\sum_{j<k} a_j \omega_j\Big) \\ &= \sum_{j<k} a_j\, \omega_1 \wedge \ldots \wedge \omega_{k-1} \wedge \omega_j \\ &= 0 \end{aligned}$$

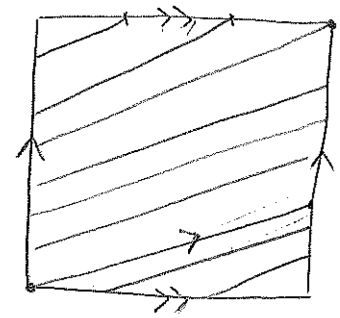
As desired.

$\square$

(5) 405 846 515

We recall that the Poincaré dual of a submanifold $W \subset M$ is a differential form η on M s.t. $\int_M w \wedge \eta = \int_W w \quad \forall\ w$.

Consider first $S = \pi(L) \subset M$. We note that, viewing M as the unit square w/ identified opposite edges, S loops 7 times in the x direction and 3 times in the y direction before returning to the origin. We recall that $H^1(M) \cong \mathbb{R}^2$ and so all 1-forms w on M can be written as $a\,dx + b\,dy$ for $a, b \in \mathbb{R}$. ~~up to~~ Then



$$\int_S w = \int_{7S^1} a\,dx + \int_{3S^1} b\,dy$$
$$= 7a + 3b$$

Now let $\eta = 7dy - 3dx$. Then $\forall$ 1-forms $w = a\,dx + b\,dy$ on M,

$$\int_M w \wedge \eta = \int_M (7a\,dx \wedge dy + 3b\,dx \wedge dy)$$
$$= 7a + 3b$$
$$= \int_S w$$

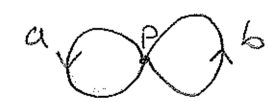
~~and~~ Therefore $w = 7dy - 3dx$ is a Poincaré dual of $S \subset M$. $\square$

(7) 405 846 515

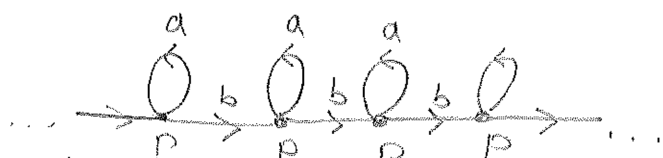
By Van Kampen's, we recall that $\pi_1(S^1 \vee S^1) \cong \langle a, b\rangle \cong \mathbb{Z} * \mathbb{Z}$.

~~We... Since $S^1 \vee S^1$ can be visualized as a graph single vertex w/ two outward~~

$S^1 \vee S^1 = X$



Viewing $S^1 \vee S^1$ as a graph, we see that all connected covering spaces of $S^1 \vee S^1$ consist of graphs where each vertex has an outgoing and ingoing a and b identified edges. Consider in particular the infinite-sheeted cover $\tilde{X}$



For any vertex p in $\tilde{X}$, it follows that $p_*\pi_1(\tilde{X}, p) = \langle a \rangle \subset \pi_1(X, p)$. Consider this subgroup $\langle a \rangle \subset \langle a, b\rangle$. Since $bab^{-1} \notin \langle a \rangle$, $\langle a \rangle$ is not a normal subgroup. Therefore $\tilde{X}$ is an irregular cover.

$\square$

(B) 405 846 515

(a) The cell decomposition produced depends on the parity of n.
Suppose that n is even. Then since we identify an even # of pairs of edges, there is only one remaining vertex. The cellular decomposition is then

$$\begin{aligned} &1 \text{ 0-cell: } p \\ &n \text{ 1-cells: } e_1, \ldots, e_n \text{ w/ } \partial_1 e_i = p - p = 0 \\ &1 \text{ 2-cell: } f \text{ w/ } \partial_2 f = e_1 + e_2 + \ldots + e_n - e_1 - \ldots - e_n \end{aligned}$$

The cellular chain complex is then

$$0 \to C_2 \cong \mathbb{Z} \xrightarrow{\partial_2} C_1 \cong \mathbb{Z}^n \xrightarrow{\partial_1} C_0 \cong \mathbb{Z} \to 0$$

Suppose instead that n is odd. Then there are 2 remaining vertices ~~after~~ after identification. ~~and so~~ Therefore we have the cellular decomposition

$$\begin{aligned} &2 \text{ 0-cells: } p, q \\ &n \text{ 1-cells: } e_1, \ldots, e_n \text{ w/ } \partial_1 e_i = (-1)^i (q - p) \\ &1 \text{ 2-cell: } f \text{ w/ } \partial_2 f = e_1 + \ldots + e_n - e_1 - \ldots - e_n \end{aligned}$$

The cellular chain complex is then

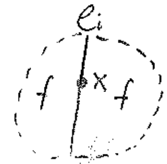
$$0 \to C_2 \cong \mathbb{Z} \xrightarrow{\partial_2} C_1 \cong \mathbb{Z}^n \xrightarrow{\partial_1} C_0 \cong \mathbb{Z}^2 \to 0$$

~~(b) By construction, every 0-cell in X_n has an even # of attached attachings, so there are no dangling~~

$\longrightarrow$

(b) Consider a point $x \in X_n$. We aim to show that there is a neighborhood of x diffeomorphic to ~~a~~ open subset of $\mathbb{R}^2$. If $x \in f \setminus \partial f$, then this diffeomorphism is clear. Therefore suppose that $x \in e_i$ or x is a vertex.

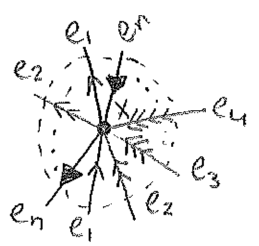
Suppose first that $x \in e_i$ for some i. By construction, part of ∂f is identified w/ e_i and part of ∂f is identified w/ $-e_i$. Therefore at x, X_n locally looks like



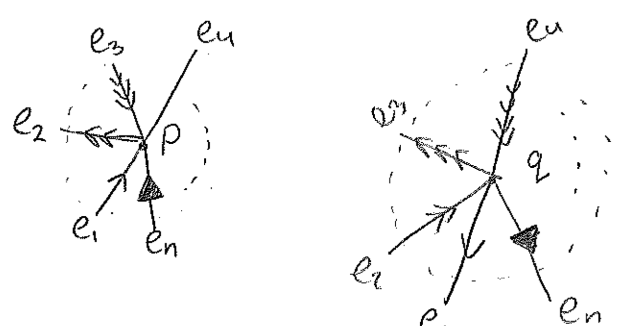
from which it is clear that a neighborhood of x is diffeomorphic to an open subset of $\mathbb{R}^2$.

Similarly, if x is a vertex, X_n locally looks like

even n



odd n



In all cases, there is a neighborhood diffeomorphic to a subset of $\mathbb{R}^2$. Additionally, it follows from the diagrams that ~~these~~ ~~transition~~ maps are smooth. Therefore X_n is a smooth 2-manifold w/o boundary.

We also note that X_n is compact as the ~~quot~~ quotient of a compact space.

Therefore, if g is the genus of X_n,

$$2-2g = \chi(X_n) = \begin{cases} 1-n+1 & n \text{ even} \\ 2-n+1 & n \text{ odd} \end{cases} = \begin{cases} 2-n & n \text{ even} \\ 3-n & n \text{ odd} \end{cases} \Rightarrow g = \begin{cases} n/2 & n \text{ even} \\ \frac{n-1}{2} & n \text{ odd} \end{cases} \qquad \square$$

⑨ 405 846515

(a) We recall that $\mathbb{RP}^3$ can be constructed as a 3-cell B^3 attached to $\mathbb{RP}^2$ via the map $\partial : \partial B^3 \cong S^2 \to \mathbb{RP}^2 : (x_0, x_1, x_2) \longmapsto [x_0 : x_1 : x_2]$.
This implies that $\mathbb{RP}^3$ w/ a deleted 'open' ~~neighbo~~ ball U, i.e. $\mathbb{RP}^3 \setminus U$ can be constructed as ~~an~~ a 3-dimensional annulus ~~att~~ w/ ~~outer~~ inner boundary attached to $\mathbb{RP}^2$ as done above. Since $S^2 \times [0, 1/2]$ is diffeomorphic to a 3-dimensional annulus and $\mathbb{RP}^2 \cong S^2/x \sim -x$, this implies that

$$\mathbb{RP}^3 \setminus U \cong S^2 \times [0, 1/2] / (x,0) \sim (-x, 0)$$

~~By~~ Similarly, $\mathbb{RP}^3 \setminus U \cong S^2 \times [1/2, 1] / (x,1) \sim (-x,1)$.
~~Thus~~ hence $\mathbb{RP}^3 \# \mathbb{RP}^3 \cong (\mathbb{RP}^3 \setminus U) \amalg (\mathbb{RP}^3 \setminus U) / \sim$
where $\sim$ identifies the ~~boundaries~~ of the two $\mathbb{RP}^3 \setminus U$,
this implies that $S \cong \mathbb{RP}^3 \# \mathbb{RP}^3$ as desired.

~~(b)~~ Alternatively, $\mathbb{RP}^3 \setminus D^3 \cong \mathbb{RP}^2$ so $\mathbb{RP}^3 \# \mathbb{RP}^3 \cong \mathbb{RP}^2 \amalg \mathbb{RP}^2$ glued along $S^2 \times [0,1]$.

(b) We recall that S^2 is a double cover of $\mathbb{RP}^2$ w/ covering map given by the quotient map $\pi : S^2 \to S^2/x \sim -x \cong \mathbb{RP}^2$.
~~Heuristically, S^2 can be $\mathbb{RP}^2$ can be viewed as half~~
~~Since~~ Since $\mathbb{RP}^3 \# \mathbb{RP}^3$ is homeomorphic to ~~S~~ Y, it suffices to show $S^2 \times S^1$ is a double cover of Y.
We view $S^2 \times S^1$ as $S^2 \times [0,4] / (x,0) \sim (x,4)$. Then define $p : S^2 \times S^1 \to Y$ by

$$\begin{cases} S^2 \times \{0\}, S^2 \times \{2\} \to \mathbb{RP}^2 \times \{0\} \subset Y & \text{via the 2-cover } \pi : S^2 \to \mathbb{RP}^2 \\ S^2 \times (0,1), S^2 \times (1,2), S^2 \times (2,3), S^2 \times (3,4) \to S^2 \times (0,1) \subset Y & \text{via the identity (1-cover)} \\ S^2 \times \{1\}, S^2 \times \{3\} \to \mathbb{RP}^2 \times \{1\} \subset Y & \text{via the 2-cover } \pi : S^2 \to \mathbb{RP}^2 \end{cases}$$

$\longrightarrow$

we note this map is ~~constructed~~ locally as an identity and each point in Y has 2 pre-images. Therefore $p : S^2 \times S^1 \to Y$ is a double cover as desired. $\square$

(10) 405 846 515

We proved by Mayer-Vietoris.

Let $U = X\times[0,1)/\sim$ and $V = X\times(0,1]/\sim$ where $\sim$ is the contraction of $X\times\{0\}$ and $X\times\{1\}$ to points. Then U, V are both contractible, $U\cap V$ deformation retracts to X w/ the straight line homotopy along $(0,1)$, and $U\cup V = S(X)$. ~~we thus~~ This yields the LES

$$\cdots \longrightarrow \underset{\substack{\cong \\ H_k(X)}}{H_k(U\cap V)} \to \underset{\substack{\cong \\ H_k(\{pt\})^2}}{H_k(U)\oplus H_k(V)} \to H_k(S(X)) \to \cdots$$

We note that by construction, $S(X)$ is connected so $H_0(S(X)) \cong \mathbb{Z}$.

For $k \geq 2$ since $H_k(\{pt\}) = 0 \ \forall\, k\neq 0$, the above LES yields a SES

$$0 \to H_k(S(X)) \to H_{k-1}(X) \to 0$$

and so $H_k(S(X)) \cong H_{k-1}(X) \ \forall\, k\geq 2$. Finally, for $k=1$, we have the SES

$$0 \to H_1(S(X)) \to H_0(X) \to (H_0\{pt\})^2 \to H_0(S(X)) \to 0$$

Letting r denote the # of connected components of X, this yields

$$0 \to H_1(S(X)) \to \mathbb{Z}^r \to \mathbb{Z}^2 \to \mathbb{Z} \to 0$$

Taking the alternating sum then implies that

$$\operatorname{rank} H_1(S(X)) = r-2+1 = r-1$$

and so $H_1(S(X)) \cong \mathbb{Z}^{r-1}$.

Summarizing,

$$H_k(S(X)) \cong \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}^{r-1} & k=1 \\ H_{k-1}(X) & k\geq 2 \end{cases}$$

$\square$

GeoTop

Fall 2013

① | 1/2 credit |

(a) No, f need not be injective or surjective.
Consider $f: \mathbb{C} \to \mathbb{C} : z \mapsto e^z$. Then $df = e^z dz$ which is nonvanishing and hence ~~nonsing~~ nonsingular everywhere. However, $f \neq 0 \Rightarrow f$ is not surjective and $f(0) = f(2\pi i) = 1$ so f is not injective.

(b) ~~Yes. This is the classic stack of records theorem, which we~~ ~~repeat~~ ~~here~~ ~~afte~~ after.

(c) Yes.
proof: Suppose that $U \subset M$ is open. Since f is nonsingular, and $\dim M = \dim N$, $\forall p \in U$ $\exists$ ~~an~~ a neighborhood $p \in V \subset U$ s.t. $f|_V : V \to f(V)$ is a diffeomorphism by the inverse function theorem.
Therefore $f(V) \subset f(U)$ is open and $f(p) \in f(V) \subset f(U)$.
Then $\forall f(p) \in f(U)$ $\exists$ an open neighborhood $\tilde{V} \subset f(U)$ s.t. $f(p) \in \tilde{V} \subset f(U)$. Therefore $f(U)$ is open $\forall$ open U and so f is open. $\square$

(d) I was going to say yes, but then fails.
No. Consider $f: (0,1) \to (0,2)$; incl; $f(x) = x$. Then $df = id \Rightarrow f$ is non-singular. However, $(0,1)$ is closed in itself but not in $(0,2)$, so f is not a closed map.

(b) No. Consider $f: (0,1) \to S^1 \cong [0,1]/0{\sim}1$. w/ $f(x) = x$.
Then S^1 is compact, $df = id$ so f is nonsingular, but f is not surjective and hence not a covering map. $\square$

(2) 405 846 515

Sard's theorem solution

Suppose for the sake of contradiction that a retraction $r: M \to \partial M$ exists. Then by Sard's theorem, r has a regular value $p \in \partial M$. Since p is a regular value, $r^{-1}(p) \subset M$ is a codimension $\dim M - 1$ submanifold and hence a 1-dimensional manifold.
Since $\{p\}$ is closed, $r^{-1}(p)$ is closed. Moreover since M is compact, $r^{-1}(p)$ is compact. B/c the only compact 1-dimensional manifolds are disjoint unions of S^1 and ~~line~~ segments, it follows that $\partial r^{-1}(p)$ has even cardinality. However, since $\partial r = r|_{\partial M} = id$, $\partial r^{-1}(p) = \{p\}$ which is odd. Therefore our supposition led to a contradiction and no such retraction exists. □

Homology proof

Suppose $\exists$ a retract $r: M \to \partial M$.
The relative homology over $\mathbb{Z}/2\mathbb{Z}$ then gives a LES

$$0 \to \underset{\substack{\| \\ 0}}{H_n(\partial M; \mathbb{Z}/2\mathbb{Z})} \to H_n(M; \mathbb{Z}/2\mathbb{Z}) \to H_n(M, \partial M; \mathbb{Z}/2\mathbb{Z}) \to H_{n-1}(\partial M; \mathbb{Z}/2\mathbb{Z}) \to \dots$$

Since M is compact and orientable over $\mathbb{Z}/2\mathbb{Z}$, Lefschetz duality implies $H_n(M; \mathbb{Z}/2\mathbb{Z}) \cong H^0(M, \partial M; \mathbb{Z}/2\mathbb{Z}) = 0$. Therefore we have the exact sequence

$$0 \to H_n(M, \partial M; \mathbb{Z}/2\mathbb{Z}) \to H_{n-1}(\partial M; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{i} H_{n-1}(M; \mathbb{Z}/2\mathbb{Z}) \to \dots$$

Since we have a retraction $M \to \partial M$, i is injective. Therefore $H_n(M, \partial M; \mathbb{Z}/2\mathbb{Z}) = 0$. However, by Lefschetz duality,

$$H_n(M, \partial M; \mathbb{Z}/2\mathbb{Z}) = H^0(M; \mathbb{Z}/2\mathbb{Z}) = H_0(M; \mathbb{Z}/2\mathbb{Z})^* \neq 0.$$

? which is a contradiction. □

③ 405 846 515

(a) Define

$$\mu: N \times M \to S^P : (y, x) \longmapsto \frac{y-x}{\|y-x\|}$$

and note that $l(N,M) = \deg \mu$ by construction.

Let $T: N \times M \to M \times N : (y,x) \mapsto (x,y)$ be the swapping map and $\phi: S^P \to S^P : x \mapsto -x$ be the anti-podal map. Then

$$\mu = \phi \circ \lambda \circ T : N \times M \to M \times N \to S^P \to S^P$$

Therefore

$$l(N,M) = \deg \mu = \deg(\phi) \deg(\lambda) \deg(T) = \deg(\phi) \deg(T)\, l(M,N).$$

We note that the antipodal map $\phi: S^P \to S^P$ has $\deg(\phi) = (-1)^{P+1}$ since $P+1$ coordinates are being inverted.

To compute $\deg T$, we note that locally,

$$T: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^n : (x_1, .., x_n, y_1, .., y_m) \to (y_1, .., y_m, x_1, .., x_n)$$

which is a process that requires nm adjacent index swaps.

Therefore $\deg T = (-1)^{nm}$ and so

$$l(N,M) = (-1)^{P+1+nm} l(M,N) = (-1)^{mn+m+n+1} l(M,N)$$
$$= (-1)^{(m+1)(n+1)} l(M,N)$$

as desired.

(b) We additionally assume N is boundaryless.

We extend λ to $\tilde{\lambda}: W \times N \to S^P : (x,y) \mapsto \frac{x-y}{\|x-y\|}$ since $W \cap N = \emptyset$.

Moreover, $\partial(W \times N) = \partial W \times N \amalg W \times \partial N = \partial W \times N = M \times N$.

Therefore $M \times N$ is the boundary of a manifold $W \times N$ and λ is a map on the boundary which can be extended to the whole manifold.

The extension theorem then implies $l(M,N) = \deg \lambda = 0$. □

(4) 405 846 515

~~Fix some $p \in M$. Since M is connected, $\forall x \in M$ $\exists$ a piecewise~~

$(\Rightarrow)$ Suppose that w is exact w/ $w = df$. Let $\exists$ a piecewise smooth curve $c: S^1 \to M$. If c is constant then $\int_c w = 0$ trivially, so we assume c is not constant.

Let $0 = t_1 < t_2 < \ldots$ be a partition of $[0,1]$ w/ $S^1 = [0,1]/0{\sim}1$ s.t. c is smooth on $[t_n, t_{n+1}]$ $\forall n$. Then $c[t_n, t_{n+1}]$ is a smooth submanifold of M $\forall n$ w/ boundary $c(t_{n+1}) - c(t_n)$. Stokes's theorem then implies

$$\int_c w = \sum_n \int_{c[t_n,t_{n+1}]} w = \sum_n \int_{\partial c[t_n,t_{n+1}]} df = \sum_n f(c(t_{n+1})) - f(c(t_n))$$

This series telescopes to

$$\int_c w = f(c(1)) - f(c(0)) = 0$$

as desired.

$(\Leftarrow)$ Suppose instead that $\int_c w = 0$ $\forall$ piecewise smooth closed c.
Fix some $p \in M$. Since M is connected, $\forall x \in M$ $\exists$ a piecewise smooth path $\gamma(x)$ from p to x. Define $f: M \to \mathbb{R}$ by

$$f(x) = \int_{\gamma(x)} w$$

We first claim that f is independent of $\gamma(x)$.
Suppose $\exists$ two piecewise smooth paths γ_1, γ_2 from p to x.
Then $\gamma_1 - \gamma_2$ is a piecewise smooth closed path and so

$$\int_{\gamma_1} w - \int_{\gamma_2} w = \int_{\gamma_1 - \gamma_2} w = 0 \Rightarrow \int_{\gamma_1} w = \int_{\gamma_2} w$$

Therefore f is independent of the choice of path and so is well-defined.
B/c w is smooth, it follows that f is smooth.
We claim that $df = w$. To show this, it suffices to show
$df_q(v) = w_q(v)$ $\forall$ $q \in M$ and $v \in T_qM$.

It suffices to work locally. Let $(x_1, \ldots, x_n)$ be coordinates at $q \in M$, on U, and let φ be the ~~atlas~~ chart. w/ $q = (0, \ldots, 0)$.
Then ω can be written

$$\omega = \sum_i g_i \, dx_i$$

Fix $v \in T_q M$. By definition, $\forall i$,

$$\left.\frac{\partial f}{\partial x_i}\right|_q = \lim_{t \to 0} \frac{f(q + t\hat{x}_i) - f(q)}{t}$$

$$= \lim_{t \to 0} \frac{1}{t} \int_{[q, q + t\hat{x}_i]} \omega$$

since we are working locally at q, we can regard this as integration on $\mathbb{R}^n$. Then

$$\left.\frac{\partial f}{\partial x_i}\right|_q = \lim_{t \to 0} \frac{1}{t} \int_0^t g_i(s) ds = g_i(0) = g_i(q)$$

Therefore $df = \omega$ locally and hence $df = \omega$ globally. $\square$

5) 405 846 515

(a) Suppose

($\Rightarrow$) Suppose ker ω is an integrable distribution. Then locally $\exists$ coordinates x, y, z s.t. $\ker\omega = \mathbb{R}\langle \partial/\partial x, \partial/\partial y\rangle$.
This implies that $\omega = f dz$ and so

$$\omega \wedge d\omega = f dz \wedge df \wedge dz = 0$$

as desired.

(b) Consider $\omega = -y dx + x dy + dz$. Then

$$d\omega = -dy\wedge dx + dx \wedge dy = 2 dx\wedge dy$$

$$\omega \wedge d\omega = (-y dx + x dy + dz)\wedge(2dx\wedge dy) = 2 dx\wedge dy \wedge dz$$

So ω is smooth and non-vanishing but $\omega\wedge d\omega \neq 0$.
Therefore ker ω is a 2-dimensional (codimension 1) ~~no~~ distribution that is not integrable.

