Fundamental Group
F17.8

(a) \( \mathbb{R}^3 \text{-A deformation} \) extends to \( T^2 \) via \( \pi_1(\mathbb{R}^3 \text{-A}) \cong \pi_1(T^2) \cong \mathbb{Z}^2 \)

(b) \( \mathbb{R}^3 \text{-A-B deformation} \) extends to \( M_2 \). So \( \pi_1(\mathbb{R}^3 \text{-A-B}) \cong \pi_1(M_2) \cong \mathbb{Z}^2 \)

(c) 

F17.10

(a) \( \mathbb{R}^3 \text{-A deformation} \) extends to a sphere via a line through the middle \( \Omega \). Therefore \( \pi_1(\mathbb{R}^3 \text{-A}) \cong \pi_1(S^2) \times \pi_1(S^1) \cong \mathbb{Z}^2 \)

(b) \( \mathbb{R}^3 \text{-A-B deformation} \) extends to the wedge of 2 spheres each via a line. Therefore \( \pi_1(\mathbb{R}^3 \text{-A-B}) \cong \pi_1(S^2) \times \pi_1(S^1) \times \pi_1(S^2) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}^2 \)

(c) Same reasoning, the deformation extends to \( S^2 \cup T^2 \) is \( \pi_1(\mathbb{R}^3 \text{-A-B}) \cong \mathbb{Z}^2 \).

S18.7

(a) Van Kampen's \( B^n \text{-ball} \) 

U \-neighborhood \( G \) of \( M \cup B \), \( V \) \-neighborhood \( \overline{M \cup B} \). Then \( \cup \cup \). 

\( \cup \cup \). 

Van Kampen's yehde \( \pi_1(M \cup N) \cong \pi_1(M) \times \pi_1(N) \).

(b) Mayer-Vietoris \( U \cup V \text{ as above. Yehde LES} \)

\[ \cdots \to H_k(S^{n-1}) \to H_k(M \cup N) \cong H_k(M) \oplus H_k(N) \to H_k(M \# N) \to \cdots \]

By a similar Mayer-Vietoris, \( H_k(M \cup N) \cong H_k(M) \). So \( H_k(S^{n-1}) \to H_k(M \cup N) \). 

For \( 0 \leq k \leq n-2 \), this gives \( H_k(M \# N) \cong H_k(M) \oplus H_k(N) \).

For 0, \( H_0(M \# N) \cong \mathbb{Z} \) since connected.

For 1, 

\[ \mathbb{Z}^2 \rightarrow H_1(M \# N) \rightarrow H_1(M) \oplus H_1(N) \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \to \mathbb{Z} \to \cdots \]

So \( H_1(M) \oplus H_1(N) \cong H_1(M \# N) \).
Recall $S^2$ is a 2-com of $\mathbb{RP}^2$ and $\pi_1(S^2) = \mathbb{Z}$, $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$

and

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & \text{else} \end{cases}$$

$$H_k(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2\mathbb{Z} & k = 1 \\ 0 & \text{else} \end{cases}$$

$$H^k(S^2) = \begin{cases} \mathbb{R} & k = 0, 2 \\ 0 & \text{else} \end{cases}$$

$$H^k(\mathbb{RP}^2) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & \text{else} \end{cases}$$

Therefore (1), (2), (3) are false.

By stereographic projection, $S^3 \setminus H$ is diffeomorphic to $\mathbb{R}^3$ excluding a line and circle around it (think removing z axis and $S^2$'s xy plane).

This deformation retracts to $T^2$. Therefore $\pi_1(S^3 \setminus H) = \pi_1(T^2) = \mathbb{Z}^2$.

$$H_k(S^3 \setminus H) = H_k(T^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ \mathbb{Z}^2 & k = 1 \\ 0 & \text{else} \end{cases}$$

$$\mathbb{R}^3 \setminus \{L_1, L_2, L_3\} \cong S^2 \setminus 6 \text{points} \cong \mathbb{R}^2 \setminus 5 \text{points} \cong \bigvee_{k=1}^{6} S^1$$

Therefore $\pi_1(\mathbb{R}^3 \setminus \{L_1, L_2, L_3\}) \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

F17.8

First $n \geq 1$, wedge $2n$ copies of $S^2$ together in a circle, $-1$ alternating identifications as $A \cdot B$ (the 2 $\mathbb{RP}^2$).

Then $p_\# \pi_1(Y, y) \cong \langle ab \rangle^n$. 

---

![Diagram of a circle with identifications](image-url)
Finally, connected closed orientable genus $H_n(M \# N) \cong \mathbb{Z} \times 0$

$$0 \to \mathbb{Z} \to \mathbb{Z} \to H_{n-1}(M) \otimes H_{n-1}(N) \to H_{n-1}(M \# N) \to 0$$

which gives $H_{n-1}(M \# N) \cong H_{n-1}(M) \otimes H_{n-1}(N)$.

Therefore

$$H_k(M \# N) = \begin{cases} \mathbb{Z} & k=0, n \\ H_k(M) \otimes H_k(N) & \text{else} \end{cases}$$

(c) when $n=2$, $M = M_j$, $N = N_h$ for $j, h$. Then

$$M \# N = M_j \# M_h \cong M_{j+h}$$

which implies

$$H_k(M \# N) = \begin{cases} \mathbb{Z} & k=0, 2 \\ \mathbb{Z}^{n+h} & \text{else} \\ 0 & \text{else} \end{cases}$$

$$\pi_k(M \# N) = \mathbb{Z}^{n+h}$$
Learning Spaces
13.1

non-regular \rightarrow local differs
\rightarrow open map

(1) No.
\((0,1) \mapsto (0,2)\) for injective
\((0,2) \rightarrow S'\) for injective

(2) No, \((0,1) \mapsto [0,2]\)

(3) yes, local differs

(4) No, \((0,1) \mapsto (0,2)\) not closed

13.7

M, deformation retracts to \(S'\) vs \(S\). Therefore

\(\chi(M) = -1\).

Any orientable 3-fold cover \(\Sigma\) of \(M\) will be a closed orientable surface. Therefore any 3-cover of \(M\) will be a genus \(g\) surface with \(n\) punctures for some \(n\), denoted \(\Sigma_{g,n}\).

We will \(\Sigma_{g,n}\) deformation retract onto the wedge of \(2g+n-1\) copies of \(S'\). Then \(\chi(\Sigma_{g,n}) = 1 - 2g - n + 1 = 2 - 2g - n\).

Therefore if \(\Sigma_{g,n}\) is a 3-fold cover, then \(\chi(\Sigma_{g,n}) = 3\chi(M)\)

\(\implies 2g - n = 3\)

\(5 = 2g + n\)

Hence \((g,n) = (0,5), (1,3), (2,1)\).

Boundary to boundary rules out \((0,5)\) and \((2,1)\).

Therefore \(\Sigma_{1,3}\) is the only option.
\(\Sigma_{1,3}\) covers \(M\) via \((0,0) \mapsto (0,3\Phi)\).
$S^1 \times S^1$ has fundamental group $\langle a, b \rangle$. 

$\langle a \rangle \cdot \langle a, b \rangle$ is a non-normal subgroup.

Define $\tilde{X} = \frac{\mathbb{Q} \oplus \mathbb{Q}}{6}$.

Thus $\rho \circ \pi_1(\tilde{X}) = \langle a \rangle$ which is not normal.

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14.9

(a) $\mathbb{RP}^3 \# \mathbb{RP}^3 \cong \mathbb{RP}^2 \sqcup S^2 \times [0,1] \sqcup \mathbb{RP}^2/\sim \cong [x]$ 

$\mathbb{RP}^3 = D^3 \sqcup \mathbb{RP}^2/\sim \cong \times \sim [x], \forall x \in S^2 = \partial D^3.$

$\lambda_0 \mathbb{RP}^3 \# \mathbb{RP}^3 \cong \mathbb{RP}^3 \sqcup S^2 \times [0,1] \sqcup \mathbb{RP}^2/\sim \cong (x,0) \sim \{x\}$ as before

$\cong Y$

(b) $S^2 \times S^1$ is a double cone of $Y$ via

$S^1 = [0,1] / 0 \sim 2$

$\lambda_1 S^1 \times S^1$ double covering first copy of $\mathbb{RP}^2$

$S^1 \times S^1$ "second"  

$\mathbb{RP}^2 \sqcup \mathbb{RP}^2 \cong S^2 \cong \mathbb{RP}^2 \sqcup \mathbb{RP}^2$  

$S^2 \times [0,1] \cong Y$
(a) Let $\Pi$ be the deck transformation on $O$. Then
\[ \Pi^* \omega = (\Pi \circ f)^* \omega = f^* \Pi^* \omega \]
and $\omega \mid_0 \Pi^* \omega = \int_0^1 f^* \Pi^* \omega = - \int_0^1 \Pi^* \omega = - \int_\Pi^* \omega = 0$.

Since $O$ is compact orientable, $\Pi^* \omega$ is exact.

(b) $H^0_{dR}(O) \cong H^0_{dR}(M)$ is an isomorphism, so $\omega$ is exact.

Injectible if and only if
\[ (\Pi^*)^{-1}(\Pi) = \frac{1}{2} \sum_{k=1,2} \text{stub} \]

F14.9

$\mathbb{R}P^k$ has cohomology $\mathbb{R}P^k$ and $S^k$. Choose them together for some of
$\mathbb{R}P^2$ or $\mathbb{R}P^3$.

F15.8

Suppose $\mathbb{C}P^n$ acts on $\mathbb{C}P^{2n}$.

To show $X = \mathbb{C}P^n$, it suffices to show $\pi_1(X) \cong \mathbb{Z}$.

Any deck transformation. Consider a deck transformation $f \in \mathbb{C}P^{2n}$

if $f$ has a fixed point, then $f \equiv 0$. Calculating the fixed number
\[ L(f) = \sum_{k=0}^{2n} \chi_k \cdot \text{tr}(f^* : H^k(\mathbb{C}P^{2n}; \mathbb{Q}) \to H^k(\mathbb{C}P^{2n}; \mathbb{Q})) \]

for all $H^k$.
\[ L(f) = \sum_{k=0}^{2n} \chi_k \cdot \text{tr}(f^* : H^{2k}(\mathbb{C}P^{2n}; \mathbb{Q}) \to H^{2k}(\mathbb{C}P^{2n}; \mathbb{Q})) \]

Recall $H^{2k}(\mathbb{C}P^{2n}; \mathbb{Q}) = \mathbb{Q}$ for all $k$. Then
\[ f^* : H^2(\mathbb{C}P^{2n}; \mathbb{Q}) \to H^2(\mathbb{C}P^{2n}; \mathbb{Q}) \text{ is multiplication by } q, \]
\[ L(f) = 1 + q + \ldots + q^{2n} \neq 0 \]
so $f$ has a fixed point $\Rightarrow f = \text{id}$.

Then $\pi_1(x) = 0 \Rightarrow x \in \mathbb{C}P^n$.

F16.7

$\mathbb{R}^n$ is the universal cover of $(S^1)^n \Rightarrow p_\# \pi_1(\mathbb{R}^n) = 0$.

Since $\pi_1(x)$ is finite, $f_\# \pi_1(x)$ is finite.

Thus $f_\# \pi_1(x) \in \pi_1((S^1)^n) \cong \mathbb{Z}^n$ is finite, and hence $f_\# \pi_1(x) = 0$.

Therefore $f$ lifts to a map $\tilde{f} : \tilde{x} \rightarrow \mathbb{R}^n$ which is nullhomotopic.

This concludes.

S17.6

$f : Y \rightarrow X$, $p : \tilde{X} \rightarrow X$ comming

$f^*(\tilde{X}) = \{ (y, \tilde{x}) : f(y) = p(\tilde{x}) \text{ is } \tilde{x} \tilde{x} \}

f^* p : f^*(\tilde{X}) \rightarrow Y : (y, \tilde{x}) \rightarrow (f(y), p(\tilde{x}))$

S17.8

(a) $G$ acts naturally on $\tilde{X}$ via deck transformation, diffeomorphism, $g : \tilde{X} \rightarrow \tilde{X}$ $\forall g \in G$. Then $\forall g \in G$, $g : \tilde{X} \rightarrow \tilde{X}$ induces a map $g^* : H_k(\tilde{X}), H_k(\tilde{X})$.