We now return to part (a).

($\Leftarrow$) Suppose $\omega \wedge d\omega = 0$. Let $\exists$ vector fields $X, Y \in \ker\omega$.
We aim to show $[X,Y] \in \ker\omega$. By direct computation

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega[X,Y] = -\omega[X,Y]$$

b/c $X, Y \in \ker\omega$. Locally, Choose $Z \notin \ker\omega$. Then

$$0 = \omega\wedge d\omega$$

$$\Rightarrow 0 = \omega \wedge d\omega(X,Y,Z)$$

$$= -\omega(Z)\, d\omega(X,Y)$$

$$= +\omega(Z)\,\omega[X,Y]$$

Since ω is nonvanishing, this implies locally $\omega[X,Y] = 0$ and so $[X,Y] \in \ker\omega$ as desired. $\square$

(6) 405 846 515

(a) In the usual way, we define the gradient of f as

$$\nabla f = \frac{\partial f}{\partial x_1} \partial/\partial x_1 + \dots + \frac{\partial f}{\partial x_n} \partial/\partial x_n$$

we note that this is dual to

$$df = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n$$

(b) We recall that the Hessian classically is given as

$$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

We note that since f is smooth, H_f is symmetric.
We then define Hess(f) to be the ~~(0,2)~~ symmetric (0,2) tensor

$$\text{Hess}(f)(X,Y) = X^T H_f Y^T$$

or equivalently

$$\text{Hess}(f) = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \otimes dx_j .$$

(c) We write $g = \sum_{i=1}^n dx_i \otimes dx_i$ w/ associated matrix I. Then by linearity and Leibniz

$$\mathcal{L}_{\nabla f} g = \sum_{i=1}^n \mathcal{L}_{\nabla f}(dx_i \otimes dx_i)$$

$$= \sum_{i=1}^n \left((\mathcal{L}_{\nabla f} dx_i) \otimes dx_i + dx_i \otimes \mathcal{L}_{\nabla f} dx_i \right)$$

By Cartan's formula,

$$\mathcal{L}_{\nabla f} dx_i = (d \circ i_{\nabla f} + i_{\nabla f} \circ d) dx_i = d \circ i_{\nabla f} dx_i = d \frac{\partial f}{\partial x_i}$$

$$= \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j$$

Then

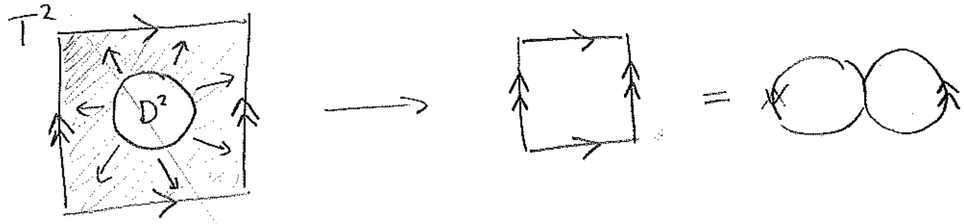
$$\mathcal{L}_{\nabla f} g = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} (dx_j \otimes dx_i + dx_i \otimes dx_j)$$

$$= 2\,\text{Hess}(f)$$

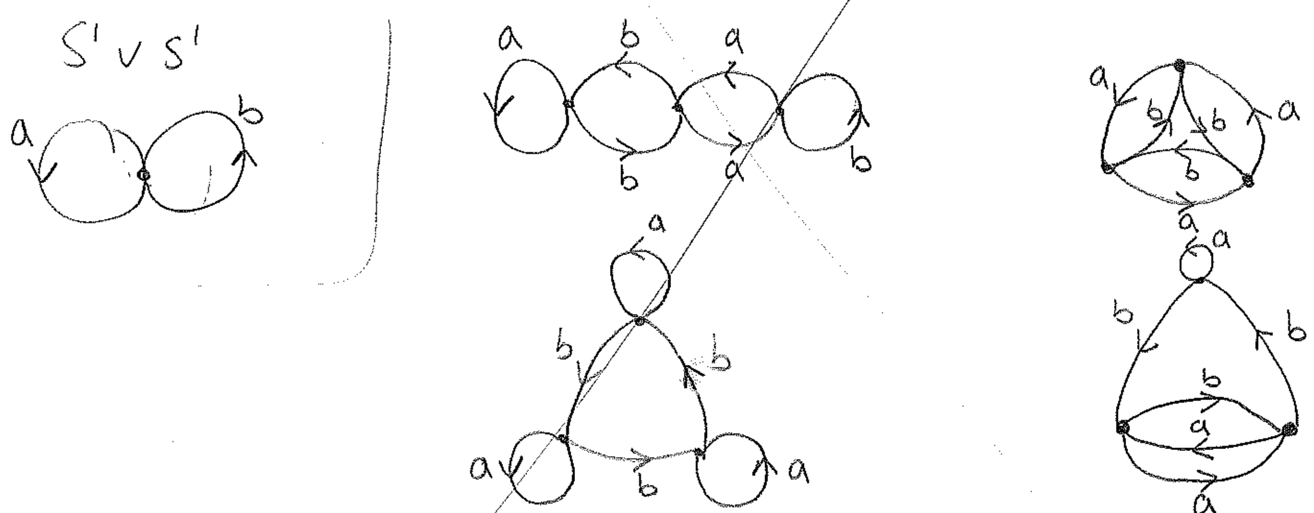
which is what was to be shown. $\square$

⑦ 405 846 515

Viewing T^2 as $[0,1]\times[0,1]/\sim$ where $(x,0)\sim(x,1)$ and $(0,x)\sim(1,x)$, we find that T^2-D^2 deformation retracts onto $S^1\vee S^1$..



We first aim to find all 3-covers of $S^1\vee S^1$.
We note that $S^1\vee S^1$ ~~consists of one vert~~ can be viewed as a graph w/ 1 vertex and 2 edges. The connected 3-fold covers of $S^1\vee S^1$ therefore consist of connected graphs w/ 3 vertices and 6 edges. Running through these, we find



and all relevant permutations of edges and orientation.
These are all possible covers since every vertex can either have one edge connected to itself and two edges connected to other vertices or 4 edges connected to other vertices. Running through all possible combinations, we find these ~~3 opts~~ options, w/ permutations

The 3-fold covers of T^2-D^2 are therefore all surfaces which deformation retract onto these graphs. □

(8) 405 846 515

Suppose that A has finite presentation

$$A \cong \langle a_1, \ldots, a_k \mid b_1, \ldots, b_\ell \rangle$$

Define a CW complex X as follows.

1 0-cell: p

k n-cells: $a_1, \ldots, a_k$ w/ a_i attached via the constant map p

ℓ $n+1$-cells: $c_1, \ldots, c_\ell$ w/ $\partial_{n+1} c_i = b_i$

Then

$$H_n(X) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}} = \frac{\langle a_1, \ldots, a_k \rangle}{\langle b_1, \ldots, b_\ell \rangle} = A$$

as desired.

Alternatively, ~~also~~ consider the boquet of k n-spheres

$$\bigvee_{j=1}^{k} S^n$$

w/ the j^{th} S^n being associated w/ a_j.

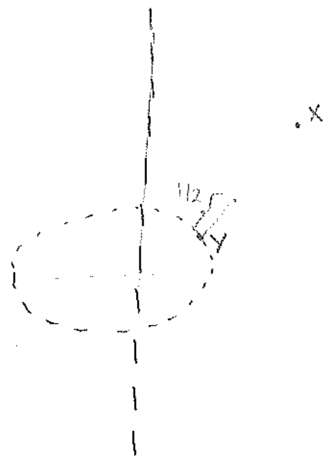
Attach ℓ $n+1$-cells to $\bigvee_{j=1}^{k} S^n$ via $b_1, \ldots, b_\ell$. Then as above,

$$H_n(X) = \langle a_1, \ldots, a_k \mid b_1, \ldots, b_\ell \rangle \cong A$$

as desired. $\square$

⑨ 405 846 515

Fix some $p \in H$. By identifying p as the north pole, we see that $S^3 \setminus H$ is equivalent to $\mathbb{R}^3 \setminus \{z \text{ axis}\} \cup \{\text{unit circle in } xy\}$ via a stereographic projection.



we claim that this deformation retracts onto $\mathbb{T}^2$.
Let Z denote the z axis and C the unit circle in the xy plane.

Then $\forall\ x \in \mathbb{R}^3 \setminus (Z \cup C)$ $\exists!\ y \in C$ s.t. $|x-y|$ is minimized.
Define $f_t : \mathbb{R}^3 \setminus (Z \cup C) \to \mathbb{R}^3 \setminus (Z \cup C)$ by

$$f_t(x) = (1-t)x + t\left(y + \frac{x-y}{2|x-y|}\right)$$

By definition f_t is continuous $\forall t$, $f_0 = id$, $\operatorname{im} f_1$ is the Torus around C of radius $1/2$, and $f_t|_T = id$ w/ T the torus.
Therefore $\mathbb{R}^3 \setminus (Z \cup C)$ deformation retracts onto T^2.

Combining these, it suffices to compute the fundamental group and homology of ~~a Torus~~ T^2. We recall that

$$\pi_1(T^2) \cong \mathbb{Z}^2 \quad \text{and} \quad H_n(T^2) \cong \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z}^2 & n = 1 \\ 0 & \text{else} \end{cases}$$

which is what was to be shown. $\square$

(10) 405 846 515

We mimic the CW construction of $\mathbb{RP}^n$ or $\mathbb{CP}^n$ to construct $\mathbb{HP}^n$ w/ a 4k-cell $\forall\, k = 0, \dots, n$

To do so, we construct $\mathbb{HP}^{n+1}$ from $\mathbb{HP}^n$ iteratively.

We start w/ $\mathbb{HP}^0$. We note that $\mathbb{HP}^0 = (\mathbb{H} - \{0\})/(\mathbb{H}-\{0\}) \cong \{0\}$ and so $\mathbb{HP}^0$ has the structure of a single point.

Suppose that we have constructed $\mathbb{HP}^n$ as described. We claim that we can construct $\mathbb{HP}^{n+1}$ by attaching a $4(n+1)$-cell to $\mathbb{HP}^n$.
For consistency in notation, we denote elements of $\mathbb{HP}^n$ by

$$[x_0 : \dots : x_n] \quad \text{for} \quad (x_0, \dots, x_n) \in \mathbb{H}^{n+1} \setminus \{0\}.$$

We first ~~claim~~ note that $\mathbb{HP}^n \hookrightarrow \mathbb{HP}^{n+1}$ via the map

$$[x_0 : \dots : x_n] \mapsto [x_0 : \dots : x_n : 0]$$

where the smoothness is obvious.
Let $U = \{[x_0 : \dots : x_n : 0] \in \mathbb{RP}^{n+1}\}$. We claim that $\mathbb{RP}^{n+1} \setminus U \cong B^{4(n+1)}$.
Consider some $[x_0 : \dots : x_{n+1}] \in \mathbb{HP}^{n+1} \setminus U$. By scaling, we may assume that $|x_0| + \dots + |x_{n+1}| = 1$. Then in particular, since $x_{n+1} \neq 0$,

$$[x_0 : \dots : x_{n+1}] = [x_0 : \dots : x_n : \sqrt{1 - \textstyle\sum_{i=0} |a_i|}]$$

Therefore $\mathbb{HP}^{n+1} \setminus U \cong B^{4(n+1)}$ via the map

$$[x_0 : \dots : x_n : \sqrt{1 - \textstyle\sum_{i=0}^{n} |a_i|}] \longmapsto (x_0, \dots, x_n) \in B^{4(n+1)}$$

where we note that $\sqrt{\sum_{i=0}^{n} |x_i|} < 1$ since $|x_{n+1}| > 0$ and x_i can be viewed as an element of $\mathbb{R}^4$ in the obvious way.
Finally, we observe that $\partial(\mathbb{HP}^{n+1} \setminus U) = U \cong \mathbb{HP}^n$.
Therefore, we can construct $\mathbb{HP}^{n+1}$ as $B^{4(n+1)}$ attached to $\mathbb{HP}^n$ via the map $\partial B^{4(n+1)} \cong S^{4n+3} \to \mathbb{HP}^n : (x_0, \dots, x_n) \to [x_0 : \dots : x_n]$ w/ $S^{4n+3} \subset (\mathbb{R}^4)^n$.

$\longrightarrow$

Iterating this construction, we find that $\mathbb{HP}^n$ consists of 1 $4k$-cell for each $k = 0, \ldots, n$. In particular, this implies

$$H_k(\mathbb{HP}^n) = \begin{cases} \mathbb{Z} & k = 0, 4, \ldots, 4n \\ 0 & \text{else} \end{cases}$$

which is what was to be found. $\square$

Algebraic Topology Prep
from Yani Notes

10F.9 If $n=0$ then $\mathbb{R}^n$ is compact $\Rightarrow$ $\mathbb{R}^m$ is compact $\Rightarrow$ $m=0=n$.

We recall that the one point compactification of $\mathbb{R}^n$ is diffeomorphic to S^n.

Suppose $\exists$ a homeomorphism $f: \mathbb{R}^n \to \mathbb{R}^m$. Then by the one point compactification, f extends to a homeomorphism

$$\tilde{f}: S^n \to S^m$$

In particular, $\tilde{f}$ induces an isomorphism $\tilde{f}^*: H_n(S^n) \to H_n(S^m)$.

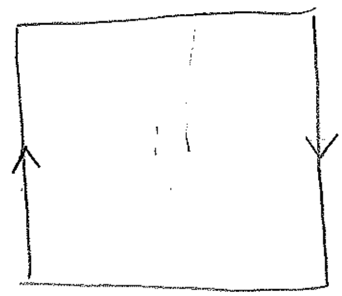
We recall that:

$$H_k(S^l) \cong \begin{cases} \mathbb{Z} & k=0,l \\ 0 & \text{else} \end{cases}$$

Therefore $\tilde{f}^*: \mathbb{Z} \to H_n(S^m)$ is an isomorphism, so $m=n$. $\square$

17F.6

We note that M is the Mobius strip and X is the Klein bottle.

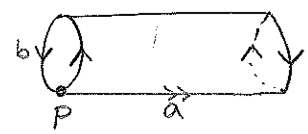


M

~~To calculate $\pi_1(X)$, we construct X as a CW complex.~~

We proceed by Van Kampen.

First, we note that M deformation retracts onto S^1 via the straight line homotopy $(x,y) \longmapsto (x, (1-t)y + t/2)$.



X

Therefore $\pi_1(M) \cong \pi_1(S^1) \cong \mathbb{Z}$.

Now let U be an open neighborhood of $M \times \{0\}$ that deformation retracts onto $M \times \{0\}$ and similarly let V be an open neighborhood of $M \times \{1\}$. Then $\pi_1(U) \cong \pi_1(V) \cong \pi_1(M) \cong \mathbb{Z}$, $U \cup V = X$, and $U \cap V$ deformation retracts onto ∂M.

Let $i_U : U \cap V \to U$ and $i_V : U \cap V \to V$ be the inclusion maps.

Then Van Kampen theorem implies that

$$\pi_1(X) \cong \pi_1(U) * \pi_1(V) / N$$

where N is generated by $i_U^*(\gamma)\, i_V^*(\gamma)^{-1}$ for $\gamma \in \pi_1(U \cap V)$

As found, $\pi_1(U) \cong \mathbb{Z}$, so let a be the generator of $\pi_1(U)$.

Since $U \cap V$ deformation retracts onto ∂M, any element of $\pi_1(U \cap V)$ is homotopic to c traversals of ∂M.

Then $i_U^*(c) = a^2$ and $i_V^*(c) = b^2$, since ∂M wraps twice around M. Therefore by Van Kampen,

$$\pi_1(X) \cong \langle a, b \mid a^2 b^{-2} \rangle$$

□

165.4

We first calculate $H_k(\partial M)$ in terms of $H_k(M)$.

Let M_1, M_2 be copies of M. Define $\tilde{M}$ by

$$\tilde{M} = M_1 \amalg \partial M \times [0,1] \amalg M_2 / \sim$$

where $x \sim (x,0)\ \forall\ x \in \partial M_1$ and $(y,0) \sim y\ \forall y \in \partial M_2$.

Let $U = M_1 \amalg \partial M \times [0,1) / \sim$ and $V = \partial M \times (0,1] \amalg M_2 / \sim$. Then U, V are open, deformation retract onto M_1, M_2 respectively.

$U \cup V = \tilde{M}$ and $U \cap V$ deformation retracts onto ∂M.

Mayer-Vietoris then yields a LES

$$\cdots \longrightarrow H_k(U \cap V) \to H_k(U) \oplus H_k(V) \to H_k(U \cup V) \to \cdots$$

which is equivalent to the LES

$$\cdots \to H_k(\partial M) \to H_k(M)^{\oplus 2} \to H_k(\tilde{M}) \to \cdots$$

Taking an alternating sum then yields

$$0 = \sum_{k=0}^{n} (-1)^k \left(\text{rank } H_k(\partial M) - 2\text{rank } H_k(M) + \text{rank } H_k(\tilde{M})\right)$$

$$= \chi(\partial M) - 2\chi(M) + \chi(\tilde{M})$$

It then remains to show that $\chi(\tilde{M}) = 0$. By construction, $\tilde{M}$ is a compact odd-dimensional w/o boundary.

By Poincare duality and the ~~unifor~~ universal coefficients theorem,

$$H_k(\tilde{M}; \mathbb{Z}/2\mathbb{Z}) \cong H^{n-k}(\tilde{M}; \mathbb{Z}/2\mathbb{Z})$$

B/c $\tilde{M}$ is orientable over $\mathbb{Z}/2\mathbb{Z}$, this implies

$$\chi(\tilde{M}) = \sum_{k=0}^{n} (-1)^k \text{ rank } \underset{\substack{\uparrow \\ \mathbb{Z}/2\mathbb{Z}}}{H_k(\tilde{M})} = -\sum_{k=0}^{n} (-1)^{n-k} \text{ rank } \underset{\substack{\uparrow \\ \mathbb{Z}/2\mathbb{Z}}}{H^{n-k}(\tilde{M})} = -\chi(\tilde{M})$$

and so $\chi(\tilde{M}) = 0$.

Thus

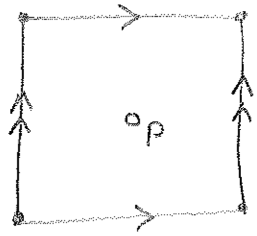
$$\chi(\partial M) = 2\chi(M)$$

as desired.

$\square$

105.6

(a) In the standard way, we may visualize the punctured torus as



From this, it can be seen that a radial straight line homotopy outwards from p will deformation retract $T^2 \setminus \{p\}$ onto $S^1 \vee S^1$. Then by deRham's theorem,

$$H^k_{dR}(T^2 \setminus \{p\}) \cong H_k(T^2 \setminus \{p\}; \mathbb{R}) \cong \begin{cases} \mathbb{R} & k=0 \\ \mathbb{R}^2 & k=1 \\ 0 & \text{else} \end{cases}$$

(b) By part (a), $H^k_{dR}(T^2 \setminus \{p\}) \cong 0$. Therefore since $dx \wedge dy$ is closed, it is exact. $\square$

16.5.7

We recall that the fundamental group of a CW complex w/ 1 0-cell, k 1-cells $e_1, \dots, e_k$, and l 2-cells attached via $a_1, \dots, a_l$ has fundamental group $\langle e_1, \dots, e_k \mid a_1, \dots, a_l \rangle$.

Therefore, we construct X as

1 0-cell: p

2 1-cells: a, b w/ $\partial a = \partial b = p - p = 0$

2 2-cells: f, g w/ $\partial f = ma$, $\partial g = nb$.

Then $\pi_1(X) \cong \langle a, b \mid a^m, b^n \rangle \cong \mathbb{Z}/m\mathbb{Z} * \mathbb{Z}/n\mathbb{Z}$

Similarly, we construct Y as

1 0-cell: p

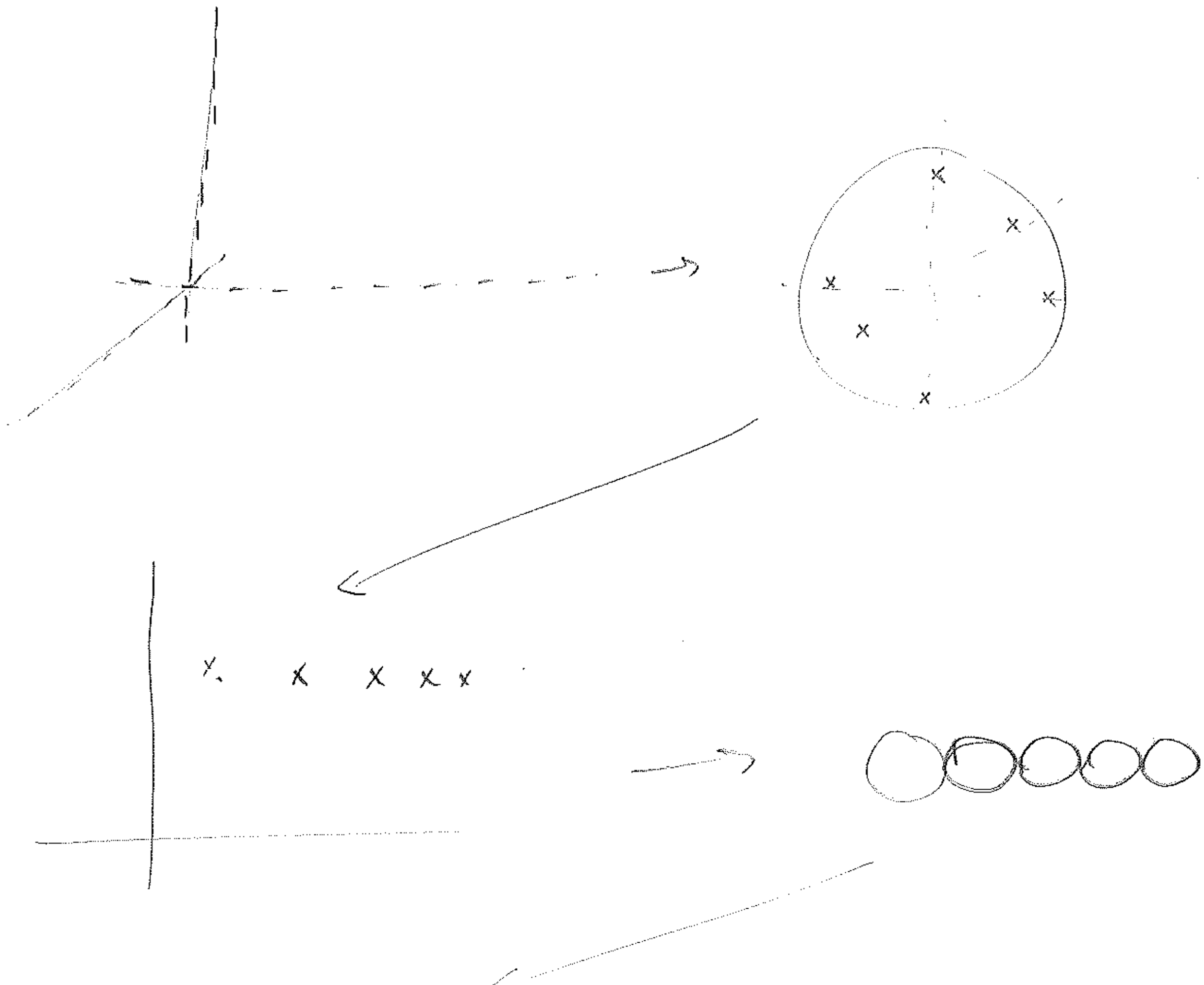
2 1-cells: a, b w/ $\partial a = \partial b = p - p = 0$

3 2-cells: f, g, h w/ $\partial f = ma$, $\partial g = nb$, $\partial h = a + b - a - b$

Then $\pi_1(Y) \cong \langle a, b \mid a^m, b^n, aba^{-1}b^{-1} \rangle \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.

As desired. $\square$

165.8



$$\pi_1\left(\bigvee_{k=1}^{5} S^1\right) \cong \mathbb{Z}*\mathbb{Z}*\mathbb{Z}*\mathbb{Z}*\mathbb{Z}.$$

14S.3/19F.3

If n is even, this violates degree.

If n odd, construct homotopy via $\frac{n+1}{2}$ rotations.

19F.8

(a) Universal cover of $S^1 \times S^1$ is $\mathbb{R}^2$ which is simply connected.
Any map $\mathbb{R}P^2 \to S^1 \times S^1$ maps $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$ to a finite subgroup of $\mathbb{Z}^2 \Rightarrow 0$ subgroup.
Therefore any map lifts. Then construct a homotopy in $\mathbb{R}^2$ then descend, QED.

(b) $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$ so $\exists$ a non-nullhomotopic path $\gamma: S^1 \to \mathbb{R}P^2$.
Define $f: S^1 \times S^1 \to \mathbb{R}P^2: (\theta, \phi) \to \gamma(\theta)$.

16F.7

Repeat $\mathbb{R}P^2 \to S^1 \times S^1$ argument.

16F. 9

As given, we ~~construct~~ X as follows

$$\left.\begin{array}{l} \left.\begin{array}{lll} 1 & \text{0-cell}: & p \\ 2 & \text{2-cells}: a, b & \text{w/ } \partial_2 a = \partial_2 b = p \end{array}\right\} S^2 \vee S^2 \\ \begin{array}{lll} 1 & \text{3-cell}: D & \text{w/ } \partial_3 D \text{ attached via } f \\ & & \Rightarrow \partial_3 D = d_1 a + d_2 b. \end{array} \end{array}\right\} S^2 \vee S^2 \cup_f D^3$$

This gives the chain complex

$$\begin{array}{ccccccccc} 0 \to & C_3(X) & \xrightarrow{\partial_3} & C_2(X) & \xrightarrow{\partial_2} & C_1(X) & \xrightarrow{\partial_1} & C_0(X) & \xrightarrow{\partial_0} 0 \\ & \cong & & \cong & & \cong & & \cong & \\ & \mathbb{Z}\langle D\rangle & & \mathbb{Z}\langle a,b\rangle & & 0 & & \mathbb{Z}\langle p\rangle & \end{array}$$

By definition, this yields the homology groups

$$H_0(X) = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \mathbb{Z}\langle p\rangle \cong \mathbb{Z}$$

$$H_1(X) = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} \cong 0$$

$$H_2(X) = \frac{\ker \partial_2}{\operatorname{Im} \partial_3} = \frac{\mathbb{Z}\langle a,b\rangle}{\mathbb{Z}\langle d_1 a + d_2 b\rangle} \cong \begin{cases} \mathbb{Z} \times \mathbb{Z}/\gcd(d_1,d_2)\mathbb{Z} & \text{else} \\ \mathbb{Z}^2 & \text{if } d_1 = d_2 = 0 \end{cases}$$

$$H_3(X) = \frac{\ker \partial_3}{\operatorname{Im} \partial_2} = 0$$

and all higher homology groups are 0.

TO STUDY

1. Distributions
2. Mapping torus

17.5.7

By the LES for relative homology, since (S^1, D^2) is a good pair,

$$\cdots \longrightarrow H_k(\partial X) \to H_k(X) \to H_k(X, \partial X) \to \cdots$$

is a LES. We recall that

$$H_k(S^1 \times S^1) \cong H_k(T^2) \cong \begin{cases} \mathbb{Z} & k = 0,2 \\ \mathbb{Z}^2 & k = 1 \\ 0 & \text{else} \end{cases}$$

To find the homology of X, we note that D^2 is contractible and hence X deformation retracts onto S^1. Therefore

$$H_k(X) \cong H_k(S^1) \cong \begin{cases} \mathbb{Z} & k = 0,1 \\ 0 & \text{else} \end{cases}$$

Therefore we have the following LES

$$0 \to H_3(X, \partial X) \hookrightarrow \mathbb{Z} \to 0 \to H_2(X, \partial X) \hookrightarrow$$

$$\hookrightarrow \mathbb{Z}^2 \to \mathbb{Z} \to H_1(X, \partial X) \to$$

$$\to \mathbb{Z} \to \mathbb{Z} \twoheadrightarrow H_0(X, \partial X) \to 0$$

Therefore $H_3(X, \partial X) \cong \mathbb{Z}$. The map $H_0(\partial X) \to H_0(X)$ is an isomorphism and so $H_0(X, \partial X) \cong 0$ by exactness. It thus remains to calculate $H_2(X, \partial X)$, $H_1(X, \partial X)$ since $H_k(X, \partial X) = 0$ $\forall k > 3$.
For $H_2(X, \partial X)$ and $H_1(X, \partial X)$ we have the SES,

$$0 \to H_2(X, \partial X) \xrightarrow{e_*} \mathbb{Z}^2 \xrightarrow{f_*} \mathbb{Z} \xrightarrow{g_*} H_1(X, \partial X) \to 0.$$

Consider the map $\mathbb{Z}^2 \to \mathbb{Z}$. This was induced by the inclusion $i: T^2 = \partial X \hookrightarrow X = S^1 \times D^2$ and the deformation retraction $S^1 \times D^2 \to S^1$. Therefore, on first homology, $\mathbb{Z}^2 \xrightarrow{f_*} \mathbb{Z}$ takes $(a, b) \to a$ since the second 1-cycle is squashed by the retraction.

$\longrightarrow$

Therefore $\operatorname{im} f_* \cong \mathbb{Z}$ and $\ker f_* \cong \mathbb{Z}$.
By exactness, this implies $\operatorname{im} e_* \cong \mathbb{Z}$.
Since e_* is injective this implies $H_1(X, \partial X) \cong \mathbb{Z}$.
Similarly, $\ker g_* \cong \mathbb{Z}$ and so $H_1(Y, \partial Y) \cong 0$ since g_* is surjective by exactness.

$\square$

175.9 / 115.8

(a) we construct $\mathbb{R}P^n$ via

$$\begin{aligned}
&1 \text{ 0-cell}: \; p \\
&1 \text{ 1-cell}: \; e_1 \text{ w/ } \partial_1 e_1 = p - p = 0 \\
&1 \text{ 2-cell}: \; e_2 \text{ w/ } \partial_2 e_2 = 2e_1 \\
&\quad \vdots \\
&1 \text{ k-cell}: \; e_k \text{ w/ } \partial_k e_k = e_{k-1} + (-1)^k e_{k-1} \\
&\quad \vdots \\
&1 \text{ n-cell}: \; e_n \text{ w/ } \partial_n e_n = e_{n-1} + (-1)^n e_{n-1}
\end{aligned}$$

S^n to $\mathbb{R}P^n$!
via
$(x_0, \dots, x_n) \mapsto [x_0 : \dots : x_n]$

Then $\quad H_0(\mathbb{R}P^n) = \dfrac{\ker \partial_0}{\operatorname{Im} \partial_1} = \dfrac{\mathbb{Z}\langle p \rangle}{0} \cong \mathbb{Z}$

For $0 < k < n$, we have two cases. If k is even then

$$H_k(\mathbb{R}P^n) = \frac{\ker \partial_k}{\operatorname{Im} \partial_{k+1}} = 0$$

If k is odd, then

$$H_k(\mathbb{R}P^n) = \frac{\ker \partial_k}{\operatorname{Im} \partial_{k+1}} = \frac{\mathbb{Z}\langle e_k \rangle}{\mathbb{Z}\langle 2e_k \rangle} \cong \mathbb{Z}/2\mathbb{Z}$$

Finally, for $k = n$,

$$H_n(\mathbb{R}P^n) = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}} = \ker \partial_n \cong \begin{cases} \mathbb{Z} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

As usual, $H_k(\mathbb{R}P^n) = 0$ for $k > n$.

(b)

(b) As above, we construct $\mathbb{RP}^2$ via

1 0-cell: p

1 1-cell: e w/ $\partial e = p - p = 0$

1 2-cell: f w/ $\partial f = 2e$.

Then we can give $\mathbb{RP}^2 \times \mathbb{RP}^2$ a corresponding CW complex via

1 0-cell: (p,p)

2 1-cells: $(e,p), (p,e)$ w/ $\partial_1(e,p) = 0$, $\partial_1(p,e) = 0$

3 2-cells: $(e,e), (f,p), (p,f)$ w/ $\partial_2(e,e) = 0$
$\partial_2(f,p) = 2(e,p)$
$\partial_2(p,f) = +2(p,e)$

2 3-cells: $(f,e), (e,f)$ w/ $\partial_3(f,e) = 2(e,e)$
$\partial_3(e,f) = -2(e,e)$

1 4-cell: (f,f) w/ $\partial_4(f,f) = 2(e,f) + 2(f,e)$

This yields the homology groups w/ $X = \mathbb{RP}^2 \times \mathbb{RP}^2$

$$H_0(X) = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \frac{\mathbb{Z}\langle (p,p) \rangle}{0} \cong \mathbb{Z}$$

$$H_1(X) = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\mathbb{Z}\langle (e,p), (p,e) \rangle}{\mathbb{Z}\langle 2(e,p), +2(p,e) \rangle} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z}$$

$$H_2(X) = \frac{\ker \partial_2}{\operatorname{Im} \partial_3} = \frac{\mathbb{Z}\langle (e,e), (f,p) - (p,f) \rangle}{\mathbb{Z}\langle 2(e,e), -2(e,e) \rangle} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$$

$$H_3(X) = \frac{\ker \partial_3}{\operatorname{Im} \partial_4} = \frac{\mathbb{Z}\langle (f,e) + (e,f) \rangle}{\mathbb{Z}\langle 2(f,e) + 2(e,f) \rangle} \cong \mathbb{Z}/2\mathbb{Z}$$

$$H_4(X) = \ker \partial_4 \cong 0$$

In particular $H_3(X) \neq 0$. $\square$

18F.6

Suppose that $G = \langle a_1, \ldots, a_n \rangle$ is a finite rank free group.

We recall that G can be realized as the fundamental group of the wedge of k copies of S^1, denoted X

Suppose ~~for the sake of contradiction~~ that $\exists$ a finite index subgroup ~~of smaller rank,~~ $H \subset G$. Suppose H has index r.

Then, corresponding to $H \subset G$, $\exists$ an r-sheeted connected covering space $\tilde{X}$ of X.

Since X is the wedge of k ~~tot~~ copies of S^1,

$\tilde{X}$ can be realized as a graph w/ r vertices and rk edges.

Then $\tilde{X}$ has a spanning tree consisting of $k-1$ edges.

Contracting along this spanning tree, we see that $\tilde{X}$ deformation retracts to the wedge of $nk-k+1$ copies of S^1.

Therefore $\pi_1(\tilde{X})$ is the free group on $nk-k+1$ generators.

In particular, this implies that the rank of H is

$$nk - k + 1 \geq n$$

Therefore $\nexists$ a finite index subgroup of smaller rank.

18F.9

We proceed by Mayer-Vietoris.

We recall that ΣX is defined as

$$X \times [0,1]/\sim$$

where $(x,0)\sim(y,0)$ and $(x,1)\sim(y,1)$ $\forall\, x,y \in X$.

Define $U = X\times[0,1)/\sim$ and $V = X\times(0,1]/\sim$. Then U, V are contractible, $U\cap V$ deformation retracts onto X, and $U\cup V = X$.
By the Mayer-Vietoris sequence for reduced homology, this yields a LES

$$\cdots \to \tilde{H}_k(X;M) \to \tilde{H}_k(U;M)\oplus \tilde{H}_k(V;M) \to \tilde{H}_k(\Sigma X;M) \to \cdots$$

Since U, V are contractible, $\tilde{H}_k(U) \cong \tilde{H}_k(V) \cong 0$ $\forall\, k$.
Therefore $\forall\, k\geq 1$ we acquire the SES

$$0 \to \tilde{H}_k(\Sigma X;M) \to \tilde{H}_{k-1}(X;M) \to 0$$

which implies $\tilde{H}_k(\Sigma X;M) \cong \tilde{H}_{k-1}(X;M)$ as desired.
For $k=0$, we have the SES

$$0 \to \tilde{H}_0(\Sigma X;M) \to 0$$

which implies $\tilde{H}_0(\Sigma X;M) \cong 0 = \tilde{H}_{-1}(X;M)$ by convention. $\square$

This should all be in terms of $\mathbb{Z}$
Then universal coefficients theorem gives M

1SF.10

(a) Let U, V be an ε-neighborhoods of $Y, S^1 \subset Y \vee S^1$ that deformation retract onto Y, S^1 respectively. Then $U \cap V$ is simply connected and so Van Kampen's implies that

$$\pi_1(Y \vee S^1) \cong \pi_1(Y) * \pi_1(S^1) \cong \mathbb{Z}$$

Since $\pi_1(S^1) \cong \mathbb{Z}$ and Y is simply connected.

(b) Since Y is simply connected, it is its own universal cover.

We recall that the universal cover of S^1 is $\mathbb{R}$ w/ each interval ~~then~~ $[n, n+1]$ mapped to S^1.

Then the universal cover of $Y \vee S^1$ will be $\mathbb{R}$ w/ a copy of S^1 wedged to each integer.

Since the deck transformations of S^1 on $\mathbb{R}$ are integer translations, it follows that the deck transformations are integer translations for $Y \vee S^1$.

(c) As in part (b), since Y has no connected multi-sheeted covers, it suffices to consider the k-sheeted cover of S^1.

We recall that the k-sheeted covers of S^1 are

$$[0, k] / 0 \sim k$$

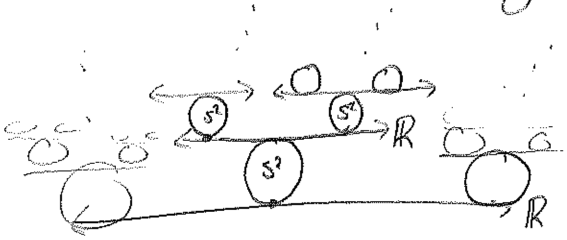
where each interval $[n, n+1]$ is mapped to S^1.

Then the k-sheeted covers of $Y \vee S^1$ are $[0, k] / 0 \sim k$ w/ a copy of Y wedged at each integer.

$\longrightarrow$

(d) Suppose now that $Y = \mathbb{RP}^2$.
Then Y is no longer simply connected, so $\pi_1(Y) \not\cong 0$.
By recalling that $\mathbb{RP}^2$ is ~~constructed~~ via
$$\begin{aligned} &1 \text{ 0-cell}: p \\ &1 \text{ 1-cell}: e \text{ w/ } \partial e = p - p = 0 \\ &1 \text{ 2-cell}: f \text{ w/ } \partial f = 2e \end{aligned}$$
it follows that $\pi_1(\mathbb{RP}^2) \cong \langle e \mid e^2 \rangle \cong \mathbb{Z}/2\mathbb{Z}$
Therefore $\pi_1(Y \vee S^1) \cong \mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$

Since $\mathbb{RP}^2$ is not simply connected, it is not its own universal cover. We recall that S^2 is the universal cover of $\mathbb{RP}^2$ via the ~~quotient~~ map $S^2 \to S^2/_{x \sim -x} \cong \mathbb{RP}^2$.
Then, the universal cover of $Y \vee S^1$ will be the universal cover of S^1, $\mathbb{R}$, w/ a universal cover of Y, S^2, wedged at each integer. Then every copy of S^2 will have a copy of $\mathbb{R}$ at the north and south pole. This process then ~~iterates~~ infinitely.
The resulting structure is then of the form



though this is a rough sketch. $\mathbb{Z}$
The deck transformation act via translation in $\mathbb{R}$ or reflections in S^2, corresponding to $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ respectively.

15.S.4

(a) Suppose n is even and $F: \mathbb{RP}^n \to \mathbb{RP}^n$ is smooth.

To show F has a fixed point, it suffices to show that $L(F) \neq 0$.

By the Lefschetz trace formula,

$$L(F) = \sum_{k=0}^{n} (-1)^k \operatorname{tr}(F^* : H^k(\mathbb{RP}^n; \mathbb{Q}) \to H^k(\mathbb{RP}^n; \mathbb{Q}))$$

We recall that for n even

$$H_k(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}/2\mathbb{Z} & k = 1, 3, \dots, n-1 \\ 0 & \text{else} \end{cases}$$

Therefore by the universal coefficients theorem, since $\mathbb{Q}$ is a field,

$$H^k(\mathbb{RP}^n; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & k=0 \\ 0 & \text{else} \end{cases}$$

Then
$$L(F) = \operatorname{tr}(F^* : H^0(\mathbb{RP}^n; \mathbb{Q}) \to H^0(\mathbb{RP}^n; \mathbb{Q}))$$
$$= \operatorname{tr}(F^* : \mathbb{Q} \to \mathbb{Q})$$

Since F^* is a cohomology ring homomorphism, $F^*(1) = 1$ so

$F^* : H^0(\mathbb{RP}^n; \mathbb{Q}) \to H^0(\mathbb{RP}^n; \mathbb{Q})$ is the identity. Then $L(F) = 1 \neq 0$

so F has a fixed point.

(b) View $\mathbb{RP}^{2k-1}$ as the quotient of $S^{2k-1} \subset \mathbb{C}^k$ under the anti-podal identification. On S^{2k-1}, we define $\tilde{F}: S^{2k-1} \to S^{2k-1}$ via $z \mapsto iz$. Then $\tilde{F}$ has no fixed points. Moreover,

$$\tilde{F}(-z) = -iz = -\tilde{F}(z)$$

and so $\tilde{F}$ factors to a map $F: \mathbb{RP}^{2k-1} \to \mathbb{RP}^{2k-1}$.

If $F[z] = [z]$ then $[iz] = [z] \Rightarrow iz = \pm z$ which is impossible.

Therefore F has no fixed points.

$\square$

155.8

(a) $\pi_1(X) \cong \langle a, b \mid ab^2, ba^2 \rangle$

$ab^2 = e \Rightarrow a = b^{-2}$

$\Rightarrow \pi_1(X) \cong \langle b \mid b^{-3} \rangle \cong \mathbb{Z}/3\mathbb{Z}$.

which is finite.

(b) $H_0(X) = \dfrac{\ker \partial_0}{\operatorname{Im} \partial_1} = \dfrac{\mathbb{Z}\langle p \rangle}{0} \cong \mathbb{Z}$

$H_1(X) = \dfrac{\ker \partial_1}{\operatorname{Im} \partial_2} = \dfrac{\mathbb{Z}\langle a, b \rangle}{\mathbb{Z}\langle a+2b, b+2a \rangle} = \dfrac{\mathbb{Z}\langle a \rangle}{\mathbb{Z}\langle -3a \rangle} \cong \mathbb{Z}/3\mathbb{Z}$

$H_2(X) = \dfrac{\ker \partial_2}{\operatorname{Im} \partial_3} \cong 0$

$H_k(X) \cong 0 \quad \forall \ k \geq 2.$

GeoTop
Fall 2022

(2) 405 846 515

Let M denote the space of $n \times n$ matrices and let S denote the space of $n \times n$ symmetric matrices. We note then that $\dim M = n^2$, $\dim S = n + (n-1) + \dots + 1 = \frac{n(n+1)}{2}$. Furthermore we recall that $T_A M \cong M$ and $T_B S \cong S$ $\forall A \in M, B \in S$.

Define $f : M \to S : A \mapsto A^T A$. Then $O(n) = f^{-1}(Id)$ by definition. We claim that Id is a regular value of f. To show this, it suffices to show that $\forall A \in O(n)$, $dF_A : T_A M \to T_{f(A)} S$ is surjective. As $T_A M \cong M$ and $T_{f(A)} S \cong S$, it suffices to view $dF_A : M \to S$.

Consider some $C \in S$. Let $B = \frac{1}{2} AC$. Then by direct computation

$$\begin{aligned}
dF_A(B) &= \lim_{t \to 0} \frac{(A+tB)^T(A+tB) - A^T A}{t} \\
&= \lim_{t \to 0} \frac{A^T A + tB^T A + tA^T B + t^2 B^T B - A^T A}{t} \\
&= B^T A + A^T B \\
&= \frac{1}{2}\left(C^T A^T A + A^T A C\right) \\
&= \frac{1}{2} C^T + \frac{1}{2} C = C \quad (\text{since } C \in S)
\end{aligned}$$

Therefore dF_A is surjective $\forall A \in O(n)$. This implies that $O(n)$ is a smooth submanifold of dimension

$$n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$$

as desired.

We recall that a tangent bundle TM is trivializable iff $\exists$ $\dim M$ linearly independent non-vanishing vector fields. Therefore, we construct $\frac{n(n-1)}{2}$ non-vanishing linearly independent vector fields on $O(n)$.

We note that $O(n)$ is a Lie group. Therefore for ease of notations we consider a Lie group G of dimension $m = \frac{n(n-1)}{2}$.