(b)
(a) orientable if \( \exists \) an atlas \( \{ (U_k, \varphi_k^3) \text{ n.l. } \det (d(\varphi_k^{-1} \circ \varphi_\beta^3)) > 0 \} \)

\( \forall U_k \cap U_\beta \neq \emptyset \).

(b) Define \( \tilde{M} = \{(p, 0) : p \in M, \text{ orientation } 0 \at \mathbb{R}^3 \} \).

\( \tilde{M} \) is given the topology \( \{ \nu, 0 \} \) where \( U \in M \) is open and can be given orientation \( 0 \) and \( \nu_{u, 0} = \{(p, 0) : p \in U \} \).

Define \( \pi : (p, 0) \mapsto p \). Then since \( M \) is locally orientable,

\( \forall p \in M \exists \ U \text{ n.l. } U \) can be given \( \pm 0 \Rightarrow \pi^{-1}(U) = \nu_{u, 0} \cup \nu_{u,-0} \)

and \( \pi|_{\nu_{u,0}} = \text{id} \) \( \pi|_{\nu_{u,-0}} = \text{id} \).

On \( \nu_{u,0} \). To \( \tilde{M} \supseteq \tilde{M} \) is an orient \( \tilde{M} \cap 0 \at (p, 0) \).

Each connected component is a cone, so if \( \tilde{M} \) is connected

the \( \tilde{M} \) is orientable and connected \( \Rightarrow \) non-orientable.

1.5.8.

(a) local diffeo \( \Rightarrow \) open \( \Rightarrow \) \( \{(u) \in U \) open

\( M \) compact \( \Rightarrow \) \( \tilde{M} \) closed \( \Rightarrow \) continuous

covering map \( \Rightarrow \) from stalk of records theorem

(b) \( \pi : (0, 1) \rightarrow \mathbb{S}^1 \equiv \mathbb{R}/\mathbb{Z} \).

1.5.8.

(a) \( \pi_1((x)) = 0 \) \( \Rightarrow \) \( \sigma, \pi_1((x)) = 0 \) \( \Rightarrow \) \( p \circ \pi_1((x)) \)

(b) \( \forall x \in \mathbb{R}^n \). Then \( p(h_i(x_0)) = p(h_2(x_0)) \). Hence \( \pi_1((x)) \) is the

universal cover, \( \pi_1((x)) \) acts transitively on \( p_{(1)}\{h_i(x_0)\} \).

\( \exists y \in \pi_1((x)) \) n.l. \( \varphi_0(h_i(x_0)) \equiv h_i(x_0), \forall y \) \( \in \gamma \) a lift of \( y \),

such that \( \gamma \) agrees at \( y \).

Let \( \zeta = \{ x \in \mathbb{R}^n \} : \varphi_0(f) = f_\gamma(x) \). Then \( \zeta \) is nonempty. Closed by continuity, and open by

1.5.8.
If $n$ even, then $H^n (\mathbb{R}^n) \cong \mathbb{R} \neq 0 \cong H^n (\mathbb{R}P^n)$.

If $n$ odd, $\pi_* : H^n (\mathbb{R}P^n) \to H^n (\mathbb{R}P^n)$ is injective and hence an isomorphism.

If $n$ even, orientation are on $\mathbb{R}P^n$.

If $n$ odd, orientation are on $\mathbb{R}P^n$.

The dark transformation $x \mapsto -x$ is orientation reversing, so $\mathbb{R}P^n \cong \mathbb{R}P^n$ is an orientation cube.

519.6

$p \in Y \implies U_p \cap p$ and $V_p, \ldots, V_k \in X \implies U_p$ a diffeo.

Then define $g : H^k (U) \to H^k (\mathbb{R})$ are locally exact. $g$ on $U_p$ via

$$g(u) = \frac{1}{k} \sum_{j=1}^k \left( (f_j u) \right)^* u$$

Is well defined on homology and is an isomorphism for $f^*$. 

519.8

$$(2k, 2k+1) \mapsto (k, k+1)$$

contracted to a point near $2k + 1$. Then $p : 2k \mapsto 1$

$$2k \mapsto S$$

$$(2k, 2k+1) \mapsto W$$

$$(2k+1, 2k) \mapsto E$$

Unoriented circle in $S^2 \cong \mathbb{R}P^1$. 

$$(1/k, 2k+1) \mapsto 1$$

and $$(k, k+1)$$ contracted to a point near.
(a) Clear by Lemma 5.6.6.

(b) \( T \) \( [\lambda \alpha \cdot \alpha] \) in a fixed point of \( \tilde{A} : \mathbb{C}P^n \to \mathbb{C}P^n \),

\[ \tilde{A} \alpha \cdot \alpha = \lambda \alpha \cdot \alpha \]

(c) \( \tilde{A} \) is a identity map if \( V \) fixed points \( p \) \( \tilde{A} \),

\[ d\tilde{A}_p \] does not have eigenvalue 1.

All eigenvalues have multiplicity 1 \( \Rightarrow \tilde{A} \) is diagonalizable.

Changing basis, \( \tilde{A} \) diagonalizes \( \lambda \alpha \cdot \alpha \).

Then \( \tilde{A} \) has fixed point \( [0, \ldots, 0, \alpha] \).

In local coordinates, around \( \alpha \), \( \mathbb{C}P^n \to \mathbb{C} \alpha \cdot \alpha \).

Then \( d\tilde{A}_\alpha = diag(\lambda_1, \ldots, \lambda_N) \).

Since these are all eigenvalues are distinct, this concludes.

(d) \( \tilde{A} \) is homotopic to \( \tilde{I} \).

\[ \lambda(\tilde{A}) = \lambda(\tilde{I}) \cdot \gamma(\partial \mathbb{C}P^n) = n+1 \]
\( f: \mathbb{RP}^n \to \mathbb{RP}^n \)

(a) Suppose \( n \) is even. By definition

\[
L(t) = \sum_{k=0}^{n} (-1)^k \text{tr}(f^*: \mathbb{H}^{k}(\mathbb{RP}^n; \mathbb{Q}) \to \mathbb{H}^{k}(\mathbb{RP}^n; \mathbb{Q}))
\]

Then \( L(1) = 1 \) as \( f^*: \mathbb{H}^{0}(\mathbb{RP}^n; \mathbb{Q}) \to \mathbb{H}^{0}(\mathbb{RP}^n; \mathbb{Q}) \) is the identity.

(b) \( 2n+1 \) odd \( \mathbb{RP}^{2n+1} \cong S^{2n+1} \times \mathbb{C} \). Then \( L(t) = 0 \) for \( n \) even, and \( L(t) = n+1 \) for \( n \) odd. Then \( L(t) = 0 \) for \( n \) even, and \( L(t) = n+1 \) for \( n \) odd.

and \( L(t) = 0 \) for \( n \) even, and \( L(t) = n+1 \) for \( n \) odd.
(1) locally 7 coordinates $x_1, x_2, \ldots, x_n$ \( \text{ker}(w) = \mathbb{R} \langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \rangle \)

Then $w = f dx_n$ if non-vanishing

\[ \mathbf{f} = \frac{df}{f} \Rightarrow \mathbf{k} \wedge \mathbf{w} = df \wedge dx_n = dw \]

(3) if $\mathbf{w} \wedge \mathbf{k} \wedge \mathbf{w} = -w \wedge k \wedge w \Rightarrow w \wedge dw = 0$

(2) $\Rightarrow$ Frobenius theorem, \( \text{ker}(w) \) integrable if $x, y \in \ker w$

$\Rightarrow [x, y] \in \ker w$.

If $x, y \in \ker w$ then $A \cdot v \notin \ker w$

$0 = w \wedge dw(u, x, y) = w(u) dw(x, y) \Rightarrow dw(x, y) = 0$

Then $dw(x, y) = x(w(y)) - y(w(x)) - w [x, y]$

$0 = -w(x, y)$

$[x, y] \in \ker w$.

(2) $\Rightarrow$ if $w = \lambda df$ then $w \wedge dw = -\lambda df \wedge df \wedge dx = 0$

If $w \wedge dw = 0$ then $\ker w$ is integrable by

locally 7 coordinates $x_1, x_2, \ldots, x_n$ \( \ker w = \mathbb{R} \langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \rangle \)

$\Rightarrow w = f dx_n$.

F18.4 $\ker w$ integrable

(1) locally, $w = f dx_n \Rightarrow w \wedge dw = 0$

(2) $x = \frac{df}{f} \Rightarrow k \wedge w = dw$

(3) $0 = d(dw) = d(k \wedge w) = d(k \wedge w) = \mathbf{k} \wedge dw = \mathbf{k} \wedge \mathbf{w}$
Poincaré Houf
Poincaré Duality

\( H_{m,n}(X) = H^k_{C}(X) \)

F 16.5

1. \[ \int_N i^* w = \int_M \eta \wedge w \]

2. Let \( N = \Sigma_i A_i \).

Then for any \( w \in \Omega \),

\[ \int_N i^* w = \int_A a dx + b dy \]

3. \( m dx \) are even, \( dy \) is odd \( \subset \Omega \)

S 19.10

(1) \( n \) odd ⇒ \( \chi(EM) = 2 \chi(M) \)

(2) \( \chi(EM) = \chi(s^{n-1}) = 2 \)
Ju Derivatives
F15.2

\[ H([0,1] \times M \to N) \text{ homotopy from } g \text{ to } f. \] Then

\[
\begin{align*}
\{t \in [0,1] \}^\ast w - g^\ast w & = H(1, \cdot)_{\ast}w - H(0, \cdot)_{\ast}w \\
& = \int_0^1 \frac{d}{dt} H(t, \cdot)^\ast w \, dt \\
& = \int_0^1 \frac{d}{dt} i_{\tau} H^\ast w \, dt \\
& = \int_0^1 L_T H^\ast w \, dt \\
& = \int_0^1 d \circ i_{\tau} H^\ast w \, dt \\
& = d \left( \int_0^1 i_{\tau} H^\ast w \, dt \right)
\end{align*}
\]

S17.4

(a) \[ L_x w = (d \circ i_x \circ i_{\text{odd}}) w \]

(b) a flow \( \Phi_t(p) \) preserves volume

\[
\begin{align*}
\Leftrightarrow & \quad \Phi_t^\ast dV = dV \\
\Rightarrow & \quad L_x dV = 0 \\
\Leftrightarrow & \quad d \circ i_x dV = 0 \\
\Leftrightarrow & \quad dw(x) dV = 0 \\
\Leftrightarrow & \quad L_x dV = 0 \\
\Rightarrow & \quad \Phi_t^\ast L_x dV = 0 \\
\Rightarrow & \quad \frac{d}{dt} \Phi_t^\ast dV = 0 \quad \forall t_0 \\
\Rightarrow & \quad \Phi_t^\ast dV = dV
\end{align*}
\]
\[
\begin{align*}
[L_x, L_y] &= L_{[x,y]} \\
[L_x, d\eta + i\omega d] &= L_x \partial \omega - d\omega \partial L_x + L_x \partial i\gamma d - i\gamma d \partial L_x \\
&= d \left( L_x \partial i\gamma - i\gamma \partial L_x \right) + \left( L_x \partial i\gamma - i\gamma \partial L_x \right) \circ d \\
&= d \circ [L_x, i\gamma] + [d_x, i\gamma] \circ d \\
&= d \circ i_{(x,y)} + i_{(x,y)} \circ d \\
&= L_{(x,y)} \\
\end{align*}
\]

\[
\begin{align*}
[w_{k+1} (V_1, \ldots, V_k)] &= \left( i_{(x,y)} \circ w \right)(V_1, \ldots, V_k) \\
&= \left( i_{(x,y)} \circ \omega \right)(V_1, \ldots, V_k) - \left( i_{(x,y)} \circ \omega \right)([x,y], V_1, \ldots, V_k) \\
&= \left( i_{(x,y)} \circ \omega \right)([x,y], V_1, \ldots, V_k) \\
&= \left( i_{(x,y)} \circ \omega \right)([x,y], V_1, \ldots, V_k) \\
&= \left( i_{(x,y)} \circ \omega \right)([x,y], V_1, \ldots, V_k) \\
&= \left( i_{(x,y)} \circ \omega \right)([x,y], V_1, \ldots, V_k) \\
&= \left( i_{(x,y)} \circ \omega \right)([x,y], V_1, \ldots, V_k) \\
&= \left( i_{(x,y)} \circ \omega \right)([x,y], V_1, \ldots, V_k) \\
&= \left( i_{(x,y)} \circ \omega \right)([x,y], V_1, \ldots, V_k) \\
&= \left( i_{(x,y)} \circ \omega \right)([x,y], V_1, \ldots, V_k) \\
&= \left( i_{(x,y)} \circ \omega \right)([x,y], V_1, \ldots, V_k) \\
&= \left( i_{(x,y)} \circ \omega \right)([x,y], V_1, \ldots, V_k) \\
\end{align*}
\]
(b) $O(n)$ has trivializable tangent bundle