Let e denote the identity in G. Consider T_eG. Since G is of dimension m, $\exists$ vectors $v_1, \dots, v_m \in T_eG$ s.t. $v_1, \dots, v_m$ are linearly independent and nonzero.

For each $g \in G$, let $m_g : G \to G : h \mapsto gh$. Since G is abelian, m_g is a diffeomorphism and so $dm_g^* : T_hG \mapsto T_{gh}G$ is a diffeomorphism.

Define $X_1, \dots, X_m$ on G via

$$X_j|_g = dm_g^*(v_j)$$

Then $X_1, \dots, X_m$ are linearly independent and non-vanishing b/c dm_g^* is a diffeomorphism. Moreover, since m_g is smooth in g, $X_1, \dots, X_m$ are smooth.

Therefore TG is trivializable and so $TO(n)$ is trivializable. $\square$

③ 405 846 515

If $k=0$ then a constant map works. Therefore assume $k\neq 0$.

First suppose that $k>0$.

Let $B_1,\dots,B_k$ be k disjoint open balls in M. Then M continuously maps to the wedge of k copies of S^n via the quotient map

$$\pi : M \to M/(M\setminus(B_1\cup\dots\cup B_k)) \cong \bigvee_{j=1}^{k} S^n$$

This follows since $\overline{B_j}/\partial B_j \cong S^n$ $\forall j$.

Define $g: \bigvee_{j=1}^{k} S^n \to S^n$ s.t. $g|_{S^n} = r\circ id$ for each copy of S^n where r is a reflection chosen that $f = g\circ\pi$ is orientation preserving on B_j for all j.

Then $\forall$ $p\in S^n$ not at the wedge point, $\# f^{-1}(p) = k$. Since f is orientation preserving on $f^{-1}(p)$ this implies that $\deg f = k$.

Now suppose $k<0$. Repeating the above argument w/ $|k|$ and then composing w/ an orientation reversing reflection yields a map f w/ deg $-|k| = k$, as desired. $\square$

④ 405 346 515

We note that since ω is non-vanishing, $\operatorname{rank}(\omega_p) = 1$ $\forall p$ consistently and so $\dim\ker(\omega_p) = n-1$ $\forall p$. Therefore $\ker(\omega)$ is consistently of dimension $n-1$ and hence is a smooth distribution.

(3=>2) Suppose $d\omega = \kappa \wedge \omega$. Then

$$\omega \wedge d\omega = \omega \wedge \kappa \wedge \omega$$

Since κ, ω are 1-forms,

$$\begin{aligned} \omega \wedge \kappa \wedge \omega &= -\kappa \wedge \omega \wedge \omega \\ &= \kappa \wedge \omega \wedge \omega \\ &= -\omega \wedge \kappa \wedge \omega \end{aligned}$$

and so $\omega \wedge d\omega = \omega \wedge \kappa \wedge \omega = 0$.

(2=>1) Suppose $\omega \wedge d\omega = 0$. We recall that by Frobenius' theorem, $\ker(\omega)$ is an integrable distribution iff $\forall$ vector fields $X, Y \in \ker(\omega)$, $[X, Y] \in \ker(\omega)$.

Suppose $\exists$ $X, Y \in \ker(\omega)$. Let $\exists$ $V_p \in T_pM \setminus \ker(\omega)$.

Then since $\omega(X) = \omega(Y) = 0$

$$0 = \omega \wedge d\omega(X_p, Y_p, V_p) = \omega(V_p)\, d\omega(X_p, Y_p)$$

and so $d\omega(X_p, Y_p) = 0$ $\forall p$. Then $d\omega(X, Y) = 0$.

Since $X, Y \in \ker(\omega)$,

$$0 = d\omega(X, Y) = X(Y(\omega)) - Y(X(\omega)) - \omega[X, Y] = -\omega[X, Y]$$

Therefore $[X, Y] \in \ker(\omega)$. As this holds $\forall$ $X, Y \in \ker(\omega)$, $\ker(\omega)$ is an integrable distribution.

(1 => 3) Suppose $\ker(\omega)$ is an integrable distribution.
We aim to define κ s.t. $d\omega = \kappa \wedge \omega$. To do so, it suffices to work locally since we may then extend globally via a partition of unity.
Since $\ker(\omega)$ is integrable, locally $\exists$ coordinates s.t.

$$\ker(\omega) = \mathbb{R}\langle \partial/\partial x_1, \ldots, \partial/\partial x_{n-1} \rangle$$

Then $\omega = f\,dx_n$ for some non-vanishing f.
Then $\kappa = \frac{df}{f}$ is well-defined and satisfies

$$\kappa \wedge \omega = df \wedge dx_n = d\omega$$

as desired. $\square$

⑤ 405 846 515

<u>Lemma</u>: Suppose that M^{2n} is a compact orientable manifold w/o boundary which admits a symplectic form. Then $H^{2k}_{dR}(M) \neq 0$ $\forall$ $k = 1, \dots, n$.

<u>proof</u>: Let ω denote ~~the~~ a symplectic form on M. Then ω is closed and so $\underbrace{\omega\wedge\dots\wedge\omega}_{k}$ is closed $\forall$ $k = 1, \dots, n$.

Suppose $\exists$ k s.t. $\underbrace{\omega\wedge\dots\wedge\omega}_{k} = d\eta$ for some η. Then

$$\begin{aligned}\underbrace{\omega\wedge\dots\wedge\omega}_{n} &= \underbrace{\omega\wedge\dots\wedge\omega}_{k}\wedge\underbrace{\omega\wedge\dots\wedge\omega}_{n-k} \\ &= d\eta\wedge\underbrace{\omega\wedge\dots\wedge\omega}_{n-k} \\ &= d(\eta\wedge\omega\wedge\dots\wedge\omega)\end{aligned}$$

Therefore $\omega\wedge\dots\wedge\omega$ is exact. By Stokes' theorem, this implies that

$$\int_M \omega\wedge\dots\wedge\omega = \int_{\partial M} \eta\wedge\omega\wedge\dots\wedge\omega = \int_{\emptyset} \eta\wedge\omega\wedge\dots\wedge\omega = 0$$

However, this contradicts the fact that $\omega\wedge\dots\wedge\omega$ is a non-vanishing volume form. Therefore no such k can exist. Then $\underbrace{\omega\wedge\dots\wedge\omega}_{k}$ is closed but not exact $\forall$ k, so $H^{2k}_{dR}(M) \neq 0$.

We recall that

$$H^k_{dR}(S^p) \cong \begin{cases} \mathbb{R} & k = 0, p \\ 0 & \text{else} \end{cases}$$

Therefore by Künneth's formula

$$H^k_{dR}(S^p \times S^l) \cong \begin{cases} \mathbb{R} & k = 0 \\ \mathbb{R} & k = p \neq l \\ \mathbb{R} & k = l \neq p \\ \mathbb{R}^2 & k = l = p \\ \mathbb{R} & k = l + p \end{cases}$$

WLOG suppose $p \geq l$. If $p + l$ is not even, then $S^p \times S^l$ cannot admit a symplectic form. $\longrightarrow$

By the lemma, this implies that if $S^p \times S^l$ admits a symplectic form, then $\{2, 4, \dots, p+l\} \subset \{p, l, p+l\}$.
This implies that

$$(p, l) \in \{(1,1), (2,2), (4,2)\}$$

We check each of these cases.

① $\underline{(p, l) = (1,1)}$

In this case $S^p \times S^l \cong T^2$. Let $w = dx \wedge dy$ be the ~~ch~~ canonical volume form on T^2. Then w is a symplectic form on $S^p \times S^l$.

② $\underline{(p, l) = (2,2)}$

Let η be the volume form on S^2.
Let η_1, η_2 be the 2-forms on $S^2 \times S^2$ corresponding to η in the first and second ~~coords~~ copy of S^2.
Then $w = \eta_1 + \eta_2$ satisfies $w \wedge w = 2\eta_1 \wedge \eta_2$. Since η_1, η_2 are linearly independent, $w \wedge w$ is non-vanishing and hence w is a symplectic form.

③ $\underline{(p, l) = (4,2)}$:

We claim that $S^4 \times S^2$ does not admit a symplectic form. *
By the Künneth formula, $H^2_{dR}(S^4 \times S^2)$ consists of forms $f\kappa$ where f is a 0-form on S^4 and κ is a 2-form on S^2. Then $f\kappa \wedge f\kappa = f^2 \kappa \wedge \kappa = 0$, and in particular $f\kappa \wedge f\kappa \wedge f\kappa = 0$. Therefore no symplectic form on $S^4 \times S^2$ exists.

To summarize, $S^p \times S^l$ admits a symplectic form iff

$$(p, l) = (1,1), (2,2)$$

$\square$

---

* We note that by the reasoning in the lemma, a symplectic form must be non-exact.

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(a) For ease of notation, let

$$0 \to \mathbb{Z}/p \xrightarrow{f} \mathbb{Z}/p^2 \xrightarrow{g} \mathbb{Z}/p \to 0 \qquad (1)$$

where f is multiplication by p and g is the quotient map.
Then $\forall\, a \in \mathbb{Z}/p$, $g(f(a)) = g(pa) = 0$. Then $g \circ f = 0$ and so
(1) is a SES.
Given ... $A_* = C_* \otimes \mathbb{Z}/p$ and $B_* = C_* \otimes \mathbb{Z}/p^2$, this gives a sequence
of chain complexes

$$0 \to A_* \xrightarrow{id \otimes f} B_* \xrightarrow{id \otimes g} A_* \to 0$$

This is a SES since $id \otimes g \circ id \otimes f = id \otimes (g \circ f) = 0$.

(b) We note that this is precisely the snake lemma.
Let $\partial_k^A, \partial_k^A$ denote the boundary maps on A_k, B_k respectively.
We aim to construct a map $\ker \partial_k^A \to \ker \partial_{k-1}^{A}$ that is well-defined
up to homology. This will then define a map $\beta_k: H_k(A_*) \to H_{k-1}(A_*)$

Define i_k, j_k s.t.

$$0 \to A_k \xrightarrow{i_k} B_k \xrightarrow{j_k} A_k \to 0 \qquad (2)$$

for ease of notation.

Fix some $a_k \in A_k$ s.t. $\partial_k^A(a_k) = 0$. Since (2) is a SES,
$\exists$ some b_k s.t. $j_k(b_k) = a_k$. We will show later that the
resulting map is independent of this choice of b_k.
Consider $\partial_k^B(b_k) \in B_k$. B/c our diagram commutes,

$$j_{k-1}(\partial_k^B(b_k)) = \partial_k^A(j_k(b_k)) = \partial_k^A(a_k) = 0$$

Therefore since (2) is exact, $\exists\, a_{k-1} \in A_{k-1}$ s.t.

$$i_{k-1}(a_{k-1}) = \partial_k^B(b_k)$$

We define $\beta_k[a_k] := [a_{k-1}]$. To show that β_k is well-defined on
homology, it must be shown that β_k is independent of the
choice of b_k, a_{k-1} on the level of homology. $\longrightarrow$

We first show the map is independent of the choice of a_{k-1}. Suppose $\exists$ a_{k-1}, a'_{k-1} s.t.

$$i_{k-1}(a_{k-1}) = i_{k-1}(a'_{k-1}) = \partial_k^B(b_k)$$

Then $i_{k-1}(a_{k-1} - a'_{k-1}) = 0$. Since (2) is a SES, this implies that $a_{k-1} - a'_{k-1} = 0 \Rightarrow a_{k-1} = a'_{k-1}$. Therefore $\exists!$ a_{k-1} s.t. $i_{k-1}(a_{k-1}) = \partial_k^B(b_k)$.

We now show the map is independent of the choice of b_k. Suppose $\exists$ b_k, b'_k s.t. $j_k(b_k) = j_k(b'_k) = a_k$.

Then $j_k(b_k - b'_k) = 0$ and so $\exists$ $\tilde{a}_k \in A_k$ s.t. $i_k(\tilde{a}_k) = b_k - b'_k$ since (2) is exact. Therefore, since our diagram commutes

$$\begin{aligned} \partial_k^B(b_k - b'_k) &= \partial_k^B(i_k(\tilde{a}_k)) \\ &= i_{k-1}(\partial_k^A(\tilde{a}_k)) \end{aligned}$$

Let $\exists$ $a_{k-1}, a'_{k-1} \in A_{k-1}$ s.t. $i_{k-1}(a_{k-1}) = \partial_k^B(b_k)$ and $i_{k-1}(a'_{k-1}) = \partial_k^B(b'_k)$.

Then

$$\begin{aligned} i_{k-1}(a_{k-1} - a'_{k-1}) &= \partial_k^B(b_k - b'_k) \\ &= i_{k-1}(\partial_k^A(\tilde{a}_k)) \end{aligned}$$

Since i_{k-1} is injective, this implies $a_{k-1} - a'_{k-1} = \partial_k^A(\tilde{a}_k)$.

Therefore a_{k-1} and a'_{k-1} are the same on the level of homology, and so the choice of b_k is irrelevant.

Therefore $B_k : H_k(A_*) \to H_k(A_*)$ is well-defined.

(c) Suppose $\exists$ x, y s.t. $d(x) = py$.
Here we view $x \in C_k \otimes \mathbb{Z}/p$ as $x \otimes 1$ and $y \in C_{k-1} \otimes \mathbb{Z}/p$ as $y \otimes 1$.
Let b_k, a_{k-1} be for x as they are in part (b).
We aim to show that a_{k-1} and y are the same on the level of homology.
By definition,

$$\begin{aligned} i_{k-1}(a_{k-1} - y) &= \partial_k^B(b_k) - i_{k-1}(y) \\ &= \partial_k^B(b_k) - py \\ &= \partial_k^B(b_k) - dx \bmod p^2 \\ &= \partial_k^B(b_k - x \bmod p^2) \end{aligned}$$

By definition, $j_k(b_k - x \bmod p^2) = x - x = 0$. Therefore y, a_{k-1} are 2 different choices for b_k in the definition of B.
As shown before, this implies that $a_{k-1} = y$ on the level of homology and so $B[x] = [a_{k-1}] = [y]$ as desired.

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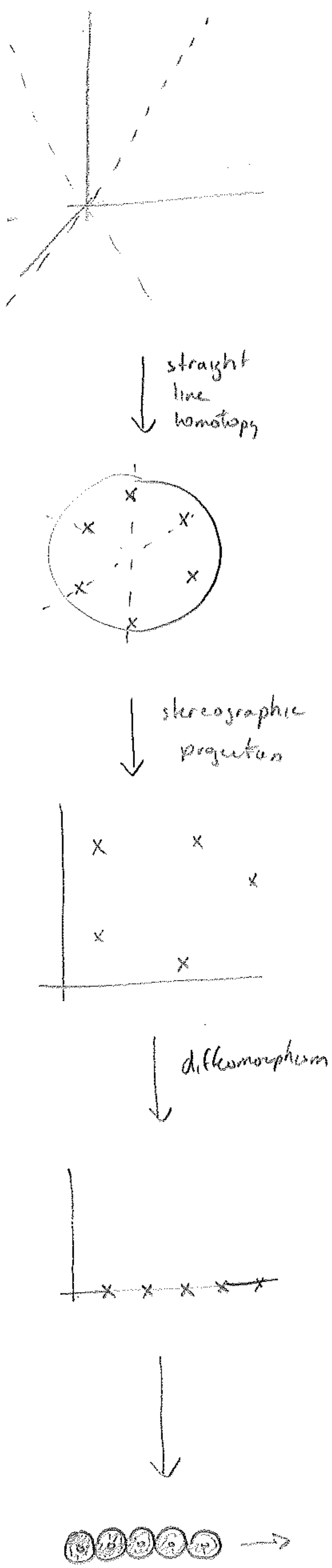
If $n=0$ then $H=\emptyset$ so, $\pi_1(\mathbb{R}^3 \setminus H) = \pi_1(\mathbb{R}^3) \cong 0$.

Assume $n>0$ and let $X = \mathbb{R}^3 \setminus H$.
We claim that X is homotopy equivalent to the wedge of $2n-1$ circles.



First, since H is radial and contains O, we may deformation retract X to the $2n$-punctured sphere S^2. This can be done explicitly by the straight line homotopy

$$x \longmapsto (1-t)x + t\frac{x}{\|x\|}$$

Second, the $2n$-punctured S^2 is equivalent to the $2n-1$-punctured $\mathbb{R}^2$ via a stereographic projection centered on one of the excluded points in S^2.

Third, we may apply a diffeomorphism so that the $2n-1$ punctured plane is precisely

$$\mathbb{R}^2 \setminus \{(1,0),(2,0),\dots,(2n-1,0)\}$$

Then we may apply a straight line homotopy to deformation retract $\mathbb{R}^2 \setminus \{(1,0),\dots,(2n-1,0)\}$ onto the wedge of $2n-1$ closed punctured disks.
Finally, on each punctured disk we may apply a deformation retract onto the boundary.
Combining all this implies that X is homotopy equivalent to $\bigvee_{j=1}^{2n-1} S^1$. Then by Van Kampen's,

$$\pi_1(X) \cong \pi_1\left(\bigvee_{j=1}^{2n-1} S^1\right) \cong \mathbb{Z}^{*2n-1}.$$

∎

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By definition, the mapping cone X is given by

$$X = \{pt\} \amalg S^1 \times [0,1] \amalg S^1 \vee S^1 / \sim$$

where $pt \sim (x,0)\ \forall x \in S^1$ and $(x,1) \sim f(x) \in S^1 \vee S^1\ \forall x \in S^1$ where f is the commutator $[a,b]$. Let c denote the constant map $S^1 \to \{pt\}$.

We recall the LES for the ~~cone~~ mapping cone, which is

$$\cdots \to H_k(S^1) \xrightarrow{f_*} H_k(S^1 \vee S^1) \xrightarrow{*} H_k(X) \to \cdots$$

This can be derived from the LES for Mayer-Vietoris by taking $U = \{pt\} \amalg S^1 \times [0,1)/\sim$, $V = S^1 \times (0,1] \amalg S^1 \vee S^1/\sim$ and noting that U ~~deformation~~ is contractible, V deformation retracts onto $S^1 \vee S^1$, and $U \cap V$ deformation retracts onto S^1.

Recalling $H_k(S^1) \cong \begin{cases} \mathbb{Z} & k=0,1 \\ 0 & \text{else} \end{cases}$ and $H_k(S^1 \vee S^1) \cong \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}^2 & k=1 \\ 0 & \text{else,} \end{cases}$

this yields the LES,

$$0 \to H_2(X) \overset{\ker=0}{\hookrightarrow} \mathbb{Z} \xrightarrow{f_*} \mathbb{Z}^2 \xrightarrow{\ker=0,\ \mathrm{im}=} H_1(X) \to \mathbb{Z} \to \mathbb{Z} \xrightarrow{f_*} \overset{\mathbb{Z}\ \cong}{H_0(X)} \to 0$$

Since X is connected, $H_0(X) \cong \mathbb{Z}$. Since X is 2-dimensional, $H_k(X) \cong 0\ \forall\ k > 2$. It thus remains to find $H_1(X)$, $H_2(X)$. By construction, we know that $H_k(S^1) \xrightarrow{f_*} H_k(S^1 \vee S^1)$ is the 0 map. Therefore we have 2 SES

$$0 \to H_2(X) \to \mathbb{Z} \to 0$$

$$0 \to \mathbb{Z}^2 \to H_1(X) \to \mathbb{Z} \to \mathbb{Z} \to 0$$

which implies $H_2(X) \cong \mathbb{Z}$ and $H_1(X) \cong \mathbb{Z}^2$ by taking alternating sums.

□

Geometry/Topology Qualifying Exam

Start each problem on a new sheet of paper.
Write your university identification number at the top of each sheet of paper.

DO NOT WRITE YOUR NAME!
Complete this sheet and staple to your answers.

Read the directions of the exam carefully.

STUDENT ID NUMBER 405 246 515

DATE: 3/15/23

EXAMINEES: DO NOT WRITE BELOW THIS LINE

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1. ____________________
2. ____________________
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Pass/fail recommend on this form.

Total score: ☆ Fuck yeah

Form revised 3/08

QUALIFYING EXAM
Geometry/Topology
March 2021

Attempt all ten problems. Each problem is worth 10 points. Justify your answers carefully.

? **1.** Without using homology groups or homotopy groups, directly derive Brouwer's fixed point theorem (any continuous map $f\colon D^2 \to D^2$ has a fixed point, where D^2 is the closed 2-disk) from the hairy ball theorem (any continuous vector field on S^2 is somewhere 0).

$\frac{1}{2}$ **2.** Solve the following problems:

(a) ~~Let $F : S^n \to S^n$ be a continuous map.~~ Show that if F has no fixed point, then the ~~degree of the map, $\deg F = (-1)^{n+1}$.~~

(b) Show that if X has S^{2n} as universal covering space, then $\pi_1(X) = \{1\}$ or $\mathbb{Z}_2$.

~~**3.** Let $p_1, \ldots, p_n$ be n distinct points in $\mathbb{R}^3$. Calculate the integral homology groups of $\mathbb{R}^3 \setminus \{p_1, \ldots, p_n\}$.~~

4. Let $\Delta_n^{(k)}$ be the k-dimensional skeleton of the n-simplex Δ_n. Calculate the reduced homology groups $\widetilde{H}_i(\Delta_n^{(k)})$ for all values of i, k, n.

~~**5.** Define the complex projective space $\mathbb{CP}^n$ to be the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the relation $x \sim \lambda x$ for all $\lambda \in \mathbb{C} \setminus \{0\}, x \in \mathbb{C}^{n+1} \setminus \{0\}$. Construct a CW complex structure on $\mathbb{CP}^n$ with no odd-dimensional cells and exactly 1 cell in each even dimension up to $2n$. Calculate the fundamental group and the integral homology groups of $\mathbb{CP}^n$.~~

~~**6.** Define the orientation double cover for any topological manifold. What is the orientation double cover of the real projective plane $\mathbb{RP}^n$?~~

~~**7.** Show that $S^2 \times S^2$ and the connected sum $\mathbb{CP}^2 \# \mathbb{CP}^2$ are not homotopy equivalent.~~

3/4 **8.** Consider a differentiable map $f : S^{2n-1} \to S^n$, with $n \geq 2$. If $\alpha \in \Omega^n(S^n)$ is a differential form of degree n such that $\int_{S^n} \alpha = 1$, let $f^*\alpha \in \Omega^n(S^{2n-1})$ be its pull-back under f.

~~(a) Show that there exists $\beta \in \Omega^{n-1}(S^{2n-1})$ such that $f^*(\alpha) = d\beta$.~~

(b) Show that the integral $I(f) = \int_{S^{2n-1}} \beta \wedge d\beta$ ~~is independent of the choices of β~~ and α.

can bullshit ~~**9.** Let $f : M \to N$ be a smooth map between smooth manifolds, X and Y be smooth vector fields on M and N, respectively, and suppose that $f_*X = Y$ (i.e., $f_*(X(x)) = Y(f(x))$ for all $x \in M$). Then prove that~~

$$f^*(L_Y\omega) = L_X(f^*\omega)$$

~~where ω is a 1-form on N. Here L denotes the Lie derivative.~~

~~**10.** Prove Cartan's lemma: Let M be a smooth manifold of dimension n. Fix $1 \leq k \leq n$. Let ω^i and φ_i be 1-forms on M. Suppose that the $\{\omega^1, \ldots, \omega^k\}$ are linearly independent and that $\sum_{i=1}^k \varphi_i \wedge \omega^i = 0$. Prove that there exist smooth functions $h_{ij} = h_{ji} : M \to \mathbb{R}$ such that for all $i = 1, \ldots, k$, $\varphi_i = \sum_{j=1}^k h_{ij}\omega^j$.~~

1) 405 846 515

Suppose for the sake of contradiction that $\exists\ f: D^2 \to D^2$ continuous s.t. $f(x) \neq x \quad \forall\ x \in D^2$.

(2) 405 846 515

(a) Since degree is homotopy invariant and $\deg(-id) = (-1)^{n+1}$, it suffices to construct a homotopy from f to $-id$.
We note that $\deg(-id) = (-1)^{n+1}$ since $S^n \subset \mathbb{R}^{n+1}$ and so we can view $-id$ as multiplying $n+1$ coordinates by -1, where each multiplication by -1 is ~~degree~~ an orientation reversing bijection and hence has degree -1. Then $\deg(-id) = (-1)^{n+1}$.

Define $H_t : S^n \to S^n$

$$H_t(x) = \frac{(1-t)f(x) + t(-x)}{|(1-t)f(x) - tx|}$$

Then $H_0 = f/|f| = f$ and $H_1 = -x/|-x| = -x$. To show that H_t is a homotopy, it must be shown that $(1-t)f(x) - tx \neq 0 \ \forall t \in [0,1], x \in S^n$.
Suppose on the contrary that $(1-t)f(x) - tx = 0$.
Then

$$f(x) = \frac{tx}{1-t}$$

Since $|f(x)| = 1 \ \forall x$ and $|x| = 1 \ \forall x$, this implies

$$\frac{t}{|1-t|} = 1 \implies t = \frac{1}{2}.$$

Then $f(x) = x$, contradicting the fact that f has no fixed point. Therefore $(1-t)f(x) - tx \neq 0 \ \forall t, x$ and so f is homotopic to $-id$. Then $\deg(f) = (-1)^{n+1}$.

(b) Suppose that X has S^{2n} as a universal covering space.

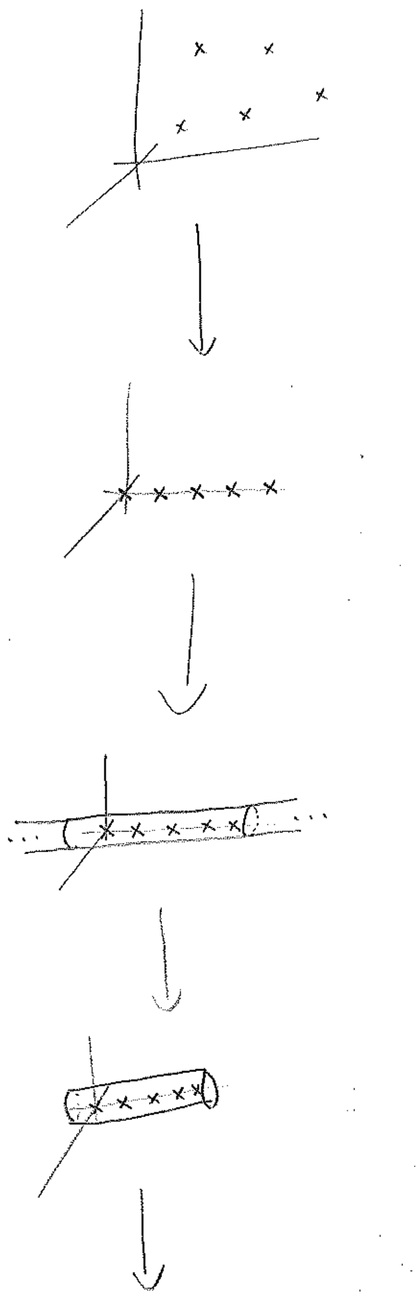
<u>idea</u>: this should follow from some lifting criterion stuff.

$$\begin{tikzcd} & S^{2n} \arrow[d, "p"] \\ S^1 \arrow[ur, dashed, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & X \end{tikzcd}$$

since $\pi_1(S^{2n}) \cong \{1\}$, any path can be lifted.

③ 405 846 515

We claim that $\mathbb{R}^3 \setminus \{p_1, \dots, p_n\}$ deformation retracts onto the wedge of n copies of S^2. First, by applying a diffeomorphism, we may assume WLOG that $p_1 = (0,0,0)$, $p_2 = (0,1,0)$, $\dots$, $p_n = (0, n-1, 0)$. This is a standard result for smooth manifolds (shown on a number of past quals) and is even more standard on $\mathbb{R}^3$.



Then, we apply a deformation retraction from $\mathbb{R}^3 \setminus \{p_1, \dots, p_n\}$ to the infinite solid cylinder w/ n points removed. This can be done explicitly via the straight line homotopy

$$H_t(x,y,z) = \begin{cases} (x,y,z) & \text{if } x^2+z^2 \leq 1 \\ (1-t)(x,y,z) + t\dfrac{(x,y,z)}{\sqrt{x^2+z^2}} & \text{if } x^2+z^2 > 1 \end{cases}$$

We then deformation retract to the finite cylinder containing $p_1, \dots, p_n$ by sending (x,y,z) to $(x,-1,z)$ if $y \leq -1$ and (x,y,z) to (x,n,z) if $y \geq n$. This again can be done via the straight line homotopy.

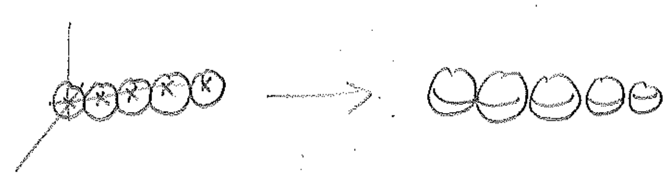
Third we deformation retract to the wedge of n punctured balls by the straight line homotopy in the radial direction.

Finally, since each ball is punctured, we deformation retract to the wedge of n spheres by radially contracting each ball individually.

Then

$$H_k(\mathbb{R}^3 \setminus \{p_1, \dots, p_n\}) \cong H_k\Big(\bigvee_{j=1}^{n} S^2\Big) \cong \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}^n & k=2 \\ 0 & \text{else} \end{cases}$$

as desired.



$\square$

(5) 405 846 515

We proceed by induction on n.

Suppose $n=0$. Then $\mathbb{CP}^n$ is a single point and so can be constructed w/ 1 0-cell.

Suppose for the sake of induction that $\mathbb{CP}^n$ can be constructed as desired. We claim that then $\mathbb{CP}^{n+1}$ can similarly be constructed as desired. To show this, it then suffices to show that $\mathbb{CP}^{n+1}$ can be constructed by attaching a $2n+2$-cell to $\mathbb{CP}^n$.

By definition, $\mathbb{CP}^{n+1} \cong \mathbb{C}^{n+1}\setminus\{0\}/z\sim\lambda z$. For any $z \in \mathbb{C}^{n+1}\setminus\{0\}$, we can normalize $|z|=1$ by $z \mapsto \frac{z}{|z|}$. Since $z \sim \frac{z}{|z|}$, this implies

$$\mathbb{CP}^{n+1} \cong S^{2n+1}/z\sim\lambda z$$

where $S^{2n+1} = \{z \in \mathbb{C}^{n+1} : |z|=1\}$.

Define $U \subset \mathbb{CP}^{n+1}$ as $U = \{[z_0:\dots:z_n:0]\}$. Then $U \cong \mathbb{CP}^n$ via the diffeomorphism $[z_0:\dots:z_n:0] \to [z_0:\dots:z_n]$.

Consider $\mathbb{CP}^{n+1}\setminus U = \{[z_0:\dots:z_n:z_{n+1}] : z_{n+1} \neq 0\}$. Since we have normalized $|z_0|^2+\dots+|z_{n+1}|^2=1$, we may write

$$\mathbb{CP}^{n+1}\setminus U = \{[z_0:\dots:z_n:\sqrt{1-|z_0|^2-\dots-|z_n|^2}]\}.$$

Then $|(z_0,\dots,z_n)|<1$ and so $\mathbb{CP}^{n+1}\setminus U \cong B^{2n+2} = \{z \in \mathbb{C}^{n+1} : |z|<1\}$ via the diffeomorphism

$$[z_0:\dots:z_n:\sqrt{1-|(z_0,\dots,z_n)|^2}] \longrightarrow (z_0,\dots,z_n)$$

By construction, $\partial(\mathbb{CP}^{n+1}\setminus U) = U$ w/ $U \cong \mathbb{CP}^n$ via $[z_0:\dots:z_n:0] \to [z_0:\dots:z_n]$.

Therefore, viewing $\mathbb{CP}^{n+1}\setminus U \cong B^{2n+2}$, we see that $\mathbb{CP}^{n+1}$ can be constructed was B^{2n+2} attached to $\mathbb{CP}^{n+1}$ by the map

$$\partial B^{2n+2} \cong S^{2n+1} \to \mathbb{CP}^n : (z_0,\dots,z_n) \to [z_0:\dots:z_n]$$

By induction, this concludes the construction.

From this construction, we immediately obtain

$$H_k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & k=0,2,\dots,2n \\ 0 & \text{else} \end{cases}$$

Since $\mathbb{CP}^n$ has no 1-cells, we also find that $\pi_1(\mathbb{CP}^n) = 0$. $\square$

(6) 405 846 515

Let M be a topological manifold. We define the orientation cover $\tilde{M}$ as follows.

Let $\tilde{M} = \{(p, O_p) : p \in M\}$ where O_p is an orientation of T_pM. Using the fact that M is locally orientable, we define the topology on $\tilde{M}$ to be generated by sets of the form $V_{U,O}$ where $U \subset M$ is open w/ local orientation O s.t. $V_{U,O} = \{(p, O_p) \cdot p \in U\}$.

From this it immediately follows that $\tilde{M}$ is locally diffeomorphic to M and so $\tilde{M}$ is a topological manifold.

We now show $\tilde{M}$ is a double cover of M. Let $\pi : \tilde{M} \to M : (p, O_p) \mapsto p$.

For any orientable $U \subset M$ $\exists$ 2 orientations $O, -O$. Therefore $\pi^{-1}(U) = V_{U,O} \amalg V_{U,-O}$. Moreover, $\pi|_{V_{U,\pm O}} : (p, \pm O_p) \to p$ is a diffeomorphism. Therefore $\tilde{M}$ is a double cover.

We recall that $\mathbb{RP}^n$ is orientable if n is odd and non-orientable is n is even.

Therefore, if n is odd, the above definition implies that $\widetilde{\mathbb{RP}^n}$ consists of 2 disjoint open sets, each w/ consistent orientation. Since $\widetilde{\mathbb{RP}^n}$ is a 2-fold cover, this implies $\widetilde{\mathbb{RP}^n} \cong \mathbb{RP}^n \amalg \mathbb{RP}^n$.

Now suppose n is even. Then $\mathbb{RP}^n$ is non-orientable and so $\widetilde{\mathbb{RP}^n}$ is ~~so~~ a connected 2-fold cover. We recall that S^n is a connected 2-fold cover of $\mathbb{RP}^n$ via the quotient map $\pi : S^n \to S^n / x \sim -x$.

Then S^n can be viewed as $S^n \cong \{(p, \pm 1) : p \in \mathbb{RP}^n\} \cong \widetilde{\mathbb{RP}^n}$ from the above definition.

□

(7) 405 846 515

we recall that $\mathbb{CP}^2$ can be constructed as a 4-ball attached to $\mathbb{CP}^1$. Therefore, since all connected sums are homotopy equivalent,

$\mathbb{CP}^2 \# \mathbb{CP}^2$ is homotopy equivalent to $\mathbb{CP}^1$ attached to $\mathbb{CP}^1$ via the identity map. Therefore $\mathbb{CP}^2 \# \mathbb{CP}^2$ is homotopy equivalent to $\mathbb{CP}^1$.

By the Kunneth formula, we find that

$$H_k(S^2) \cong \begin{cases} \mathbb{Z} & k=0,2 \\ 0 & \text{else} \end{cases} \Rightarrow H_k(S^2 \times S^2) \cong \begin{cases} \mathbb{Z} & k=0,4 \\ \mathbb{Z}^2 & k=2 \\ 0 & \text{else} \end{cases}$$

From problem 5, we recall

$$H_k(\mathbb{CP}^1) = \begin{cases} \mathbb{Z} & k=0,2 \\ 0 & \text{else} \end{cases}$$

Therefore $S^2 \times S^2$ and $\mathbb{CP}^1$ are not homotopy equivalent and so $S^2 \times S^1$ and $\mathbb{CP}^2 \# \mathbb{CP}^2$ are not homotopy equivalent. $\square$

(8) 405 846 515

(a) We recall that $H^{n-1}_{dR}(S^{2n-1}) \cong 0$. Therefore to show that $f^*\kappa$ is exact, it suffices to show that $f^*\kappa$ is closed. Since pullbacks commute w/ the exterior derivative,

$$d(f^*\kappa) = f^*(d\kappa)$$

Since $\kappa \in \Omega^{n-1}(S^{n-1})$, $d\kappa = 0$ so $d(f^*\kappa) = f^*0 = 0$.

Therefore $f^*\kappa$ is closed and so $f^*\kappa$ is exact.

Then $\exists\ \beta \in \Omega^{n-1}(S^{2n-1})$ s.t. $f^*\kappa = d\beta$ as desired.

(b) Suppose $\exists\ \beta, \beta'$ s.t. $d\beta = d\beta' = f^*\kappa$.

Then $d\beta' - d\beta = 0 \implies d(\beta' - \beta) = 0$. Since $\beta, \beta' \in \Omega^{n-1}(S^{2n-1})$, closedness implies exactness and so $\beta' - \beta = d\varphi$ for some φ.

Then $\beta' = \beta + d\varphi$. It then suffices to show that

$$\int_{S^{2n-1}} \beta \wedge d\beta = \int_{S^{2n-1}} (\beta + d\varphi) \wedge d(\beta + d\varphi)$$

for all φ. By direct computation,

$$\int_{S^{2n-1}} (\beta + d\varphi) \wedge d(\beta + d\varphi) = \int_{S^{2n-1}} (\beta + d\varphi) \wedge d\beta$$

$$= \int_{S^{2n-1}} \beta \wedge d\beta + d\varphi \wedge d\beta$$

Since $d\varphi \wedge d\beta = d(\varphi \wedge d\beta)$, Stokes' theorem implies $\int_{S^{2n-1}} d\varphi \wedge d\beta = \int_{\phi} \varphi \wedge d\beta = 0$.

Therefore $\int_{S^{2n-1}} (\beta + d\varphi) \wedge d(\beta + d\varphi) = \int_{S^{2n-1}} \beta \wedge d\beta$ as desired.

Now suppose $\exists\ \kappa, \kappa' \in \Omega^n(S^n)$ s.t. $\int_{S^n} \kappa = 1 = \int_{S^n} \kappa'$.

Then $\int_{S^n} \kappa - \kappa' = 0$ and so $\kappa - \kappa' = d\theta$ for some $\theta \in \Omega^{n-1}(S^n)$ since integration is an isomorphism $H^n_{dR}(S^n) \to \mathbb{R}$.

Then $f^*\kappa - f^*\kappa' = f^*(\kappa - \kappa') = f^*(d\theta) = d f^*\theta$.

Taking β s.t. $d\beta = f^*\kappa$ and β' s.t. $d\beta' = f^*\kappa'$, we see that $d\beta - d\beta' = d f^*\theta$.

⑨ 405 346 515

~~By Cartan's formula and since ω is a 1-form,~~
~~we find that~~

$$\cancel{\mathcal{L}_X(f^*\omega) = (i_X \circ d + d \circ i_X)(f^*\omega)}$$
$$\cancel{= i_X(f^*\omega) + d([f^*\omega](X))}$$

~~we now specify to some $p \in M$. Rewriting this,~~

$$\cancel{\mathcal{L}_X(f^*\omega)_p = X_p((f^*\omega)_p) + d_p([f^*\omega](X))}$$
$$\cancel{= X_p(}$$

By Cartan's formula, since ω is a 1-form,

$$f^*(\mathcal{L}_Y \omega) = f^*\big(d(\omega(Y)) + i_Y(d\omega)\big)$$
$$= d\big(f^*(\omega(Y))\big) + f^*(i_Y(d\omega))$$

By definition, $\omega(Y) : N \to \mathbb{R} : q \mapsto \omega_q(Y(q))$.
Therefore $f^*(\omega(Y))(p) = \omega(Y)(f(p)) = \omega_{f(p)}(Y(f(p)))$ for $p \in M$.
As given, this implies $f^*(\omega(Y))(p) = \omega_{f(p)}(f_*(X(p))) = (f^*\omega)(X)(p)$.
Then $f^*(\omega(Y)) = (f^*\omega)(X) = i_X(f^*\omega)$.
Similarly, $(i_Y(d\omega))_q : T_qN \to \mathbb{R}$.
Therefore $f^*(i_Y(d\omega))_p : T_pM \to \mathbb{R}$ and so

$$f^*(i_Y(d\omega))_p(v) = (i_Y(d\omega))_{f(p)}(f_* v)$$
$$= d\omega_{f(p)}(Y_{f(p)}, f_* v)$$
$$= (f^* d\omega)_p((f_* X)_p, f_* v)$$

and so $f^*(i_Y(d\omega)) = i_X(f^* d\omega) = i_X \circ d(f^*\omega)$.
Combining these,

$$f^*(\mathcal{L}_Y \omega) = d(i_X(f^*\omega)) + i_X(d(f^*\omega)) = \mathcal{L}_X(f^*\omega)$$

as desired.

⑯ 405 846 515

Idea: Extend $\{\omega^1, \dots, \omega^k\}$ to a basis $\omega^1, \dots, \omega^k, \omega^{k+1}, \dots, \omega^n$ of T^*M.
Then $\varphi_i = \sum_{j=1}^{n} h_{ij}\omega^j$.

It suffices to work locally as then we can extend h_{ij} to all of M via a partition of unity.

Locally, we may extend $\omega^1, \dots, \omega^k$ to a basis $\omega^1, \dots, \omega^k, \omega^{k+1}, \dots, \omega^n$ of T^*M. Then $\forall\, i, j$, $\exists\, h_{ij}$ s.t.

$$\varphi_i = \sum_{j=1}^{n} h_{ij}\omega^j.$$

As given, $\sum_{i=1}^{k} \varphi_i \wedge \omega^i = 0$. Therefore

$$\sum_{i=1}^{k}\sum_{j=1}^{n} h_{ij}\,\omega^j \wedge \omega^i = 0$$

Expanding and associating like terms, we find that

$$\sum_{i=1}^{k}\sum_{j=i}^{k} (h_{ij} - h_{ji})\,\omega^j \wedge \omega^i + \sum_{i=1}^{k}\sum_{j=k+1}^{n} h_{ij}\,\omega^j \wedge \omega^i = 0$$

Since $\omega^1, \dots, \omega^n$ are linearly independent, $\{\omega^i \wedge \omega^j\}$ are linearly independent. Therefore

$$h_{ij} = h_{ji} \quad \forall\ i, j \in \{1, \dots, k\}$$

$$h_{ij} = 0 \quad \forall\ j \geq k+1$$

Therefore

$$\varphi_i = \sum_{j=1}^{k} h_{ij}\omega^j \qquad \forall i$$

w/ $h_{ij} = h_{ji}$, as desired. $\square$

18F. 4

(a) Since $\ker(\omega)$ is 2-dimensional and integrable, $\exists$ vector fields X, Y s.t. $\ker(\omega) = \mathrm{span}(X, Y)$.

About ~~For~~ any p, we can locally extend (X, Y) to a basis X, Y, Z.

Then since $X, Y \in \ker \omega$

$$\omega \wedge d\omega(X, Y, Z) = \omega(Z)\, d\omega(X, Y) - \omega(Z)\, d\omega(Y, X)$$

$$= 2\omega(Z)\, d\omega(X, Y)$$

We recall that $d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega[X, Y]$.

Since $\ker(\omega)$ is integrable, $[X, Y] \in \ker(\omega)$ and so

$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) = 0$. Then

$$\omega \wedge d\omega(X, Y, Z) = 0$$

Since X, Y, Z are a local basis, this implies $\omega \wedge d\omega = 0$ locally and hence everywhere.

(b), (c) follow as before

GeoTop

Spring 2018

① 405 846 515

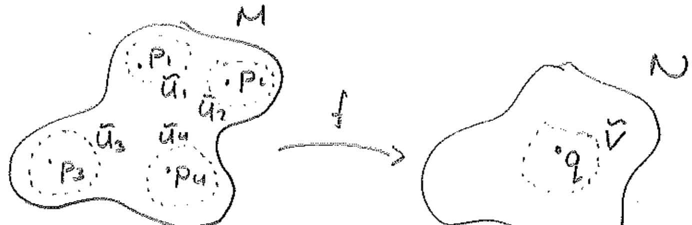
(a) We first show onto.

Let $n = \dim M = \dim N$. Since f is a submersion, $\forall p \in M$,
$$\operatorname{rank} df_p = n$$
Since $n = \dim M$, this implies that df_p is an isomorphism $\forall p \in M$. Therefore by the inverse function theorem, f is a local diffeomorphism. In particular f is an open map.

Since f is open, $f(M) \subset N$ is open. Since f is continuous and M is compact, $f(M)$ is compact. Therefore $f(M)$ is both open and closed. Because N is connected this implies that $f(M) = N$ and so f is surjective.