$V_1, \ldots, V_n$ bases for $T_x O(n)$

For $p \in O(n)$, define $X_i(p) = p^*(v_i)$

\[ (L_g) \ast \{x, y\} (t) = (L_g) \ast (x(y(t)) - y(x(t))) \]

\[ = x(y(tL_g)) - y(x(tL_g)) \]

\[ = x(L_g \ast y(t)) - y(L_g \ast x(t)) \]

\[ = x(y(t)) - y(x(t)) \]

\[ = [x, y] (t) . \]
Stokes' Theorem
Divergence Theorem

\[ \int_M \text{div}(X) \, dV = \int_{\partial M} \langle X, N \rangle \, dA \]

* N = normal VF to \( \partial M \) outwardly
* \( \langle X, N \rangle \) = standard inner product
* \( X = f \, \frac{\partial}{\partial x} + g \, \frac{\partial}{\partial y} + h \, \frac{\partial}{\partial z} \)
* \( \text{div}(X) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \)

\[ dA = c^* \text{ind} dV \quad \text{w} \quad c : \partial M \to \mathbb{M} \text{ immersion} \]

Define \( T = X - \langle X, N \rangle N \), tangent component of \( X \) to \( \partial M \)

Then \( c^* T dV = 0 \) since \( T, X, Y \) are linearly dependent \( \forall x, y \in T \partial M \)

\[ c^* i_N dV = \langle X, N \rangle \quad c^* i_N dV = \langle X, N \rangle dV \]

Then by Ostrogradski's,

\[ \int_{\partial M} \langle X, N \rangle \, dA = \int_M c^* i_N dV \]

\[ = \int_M d i_X dV \]

\[ = \int_M \text{div}(X) \, dV \]

\[ \square \]
Parallelizable

\( p \text{ odd, } q \text{ whatever,} \)

\[
T(s^p \times s^q) \cong \pi_p^*(Ts^p) \oplus \pi_q^*(Ts^q)
\]

\[
\cong \pi_p^* (E \oplus E^\perp) \oplus \pi_q^*(Ts^q) \oplus \epsilon
\]

\[
\cong \pi_p^* (E^\perp) \oplus \pi_q^*(Ts^q) \oplus \epsilon
\]

\[
\cong \pi_p^* (E^\perp) \oplus \pi_q^*(Ts^q) \oplus \epsilon
\]

\[
\cong \pi_p^* (E^\perp) \oplus \pi_q^*(Ts^q) \oplus \epsilon
\]

\[
\cong \pi_p^* (E^\perp) \oplus \pi_q^*(Ts^q) \oplus \epsilon
\]

\[
\cong \pi_p^*(T\mathbb{R}^q) \oplus \epsilon
\]

\[
\cong \epsilon
\]

as desired.
Diff Geo Prep
from Yan's notes.
(a) We claim that this holds if \( K = 1 \).

Suppose \( K = 1 \). Then

\[
\int_{\gamma'} w = \int_{\gamma} w = \int_{S^1} H_0^* w - \int_{S^1} H_0^* w
\]

\[
= \int_{\partial(S^1 \times [0,1])} d(H_0^* w) = \int_{S^1 \times \{0,1\}} H_0^* (d\omega) = 0
\]

Hence, \( \int_{\gamma'} w = \int_{\gamma} w \) for all smoothly homotopic curves.

Suppose instead that \( \int_{\gamma'} w = \int_{\gamma} w \) for all closed loops \( \gamma_0, \gamma_1 \). Let \( \gamma_0 \) be the unit circle and \( \gamma_1 \) be the circle of radius \( r \).

We parametrize \( Y_0 \) by \((\cos t, \sin t)\) for \( 0 \leq t \leq 2\pi \) and \( Y_1 \) by \( r(\cos t, \sin t) \) for \( 0 \leq t \leq 2\pi \). Then

\[
\int_{Y_0} w = \int_{0}^{2\pi} \overbrace{-\sin t(-sint+cost)}^{1} dt = \int_{0}^{2\pi} dt = 2\pi
\]

and

\[
\int_{Y_1} w = \int_{0}^{2\pi} \overbrace{-r\sin t(-r\sin t + \cos r\cos t)}^{r^2 \cos t} dt = \int_{0}^{2\pi} \frac{r^2 - 2\pi}{r^2} dt = 2\pi \int_{0}^{2\pi}
\]

Hence, \( \int_{Y_0} w = \int_{Y_1} w \) for all \( r \), which implies \( K = 1 \).
we recall that $\mathbb{R}^2 \setminus \{0\}$ deformation retracts onto $S^1$.

Therefore $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \pi_1(S^1) \cong \mathbb{Z}$.

Then any closed path $\gamma$ in $\mathbb{R}^2 \setminus \{0\}$ is homotopic to $kS^1$ for $k \in \mathbb{Z}$, where negative corresponds to clockwise loops and $0S^1$ is a point.

Then by the calculation in part a,

$$\int_{\gamma} w = \int_{kS^1} w = k \int_{S^1} w = 2\pi k$$

for $k \in \mathbb{Z}$ when $x=1$. 

\[\square\]
(a) Cartan's formula states
\[ L_X = d o i_X + i_X d \]
for a vector field \( X \).

**Proof:** We recall the definition of \( L_X \). If \( \phi \) is a local
flow of \( X \) at \( p \), then
\[ (L_X w)_p = \lim_{h \to 0} \frac{1}{h} \left( (\phi^h \circ w)_p - w_p \right) \]
In particular, this formula implies that \( L_X d \omega = d o L_X \) since
pullbacks commute with the exterior derivative \( d \).

We first show the claim in the case that \( \omega \) is a 0-form.

Thus \( L_X w \circ = \chi(f) = \lim_{h \to 0} \frac{1}{h} \left( \omega \circ (\phi^h \circ f) - \omega \circ f \right) = d \omega(X) \)

Since \( i_X \omega = 0 \), this implies the Cartan formula holds for \( \omega \) a smooth function.

We now show that the claim extends through the wedge product.

Suppose that the formula holds for \( \omega, \eta \). Then by the shuffling rule
\[ d o i_X (\omega \wedge \eta) + i_X d (\omega \wedge \eta) = d (i_X \omega \wedge \eta + (-1)^k \omega \wedge i_X \eta) + i_X (d \omega \wedge \eta + (-1)^{k+1} \omega \wedge d \eta) \]
\[ = d (i_X \omega) \wedge \eta + (-1)^k i_X d \omega \wedge \eta + i_X d \eta + i_X \omega \wedge d \eta \]
\[ = (d i_X \omega) \wedge \eta + i_X (d \omega) \wedge \eta + \omega \wedge (d i_X \eta + i_X d \eta) \]
\[ = (o i_X \omega) \wedge \eta + \omega \wedge (L_X \eta) \]
\[ = L_X (\omega \wedge \eta) \]

Finally, we show the formula extends through the exterior derivative.

Suppose it holds for \( \omega \). Then
\[ L_X (d \omega) = d o L_X (\omega) = (d o d i_X + d o i_X d) \omega = d \omega(X) = (o i_X i_X X) (d \omega) \]

As any form can locally be written as the wedge product of differentials,
If $\omega$-form (local coordinate), then implies that the formula holds locally for any form. Therefore Cartan's formula holds for form.

(b) Define $d^\omega x$ to be the volume form.

Suppose that $X$ has a flow $\Phi$ that preserves volume.

Then $\Phi^*d\omega = d\omega$ and so

$$L_x d\omega = \lim_{n \to \infty} \frac{1}{n} (\Phi^n_* d\omega - d\omega) = 0$$

Therefore by Cartan's formula, if $X = X_1 \partial_x + X_2 \partial_y + X_3 \partial_z$, then

$$0 = (dox + i_{x_0} \omega) d\omega$$

As $d\omega$ is non-vanishing, this implies that $\Phi^n(x) = 0$ as desired.

Now suppose that $\Phi^n(x) = 0$. Then by the above computation,

$$L_x d\omega = 0.$$ Let $\Phi$ be a flow of $X$. Then for $\forall h$,

$$L_x (\Phi^*_h d\omega) = (dox + i_{x_0} \omega) (\Phi^*_h d\omega)$$

$$= \Phi^*_h (dox + i_{x_0} \omega) d\omega$$

$$= 0$$

Therefore $\Phi^*_h d\omega$ is constant in $h$ and hence $\Phi^*_h d\omega = \Phi^*_0 d\omega = d\omega$.

Therefore $\Phi$ preserves volume.
Define \[ D = \ker \left( d\mathbf{x}_3 - x_3 d\mathbf{x}_2 \right) \cap \ker \left( d\mathbf{x}_1 - x_1 d\mathbf{x}_3 \right). \]

Since \( T_p \mathbb{R}^4 \cong \mathbb{R}^4 \), we can view \( x_p, B_p \) as maps \( (x_3, x_2, x_1) \rightarrow \mathbb{R} \).

Then \( x_p = [0, -x_3, 1, 0] \) and \( B_p = [1, -x_4, 0, 0] \) for \( p = (x_1, x_2, x_3, x_4) \).

We can then view \( \ker x_p \cap \ker B_p = \ker \begin{bmatrix} 1 & -x_4 & 0 & 0 \\ 0 & -x_3 & 1 & 0 \end{bmatrix} = \ker A_p \)

Analyzing the pull-backs, we find that \( A_p \) has rank 2 and so has \( \dim \ker A_p = 2 \) at \( p \). Therefore, \( D = \{ \ker A_p \}_p \) is a smooth distribution of rank 2 since \( A_p \) depends smoothly on \( p \).

We claim that \( D \) is not an integrable distribution.

To show this, it suffices to show that for every \( X, Y \in D \) s.t. \( [X, Y] \notin D \).

Consider

\[
X = \partial / \partial x_4 \\
Y = x_4 \partial / \partial x_1 + x_2 \partial / \partial x_2 + x_1 \partial / \partial x_3
\]

Then \( \alpha(X) = B(X) = 0 \) and \( \alpha(Y) = x_1 - x_1 = 0 \)
\[ \beta(Y) = -x_4 + x_4 = 0 \]
so \( X, Y \in D \).

However,

\[
[X, Y] = XY - YX = 2 \partial / \partial x_4 \left( x_4 \partial / \partial x_1 + x_2 \partial / \partial x_2 + x_1 \partial / \partial x_3 \right) - Y \left( \partial / \partial x_4 \right)
\]
\[ = \partial / \partial x_1 \]
\[ = \alpha(X)
\]

Then \( \alpha(X, Y) = 0 \) but \( \beta(X, Y) = 1 \) and so \( [X, Y] \notin D \).

Therefore, \( D \) is not integrable.
For \( x_1, \ldots, x_n, y_1, \ldots, y_n \in M \), have \( M \) is connected.

If a smooth path \( Y_1 \) from \( x_1 \) to \( y_1 \). Since \( M \) has dimension \( \geq 2 \), and \( Y_1 \) is a smooth 1-dimensional submanifold, \( M \setminus Y_1 \) is connected.

Therefore \( \exists \) a path \( Y_2 \) from \( x_2 \) to \( y_2 \) in \( M \setminus Y_1 \), so \( Y_1 \cap Y_2 = \emptyset \).

Iterating this process, we can construct \( Y_i : [0, 1] \rightarrow M \) from \( x_i \) to \( y_i \)

s.t. \( Y_i \cap Y_j = \emptyset \) \( \forall i \neq j \).

Now \( Y_1, \ldots, Y_n \) are compact subsets of \( M \), for each \( i \) we may choose an open neighborhood \( U_i \) of \( Y_i \) s.t. \( U_i \cap U_j = \emptyset \) \( \forall i \neq j \).

We claim that we may construct a diffeomorphism \( f_i : M \rightarrow M \)

s.t. \( f_i = \text{id} \) on \( M \setminus U_i \) and \( f_i(Y_i) = y_i \).