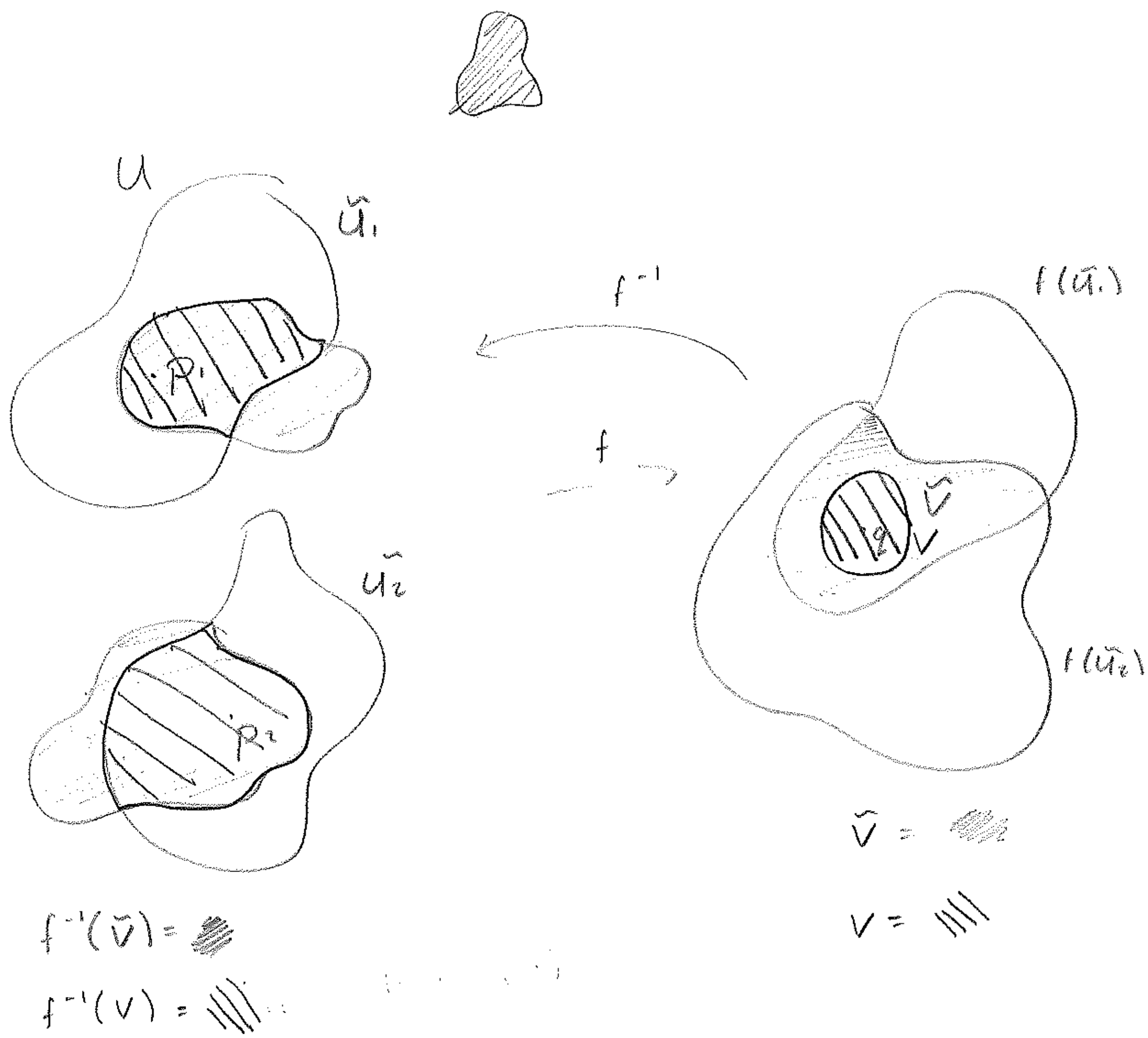
We now show f is a covering map. We note that this is ~~[illegible]~~ ~~[illegible]~~ exactly the statement of Ehresmann's theorem.
Since f is smooth and onto, it suffices to show that $\forall q \in N$ $\exists$ a neighborhood V of q and disjoint neighborhoods $U_1 \dots U_n$ of the k pre-images of q under f s.t. $f|_{U_i}$ is a diffeo onto V, and $f^{-1}(V) = \coprod_{i=1}^{k} U_i$

Fix some $q \in N$. Since f is a ~~local~~ local diffeomorphism, $f^{-1}(q)$ is a 0-dimensional submanifold of M. Since M is compact, this implies that $f^{-1}(q)$ is finite. We enumerate $f^{-1}(q)$ as $p_1, \dots, p_k$.



For each $i = 1, \dots, k$, $\exists$ a neighborhood $\tilde{U}_i$ of p_i s.t. $f : \tilde{U}_i \to f(\tilde{U}_i)$ is a diffeomorphism. By shrinking $\tilde{U}_i$ if necessary, we may assume that $\tilde{U}_i \cap \tilde{U}_j = \emptyset$ if $i \neq j$. Let $\tilde{V} = f(\tilde{U}_1) \cap \dots \cap f(\tilde{U}_k)$. Then $f(\tilde{V})$ is a neighborhood of q.

$\longrightarrow$



U
$\tilde{U}_1$
p_1
$\tilde{U}_2$
p_2
f^{-1}
f
$f(\tilde{U}_1)$
$f(\tilde{U}_2)$
$\tilde{V}$
q
V
$f^{-1}(\tilde{V}) =$
$f^{-1}(V) =$
$\tilde{V} =$
$V =$

figure 1

Now define $V = \tilde{V} \setminus f(M \setminus \bigcup_{i=1}^{k} \hat{U}_i)$. Since M is compact, $M \setminus \bigcup_{i=1}^{k} \hat{U}_i$ is compact. Therefore $f(M \setminus \bigcup_{i=1}^{k} \hat{U}_i)$ is closed and so V is open. Since all pre-images of q are contained in $\bigcup_{i=1}^{k} \hat{U}_i$, V is an open neighborhood of q. See figure 1 for visual.

Finally, take $U_i = \hat{U}_i \cap f^{-1}(V)$. Then U_i is a neighborhood of p_i, $f(U_i) = V$, and $U_i \cap U_j = \emptyset$ $\forall i \neq j$. Moreover, since $U_i \subset \hat{U}_i$, $f: U_i \to V$ is a diffeomorphism. Finally, by construction, $f^{-1}(V) = \coprod_{i=1}^{k} U_i$.

Therefore f satisfies the covering property. ~~and so is a covering map~~. To show that f is a covering map, it only remains to show that $\#\{f^{-1}(q)\}$ is independent of q.

Fix some $q_0 \in N$. Define $k_0 = \# f^{-1}(q_0)$ and

$$\Omega = \{ q \in N : \#(f^{-1}(q)) = k_0 \}$$

Since $q_0 \in \Omega$, Ω is non-empty.

We claim that Ω is also open and closed.

Consider some $q \in \Omega$. Then $\exists$ a neighborhood V of q and ~~no~~ disjoint neighborhoods $U_1, \ldots, U_{k_0} \subset M$ s.t. $f: U_i \to V$ is a diffeomorphism, and $f^{-1}(V) = U_1 \amalg \ldots \amalg U_{k_0}$. Then $\forall \tilde{q} \in V$, $f^{-1}(\tilde{q}) \subset U_1 \amalg \ldots \amalg U_{k_0}$. Since $f: U_i \to V$ is bijective, $\#(f^{-1}(\tilde{q}) \cap U_i) = 1$ $\forall i$. Therefore $\# f^{-1}(\tilde{q}) = k_0$. Then $V \subset \Omega$ and so Ω is open.

Repeating the argument for $q \notin \Omega$ implies that ~~$\Omega \setminus V$~~ $N \setminus \Omega$ is open and hence Ω is closed.

Since N is connected this implies $\Omega = N$ and so $\# f^{-1}(q)$ is constant in q.

Therefore f is a covering map.

(b) ~~Define~~ Define

$$N = S^1 = \mathbb{R}/\mathbb{Z}$$
$$M = (0,2) \subset \mathbb{R}$$

Then N, M are connected, of the same dimension, and N is compact.
Let π be the quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$
and $f = \pi|_M$. Then f is onto, smooth, and a local diffeomorphism.
In particular, f is a smooth submersion.
However, $f^{-1}[0] = 1$ and $f^{-1}[1/2] = \{1/2, 3/2\}$.
Therefore $f^{-1}(q)$ is not constant and so f is not a covering map. $\square$

(3) 405 846 515

(a) By construction X is diffeomorphic to $\mathbb{CP}^n$ via the map

$$[z_0 : \dots : z_n : 0 : \dots : 0] \longleftrightarrow [z_0 : \dots : z_n] \in \mathbb{CP}^n$$

Therefore X is a smooth submanifold.

~~Alternatively, we may map~~

(b) We claim that $I_2(X, X) = 1$.
We recall that the mod 2 intersection # is invariant under homotopies. We then aim to define a homotopy of X.

Define H_t by

$$H_t[z_0 : \dots : z_n : 0 : \dots : 0] = [(1-t)z_0 : \dots : (1-t)z_{n-1} : z_n : t z_0 : \dots : t z_{n-1}]$$

Then H_0 is the identity on X and ~~$H_1 : [z_0 : \dots z_n : 0 : \dots 0] \to$~~

$$H_1 : [z_0 : \dots : z_n : 0 : \dots : 0] \to [0 : \dots : 0 : z_n : z_0 : \dots : z_{n-1}]$$

Therefore H_t is a homotopy from X to $\hat{X} = \{[0 : \dots : 0 : z_n : z_0 : \dots : z_{n-1}]\}$.
By homotopy invariance, this implies that

$$I_2(X, X) = I_2(X, \hat{X})$$

By definition,

$$X \cap \hat{X} = \{[0 : \dots : 0 : z : 0 : \dots : 0]\} = \{[0 : \dots : 0 : 1 : 0 : \dots : 0]\}$$

and so $I_2(X, \hat{X}) = 1$. $\square$

(4) 405 846 515

(a) It suffices to show the result locally.
~~By the implicit function~~ Since any smooth embedding is locally the canonical embedding, we may choose local coordinates $x_1, \dots, x_n$ of M s.t. $x_1, \dots, x_k$ are local coordinates for N.
Doing so, a vector field V is tangent to N iff it can be written as $V = V_1 \partial/\partial x_1 + \dots + V_k \partial/\partial x_k$.

Suppose $\exists$ vector fields X, Y tangent to N. Then we may write, in the coordinates above,

$$X = X_1 \partial/\partial x_1 + \dots + X_k \partial/\partial x_k = \sum_{j=1}^{k} X_j \partial/\partial x_j$$
$$Y = Y_1 \partial/\partial x_1 + \dots + Y_k \partial/\partial x_k = \sum_{i=1}^{k} Y_i \partial/\partial x_i$$

Then

$$XY = \sum_{j=1}^{k} \sum_{i=1}^{k} X_j \frac{\partial Y_i}{\partial x_j} \partial/\partial x_i$$
$$YX = \sum_{i=1}^{k} \sum_{j=1}^{k} Y_i \frac{\partial X_j}{\partial x_i} \partial/\partial x_j$$

and so

$$[X,Y] = \sum_{j=1}^{k} \left[\sum_{i=1}^{k} \left(X_j \frac{\partial Y_i}{\partial x_j} - Y_j \frac{\partial X_i}{\partial x_j} \right) \right] \partial/\partial x_j$$

Therefore $[X,Y]$ is ~~tangent~~ to N.

(b) Define

$$X = x\, \partial/\partial y - y\, \partial/\partial x$$
$$Y = x\, \partial/\partial z - z\, \partial/\partial x$$

~~we note that these are continuous but not smooth.~~

~~If we desire smooth vector fields~~

To check that these are ~~tangent~~ to S^2, it suffices to check that $\langle X, N\rangle = \langle Y, N\rangle = 0$ where N is a normal vector field: $N = x\,\partial/\partial x + y\,\partial/\partial y + z\,\partial/\partial z$ to S^2.

→

By direct computation, on S^2

$$\langle X, N\rangle = -xy + xy = 0$$

$$\langle Y, N\rangle = -zx + xz = 0$$

Therefore X, Y are tangent to S^2.

By direct computation,

$$[X,Y] = XY - YX = -y\,\partial/\partial z - (-z\,\partial/\partial y)$$

$$= z\,\partial/\partial y - y\,\partial/\partial z.$$

and similarly,

$$\langle [X,Y], N\rangle = 0 + zy - zy = 0$$

~~so [X,Y] is tan~~ Therefore $[X,Y]$ is tangent to S^2. $\square$

⑤ 405 846 515

Lemma: Suppose that a compact orientable, connected M^{2n} admits a symplectic form ω.

Then $H^{2k}_{dR}(M^{2n}) \neq 0 \quad \forall\, k = 0, \dots, n$.

Proof: Suppose for the sake of contradiction that $\underbrace{\omega \wedge \dots \wedge \omega}_{k} = d\eta$ some η, k. Then $\underbrace{\omega \wedge \dots \wedge \omega}_{n \text{ times}} = d(\eta \wedge \underbrace{\omega \wedge \dots \wedge \omega}_{n-k \text{ times}})$. Since M is compact orientable, $H^{2n}_{dR}(M) \cong \mathbb{R}$ w/ isomorphism $\theta \mapsto \int_M \theta$. Then $\int_M \omega \wedge \dots \wedge \omega = 0$, contradicting the fact that $\omega \wedge \dots \wedge \omega$ is non-vanishing. Therefore $\underbrace{\omega \wedge \dots \wedge \omega}_{k}$ is not exact for all $k = 1, \dots, n$. Therefore, since ω is closed, $H^{2k}_{dR}(M) \neq 0$ for $k = 1, \dots, n$. Since M is connected, $H^0_{dR}(M) \cong \mathbb{R} \neq 0$. $\square$

(a) We recall that

$$H^k_{dR}(S^8) \cong \begin{cases} \mathbb{R} & k = 0, 8 \\ 0 & \text{else} \end{cases}$$

therefore S^8 does not admit a symplectic form by the lemma

(b) By the Künneth formula,

$$H^k_{dR}(S^2 \times S^6) \cong \begin{cases} \mathbb{R} & k = 0, 2, 6, 8 \\ 0 & \text{else} \end{cases}$$

Therefore $H^4_{dR}(S^2 \times S^6) \cong 0$ and so $S^2 \times S^6$ does not admit a symplectic form.

$\longrightarrow$

(c) We claim that $S^2 \times S^2 \times S^2 \times S^2$ does admit a symplectic form.

Let Θ denote a volume form on S^2.
For each ~~coordinate~~ "copy" of S^2 in $S^2 \times S^2 \times S^2 \times S^2$, we let $\Theta_1, \Theta_2, \Theta_3, \Theta_4$ denote their volume form in $S^2 \times S^2 \times S^2 \times S^2$.
Then $\Theta_i \wedge \Theta_j \neq 0$ for $i \neq j$.
Let $w = \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4$. Then since Θ_i is a 2-form

$$w \wedge w = 2(\Theta_1 \wedge \Theta_2 + \Theta_1 \wedge \Theta_3 + \Theta_1 \wedge \Theta_4 + \Theta_2 \wedge \Theta_3 + \Theta_2 \wedge \Theta_4 + \Theta_3 \wedge \Theta_4)$$

$$\Rightarrow w \wedge w \wedge w \wedge w = 4\, \Theta_1 \wedge \Theta_2 \wedge \Theta_3 \wedge \Theta_4$$

which is non-vanishing. Therefore w is a symplectic form on $S^2 \times S^2 \times S^2 \times S^2$.

$\blacksquare$

⑥ 405 846 515

The classical divergence theorem states

$$\iiint_U \operatorname{div}(V)\, d(\mathrm{vol}) = \iint_{\partial U} \langle V, N\rangle\, dA$$

where N is the unit normal to ∂U w/ the boundary orientation, $\langle V, N\rangle$ is the standard inner product, and dA is the induced ~~surface~~ ~~measure~~ area form on ∂U given by

$$dA = \iota^* i_N d(\mathrm{vol})$$

where $\iota : \partial U \to U$ is the inclusion map.

Define $T = V - \langle V, N\rangle N$ on ∂U. Then T is the tangential component of V to ∂U. We claim that $\iota^* i_T d(\mathrm{vol}) = 0$ on ∂U. Suppose $\exists$ 2 ~~linearly independent~~ vector fields on ∂U. Then $\iota^* i_T d(\mathrm{vol})(X, Y) = \iota^* d(\mathrm{vol})(T, X, Y)$. Since ∂U is 2 dimensional and $T \subset T\partial U$, $\{T, X, Y\}$ must be linearly dependent. Therefore $d(\mathrm{vol})(T, X, Y) = 0$ and so $\iota^* i_T d(\mathrm{vol})(X, Y) = 0$. As this holds $\forall X, Y$, this implies $\iota^* i_T d(\mathrm{vol}) = 0$.

By linearity, this implies

$$\begin{aligned} 0 = \iota^* i_T d(\mathrm{vol}) &= \iota^* \left(i_V d(\mathrm{vol}) - \langle V, N\rangle i_N d(\mathrm{vol}) \right) \\ &= \iota^* i_V d(\mathrm{vol}) - \langle V, N\rangle dA \\ \Rightarrow \langle V, N\rangle dA &= \iota^* i_V d(\mathrm{vol}) \end{aligned}$$

By Stokes' theorem, this implies

$$\iint_{\partial U} \langle V, N\rangle\, dA = \iint_{\partial U} \iota^* i_V d(\mathrm{vol}) = \iiint_U d(i_V d(\mathrm{vol}))$$

Let $V = V_1 \partial/\partial x + V_2 \partial/\partial y + V_3 \partial/\partial z$, and note $d(vol) = dx \wedge dy \wedge dz$.

Then by direct computation

$$i_V d(vol) = V_1 \, dy \wedge dz - V_2 \, dx \wedge dz + V_3 \, dx \wedge dy$$

$$\Rightarrow d(i_V d(vol)) = \left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \right) dx \wedge dy \wedge dz$$

$$= \mathrm{div}(V) \, d(vol)$$

Therefore

$$\iint_{\partial U} \langle V, N \rangle \, dA = \iiint_U \mathrm{div}(V) \, d(vol)$$

as desired.

$\square$

(7) 405 846 515

Part (a) is an immediate application of Van Kampen
part (b) is an immediate application of Mayer-Vietoris

(a) Let B_1, B_2 be open balls in M, N respectively.
We define $M \# N$ to be $M \setminus B_1 \amalg N \setminus B_2$ glued at the boundaries of B_1, B_2.

Take U to be an ε-neighborhood of $M \setminus B_1$ which deformation retracts to M and V similarly for $N \setminus B_2$.

Then $U \cap V$ is simply connected, $\pi_1(U) \cong \pi_1(M \setminus B_1)$ and $\pi_1(V) \cong \pi_1(N \setminus B_1)$. Therefore by Van Kampen, $\pi_1(M \# N) \cong \pi_1(M \setminus B_1) * \pi_1(N \setminus B_2)$.

Since $n \geq 3$, $\dim(B_i) \geq 3$ and so $\pi_1(B_i) \cong 0$.
Therefore $\pi_1(M \setminus B_1) \cong \pi_1(M)$ and $\pi_1(N \setminus B_2) \cong \pi_1(N)$.
Thus $\pi_1(M \# N) \cong \pi_1(M) * \pi_1(N)$.

(b) Mayer-Vietoris.

↑ original attempt

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(7) 405 646 515

(a) Let B_1 be an open ball in M and B_2 an open ball in N. Then we define the connected sum $M \# N$ as

$$M \# N = (M \setminus B_1) \cup (N \setminus B_2) / \sim$$

where $\sim$ associates ∂B_1 w/ ∂B_2.

Let U be an open neighborhood of $M \setminus B_1$ that deformation retracts onto $M \setminus B_1$ and similarly V for $N \setminus B_2$.

Then $U \cup V = M \# N$ and $U \cap V$ deformation retracts onto $\partial B_1 = \partial B_2 \cong S^{n-1}$!

Since $n \geq 3$, S^{n-1} is simply connected. Therefore $U \cap V$ is simply connected and Van Kampen's implies that, since the base point is in $U \cap V$,

$$\pi_1(M \# N) \cong \pi_1(M \setminus B_1) * \pi_1(N \setminus B_2)$$

Now consider M. Let U_1 be an open neighborhood of $M \setminus B_1$ and $V = B_1$. Then $M = U \cup V$ and $U \cap V$ deformation retracts onto S^{n-1}. Then by Van Kampen's, $\pi_1(M) \cong \pi_1(M \setminus B_1) * \pi_1(B_1) \cong \pi_1(M \setminus B_1)$ since B_1 is simply connected. Therefore $\pi_1(M) \cong \pi_1(M \setminus B_1)$.

Similarly $\pi_1(N) \cong \pi_1(N \setminus B_2)$. Then

$$\pi_1(M \# N) \cong \pi_1(M) * \pi_1(N)$$

as desired.

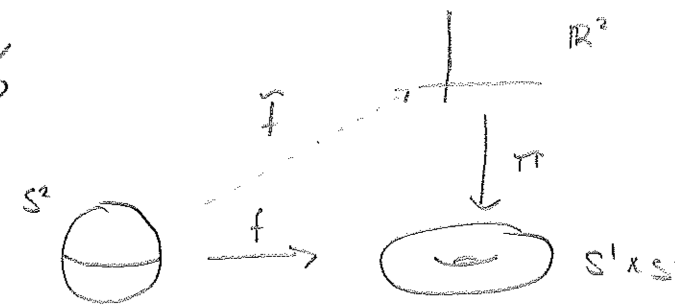
(b) Since M, N are connected and orientable, $M \# N$ is connected and orientable. Therefore

$$H_0(M \# N) \cong H_n(M \# N) \cong \mathbb{Z}.$$

(8) 405 846 515

We first claim that all maps $f: S^2 \to S^1 \times S^1$ are nullhomotopic.

We recall that the universal cover of $S^1 \times S^1$ is $\mathbb{R}^2$ via the quotient map $\pi: \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$. We claim that $f: S^2 \to S^1 \times S^1$ lifts to a map $\tilde{f}: S^2 \to \mathbb{R}^2$.



Indeed, since S^2 is simply connected, $\pi_1(S^2) \cong 0$ so $f_* \pi_1(\mathbb{R}^2) \cong 0$ and in particular $\pi_* \pi_1(\mathbb{R}^2) \supset f_* \pi_1(S^2) \cong 0$.