Assuming the claim, taking \( f = f_{1} \circ \cdots \circ f_n \) would be a diffeomorphism

\( M \rightarrow M \) s.t. \( f(Y_i) = y_i \) \( \forall i \).

We now show the claim. Fix some \( i \). Have \( M \) is smooth and \( U_i \) is open, for each \( p \in Y_i \) \( \exists \) a neighborhood \( V_p \) of \( p \) s.t. \( V_p \cap U_i \) and \( V_p \) is diffeomorphic to an open ball in \( \mathbb{R}^n \). Since \( Y_i \) is compact, we may take a finite partition \( 0 = t_0 < t_1 < \cdots < t_k = 1 \in [0, 1] \) s.t. \( V_{Y_i(t_0)}, \ldots, V_{Y_i(t_k)} \) cover \( Y_i \). By adding finitely many neighborhoods, we may assume that \( V_{Y_i(t_j)} \cap V_{Y_i(t_j')} = \emptyset \) \( \forall j \neq j' \).

Define \( V_j = V_{Y_i(t_j)} \) for ease of notation.

Let \( a_0 = x_i \) and \( a_k = y_i \). For each \( j = 1, \ldots, k-1 \), choose \( a_j \in V_j \cap V_{j+1} \cap Y_i \).

We claim that \( V_j = 1, \ldots, k \) \( \exists \) a diffeomorphism

\( f_j : M \rightarrow M \) s.t. \( f_j = \text{id} \) on \( M \setminus U_i \) and

\( f_j(a_{j-1}) = a_j \).

Assuming the claim, taking \( f = f_1 \circ \cdots \circ f_n \) would then be a diffeomorphism \( M \rightarrow M \) s.t.

\( f(Y_i) = y_i \) and \( f = \text{id} \) on \( M \setminus U_i \).

To complete the proof, it thus suffices to construct \( f_i \).
For i,j, by construction, \( a_{j-1}, a_j \in V_j \) and \( V_j \) is diffeomorphic to an open ball in \( \mathbb{R}^3 \), via a diffeomorphism \( \varphi \).

Define \( \Psi \) to be the vector field on \( \varphi(V_j) \) given by the vector \( \varphi(a_j) - \varphi(a_{j-1}) \).

Let \( \psi \) be a bump function s.t.
\[ \psi = 1 \] on a neighborhood of the line segment \( [\varphi(a_{j-1}), \varphi(a_j)] \) and \( \psi = 0 \) outside of a compact neighborhood of \( [\varphi(a_{j-1}), \varphi(a_j)] \)
inside \( \varphi(V_j) \). Then \( \psi \Psi \) is a smooth vector field on \( \varphi(V_j) \). Let \( \Xi \) be the global flow of \( \psi \Psi \). Assume that \( \Xi \). Then \( \Xi \) is a diffeomorphism of \( \varphi(V_j) \) that is the identity near \( \partial \varphi(V_j) \). WLOG, assume that \( \Xi(a_{j-1}) = a_j \).

We define \( f_i^j = \psi^{-1} \circ \Xi_i \circ \psi \) on \( V_j \) and \( f_i = \text{id} \) elsewhere.

Then \( f_i \) is a diffeomorphism \( \mathbf{M} \times \mathbf{M} \) s.t. \( f_i(a_{j-1}) = a_j \) and \( f_i = \text{id} \) outside of \( U_i \).

Letting \( f = f_1 \circ \cdots \circ f_k \) and \( f = f_1 \circ \cdots \circ f_k \), then concludes. \( \square \)
Let $K$ denote the space of skew-symmetric $2n \times 2n$ matrices. We note that
$$\Omega^2 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = -\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} = -\Omega$$ and $\Omega \in K$.
We also recall that $K$ is a smooth submanifold of $M_{2n}(\mathbb{R})$ of dimension $(2n-1)+(3n-2)+\ldots+1+0 = 2n-1 \cdot \frac{2n-1+1}{2} = n(2n-1)$.

Define $f : M_{2n} \to K$ by $f(A) = A^T \Omega A$. Then $f^{-1}(\Omega) = S$.

To show that $S$ is a submanifold, it thus suffices to show that $\Omega$ is a regular value of $f$.

We recall that $T_A M_{2n} \cong M_{2n}$ and similarly that $T_B K \cong K$.

We can then write $d F_A : M_{2n} \to K$ for all $A \in M_{2n}$.

Fix $A \in S$ and $M \in M_{2n}$. Then by definition,
$$d F_A (M) = \lim_{t \to 0} \frac{1}{t} \left( (A+tm)^T \Omega (A+tm) - A^T \Omega A \right)$$
$$= \lim_{t \to 0} \frac{1}{t} \left( \Omega + tm^T \Omega A + tA^T \Omega M + t^2 M^T \Omega M - \Omega \right)$$
$$= M^T \Omega A + A^T \Omega M$$

Since $\Omega$ is skew-symmetric, we note that $d F_A (M) \in K \forall A, M$.

We claim that $d F_A$ is negative $\forall A \in S$.

Fix some $B \in K$. Then $B$ can be written in block form as
$$B = \begin{bmatrix} C & D \\ D & C \end{bmatrix}$$

where $C, D$ are skew-symmetric. Then
$$d F_A \left( \frac{1}{2} A \begin{bmatrix} C & D \\ D & C \end{bmatrix} \right) = \begin{bmatrix} C & -D \\ D & C \end{bmatrix} A^T \Omega A + A^T \Omega A \begin{bmatrix} D & -C \\ C & D \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} C & -D \\ D & -C \end{bmatrix} \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} \begin{bmatrix} -C & D \\ -D & C \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} C & D \\ D & C \end{bmatrix} + \frac{1}{2} \begin{bmatrix} C & D \\ D & C \end{bmatrix}$$
$$= B$$

Therefore $d F_A$ is negative. As this holds $\forall A \in S$, the complex $S$ is a smooth submanifold of dimension
$$n^2 - n(2n-1) = 2n^2 + n.$$

$\square$
We recall that \( S^n \) has a non-vanishing vector field if and only if \( n \) is odd.
Therefore by Poincaré-Hopf, \( \chi(S^n) = 0 \) if \( n \) is odd.

We claim that \( \chi(S^n) = 2 \) if \( n \) is even.

By Poincaré-Hopf, it suffices to construct a vector field \( X \) on \( S^n \) such that \( X \) has \( 2 \) zeros, each with index \( 1 \).

For even \( n \geq 2 \), we can view

\[
S^n \cong \mathbb{R}^{2k+1} \cong \mathbb{C}^k \times \mathbb{R}
\]

We recall that \( T_p S^n = \{ v \in \mathbb{C}^k \times \mathbb{R} : (v, p) = 0 \} \) (i.e., \( p + iv \)).

Therefore we construct \( X \) on \( S^n \) via

\[
X(p, x) = (ip, 0)
\]

Then since \( (ip, 0) \cdot (p, x) = ip \cdot p = 0 \), this is well-defined.
Moreover, \( X \) only has zeros at \( (0, \pm 1) \).

We now compute the index of \( X \) at \( (0, \pm 1) \).

Let \( U_\pm \) be the neighborhood \( \{ (p, x) \in S^n : \pm x > 0 \} \).

Then \( U_\pm \) is a neighborhood of \( (0, \pm 1) \) that does not contain \( (0, \pm 1) \).

Therefore \( \text{ind}(0, \pm 1) X = \deg X|_{U_\pm} \).

where \( X|_{U_\pm} \) can be realized as the map \( \partial U_\pm \subseteq S^{n-1} \rightarrow S^{n-1} \).

By construction, \( X|_{U_\pm} : p \rightarrow ip \)

and \( \deg X|_{U_\pm} = 1 \). Therefore

\[
\chi(S^n) = \text{ind}(0, +1) X + \text{ind}(0, -1) X = 1 + 1 = 2
\]

as claimed.

\[\square\]
Consider the open cover \( M \setminus A, M \setminus B \) of \( M \).

Let \( \psi, \varphi \) be a partition of unity subordinate to \( \{ M \setminus A, M \setminus B \} \).

Then \( \psi = 0 \) on \( A \) and \( \varphi = 0 \) on \( B \).

Since \( \psi + \varphi = 1 \), this implies

\( \psi = 0 \) on \( A \) and
\( \varphi = 1 \) on \( B \)

as desired.
Let $M_n(\mathbb{R})$ be the space of $n \times n$ real matrices.

(a) Define $F: M_n(\mathbb{R}) \to S_n(\mathbb{R})$ where $S_n(\mathbb{R})$ is the space of $n \times n$ symmetric real matrices by

$$F: A \mapsto AA^T$$

Then $F^{-1}(\text{Id}) = O(n)$. To show $O(n) \setminus \text{coin}(n)$ is a smooth submanifold, it then suffices to show that $\text{Id}$ is a regular value of $F$.

Fix some $A \in O(n)$. We claim that $dF_A$ is negative. We recall that $T_M M_n(\mathbb{R}) = M_n(\mathbb{R})$ and similarly $T_M S_n(\mathbb{R}) = S_n(\mathbb{R})$. Fixing $M \in M_n(\mathbb{R})$, we compute

$$dF_A(M) = \lim_{t \to 0} \frac{(A+tM)(A+tM)^T - AA^T}{t}$$

$$= \lim_{t \to 0} \frac{t(MA^T + AM^T) + t^2 MM^T}{t}$$

$$= MA^T + AM^T$$

Now fix some $C \in S_n(\mathbb{R})$. Then

$$dF_A(\frac{1}{2}CA) = \frac{1}{2} CA A^T + \frac{1}{2} A A^T C$$

$$= \frac{1}{2} C + \frac{1}{2} C = C$$

Therefore $dF_A: T_M M_n(\mathbb{R}) \to T_C S_n(\mathbb{R})$ is negative for all $A \in O(n)$. Then regular value $\text{Id}$ is a regular value of $F$ and $O(n) = F^{-1}(\text{Id})$ is a smooth submanifold. \(\square\)
(b) It suffices to consider vector fields
\[ X_1, \ldots, X_m \quad \text{mod} \dim O(n) \]
\[ 1 \leq m = \dim O(n) \]

i.e. \( X_1, \ldots, X_m \) is a basis for \( T_p O(n) \) at all \( p \).

Consider \( T_{Id} O(n) \). Let \( v_1, \ldots, v_m \) be a basis for \( T_{Id} O(n) \). Define \( X_1, \ldots, X_m \) by
\[ X_j(p) = dp_{Id} v_j \]

Hence \( O(n) \) is a real Lie group, \( dp_{Id} : T_{Id} O(n) \to T_p O(n) \) is an isomorphism. Therefore \( X_1(p), \ldots, X_m(p) \) is a basis for \( T_p O(n) \) at \( p \).

Therefore \( O(n) \) has trivial tangent bundle. \( \square \)
Home $S^2$ is compact and orientable, we recall that integration defines an isomorphism $H^2_{dR}(S^2) \rightarrow \mathbb{R}$ via
\[ H^2_{dR} \rightarrow \mathbb{R}: w \mapsto \int_{S^2} w. \]

Therefore, since all 2-forms on $S^2$ are closed,

\[ 2^n dA \text{ is closed } \iff \int_{S^2} 2^n dA = 0. \]

If $n$ is odd, then $2^n dA$ is odd under reflections across the $xy$ plane, and hence $\int_{S^2} 2^n dA = 0$.

If $n$ is even, then $2^n$ is non-negative and strictly positive outside of the $xy$ plane. Therefore $\int_{S^2} 2^n dA > 0$.

This concludes that $2^n dA$ is closed if and only if $n$ is even.
(a) A manifold \( M \) is orientable if \( \exists \) an atlas \( \{ U_k, \phi_k \} \) such that every transition map \( \phi_k \circ \phi_k^{-1} \) is orientation preserving. We define orientation preserving to be the requirement \( \det(\phi_k \circ \phi_k^{-1}) > 0 \).

(b) Define
\[
\tilde{M} = \{(p, O_p) : p \in M \text{ and } O_p \text{ is an orientor of } T_pM\}
\]
we define a topology on \( \tilde{M} \) as follows.

For each open set \( U \subseteq M \) with consistent orientation \( O \) on \( U \), we define \( V(U, O) = \{(p, O_p) : O_p = O\} \). These sets define a basis for the topology on \( \tilde{M} \).

Let \( \pi \) be the projection \( \tilde{M} \to M : (p, O_p) \mapsto p \).

Since any manifold is locally orientable, \( \forall (p, O_p) \in \tilde{M} \) if an open neighborhood \( U \) of \( p \) and orientation \( O' \) on \( U \) s.t. \( O'_p = O_p \).

Then \( \pi|_{\tilde{M} \setminus \{O_p\}} \) is a diffeomorphism onto \( U \).

Therefore \( \tilde{M} \) has a smooth structure.

Since any \( p \in M \) has 2 orientations \( O_p, -O_p \), this also implies that \( \tilde{M} \) is a double cover for \( M \).

Additionally, by construction of the topology on \( \tilde{M} \), we can define an orientation on \( \tilde{M} \) via orienting \( T_p, O_p \tilde{M} = T_pM \) \( U \), \( O_p \) and \( T_p, -O_p \tilde{M} = T_pM \) \( U \), \( -O_p \).

Therefore \( \tilde{M} \) is an orientable double cover.

Suppose that \( \tilde{M} \) is disconnected. Then \( \tilde{M} = U \cup V \), where \( U, V \) are open. Since \( \tilde{M} \) is a double cover and \( M \) is connected, \( \pi|_{\tilde{M} \cup U} \to M \) is a diffeomorphism. Then \( M \) is orientable \( \blacksquare \).
It suffices to work locally. Therefore it suffices to consider \( w = f dx \).

By direct computation, \( dw = df \wedge dx \) and so
\[
dw(x,y) = (df \wedge dx)(x,y) \\
= df(x)dx(y) - df(y)dx(x) \\
= X(f)Y(x) - Y(f)X(x)
\]

We also compute
\[
X(w(y)) = X(fdx(y)) \\
= X(fY(x)) \\
= X(f)Y(x) + fXY(x)
\]

and similarly \( Y(w(x)) = Y(f)X(x) + f YX(x). \)

Then
\[
dw(x,y) - X(w(y)) + Y(w(x)) = f YX(x) - f XY(x) \\
= -f [X,Y](x) \\
= -df_x [X,Y] \\
= -w([X,Y])
\]

which implies
\[
dw(x,y) = X(w(y)) - Y(w(x)) - w([X,Y])
\]
as desired. \( \square \)
By Stokes' theorem, since \( \kappa \wedge \beta \) is an \( n-1 \)-form,

\[
\int_M d(\kappa \wedge \beta) = \int_M (\delta \kappa \wedge \beta) + \int_{\partial M} (\kappa \wedge \beta).
\]

Blake pullbacks commute with wedge products,

\[
\int_M d(\kappa \wedge \beta) = \int_{\partial M} (i_{\delta \kappa} \wedge (i_{\delta \beta} \kappa)) + \int_{\partial M} (i_{\delta \kappa} \wedge (i_{\delta \beta} \kappa))
\]

\[
= 0.
\]

Then by the lemma rule, since \( \kappa \) is a \( p \)-form

\[
0 = \int_M d(\kappa \wedge \beta) + (-1)^p \kappa \wedge d\beta
\]

\[
\Rightarrow \int_M d(\kappa \wedge \beta) = (-1)^{p+1} \int_M \kappa \wedge d\beta
\]

As claimed. \( \square \)
Suppose that \( X, Y \) are submanifolds of \( \mathbb{R}^n \). We claim that 
\[ X + a Y \] 
for \( a \in \mathbb{R}^n \). To show this, we use the transversality theorem.

Define \( F : X \times \mathbb{R}^n \to \mathbb{R}^n \) by \( (x, a) \mapsto x + a \).

We claim that \( F \) is transversal to \( Y \). To show this, it suffices to show that \( dF(x, a) \) is surjective \( \forall (x, a) \).

Working in local coordinates, we may write \( dF(x, a) \) as a 
\[ n \times (\min(n, m)) \] 
matrix where \( m = \dim X \). Doing so,
\[ dF(x, a) = \begin{bmatrix} * & I \end{bmatrix} \] 
where \( I \) is the \( n \times n \) identity matrix.

Therefore \( dF(x, a) \) has rank \( n \) and \( \omega \) is surjective.

In particular, \( \forall (x, a) \) s.t. \( F(x, a) \in Y \),
\[ \text{Im}(dF(x, a)) + T_{F(x, a)} Y = \mathbb{R}^n \leq T_{F(x, a)} \mathbb{R}^n \]
and \( \omega \) \( F \) \( Y \).

By the transversality theorem, this implies that \( fa = F(\cdot, a) \) is transversal to \( Y \) for \( a \in \mathbb{R}^n \).

Consider some \( a \) s.t. \( fa \) \( Y \). Then \( \forall x \in X \) s.t. \( x + a \in Y \),
\[ \text{Im}(dx fa) + T_{x+a} Y = T_{x+a} \mathbb{R}^n \]
\[ \implies T_{x+a} (x + a) + T_{x+a} Y = T_{x+a} \mathbb{R}^n \]
As this holds \( \forall x + a \in Y \), this implies that \( T_x (x + a) + T_2 Y = T_2 \mathbb{R}^n \)
\( \forall x \in X \) \( Y \). Therefore \( (x + a) \) \( Y \).

As this holds for \( a \in \mathbb{R}^n \), \( x \) concludes.
Define $F: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ by $(x,y) \mapsto y-x$. Then $F$ is smooth and so locally thicken implies that for $a \in \mathbb{R}^n$, $a$ is a regular value of $F$.

Consider such an $a$. We claim $(X+a) \cap Y$.

Suppose $\exists \; (x,y)$ s.t. $F(x,y) = a \iff x + a = y$.

Since $a$ is a regular value of $F$, 

$$n = \dim \left( \frac{dF(x,y)}{(T_{x,y}) (x,y)} \right)$$

$$= \dim \left( \frac{dF(x,y)}{T_x X + T_y Y} \right)$$

Thus, since $x + a = y$,

$$T_{x+a} (X+a) \cap T_y Y = T_y \mathbb{R}^n$$

As this holds $\forall \; (x,y)$ s.t. $x + a = y$, this implies that $(X+a) \cap Y$.

Therefore $(x+a) \cap Y$ for $a \in \mathbb{R}^n$. $\blacksquare$
Let $M$ be a smooth compact $3$-dimensional submanifold of $\mathbb{R}^3$ without smooth boundary $\partial M$. Let $X$ be a smooth vector field on $\mathbb{R}^3$.

The classical divergence theorem states that

$$\int_M \text{div}(X) \, dV = \int_{\partial M} \langle X, N \rangle \, dA$$

where $\text{div}(X) = \frac{\partial X_1}{\partial x} + \frac{\partial X_2}{\partial y} + \frac{\partial X_3}{\partial z}$, $N$ is the unit normal of $\partial M$ with the boundary orientation, and $dA = \mathbf{c}^{*} \, i_{\mathbf{n}} \, dV$ where $\mathbf{c} : \partial M \to M$ is the inclusion.

Define $T = X - \langle X, N \rangle N$ on $\partial M$. Then $T$ is the component of $X$ tangent to $\partial M$, and we can be regarded as a vector on $T \partial M$.

We claim that $\mathbf{c}^{*} i_{T} \, dV = 0$ on $\partial M$. For some $p \in \partial M$ and vectors $V_1, V_2 \in T_p(\partial M)$, since $T_p(\partial M)$ is $2$-dimensional, $T, V_1, V_2$ are linearly dependent. Thus

$$\mathbf{c}^{*} i_{T} \, dV(V_1, V_2) = \mathbf{c}^{*} \, dV(T, V_1, V_2) = 0$$

Therefore $\mathbf{c}^{*} i_{T} \, dV = 0$. By linearity, this implies that

$$0 = \mathbf{c}^{*} i_{T} \, dV = \mathbf{c}^{*} \left( i_{X} - \langle X, N \rangle \, i_{N} \right) \, dV$$

$$\Rightarrow \quad \mathbf{c}^{*} i_{X} \, dV = \langle X, N \rangle \, \mathbf{c}^{*} i_{N} \, dV = \langle X, N \rangle \, dA$$

By direct computation,

$$\mathbf{c}^{*} i_{X} \, dV = i_{X} (dx \wedge dy \wedge dz) = X_1 \, dy \wedge dz - X_2 \, dx \wedge dz + X_3 \, dx \wedge dy$$

and

$$d \left( \mathbf{c}^{*} i_{X} \, dV \right) = \left( \frac{\partial X_1}{\partial x} + \frac{\partial X_2}{\partial y} + \frac{\partial X_3}{\partial z} \right) dx \wedge dy \wedge dz = \text{div}(X) \, dV,$$

Then by Stokes' theorem,

$$\int_M \text{div}(X) \, dV = \int_{\partial M} \mathbf{c}^{*} i_{X} \, dV = \int_{\partial M} \langle X, N \rangle \, dA$$

as claimed.

$\blacksquare$
Suppose first that \( n = 2k - 1 \) is odd.

Consider \( S^n \subset \mathbb{R}^{2k} \). For each \( p \in S^n \), we can view \( T_p S^n \) in the canonical way as \( T_p S^n \cong \{ e \in \mathbb{R}^{2k} : e \cdot p = 0 \} \).

We then define a non-vanishing vector field \( V \) on \( S^n \) by

\[
V : (x_1, x_2, \ldots, x_n, y_1) \mapsto (-y_1, x_1, -y_2, x_2, \ldots, -y_n, x_n).
\]

Then \( V \) is well-defined and is non-vanishing \( \forall p \in S^n \).

We recall that \( \mathbb{RP}^n = S^n / \{ x \sim -x \} \). We claim that \( V \) factors through the quotient map \( \pi : S^n \rightarrow \mathbb{RP}^n \) to a nonvanishing vector field on \( \mathbb{RP}^n \). To show this, it must be shown that

\[
\frac{d}{d\tau} V_p = d\pi \circ \frac{d}{d\tau} V_p = d\pi \circ V_p.
\]

Let \( f \) be the antipodal map \( p \mapsto -p \) on \( S^n \).

Then \( \pi f = \pi \) and so

\[
\frac{d}{d\tau} V_p = d(\pi f) V_p = d\pi \circ \frac{d}{d\tau} V_p = d\pi \circ (-V_p).
\]

By construction, \( -V_p = V_p \) and so \( d\pi \circ V_p = d\pi \circ V_p \).

Therefore, \( V \) descends to a non-vanishing vector field on \( \mathbb{RP}^n \).

Suppose instead that \( n \) is even. We recall that \( \mathbb{RP}^n \) can be constructed \( \cup \{ H_k \} \) for \( k = 0, 1, \ldots, n \). Then since \( n \) is even,

\[
\chi(\mathbb{RP}^n) = \sum_{k=0}^{n} (-1)^k = 1.
\]

By Poincaré-Hopf, this implies that any vector field on \( \mathbb{RP}^n \) must have a zero.

Therefore, \( \mathbb{RP}^n \) has a non-vanishing vector field if \& only if \( n \) is odd.
Suppose that \( X, Y \) are left-invariant vector fields on \( G \).

We recall that \( \exp: \mathfrak{g} \to G \) is a Lie group. \( X \) and \( Y \) are vector fields on \( G \),
\[
(\exp)_* Z(t) = Z(t \circ \exp)
\]

Then in particular, \( \forall f \)
\[
(\exp)_*[X,Y](t) = (\exp)_* (X \circ Y(t) - Y \circ X(t))
= X \circ Y(t \circ \exp) - Y \circ X(t \circ \exp)
= X \circ (\exp)_* Y(t) - Y \circ (\exp)_* X(t)
\]

Since \( X, Y \) are left-invariant,
\[
(\exp)_*[X,Y](t) = X \circ Y(t) - Y \circ X(t)
= [X,Y](t)
\]

As this holds \( \forall f \), this implies that \([X,Y] \) is left-invariant.
we recall that all straight lines in $\mathbb{R}^2$ can be written as $ax+by=c$ for some $a, b, c \in \mathbb{R}$ with $(a, b) \neq (0, 0)$. Moreover, $\forall \lambda, \varepsilon \in \mathbb{R}$, $\forall x, y \in \mathbb{R}$, $\lambda a x + \varepsilon b y = c \lambda$ defines the same line as $ax + by = c$.