Therefore f lifts to a map $\tilde{f}: S^2 \to \mathbb{R}^2$.

Since $\mathbb{R}^2$ is contractible, $\tilde{f}$ is nullhomotopic and $\exists$ a homotopy $\tilde{f}_t$ from $\tilde{f}_0 = \tilde{f}$ to $\tilde{f}_1 = 0$. Then $\pi \circ \tilde{f}_t$ is a homotopy from $f = \pi \circ \tilde{f}$ to the constant map $\pi(0)$. Therefore f is nullhomotopic.

We recall that degree is homotopy invariant and is ~~ge~~ defined by the induced map $f_*: H_2(S^2) \to H_2(S^1 \times S^1)$.

Since the constant map induces the 0 map on top homology, this implies that $\deg f = 0$ $\forall$ continuous $f: S^2 \to S^1 \times S^1$. $\square$

(9) 405 846 515

(a) We recall that S^2 can be constructed via 1 0-cell and 1 2-cell attached to the 0-cell. Also, we recall that a cartesian product of CW complexes can be constructed via the cartesian product of the cellular complexes. Therefore, by giving one copy of S^2 the cell structure 1 0-cell: p_1 1 2-cell: f_1 and the other p_2, f_2, we find that $S^2 \times S^2$ has the ~~cellu~~ following cellular structure

1 0-cell: $p = (p_1, p_2)$

2 2-cells: $a = (f_1, p_2)$, $b = (p_1, f_2)$ w/
$$\partial a = (\partial f_1, p_2) = (p_1, p_2) = p$$
$$\partial b = (p_1, \partial f_2) = (p_1, p_2) = p$$

1 4-cell: $A = (f_1, f_2)$ w/
$$\partial A = (\partial f_1, f_2) - (f_1, \partial f_2)$$
$$= b - a = 0$$

By assuming p_1, p_2 are the south poles of their respective copies of S^2, this implies that $S^2 \vee S^2 \hookrightarrow S^2 \times S^2$ via $S^2 \vee S^2 \cong a \vee b$ w/ the wedge at point p, as done in $S^2 \times S^2$.

(b) As given, we attach a 3-cell to $S^2 \times S^2$ via the map $S^2 \to S^2 \vee S^2$ which crushes a great circle connecting the north and south poles. Therefore, using the structure from part (a), we now have the structure

1 0-cell: p

2 2-cells: a, b w/ $\partial a = \partial b = p = 0$

1 3-cell: g w/ $\partial g = a + b$

1 4-cell: A w/ $\partial A = a - b = 0$.

$\longrightarrow$

$\cancel{a+b=0 \quad a=-b}$

Our homology groups are thus

$$H_0(X) = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \mathbb{Z}\langle p \rangle \cong \mathbb{Z}$$

$$H_2(X) = \frac{\ker \partial_2}{\operatorname{Im} \partial_3} = \frac{\mathbb{Z}\langle a, b \rangle}{\mathbb{Z}\langle a+b \rangle} = \mathbb{Z}\langle a, -a \rangle \cong \mathbb{Z}$$

$$H_3(X) = \frac{\ker \partial_3}{\operatorname{Im} \partial_4} = 0$$

$$H_4(X) = \frac{\ker \partial_4}{\operatorname{Im} \partial_5} = \mathbb{Z}\langle A \rangle \cong \mathbb{Z}$$

$$H_1(X) = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = 0$$

$\square$

and all other homology groups are 0.

(10) 405 846 615

~~(a) By definition, the universal cover $\hat{X}$ is simply connected. Therefore $\pi_1(\hat{X}) \cong 0$ and so if $p: \hat{X} \to X$ is the covering map then $p_*(\pi_1(\hat{X})) \cong 0 \subset \sigma_*(\pi_1(\Delta$~~

(a) We recall that the n-simplex Δ^n is simply connected. Therefore $\pi_1(\Delta^n) \cong 0$ and in particular $\sigma_*(\pi_1(\Delta^n)) \cong 0$. Then $\sigma_*(\pi_1(\Delta^n)) \subset p_*(\pi_1(\hat{X}))$ where $p: \hat{X} \to X$ is the covering map. The lifting criterion then implies $\exists$ a lift $\hat{\sigma}: \Delta^n \to \hat{X}$ as desired.

(b) We recall that

$$\sigma = p \circ \hat{\sigma}_1 = p \circ \hat{\sigma}_2$$

$$\begin{tikzcd}
 & \hat{X} \arrow[d, "p"] \\
\Delta^n \arrow[ur, dashed, bend left, "\hat{\sigma}_2"] \arrow[ur, dashed, "\hat{\sigma}_1"'] \arrow[r, "\sigma"'] & X
\end{tikzcd}$$

① 405 846 515

a) $\Rightarrow$ Suppose $\Theta = dF$ for some F. We recall that $H^1_{dR}(S^1) \cong \mathbb{R}$ ~~induced by~~ w/ the isomorphism given by $\omega \mapsto \int_{S^1} \omega$. Then since $d\, i^*F$ is exact on S^1, it follows that $\int_{S^1} i^*\Theta = \int_{S^1} d\, i^*F = 0$ as desired.

$\Leftarrow$: Suppose instead that $\int_{S^1} i^*\Theta = 0$. Let γ be closed path on $S^1\times(-1,1)$. Then since $S^1\times(-1,1)$ contracts to $S^1\times\{0\}$ either γ is contractible or is homotopic to k copies of $S^1\times\{0\}$. In either case, since $\int_{S^1} i^*\Theta = 0$, it follows that $\int_\gamma \Theta = 0$. We now define F s.t. $\Theta = dF$. Fix some $x_0 \in S^1\times(-1,1)$.

Since $S^1\times(-1,1)$ is path connected, $\forall x \in S^1\times(-1,1)$, $\exists$ a path γ_x from x_0 to x. Define $F(x) = \int_{\gamma_x} \Theta$. Since $\int_\gamma \Theta = 0$ $\forall$ closed paths, if γ_1, γ_2 are two paths from x_0 to x, then $\int_{\gamma_1 - \gamma_2} \Theta = 0 \Rightarrow \int_{\gamma_1}\Theta = \int_{\gamma_2}\Theta$. Therefore F is ~~well-d~~ independent of the path chosen and so is well-defined. The smoothness of the integral ~~the~~ and Θ then imply that F is smooth.

It remains to show $dF = \omega$. For any piecewise smooth γ curve, the FTC implies

$$\int_\gamma dF = F\circ\gamma(1) - F\circ\gamma(0) = F(x) - F(x_0) = 0$$

(4) 405 346 515

a) It suffices to ~~construct~~ C^∞ linear $f_K : \mathcal{F}^k(U\cup V) \to \mathcal{F}^k(U)\oplus\mathcal{F}^k(V)$ and $g_K : \mathcal{F}^k(U)\oplus\mathcal{F}^k(V) \to \mathcal{F}^k(U\cap V)$ s.t. f_K, g_K commute w/ d in the sense that $d\circ f_K = f_{K+1}\circ d$, and $0 \to \mathcal{F}^k(U\cup V) \xrightarrow{f_K} \mathcal{F}^k(U)\oplus\mathcal{F}^k(V) \xrightarrow{g_K} \mathcal{F}^k(U\cap V) \to 0$ is exact.

Define $f_K : \mathcal{F}^k(U\cup V) \to \mathcal{F}^k(U)\oplus\mathcal{F}^k(V)$ by $\mathcal{F}^k(\omega) = (\omega|_U, \omega|_V)$.

Clearly f_K is C^∞-linear and $d(f_K(\omega)) = d(\omega|_U, \omega|_V) = (d\omega|_U, d\omega|_V)$
$= f_{K+1}(d\omega)$.

Define $g_K : \mathcal{F}^k(U)\oplus\mathcal{F}^k(V) \to \mathcal{F}^k(U\cap V)$ by $g_K(\omega, \eta) = \omega|_{U\cap V} - \eta|_{U\cap V}$.

Then g_K is C^∞-linear and $d\, g_K(\omega,\eta) = d\omega|_{U\cap V} - d\eta|_{U\cap V}$
$= g_{K+1}(d\omega, d\eta)$.

We now show the sequence is exact. Consider $\omega \in \mathcal{F}^k(U\cap V)$ then it suffices to show $g_K(f_K(\omega)) = 0$. By construction,

$$g_K(f_K(\omega)) = g_K((\omega|_U, \omega|_V)) = \omega|_{U\cap V} - \omega|_{U\cap V} = 0$$

Therefore the sequence is exact.

(6) 405 846 515

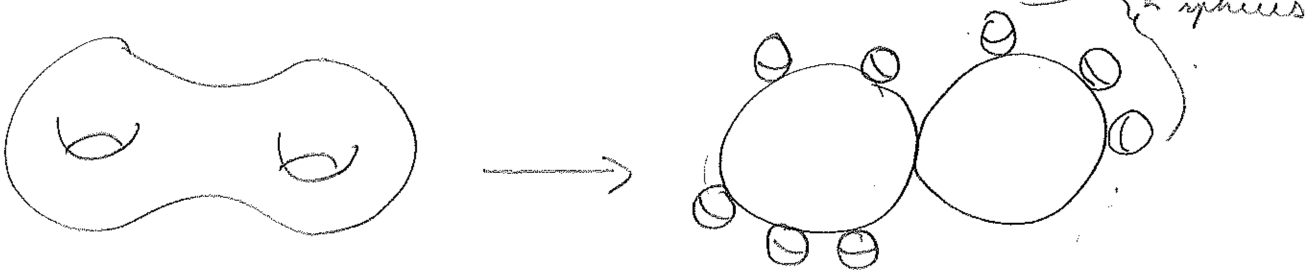
a) Since $GL^+ \subset \mathbb{R}^{n^2}$ is open, it has the structure of a smooth manifold. Therefore to show SL^+ is a submanifold it suffices to ~~show it is the preimage~~ use the ~~regular~~ regular value theorem. Define $F: GL^+ \to \mathbb{R}$ by $A \mapsto \det(A)$.

Then $SL^+ = F^{-1}(1)$. We aim to show 1 is a regular value of F.

(10) 405 846 515

a) ~~[illegible] each~~

Let $x_1, \dots, x_k$ denote the removed points. Around each x_i, there exists a neighborhood which deformation retracts onto S^2 centered at x_i. Then, since the solid two-holed torus deformation retracts onto $S^1 \vee S^1$, it follows that X deformation retracts onto $S^1 \vee S^1 \vee (\bigvee_{i=1}^{k} S^2)$



Van Kampen's then implies that $\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$.

b) Since X is connected, $H_0(X) \cong \mathbb{Z}$. Since $H_1(X)$ is the abelianization of $\pi_1(X)$, $H_1(X) \cong \mathbb{Z}^2$.
Since X deformation retracts onto a cell structure ~~to 3-cell~~ only containing 0, 1, 2 cells, $H_3(X) = 0$.
Finally, since X deformation retracts onto $S^1 \vee S^1 \vee (\bigvee_{i=1}^{k} S^2)$, it follows that $H_2(X) \cong \mathbb{Z}^k$. $\forall n > 3$, $H_n(X) = 0$.

This follows more rigorously by constructing $S^1 \vee S^1 \vee (\bigvee_{i=1}^{k} S^2)$ as

1 0-cell p
2. 1-cells a, b attached via $\partial a = \partial b = p - p$
k 2-cells $f_1, \dots, f_k$ attached via $\partial f_i = p$

So $\partial_2 = 0$, $\partial_1 = 0$ $\Rightarrow$ $H_2(X) \cong \mathbb{Z}^k$, $H_1(X) \cong \mathbb{Z}^2$, $H_0(X) \cong \mathbb{Z}$. $\square$

(2) 405 846 515

(a) Fix two points $p, q \in \mathbb{R}^n \setminus M^m$ and let $\gamma : [0,1] \to \mathbb{R}^n$ be a smooth path from $\gamma(0) = p$ to $\gamma(1) = q$.

~~Then γ is a 1 dimensional submanifold of $\mathbb{R}$~~

Since $p, q \notin M^m$, $\gamma|_{\{0,1\}}$ is transversal to M^m.

Therefore since $\{0,1\} \subset [0,1]$ is closed, the extension theorem implies that $\exists\ \tilde{\gamma} : [0,1] \to \mathbb{R}^n$ homotopic to γ s.t. $\tilde{\gamma}|_{\{0,1\}} = \gamma|_{\{0,1\}}$ and $\tilde{\gamma}$ is transversal to M^m.

Suppose $\exists\ t$ s.t. $\tilde{\gamma}(t) \in M^m$. Then

$$\operatorname{Im} d_t\tilde{\gamma} + T_{\tilde{\gamma}(t)} M^m = T_{\tilde{\gamma}(t)} \mathbb{R}^n \cong \mathbb{R}^n$$

$\therefore$ Then
$$\dim(\operatorname{Im} d_t \tilde{\gamma}) + \dim T_{\tilde{\gamma}(t)} M^m \geq n$$
$$\Rightarrow \quad 1 + m \geq n \quad \Rightarrow \quad m \geq n-1$$

which contradicts the fact that $m \leq n-2$.

Therefore $\nexists\ t$ s.t. $\tilde{\gamma}(t) \in M^m$ and so $\tilde{\gamma} \subset \mathbb{R}^n \setminus M^m$!

As such a path $\tilde{\gamma}$ can be constructed $\forall\ p, q \in \mathbb{R}^n \setminus M^m$, this implies that $\mathbb{R}^n \setminus M^m$ is connected.

(b)

(3) 405 846 515

(a) By definition, $\mathcal{O}$ is ~~a~~ a closed connected orientable n-manifold. Therefore $H^n_{dR}(\mathcal{O}) \cong \mathbb{R}$ w/ isomorphism given by

$$\eta \longmapsto \int_{\mathcal{O}} \eta$$

Let ω be an n-form on M. To show that $\pi^*\omega$ is exact, it then suffices to show that $\int_{\mathcal{O}} \pi^*\omega = 0$.

Let f be the deck transformation on $\mathcal{O}$ corresponding to the covering map π. Since $\mathcal{O}$ is an orientation cover, f has degree -1, as it takes positive orientations to negative orientation. Additionally, $\pi \circ f = \pi$. Therefore

$$\int_{\mathcal{O}} \pi^*\omega = \int_{\mathcal{O}} (\pi \circ f)^*\omega = \int_{\mathcal{O}} f^*\pi^*\omega$$

$$= \int_{f_*\mathcal{O}} \pi^*\omega = -\int_{\mathcal{O}} \pi^*\omega$$

and so $\int_{\mathcal{O}} \pi^*\omega = 0$. Then $\pi^*\omega$ is exact.

(b). To show that ω is exact, we construct an n-1-form η on M s.t. $d\eta = \omega$.

Since π is a 2-fold cover, $\forall\, p \in M\ \exists\ q_1, q_2 \in \mathcal{O}$ s.t. $\pi(q_1) = \pi(q_2) = p$ and $\exists$ neighborhoods U_p of p, V_i of q_i s.t. $\pi|_{V_i}$ is a diffeomorphism onto U_p. Define θ s.t. $d\theta = \pi^*\omega$. We then define η on U_p by

$$\eta|_{U_p} = \frac{1}{2}\sum_{i=1}^{2} (\pi|_{V_i})^{-1*} \theta|_{V_i}$$

$\longrightarrow$

Then, on U_p,

$$d\eta|_{U_p} = \frac{1}{2}\sum_{i=1}^{2}\left(\left(\pi|_{V_i}\right)^{-1}\right)^* d\theta$$

$$= \frac{1}{2}\sum_{i=1}^{2}\left(\left(\pi|_{V_i}\right)^{-1}\right)^* \pi^* \omega$$

$$= \frac{1}{2}\sum_{i=1}^{2}\omega$$

$$= \omega$$

We then extend η to all of M via a standard locally finite cover and partition of unity.

$\square$

Alternatively, we recall that since $\mathcal{O}$ is a finite sheeted cover, π induces an injection on deRham cohomology.
Therefore $\pi^*: H^n_{dR}(M) \hookrightarrow H^n_{dR}(\mathcal{O})$ is injective. As shown in part a, $\pi^*\omega = 0$ on the level of cohomology $\forall\, \omega \in H^n_{dR}(M)$.
Therefore, $H^n_{dR}(M) = 0$ and so all n-forms are exact since all n forms are closed.

$\square$

④ 405 846 515

Let $\varphi : [0,1] \to [0,1]$ be smooth such that $\varphi = 0$ on $[1, 1/3)$ and $\varphi = 1$ on $(1/3, 1]$.
Define

$$F : D^n \to D^n \quad \text{s.t.} \quad F(x) = \begin{cases} \varphi(|x|)\, f\left(\frac{x}{|x|}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

For $x \neq 0$, $x \mapsto \frac{x}{|x|}$ is smooth and $x \mapsto |x|$ is smooth. Therefore by smoothness of f, φ F is smooth for $x \neq 0$.
By construction of φ, $F \equiv 0$ on a neighborhood of 0. Therefore F is smooth at 0, and hence smooth on D^n.
Finally, by construction, for $x \in S^{n-1} \subset D^n$, $\frac{x}{|x|} = x$ and $\varphi(|x|) = 1$ so $F(x) = f(x)$ as desired.

$\square$

5) 405 846 515

($\Rightarrow$) Suppose that $\omega = \lambda df$ locally.
Then locally,

$$\omega \wedge d\omega = \lambda df \wedge d\lambda \wedge df$$
$$= -\lambda d\lambda (df \wedge df)$$

Since df is a 1-form
$$df \wedge df = -df \wedge df$$
and so $df \wedge df = 0 \Rightarrow \omega \wedge d\omega = 0.$

($\Leftarrow$) Suppose instead that $\omega \wedge d\omega = 0.$
~~For each $p \in M$, $\exists$ a neighborhood U_p of p and coordinates $x_1, \dots, x_n$ s.t. $\omega =$~~
~~Then~~ We claim then that $\ker \omega$ is integrable.
It suffices to show that if $X, Y \in \ker \omega$ then $[X,Y] \in \ker \omega$.
We recall that

$$d\omega(X,Y) = \cancel{X\omega(Y)} - \cancel{Y\omega(X)} - \omega([X,Y])$$
$$= -\omega([X,Y])$$

If X, Y are linearly dependent, then $d\omega(X,Y) = -\omega([X,Y]) = 0$ which concludes. Otherwise ~~$\exists\, Z$ s.t. locall~~ locally $\exists\, Z$ s.t. $X, Y, \cancel{Z}$ ~~form a basis.~~ Then are linearly independent and $Z \notin \ker \omega$

$$0 = \omega \wedge d\omega(X,Y,Z) = \omega(Z) d\omega(X,Y) \Rightarrow d\omega(X,Y) = 0$$

~~Since $\ker\omega$ has codimension~~ Then $\ker \omega$ is integrable.
By definition, this implies $\exists$ local coordinates so that
$$\ker \omega = \mathbb{R}\langle \partial/\partial x_1, \dots, \partial/\partial x_{n-1} \rangle \Rightarrow \omega = \lambda dx_n$$
for some λ. Taking $f = x_n$ ~~and extending via~~ a then concludes.

$\square$

(7) 405 846 515

(a) Let $\{U_\alpha, \varphi_\alpha\}$ be an atlas for H. We aim to use $\{U_\alpha, \varphi_\alpha\}$ to construct an atlas $A = \{V_\beta, \psi_\beta\}$ for M. Let H_1, H_2 denote the 2 handlebodies in M and let $\{U_\alpha^1, \varphi_\alpha^1\}, \{U_\alpha^2, \varphi_\alpha^2\}$ be their associated atlases.

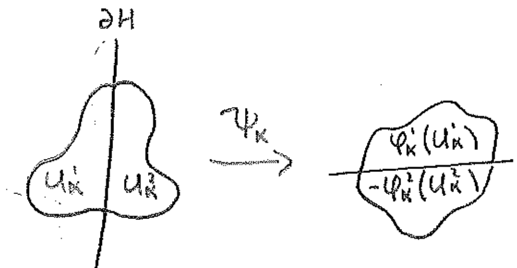
~~For all α s.t. $U_\alpha \cap \partial H = \emptyset$,~~

Define

$$A = \begin{cases} (U_\alpha^1, \varphi_\alpha^1), (U_\alpha^2, \varphi_\alpha^2) & \forall \alpha \text{ s.t. } U_\alpha \cap \partial H = \emptyset \\ (U_\alpha^1 \cup U_\alpha^2, \psi_\alpha) & \forall \alpha \text{ s.t. } U_\alpha \cap \partial H \neq \emptyset \end{cases}$$

where $\psi_\alpha : U_\alpha^1 \cup U_\alpha^2 \to \mathbb{R}^n$ is defined by

$$\psi_\alpha(p) = \begin{cases} \varphi_\alpha^1(p) & p \in U_\alpha^1 \\ -\varphi_\alpha^2(p) & p \in U_\alpha^2 \end{cases}$$



Since $\varphi_\alpha^1, \varphi_\alpha^2$ map diffeomorphically onto $\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n \geq 0\}$, $\psi_\alpha : U_\alpha^1 \cup U_\alpha^2 \to \psi_\alpha(U_\alpha^1 \cup U_\alpha^2)$ is diffeomorphic (and symmetric across ~~the~~ $\partial \mathbb{H}^n$). ~~We note that in A, the only overlaps of atlases~~

Since the ~~transition~~ maps in H_1, H_2 are smooth.

this gives smooth ~~transition~~ maps in M.

Therefore M is a smooth 3-manifold.

Since H_1, H_2 are compact, M is compact. Since H_1, H_2 are identified at their boundary, M is boundaryless.

Therefore M is closed.

$\longrightarrow$

(b) We proceed by Mayer-Vietoris.

Let U be an ε-neighborhood of $H_1 \subset M$ that deformation retracts onto H_1 and similarly for V w/ H_2.

Then $U \cup V = M$ and $U \cap V$ deformation retracts onto $\partial H_1 = \partial H_2 \cong M_g$. Then we get a long exact sequence

$$\cdots \longrightarrow H_k(M_g) \xrightarrow{\binom{i_*}{j_*}} H_k(H) \oplus H_k(H) \xrightarrow{k_* - l_*} H_k(M) \xrightarrow{\partial_*} \cdots$$

where $i: U \cap V \hookrightarrow U$, $j: U \cap V \hookrightarrow V$, $k: U \hookrightarrow M$, $l: V \hookrightarrow M$ and ∂ is the ~~max~~ boundary map from the snake lemma.