Therefore we can view the space of all lines in $\mathbb{R}^2$ as a subset of $\mathbb{RP}^2$, specifically

$$U = \{ [a:b:c] : (a, b) \neq (0, 0) \} \subset \mathbb{RP}^2$$

Equivalently,

$$U = \mathbb{RP}^2 \setminus \{ [0:0:1] \}$$

and as $U$ is open subset of $\mathbb{RP}^2$, therefore $U$ can be given a smooth structure via the smooth structure on $\mathbb{RP}^2$.

we claim that $U$ is not orientable. we recall that $\mathbb{RP}^2$ is non-orientable and hence has a connected 2-sheeted orientation cover $M$. Let $p_1, p_2$ be the pre-images of $[0:0:1]$ in $M$. Then $M \setminus \{ p_1, p_2 \}$ is an orientation cover of $U$. Since $M$ is 2-dimensional, $M \setminus \{ p_1, p_2 \}$ is connected and so $U$ has a connected orientation cover. Therefore $U$ is not orientable.
Let $a_1, \ldots, a_n$ denote the zeros of $X$.
For each $i$, let $k_i$ denote the order of the zero at $a_i$.
Then we can write

$$X(z) = (z-a_i)^{k_i} q_i(z)$$

for each $i$ where $q_i$ is a polynomial and $q_i(a_i) \neq 0$.
Then $X(z)$ is of degree $n$ and has index $k_i$ at $a_i$ for all $i$.

Thus

$$\sum_{i=1}^{n} \text{ind}_{a_i} X = \sum_{i=1}^{n} k_i = 2016$$

by the fundamental theorem of algebra.
(a) As given,

\[ \mathbf{x} = x_1 \, dx_2 \wedge dx_3 \wedge dx_4 - x_2 \, dx_1 \wedge dx_3 \wedge dx_4 + x_3 \, dx_1 \wedge dx_2 \wedge dx_4 - x_4 \, dx_1 \wedge dx_2 \wedge dx_3 \]

By Stokes' theorem,

\[ \int_{S^3} i^* \mathbf{x} = \int_{B^3} d\mathbf{v} = 4 \int_{B^3} d\mathbf{v} = 4 \text{vol}(B^4) \]

where \( B^3 \) is the open unit ball in \( \mathbb{R}^4 \).

(b) By direct calculation,

\[ d\mathbf{y} = -2k \left( \frac{x_1 \, dx_1 + x_2 \, dx_2 + x_3 \, dx_3 + x_4 \, dx_4}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^{k+1}} \right) \wedge \mathbf{x} + \frac{dx}{(x_1^2 + \ldots + x_4^2)^k} \]

\[ = \frac{2}{(x_1^2 + \ldots + x_4^2)^k} \left( d\mathbf{v} - \frac{k(x_1 \, dx_1 + \ldots + x_4 \, dx_4) \wedge \mathbf{x}}{(x_1^2 + \ldots + x_4^2)} \right) \]

\[ = \frac{2}{(x_1^2 + \ldots + x_4^2)^2} \left( d\mathbf{v} - \frac{k(x_1 \, dx_1 + \ldots + x_4 \, dx_4) \, d\mathbf{v}}{(x_1^2 + \ldots + x_4^2)} \right) \]

\[ = \frac{2(2-k)}{(x_1^2 + \ldots + x_4^2)^2} d\mathbf{v} \]

Therefore \( d\mathbf{y} = 0 \) if \( k = 2 \).

We recall that \( \mathbb{R}^4 \) isomorphs deformation retracts onto \( S^3 \times \mathbb{R}^4 \times 0 \) via the homotopy

\[ (x_1, x_2, x_3, x_4) \mapsto \frac{x}{|x|^2} \]

for \( x \in \mathbb{R}^4 \).

Suppose that \( \mathbf{y} \) is exact for some \( k \). Then \( \mathbf{y} = d\mathbf{z} \).

Then since \( d \) commutes with pullbacks, \( i^* \mathbf{y} \) is exact on \( S^3 \).

Hence \( \frac{1}{S^3} \text{H}^2(S^3) = \mathbb{R} \) is an isomorphism \( \mathbf{i}_* : \mathbb{R} \to \mathbb{R} \), the complex that

\[ \int_{S^3} i^* \mathbf{y} = 0 \]. However, by construction, \( i^* \mathbf{x} = i^* \mathbf{k} \). Therefore

\[ 0 = \int_{S^3} i^* \mathbf{y} = \int_{S^3} i^* \mathbf{x} = 4 \text{vol}(B^4) \neq 0 \]

and so \( \mathbf{y} \) is never exact.
Suppose that \( w_1, \ldots, w_n \) are linearly dependent. Then \( \exists f_1, \ldots, f_{n-1} \in k \) s.t. \( w_n = f_1 w_1 + \ldots + f_{n-1} w_{n-1} \).

Therefore

\[
\sum_{i=1}^{n-1} f_i w_i = w_n
\]

For all \( i \), since \( w_i \) is a 1-form, the Jacobian

\[
\begin{pmatrix}
 w_i \\
 w_i \\
\end{pmatrix}
\]

and so \( w_i \wedge w_i = 0 \). Then \( w_1 \wedge \ldots \wedge w_n = 0 \) as desired.

Now suppose that \( w_1, \ldots, w_n \) are linearly independent. Then \( \forall \ p \in M \), \( (w_1)_p, \ldots, (w_n)_p \in T^*_p M \) are linearly independent. This implies \( \exists \) dual vectors \( v_1, \ldots, v_n \in T_p M \) s.t. \( (w_i)_p(v_j) = \delta_{ij} \).

Then \( (w_1 \wedge \ldots \wedge w_n)_p(v_1, \ldots, v_n) = (w_1)_p(v_1) \ldots (w_n)_p(v_n) = 1 \neq 0 \).

Therefore \( w_1 \wedge \ldots \wedge w_n \neq 0 \). \( \square \)
We recall that $S'$ is parallelizable since it admits a nonzero non-vanishing vector field. Therefore

$$TS' \cong S' \times \mathbb{R}$$

Since $M = f^{-1}(03)$, $M$ is a dimension $n$ manifold (codimension 1). The normal bundle is defined by $\nu f$. Moreover, $\nu f$ is a non-vanishing normal vector field on $M$ and so $M$ has a trivializable normal bundle. Then

$$NM \cong M \times \mathbb{R}$$

From this, it follows that $TM \times \mathbb{R} \cong M \times \mathbb{R}^{n+1}$ since

$$NM \oplus TM \cong \mathbb{R}^{n+1}.$$ Therefore, combining all these facts,

$$T(M \times S') \cong TM \oplus TS'$$

$$\cong TM \oplus (S' \times \mathbb{R})$$

$$\cong (TM \times \mathbb{R}) \oplus S'$$

$$\cong M \times \mathbb{R}^{n+1} \times S'$$

$$\cong M \times S' \times \mathbb{R}^{n+1}$$

Therefore $T(M \times S')$ is trivializable and so $M \times S'$ is parallelizable. \( \square \)
By Cartan's formula,
\[
[L_x, L_y] = [L_x, do_i + iy o d]
\]
\[
= [L_x, do_i] + [L_x, iy o d]
\]
By Cartan's formula, we also see that \(L_x\) commutes with \(d\) since
\[
L_x o d = (d o i x + i x o d) = d o i x o d = d (i x o d + d o i x) = d o L_x
\]
Therefore
\[
[L_x, L_y] = d o [L_x, i y] + [L_x, i y] o d
\]
we recall that \([L_x, i y] = i [x, y]\). (Shown in lemma 6.) Therefore
\[
[L_x, L_y] = d o [x, y] + i [x, y] o d = L [x, y]
\] as desired.
Suppose \( w \) is exact. Then \( w = \omega \) and so if \( f: S' \to M \),
\[
\int_{S'} f^* w = \int_{S'} d(f^* \eta)
\]
we recall that since \( S' \) is compact and orientable,
\[
H^1_{dR}(S') \cong \mathbb{R} \text{ via isomorphism } \Theta \mapsto \int_{S'} \Theta.
\]
Therefore
\[
\int_{S'} f^* w = \int_{S'} d(f^* \eta) = 0
\]
as desired, since \( w \) is closed.

Now suppose \( \int_{S'} f^* w = 0 \) for all \( f: S' \to M \).

Restrict to path-connected \( S' \).

Fix a base point \( p_0 \). For each \( p_0 \in M \), let \( \gamma(p) \) from
\( p_0 \) to \( p \) such that \( \int_{S'} f^* w = 0 \).

Define \( g(p) = \int_{\gamma(p)} w \).

Since \( \int_{S'} f^* w = 0 \) for all \( f \), \( g \) is independent of the choice of \( \gamma(p) \). In particular, \( g \) is smooth by working locally.

Then by the fundamental theorem of calculus,
\[
dg = \frac{1}{\partial t} |_{t=1} \int_{\gamma(t)} w = w_p
\]
so and \( w = dg \) as desired.

\( \square \)
We continued an inverse map to show injectivity.

Since $f$ is a covering map, for each $p \in U$ of $p$
$s.t. \ f^{-1}(U) = \bigcup_{j=1}^{k} U_j$; for $k$ independent of $p$, and $f: U_j \rightarrow U$ is a

Define $g$ locally on $U$ via

$$ g \ |_U (w) = \frac{1}{k} \sum_{j=1}^{k} \left((f \mid U_j)^{-1}\right)^* (w | U_j) $$

Since $f$ is a smooth covering map, this is well-defined and extends globally.

We claim that $g$ is well-defined on the level of cohomology.

Consider $w + dk$. Then $\forall$ $p, U$ as above

$$ g \ (w + dk) = \frac{1}{k} \sum_{j=1}^{k} \left((f \mid U_j)^{-1}\right)^* (w | U_j + dk | U_j) $$

$$ = g \ (w) + \frac{1}{k} \sum_{j=1}^{k} \left((f \mid U_j)^{-1}\right)^* (dk | U_j) $$

$$ = g \ (w) + d \left(\frac{1}{k} \sum_{j=1}^{k} \left((f \mid U_j)^{-1}\right)^* (k | U_j)\right) $$

Therefore $g$ is well-defined on cohomology.

Finally, for any use $H_{\bullet, k}(X)$, $\forall$ $p, U$,

$$ g \circ f^* | U (w) = \frac{1}{k} \sum_{j=1}^{k} \left((f \mid U_j)^{-1}\right)^* (f^* w | U_j) $$

$$ = \frac{1}{k} \sum_{j=1}^{k} \left((f \mid U_j)^{-1}\right)^* (f^* w | U_j) $$

$$ = \frac{1}{k} \sum_{j=1}^{k} w | U_j $$

$$ = w | U $$

and $w \circ g = 1_U$. Therefore $f^*$ is injective.
To show that $\ker(\Theta)$ is a distribution, it must be shown that $\ker(\Theta)$ is of consistent dimension. To show this, it suffices to show that $\text{rank} \Theta_p = 1 \text{ } \forall p \in S^{2n-1}$.

Since $\Theta$ maps into $\mathbb{R}$, it suffices to find $v \in T_p S^{2n-1} \forall v \text{ s.t. } \Theta_p(v) = 0$.

In fact, we may find a smooth vector field $X$ s.t. $\Theta(X) = 1 = 0$.

Define $X = x^2 \partial / \partial x^2 - x^1 \partial / \partial x^1 + \ldots + x^{2n} \partial / \partial x^{2n} - x^{2n-1} \partial / \partial x^{2n}$.

Then $X \in T S^{2n-1}$ since $X \perp p \text{ } \forall p$. Moreover, by direct calculation,

$$\Theta(X) = \left( x^2 \right)^2 + \left( x^1 \right)^2 + \ldots + \left( x^{2n} \right)^2 + \left( x^{2n-1} \right)^2 = 1 = 0$$

and so $\ker(\Theta)$ is of consistent dimension.

By direct computation,

$$d\Theta = -2d\varphi \wedge dx^1 - \ldots - 2d\varphi \wedge dx^{2n}$$

$$\Rightarrow \Theta \wedge d\Theta = -2 \sum_{j=1}^{n} \sum_{i=1}^{n} \left( x^j \partial / \partial x^j \wedge dx^i \wedge dx^{i+1} - x^{2n-1} \partial / \partial x^{2n-1} \wedge dx^i \wedge dx^{i+1} \right)$$

We note that there are no repeated terms $dx^a \wedge dx^b \wedge dx^c$ in this sum and so $(\Theta \wedge d\Theta)_p = 0 \text{ } \forall p \in S^{2n-1}$ implies $\Theta \wedge d\Theta \equiv 0$. Frobenius' theorem then implies $\ker(\Theta)$ is not integrable.
On $S^3$, $x^2+y^2+z^2=1$. Therefore by Stokes' theorem,

$$\int_{S^3} i^* w = \int_{S^3} \left( \frac{x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy}{(x^2+y^2+z^2)^{3/2}} \right)$$

$$= \int_{S^3} i^* (x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy)$$

$$= \int_{B^3} d (x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy)$$

$$= \int_{B^3} 3\,dV$$

$$= 3\,\text{vol}(B^3) = 4\pi$$

depending on chosen normalization.

By direct computation $A \cdot (v_1,v_2) = 0$,

$$d\omega = 3r^3 \frac{dxdydz}{r^6} - \left( x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy \right) \frac{\sqrt{2}\,dx+2y\,dy+2z\,dz}{r^6}$$

$$= \frac{3r^3dV - \frac{3}{2}r \left( 2x^2\,dxdydz + 2y^2\,dydzdx + 2z^2\,dzdx \right)}{r^6}$$

$$= \frac{(3r^3 - 3r^3)dV}{r^6} = 0$$

where $r = \sqrt{x^2+y^2+z^2}$ for some $B$ in $\omega$ computation.

By definition, $J : (x,y,z) = (3x,2y,3z)$ pulls back $S^3$ diffeomorphically onto the ellipse $E = \{ \frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{9} = 1 \}$.

We note that $E$ contains $S^2$ and so we can consider the annular region $A$ between $S^2$ and $E$.

Then by Stokes' theorem,

$$0 = \int_A d\omega = \int_{S^2} i^* \omega - \int_{E} k^* \omega = 4\pi - \int_{S^2} j^* \omega$$

and $\omega = \int_{S^2} i^* \omega = 4\pi$ where $k : E \to R^3 \setminus S^2$ is the inclusion.
we say that a differential form is closed if its exterior derivative $d\omega = 0$ and exact if it comes from $\Omega$ i.e. $d\eta = \omega$. Now $d^2 = 0$, if $\omega$ is exact then it is closed.

Therefore we may consider the quotient

$$H^k_{dR} = \frac{\text{closed k-forms}}{\text{exact k-forms}} = \frac{\ker d_k}{\im d_{k-1}}$$

where $d_k: \Omega^k \to \Omega^{k+1}$ is the exterior derivative on k-forms.

By convention we take $d_{-1} = 0$.

We claim that $H^k_{dR}(S') = \mathbb{R}$ if $k = 0, 1$ and $0$ otherwise. Since $S'$ has no non-trivial k-form for $k \geq 2$, it follows that $H^k_{dR}(S') = 0$ if $k \geq 2$.

We first consider $k = 0$. Suppose that $f$ is a 0-form, i.e., a smooth function, on $S'$, such that $df = 0$.

Then, since $S'$ has a global coordinate $\Theta$,

$$0 = df = \frac{\partial f}{\partial \theta} d\theta$$

and so $\frac{\partial f}{\partial \theta} = 0$. Therefore $f$ is a constant. Hence any constant is closed, this implies that $H^0_{dR}(S') = \mathbb{R}$.

Now consider $k = 1$. We claim that $i: [\omega] \mapsto \int_{S'} \omega$ for $[\omega] \in H^1_{dR}(S')$ is a well-defined isomorphism $H^1_{dR}(S') \to \mathbb{R}$.

To show it is well-defined on cohomology, it suffices to show that

$$\int_{S'} \omega + d\eta = \int_{S'} \omega.$$  By Stokes' theorem, $\int_{S'} d\eta = 0$ and so $\int_{S'} \omega + d\eta = \int_{S'} \omega$.

Therefore the map is well-defined on cohomology.
we now show injectivity. Suppose that $\int_{S^1} w = \int_{S^1} \psi$. Then $\int_{S^1} (w - \psi) = 0$. Let $w - \psi = g \, d\theta$. Then $\int_{S^1} g \, d\theta = 0$ which implies that $g = f'$ for some $f$. Then $g \, d\theta = df$ and

$w = \psi + df$. Therefore if $\int_{S^1} w = \int_{S^1} \psi$, then $w = \psi$ on the level of cohomology. Thus the map is indeed injective.

Finally, for surjectivity, we note that $\int_{S^1} d\theta = 2\pi$ and $\omega \forall \omega \in \mathbb{R}$

\[ \int_{S^1} \omega \, d\theta = 0. \]

Therefore the map is surjective and hence $H^1_{dR}(S^1) \cong \mathbb{R}$. 

0
Suppose that $H_{dR}^{2n+1}(M) \cong \mathbb{R}^k$, hence $M$ is compact and orientable, $H_{dR}^{2n+2}(M) \cong \mathbb{R}$.

We recall that the wedge product $\wedge : H_{dR}^{2n+1}(M) \times H_{dR}^{2n+1}(M) \to H_{dR}^{2n+2}(M)$ acts as an alternating bilinear map, on cohomology. Therefore, there exists an alternating bilinear map $\tilde{\wedge} : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$.

Since $\tilde{\wedge}$ is bilinear, there exists an anti-symmetric matrix $A$ s.t.

$$\tilde{\wedge}(v, u) = v^T A u$$

Since $\tilde{\wedge}$ is alternating, $A$ is anti-symmetric.

We claim that $A$ is invertible. Suppose $\exists u \ s.t. \ A u = 0$.

Then $v^T A u = 0$ for all $v \in \mathbb{R}^k$. In particular, letting $w$ be the $2n+1$-form equivalent to $u$, this implies that

$$\eta \wedge w = 0$$

for any $2n+1$-form $\eta$. Thus $w, \eta$ are linearly dependent for any $\eta$.

Since $A$ is a $k \times k$ matrix, this implies $A$ is invertible.

Thus by anti-symmetry,

$$0 \neq \det A = (-1)^k \det(A^T) = (-1)^k \det A$$

and $k$ is even.
We recall that $\mathbb{R}P^n$ is orientable if $n$ is odd.

In particular, this implies that $H_0^\text{or}(\mathbb{R}P^n) \cong \mathbb{R}$ if $n$ is odd. Since $H_0^\text{or}(S^n) \cong \mathbb{R}$ for all $n$, this implies that $\mathbb{R}P^n \cong S^n$ if $n$ is odd.

Now suppose that $n$ is odd. We aim to show that $\pi$ induces an isomorphism on cohomology.

We recall that covering maps induce injections on cohomology. This can be shown in a standard way by constructing an inverse.

Since all de Rham cohomology groups of $S^n$ and $\mathbb{R}P^n$ are finite dimensional, this implies that $\pi^*$ is an isomorphism.

If $n$ is odd, then $\mathbb{R}P^n$ is orientable and $\pi$ has orientation cover $\mathbb{R}P^n_2 \cong S^n$.

If $n$ is even, then $S^n$ is a connected double cover of $\mathbb{R}P^n$, this implies that $\pi$ has the orientation cover $\mathbb{R}P^n_2 \cong S^n$.

By definition, the orientation cover is the 2-fold cover such that the non-trivial deck transformation reverses orientation. If $n$ is even then $x \mapsto -x$ is the deck transformation $x \mapsto -x$ is of degree $(-1)^n = -1$ and so reverses orientation. Therefore $S^n$ is the orientation cover of $\mathbb{R}P^n_2$ if $n$ is even. \[\square\]
It suffices to show the result locally since we may then extend \( f_{ij} \) globally via a partition of unity. Since \( \phi_1, \ldots, \phi_k \) is non-vanishing, \( \{\phi_1, \ldots, \phi_n\} \) are linearly independent.

Locally, we may extend \( \phi_1, \ldots, \phi_k \) to a basis \( \phi_1, \ldots, \phi_n \). Then \( A \in \mathbb{R}^n \) smooth functions \( f_{ij} \) n.l.c.

\[
\omega_i = \sum_{j=1}^n f_{ij} \phi_j
\]

Taking the wedge 1-form \( \phi_1 \wedge \ldots \wedge \phi_k = \omega_i \wedge \ldots \wedge \omega_k \), we find that

\[
\sum_{j=1}^n f_{ij} \phi_j \wedge \phi_1 \wedge \ldots \wedge \phi_k = \omega_i \wedge \omega_1 \wedge \ldots \wedge \omega_k
\]

\[
\Rightarrow \sum_{j=k+1}^n f_{ij} \phi_j \wedge \phi_1 \wedge \ldots \wedge \phi_k = 0
\]

Since \( \phi_1 \wedge \ldots \wedge \phi_k \) is non-vanishing and \( \phi_1, \ldots, \phi_n \) are linearly independent, this implies \( f_{ij} = 0 \forall j \geq k+1 \). Therefore \( V_i \)

\[
\omega_i = \sum_{j=1}^k f_{ij} \phi_j
\]

as desired. \( \square \)
(a) Define \( F : \mathbb{M}^n \to \mathbb{R} : A \mapsto \det A \). Then \( F \) is smooth and 
\( F^{-1}(1) = S\mathbb{L}^n \). To show \( S\mathbb{L}^n \subset \mathbb{M}^n \) is a smooth submanifold, it then suffices to show that 1 is a regular value of \( F \).

Consider \( A \in S\mathbb{L}^n \). We aim to show that \( dF_A : T\mathbb{M}^n \to T\mathbb{R} \) is negative. We recall \( T\mathbb{M}^n \equiv \mathbb{M}^n \) and \( T\mathbb{R} \equiv \mathbb{R} \). Then \( dF_A : \mathbb{M}^n \to \mathbb{R} \).

Take some \( k \in \mathbb{R} \). Then \( V \in S\mathbb{L}^n \)
\[
dF_A ( \frac{k}{n} A) = \lim_{t \to 0} \frac{\det (A + t \frac{k}{n} A) - \det A}{t}
= \lim_{t \to 0} \frac{\det A \det (I + t \frac{k}{n} I) - 1}{t}
= \lim_{t \to 0} \frac{(1 + k/n)^n - 1}{t}
= \frac{d}{dt} (1 + \frac{k}{n}) \Big|_{t=0} = \frac{k}{n} n (1 + \frac{k}{n})^{n-1} \Big|_{t=0} = k
\]

with \( dF_A \) negative. Therefore \( dF_A \) is negative \( \forall A \in S\mathbb{L}^n \).

Then 1 is a regular value of \( F \) and \( S\mathbb{L}^n \) is a smooth submanifold of \( \mathbb{M}^n \).

(b) Identify the tangent space of \( S\mathbb{L}^n \) at \( F^{-1}(1) \).

Since \( S\mathbb{L}^n = F^{-1}(1) \), it follows that \( T_{F^{-1}(1)} S\mathbb{L}^n \equiv \ker dF_{F^{-1}(1)} \).

We recall that \( \det (I + tB) \) can be expanded as
\[
\det (I + tB) = 1 + tTrB + O(t^2)
\]

Therefore
\[
dF_{F^{-1}(1)} (B) = \lim_{t \to 0} \frac{1 + tTrB + O(t^2) - 1}{t} = Tr B.
\]

Then \( B \in \ker dF_{F^{-1}(1)} \) if and only if \( Tr(B) = 0 \).

Therefore \( T_{F^{-1}(1)} S\mathbb{L}^n \equiv \{ B \in \mathbb{M}^n : Tr B = 0 \} \).
(c) We recall Pontrjagin-Hopf, we state that the Euler Characteristic of a compact orientable orientable manifold is the sum of indices of a vector field, finite remaining.

In order to apply Pontrjagin-Hopf, we show that $SL_n$ is homotopic to $SO_n$, the group of orthogonal matrices. We recall that $SO_n$ is a compact Lie group. Since $SO_n$ is a Lie group, $SO_n$ admits a non-vanishing vector field $v$. Let $v \in T_{e} SO_n$ be nonzero, and let $m_{j}(h) = ghV_{j}e SO_n$.

Then $d(m_{j})_{e} : T_{e} SO_n \rightarrow T_{e} SO_n$. We define $V_{j}$ on $T_{e} SO_n$ via

$$V_{j} = d(m_{j})_{e} v$$

which is non-vanishing. Therefore $X SO_n = 0$.

We now show $SL_n$ is homotopic to $SO_n$.

For $A \in SL_n$, we recall the polar decomposition $A = OP$ where $O$ is non-orthogonal and $P$ is positive definite. Note $O, P$ depend smoothly on $A$.