We recall that M_g has homology

$$H_k(M_g) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ \mathbb{Z}^{2g} & k = 1 \\ 0 & \text{else} \end{cases}$$

For H, we note that H is the inside of a genus g surface. Therefore H deformation retracts onto the wedge of g circles $\bigvee_{i=1}^{g} S^1$. Then

$$H_k(H) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}^g & k = 1 \\ 0 & \text{else} \end{cases}$$

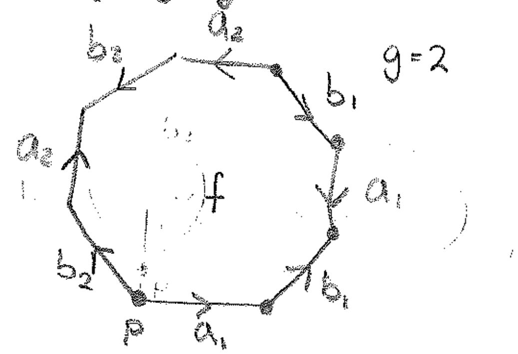
which gives an exact sequence

$$0 \to \mathbb{Z} \to 0 \to H_2(M) \hookrightarrow$$
$$\hookrightarrow \mathbb{Z}^{2g} \to \mathbb{Z}^{2g} \to H_1(M) \to$$
$$\to \mathbb{Z} \to \mathbb{Z} \to H_0(\cancel{M})^{\mathbb{Z}} \to 0$$

We note $H_0(M) \cong \mathbb{Z}$ since M is connected.

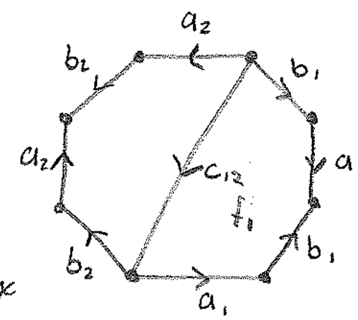
(b) We can construct a genus g surface by

- 1 0-cell: p
- $2g$ 1-cells: $a_1, b_1, \dots, a_g, b_g$ w/ $\partial_1 a_i = \partial_1 b_i = p - p$
- 1 2-cells: f w/ $\partial_2 f = a_1 + b_1 - a_1 - b_1 + \dots + a_g + b_g - a_g - b_g$



To construct a handlebody H, it is necessary to add additional 1-cells and 2-cells. We do this via

- 1 0-cell p
- $3g$ 1-cells $a_1, b_1, c_1, \dots, a_g, b_g, c_g$ w/ $\partial a_i = \partial b_i = \partial c_i = p - p$
- g 2-cells $f_1, \dots, f_g$ w/ $\partial_2 f_i = a_i + b_i - a_i - b_i + c_i$
- g 3-cells $A_1, \dots, A_g$ w/ $\partial_3 A_i = f_i$



This gets to notation/index hell

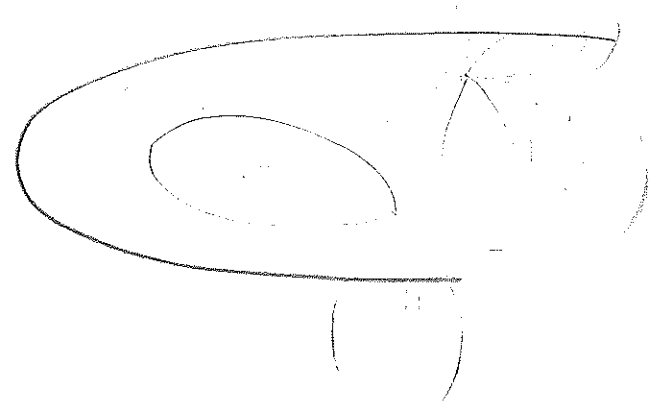
(b)

We can construct a genus g surface via

1 0-cell: p

$2g$ 1-cells: $a_1, b_1, \ldots, a_g, b_g$ w/ $\partial a_i = \partial b_i = p - p$

g 2-cells: $f_1, \ldots, f_g$ w/ $\partial f_i =$



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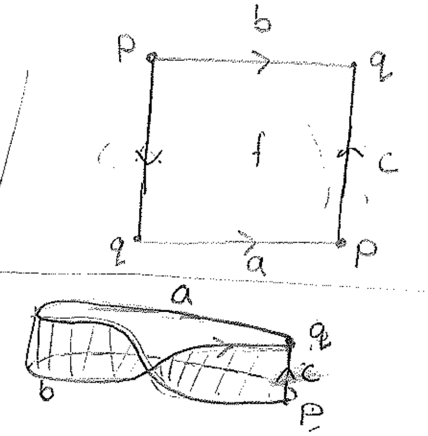
(a) We recall that the Mobius strip has a cellular decomposition as follows



M: 2 0-cells : p, q

3 1-cells : a, b, c w/ $\partial_1 a = p - q$, $\partial_1 b = q - p$, $\partial_1 c = q - p$

1 2-cell : f w/ $\partial_2 f = a + c - b + c = a - b + 2c$

B/c the boundary of M is $a+b$, ~~Therefore~~ we can express X via the cellular decomposition

2 0-cells : p, q

4 1-cells : a, b, c, d as above w/ $\partial d = q - p$

2 2-cells : f, g w/ $\partial_2 f = a - b + 2c$

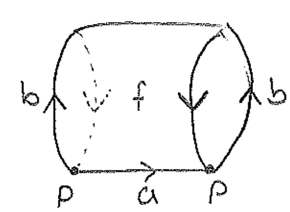
$\partial_2 g = a - b + 2d$

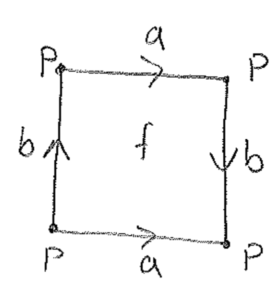
We note that in the above decomposition, $f + g$ form a cylinder which is attached on one end to $c - d$ and the other end to $d - c$. Therefore we may simplify the cellular decomposition to

1 0-cell : p

2 1-cell : a, b w/ $\partial_1 a = \partial_1 b = p - p = 0$

1 2-cell : f w/ $\partial f = a - b - a - b = -2b$





hence X then only has 1 0-cell, $\pi_1(X)$ has a representation w/ generators a, b and relations given by ∂f. Then

$$\pi_1(X) \cong \langle a, b \mid ab^{-1}a^{-1}b^{-1} \rangle \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

which is what was to be found.

⟶

(b) ~~We From the~~ Let M_g denote the compact orientable surface of genus g. We recall that $\chi(M_g) = 2-2g$.

From the above cellular decomposition, $\chi(X) = 1-2+1=0$.

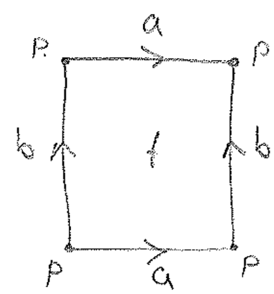
Therefore if X is homotopy equivalent to M_g then $g=1$ since Euler characteristic is homotopy ~~equivalent~~ invariant.

However, we can construct M_1 as

1 0-cell p:

2 1-cells: a, b w/ $\partial a = \partial b = p - p$

1 2-cell: f w/ $\partial f = a+b-a-b=0$



Therefore $\pi_1(M_1) \cong \langle a,b \mid aba^{-1}b^{-1}\rangle \cong \mathbb{Z}^2$

However $\mathbb{Z}^2 \not\cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and so M_1 and X are not homotopy equivalent since π_1 is homotopy invariant.

□

Alternately, from (a),

$$H_1(X) \cong \cancel{\langle a,b \mid ab^{-1}a^{-1}b^{-1}\rangle} \ \langle a,b \mid ab^{-1}a^{-1}b^{-1}, aba^{-1}b^{-1}\rangle$$

so $ab=ba$ and $a = bab \Rightarrow b^2 = e$. Then

$$H_1(X) \cong \langle a,b \mid b^2, aba^{-1}b^{-1}\rangle \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

However $H_1(M_g) \cong \mathbb{Z}^{2g}$. Since homology is homotopy invariant, this concludes.

□

(9) 405 846 515

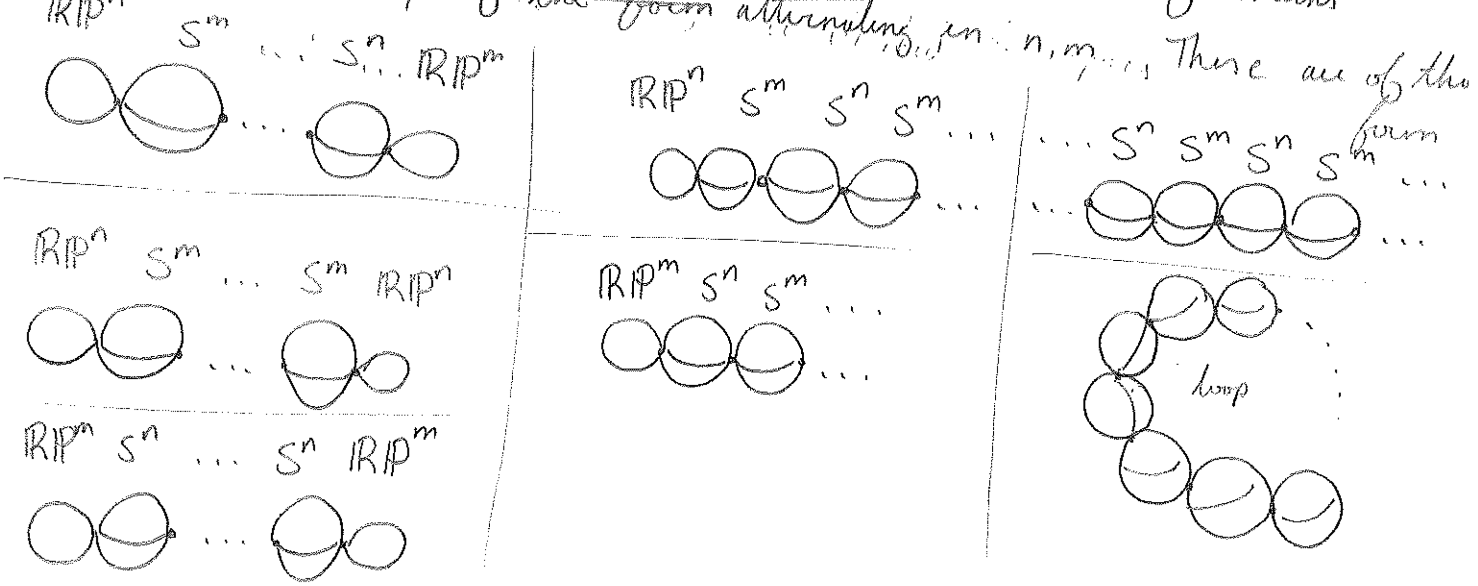
$n, m > 1$

We recall that the connected covers of $\mathbb{RP}^n$ are S^n, which is a double cover via the antipodal identification, and $\mathbb{RP}^n$, which is trivially a single cover via the identity. We note that these are the only covers of $\mathbb{RP}^n$ because the universal cover of S^n is S^n and so any additional covers of $\mathbb{RP}^n$ would contradict this.

Consider $\mathbb{RP}^n \vee \mathbb{RP}^m$, w/ wedge point p. Any cover of $\mathbb{RP}^n \vee \mathbb{RP}^m$ will consist of wedges of covers of $\mathbb{RP}^n, \mathbb{RP}^m$ respectively, where the wedge points are pre-images of p. Each pre-image of p must wedge a region covering $\mathbb{RP}^n$ to a region covering $\mathbb{RP}^m$.

Since S^n is a double cover of $\mathbb{RP}^m$, p has two pre-images in S^n, each of which can be attached to a cover of $\mathbb{RP}^m$. On the other hand, since $\mathbb{RP}^n$ is a single cover of $\mathbb{RP}^n$, p only has a single pre-image in $\mathbb{RP}^n$, and so each copy of $\mathbb{RP}^n$ in the cover can only be attached to a single cover of $\mathbb{RP}^m$. The same reasoning holds for m.

Therefore, all connected covers of $\mathbb{RP}^n \vee \mathbb{RP}^m$ will consist of chains of $\mathbb{RP}^n, S^n, S^m, \mathbb{RP}^m$, ~~of the form~~ alternating in n, m. These are of the form



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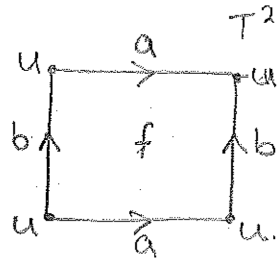
We note that D_1 is attached to T^2 via a degree p map along one coordinate

D_2 is attached to T^2 via a degree q map along other coordinate.

We can thus construct X as follows

1 0-cell : u



2 1-cell : a, b w/ $\partial_1 a = \partial_1 b = u - u = 0$

3 2-cell : f, D_1, D_2 w/ $\partial_2 f = a + b - a - b = 0$

$$\partial_2 D_1 = pa$$

$$\partial_2 D_2 = qb$$

Which yields the chain complex

$$\begin{array}{ccccccccc} 0 \to & C_2(X) & \xrightarrow{\partial_2} & C_1(X) & \xrightarrow{\partial_1} & C_0(X) & \to 0 \\ & \cong & & \cong & & \cong & \\ & \mathbb{Z}^3 & \xrightarrow[(x,y,z)\mapsto(0,py,qz)]{} & \mathbb{Z}^2 & \xrightarrow[0]{} & \mathbb{Z} & \end{array}$$

This yields the following homology groups

$$H_0(X) = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \frac{\mathbb{Z}\langle u \rangle}{0} \cong \mathbb{Z}$$

$$H_1(X) = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\mathbb{Z}\langle a, b \rangle}{\mathbb{Z}\langle 0, pa, qb \rangle} \cong \frac{\mathbb{Z}\langle a \rangle}{\mathbb{Z}\langle pa \rangle} \oplus \frac{\mathbb{Z}\langle b \rangle}{\mathbb{Z}\langle qb \rangle} \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}$$

$$H_2(X) = \frac{\ker \partial_2}{\operatorname{Im} \partial_1} = \frac{\mathbb{Z}\langle f \rangle}{0} \cong \mathbb{Z}$$

and 0 for all higher homologies. Therefore

$$H_k(X) \cong \begin{cases} \mathbb{Z} & k = 0, 2 \\ \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z} & k = 1 \\ 0 & \text{else} \end{cases}$$

$\square$

Geometry Topology Qualifying Exam, Fall 2018

(Q-1) Let M be a compact smooth n-manifold, and $f: M \to \mathbb{R}^n$ a smooth map. Let

$$S = \{p \in M \mid \operatorname{rank}(df_p) < n\}.$$

(a) Prove $S \neq \varnothing$.

(b) Prove $f(S) \subset \mathbb{R}^n$ has empty interior.

(Q-2) Let M_n be the space of $n \times n$ real matrices, viewed as the smooth manifold $\mathbb{R}^{n^2}$. Let M_n^k be the subset of matrices of rank k. Prove that M_n^k is a smooth submanifold of M_n. (Hint: First prove the subset of M_n^k where the top-left $k \times k$ minor is non-singular is a smooth submanifold M_n.)

(Q-3) Let θ be the restriction of

$$(x^2 dx^1 - x^1 dx^2) + (x^4 dx^3 - x^3 dx^4) + \cdots + (x^{2n} dx^{2n-1} - x^{2n-1} dx^{2n})$$

to the unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$. Prove $\ker(\theta)$ is a distribution on S^{2n-1}. Is it integrable?

(Q-4) Let M be a compact smooth 3-manifold and $\omega \in \Omega^1(M)$ a nowehere zero 1-form, so that $\ker(\omega)$ is an integrable distribution. Prove the following.

(a) $\omega \wedge d\omega = 0$.

(b) There exists some 1-form α with $d\omega = \alpha \wedge \omega$.

(c) $d\alpha \wedge \omega = 0$.

(Q-5) Let $M \subset \mathbb{R}^n$ be a compact $(n-1)$-dimensional submanifold, let $\iota: M \hookrightarrow \mathbb{R}^n$ be the inclusion map, and let $D \subset \mathbb{R}^n$ be the n-dimensional compact region with $\partial D = M$. Let $dV = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \in \Omega^n(\mathbb{R}^n)$ be the standard volume form on $\mathbb{R}^n$.

(a) Define $dA \in \Omega^{n-1}(M)$, the standard volume form on M, induced by the embedding ι.

(b) Prove $\iota^*(i_X dV) = \langle X, N\rangle\, dA$, for any smooth vector field X on $\mathbb{R}^n$. (Here, N is the unit normal vector field along M, pointing outward from D.)

(c) Prove

$$\int_D L_X(dV) = \int_M \langle X, N\rangle\, dA.$$

(d) Derive Gauss' Divergence Theorem from the case $n = 3$.

(Q-6) Can a finite rank free group have a finite index subgroup of smaller rank?

(Q-7) Prove that the covering map $S^n \to \mathbb{RP}^n$ induces an isomorphism on de Rham cohomology if and only if n is odd. What is the orientable double cover of $\mathbb{RP}^n$?

(Q-8) Assume the integral homology of a space is $\mathbb{Z}$ in grading 0, $\mathbb{Z}$ in grading 2, $\mathbb{Z}/2$ in grading 3, and 0 in all other gradings.

(a) What is its integral cohomology group?

(b) Construct a simply connected CW complex X with the given homology.

(c) Construct another simply connected CW complex Y with the same homology, which is not homotopy equivalent to X.

(Q-9) Let X be a connected CW-complex. Show that there is a natural isomorphism

$$\widetilde{H}_k(\Sigma X; M) \cong \widetilde{H}_{k-1}(X; M)$$

for all k and for all abelian groups M.

(Q-10) Let Y be a connected and simply connected CW-complex.

(a) Compute the fundamental group of $Y \vee S^1$.

(b) Describe the universal cover of $Y \vee S^1$, together with the action of the deck transformations.

(c) Describe all finite covers of $Y \vee S^1$, again with the action of the deck transformations.

(d) Describe what changes in the first two parts for $Y = \mathbb{RP}^2$.

QUALIFYING EXAM
Geometry/Topology
March 2018

Attempt all ten problems. Each problem is worth 10 points. Justify your answers carefully.

1. Suppose that M and N are connected smooth manifolds of the same dimension and $f : M \to N$ is a smooth submersion.

(a) Prove that if M is compact, then f is onto and f is a covering map.

(b) Give an example of a smooth submersion $f : M \to N$ such that M and N have the same dimension, N is compact, and f is onto, but f is not a covering map.

2. Let $\Phi_N, \Phi_S : \mathbb{R} \times S^2 \to S^2$ be two global flows on the sphere S^2. Show that there exist $\epsilon > 0$, a neighborhood U of the North pole, a neighborhood V of the South pole, and a global flow $\Phi : \mathbb{R} \times S^2 \to S^2$ such that $\Phi(t, q) = \Phi_N(t, q)$ for all $t \in (-\epsilon, \epsilon), q \in U$, and $\Phi(t, q) = \Phi_S(t, q)$ for all $t \in (-\epsilon, \epsilon), q \in V$.

3. For $n \geq 1$, consider the subset $X \subset \mathbb{CP}^{2n}$ given by

$$X = \{[z_0 : z_1 : \cdots : z_{2n}] \in \mathbb{CP}^{2n} \mid z_{n+1} = z_{n+2} = \cdots = z_{2n} = 0\}.$$

(a) Show that X is a smooth submanifold.

(b) Calculate the mod 2 intersection number of X with itself.

4. Suppose N is a smoothly embedded submanifold of a smooth manifold M. A vector field X on M is called tangent to N if $X_p \in T_pN \subset T_pM$ for all $p \in M$.

(a) Show that if X and Y are vector fields on M both tangent to N, then $[X, Y]$ is also tangent to N.

(b) Illustrate this principle by choosing two vector fields X, Y tangent to $S^2 \subset \mathbb{R}^3$ (such that $[X, Y]$ is not identically zero), computing $[X, Y]$ and checking that it is tangent to S^2.

5. A symplectic form on an eight-dimensional manifold is defined to be a closed two-form ω such that $\omega \wedge \omega \wedge \omega \wedge \omega$ is a volume form (that is, everywhere nonvanishing). Determine which of the following manifolds admit symplectic forms: (a) S^8; (b) $S^2 \times S^6$; (c) $S^2 \times S^2 \times S^2 \times S^2$.

6. Let U be a bounded open set in $\mathbb{R}^3$ with smooth boundary, and let V be a smooth vector field on $\mathbb{R}^3$. The classical divergence theorem expresses the triple integral $\iiint_V \operatorname{div} V d(\mathrm{vol})$ as a surface integral over the boundary of V. State this theorem, and show how it can be obtained as a particular case of Stokes' Theorem for differential forms.

7. Let M and N be smooth, connected, orientable n-manifolds for $n \geq 3$, and let $M \# N$ denote their connect sum.

(a) Compute the fundamental group of $M \# N$ in terms of that of M and of N (you may assume that the basepoint is on the boundary sphere along which we glue M and N).

(b) Compute the homology groups of $M\#N$. (You may use without proof that $H_n(-;\mathbb{Z})$ of a connected orientable n-manifold is always isomorphic to $\mathbb{Z}$).

(c) For part (a), what changes if $n = 2$? Use this to describe the fundamental groups of orientable surfaces.

8. Determine all of the possible degrees of maps $S^2 \to S^1 \times S^1$.

9. Point S^2 via the south pole, and consider the Cartesian product $S^2 \times S^2$.

(a) Describe a cell structure on $S^2 \times S^2$ that is compatible with the inclusion of

$$S^2 \vee S^2 \hookrightarrow S^2 \times S^2$$

as those pairs where one coordinate is the south pole.

(b) Let X be $(S^2 \times S^2) \cup_{S^2} D^3$, where we attach the 3-disk via the map

$$S^2 \to S^2 \vee S^2$$

which crushes a great circle connecting the north and south poles. Compute the homology groups of X.

10. Let X be a semi-locally simply connected space and let $\tilde{X} \to X$ be the universal cover.

(a) Show that any map $\sigma\colon \Delta^n \to X$ lifts to a map $\tilde{\sigma}\colon \Delta^n \to \tilde{X}$, where Δ^n is the standard n-simplex.

(b) Show that if $\tilde{\sigma}_1, \tilde{\sigma}_2\colon \Delta^n \to \tilde{X}$ are two lifts of σ, then there is an element g of the fundamental group of X such that $g \circ \tilde{\sigma}_1 = \tilde{\sigma}_2$, where we view g as an automorphism of $\tilde{X}$ via the deck transformations.