Define $H_{j} : B_{0}$

$$H_{j} : A \rightarrow \begin{pmatrix} (1-t)A + O & 0 \\ 0 & dtO \end{pmatrix}$$

Then $H_{0} = \text{id}$ and $H_{1} : A \rightarrow O + \text{SO}_n$. Therefore, $H_{1}$ is a homotopy $SL_n \rightarrow SO_n$.

Alternate: $H_{j} : A \rightarrow \frac{(1-t)A + O}{dt((1-t)A + O)}$. This is well defined since $\det((1-t)A + O) = \det((1-t)P + tJ)\det U$ which has positive eigenvalues since $P$ is positive definite and $\det U \neq 0$.

Therefore $X SL_n = X(SO_n) = 0$. 

\[\text{\square}\]
16F.3 Suppose \( \omega \) is a \( k+1 \)-form.
Let \( Z_1, \ldots, Z_k \) be vector fields on \( M \). Then by Cartan's formula,

\[
([L_x, i_Y] \omega)(Z_1, \ldots, Z_k) =
\]

\[
(L_x \circ i_Y \omega - i_Y L_x \omega)(Z_1, \ldots, Z_k)
\]

We recall that \( \forall \) \( m \)-forms \( \eta \) and vector fields \( V_1, \ldots, V_m \)

\[
(L_x \eta)(V_1, \ldots, V_m) = L_x(\eta(V_1, \ldots, V_m)) - \sum_{i=1}^m \eta(V_1, \ldots, L_x V_i, \ldots, V_m)
\]

Therefore,

\[
(L_x \circ i_Y \omega)(Z_1, \ldots, Z_k) = L_x(i_Y \omega(Z_1, \ldots, Z_k)) - \sum_{j=1}^k i_Y \omega(Z_1, \ldots, [X_j Z_j], \ldots, Z_k)
\]

\[
= L_x(i_Y \omega(Z_1, \ldots, Z_k)) - \sum_{j=1}^k \omega(Y, Z_1, \ldots, [X_j Z_j], \ldots, Z_k)
\]

and similarly,

\[
(i_Y \circ L_x \omega)(Z_1, \ldots, Z_k) = (L_x \omega)(Y, Z_1, \ldots, Z_k)
\]

\[
= L_x(i_Y \omega(Z_1, \ldots, Z_k)) - \omega([X_1 Y], Z_1, \ldots, Z_k) - \sum_{j=1}^k \omega(Y, Z_1, \ldots, [X_j Z_j], \ldots, Z_k)
\]

Therefore,

\[
([L_x, i_Y] \omega)(Z_1, \ldots, Z_k) = \omega([X_1 Y], Z_1, \ldots, Z_k) = i_{[X_1 Y]} \omega(Z_1, \ldots, Z_k)
\]

and \( \omega([L_x, i_Y] \omega) = i_{[X_1 Y]} \omega \) as desired.
we recall that the Poincare dual of $C$ is $\eta \in H^2_0(M)$ i on $M$ s.t.

$$\forall \theta \in H_0^1(M)$$

$$[\iota^*_C \theta] = [\theta \wedge \eta]_M$$

where $\iota: C \to M$ is the inclusion.

we recall that $H^1_0(M) \cong \mathbb{R}^3$ and that all 1-forms on $M$ can be written as $a \, dx + b \, dy + c \, dz$ in the kernel of cohomology.

Moreover, since $L$ goes from $(0,1,1) \to (1,3,5)$, we see that $\Pi(L)$ maps 1 time in the $x$ direction, 2 times in the $y$ direction and 4 times in the $z$ direction. Therefore, $\forall \theta = a \, dx + b \, dy + c \, dz$,

$$[\iota^*_C \theta] = a \int_C \iota^* dx + b \int_C \iota^* dy + c \int_C \iota^* dz$$

$$= a + 2b + 4c$$

Taking $\eta = dy \wedge dz + 2dz \wedge dx + 4dx \wedge dy$ then implies that

$\forall \omega = a \, dx + b \, dy + c \, dz$,

$$[\omega \wedge \eta] = (a+2b+4c) \int_M dx \wedge dy \wedge dz = a + 2b + 4c = \int_C \iota^*_C \omega$$

and so $\phi \eta$ is the Poincaré dual of $\omega$ in $C$. 

$$\Box$$
We first show connected. To do so, it suffices to show path connected.

Suppose \( \exists a, b \in \mathbb{R}^n \setminus M \). Then since \( \mathbb{R}^n \) is path connected, \( \exists \) a path \( \tilde{Y} : [0,1] \rightarrow \mathbb{R}^n \) s.t. \( \tilde{Y}(0) = a, \tilde{Y}(1) = b \).

By the extension theorem, since \( \tilde{Y} \) is closed, there exists a homotopic path \( Y : [0,1] \rightarrow \mathbb{R}^n \) s.t. \( Y(0) = \tilde{Y}(0) = a, Y(1) = \tilde{Y}(1) = b \) and \( Y \cap M \).

Suppose \( \exists \) \( p \in Y \cap M \). Then \( T_p M + \text{im}(dY_p) = T_p \mathbb{R}^n \) and \( n = \dim T_p \mathbb{R}^n = \dim (T_p M + \text{im}(dY_p)) = m + 1 < n - 1 \)

which is a contradiction. Therefore \( Y \cap M = \emptyset \) and \( \mathbb{R}^n \setminus M \) connected.

To show simply connected, we proceed similarly.

Suppose \( \exists \) a closed loop \( Y : S^1 \rightarrow \mathbb{R}^n \setminus M \). Since \( \mathbb{R}^n \) is simply connected, \( \exists \) a homotopy \( \tilde{Y} : S^1 \rightarrow \mathbb{R}^n \) s.t. \( \tilde{Y}_0 = Y \) and \( \tilde{Y}_1 = \tilde{Y} \in \mathbb{R}^n \). Hence \( \tilde{Y} \) is a homotopy in \( \mathbb{R}^n \setminus M \).

By \( S^1 \times [0,1] \subset S^1 \times [0,2] \) is closed in \( S^1 \times [0,1] \), the extension theorem implies \( \exists \) a homotopic map \( Y_t \) s.t. \( Y_0 = \tilde{Y}_0 = Y \) and \( Y_1 = \tilde{Y}_1 = \tilde{Y} \).

Suppose \( \exists \tilde{p} \in Y_t \cap M \). Then by transversality,
\[
0 = \dim T_{\tilde{p}} \mathbb{R}^n = \dim (T_{\tilde{p}} M + \text{im}(dY_{\tilde{p}})) = m + 2 < n
\]
which is a contradiction. Therefore \( Y_t \cap M = \emptyset \) and \( Y_t : S^1 \rightarrow \mathbb{R}^n \setminus M \). Therefore \( Y \) is homotopic to \( p \in M \setminus \mathbb{R}^n \) and \( \mathbb{R}^n \setminus M \) is simply connected.
Let $M$ denote the space of $n 	imes k$ matrices, and let $S$ denote the space of $k 	imes k$ symmetric matrices. We note that $\dim M = nk$ and $\dim S = k(k+1) + \cdots + 1 = \frac{k(k+1)}{2}$.

We view $V_k \subset M$ via the isomorphism inclusion $V_k \to M: (v_1, \ldots, v_k) \mapsto [v_1 \ldots v_k]$.

Define $F: M \to S: A \mapsto A^T A$. Then by definition of orthogonal,

$$V_k = F^{-1}(I).$$

To show that $V_k$ is a smooth manifold, it then suffices to show $I$ is a regular value of $F$. We recall that $T_0 M = M$ and $T_0 S = S$, so it suffices to show that $dF_A: M \to S$ is surjective for all $A \in V_k$.

For some $A \in V_k$ and $B \in S$. Define $C = \frac{1}{2} AB$. Then

$$dF_A(C) = \lim_{t \to 0} \frac{(A + tC)^T (A + tC) - A^T A}{t}$$

$$= \lim_{t \to 0} \frac{A^T A + tC^T A + tA^T C + t^2 C^T C - A^T A}{t}$$

$$= C^T A + A^T C$$

$$= \frac{1}{2} B^T + \frac{1}{2} B = B \quad \text{(since $B \in S$)}$$

Therefore $dF_A$ is surjective for all $A \in V_k$ and so $I$ is a regular value of $F$. Thus $V_k = F^{-1}(I)$ is a smooth submanifold of dimension

$$\dim V_k = nk - \frac{k(k+1)}{2} = n(k - \frac{k+1}{2})$$

as desired.
WLOG suppose that $p$ is odd.

Let $\pi_p, \pi_q$ denote the projections from $S^p \times S^q$ to the respective factors.

Then

$$T(S^p \times S^q) \cong \pi_p^*(TS^p) \oplus \pi_q^*(TS^q).$$

Since $p$ is odd, $S^p$ admits a non-vanishing vector field. Therefore $TS^p \cong E \oplus E^1$ where $E$ is a tautological line bundle.

Then

$$T(S^p \times S^q) \cong \pi_p^*(E^1) \oplus \pi_q^*(TS^q).$$

Viewing $S^q \cong \mathbb{R}^{q+1}$, we note that $NS^q$ is trivializable.

Therefore $E \cong \pi_q^*(NS^q)$ and

$$T(S^p \times S^q) \cong \pi_p^*(E^1) \oplus \pi_q^*(NS^q \oplus TS^q)$$

$$\cong \pi_p^*(E^1) \oplus \pi_q^*(TR^{q+1}).$$

Since $\mathbb{R}^{q+1}$ is parallelizable, $TR^{q+1} \cong E^{q+1}$. Therefore

$$T(S^p \times S^q) \cong \pi_p^*(E^1) \oplus E^{q+1}$$

$$\cong \pi_p^*(E^1 \oplus E) \oplus E \oplus E^{q-1}$$

$$\cong \pi_p^*(TS^p) \oplus E \oplus E^{q-1}.$$

Repeating the same steps for $\pi_p^*(TS^p) \oplus E$, we see that

$$\pi_p^*(TS^p) \oplus E \cong E^{p+q}.$$

Therefore

$$T(S^p \times S^q) \cong E^{p+q},$$

and so $S^p \times S^q$ is parallelizable.
(⇒) Suppose first that \( w = \nabla \eta \) for some \( \eta \in C^1_c(\mathbb{R}^n) \).

Let \( B \) be a ball in \( \mathbb{R}^n \) containing the support of \( \eta \).

Then, by Stokes' theorem,
\[
\int_{\mathbb{R}^n} \omega = \int_{\partial B} \omega = \int_{\partial B} \eta = 0
\]
as claimed.

(⇐) Now suppose instead that \( \int_{\mathbb{R}^n} \omega = 0 \).

Let \( B \) be a ball containing the support of \( w \).

Then, \( \int_B \omega = \int_{\mathbb{R}^n} \omega = 0 \).

(⇒) Define \( f : H^1_c(\mathbb{R}^n) \to \mathbb{R} \) by \( w \mapsto \int_{\mathbb{R}^n} \omega \). By the previous discussion, this is well-defined on cohomology.

By the linearity of integration, \( f \) is linear.

We claim that \( f \) is an isomorphism.

First we show surjectivity. Let \( \psi \geq 0 \) be a smooth bump function, i.e., \( \psi = 1 \) on \( B(0,1) \) and \( \psi = 0 \) on \( \mathbb{R}^n \setminus B(0,2) \).

Then, \( f(\psi dV) = \int_{\mathbb{R}^n} \psi dV = \int_{B(0,1)} \psi dV = \text{vol}(B(0,1)) > 0 \).

By linearity, this then implies surjectivity.

We will use Poincaré duality to show \( H^1_c(\mathbb{R}^n) \cong H^0_c(\mathbb{R}^n) \cong \mathbb{R} \).

Therefore, \( f \) is a linear map surjective map between same dimension spaces and hence is an isomorphism.

Then \( \omega \) is exact if \( \int_{\mathbb{R}^n} \omega = 0 \), as claimed. \( \Box \)
We show (b) only as (a) is a specific instance of (b).

Fix $k > 1$ and let $M$ be the set of rank $k$ $m \times n$ matrices.

We recall that a matrix $M$ is rank $k$ iff $M$ contains an invertible $k \times k$ submatrix (in the sense of exchanging columns from $M$) and no $j \times j$ invertible submatrices for $j > k$.

Let $M_k$ denote the matrix of the block form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^k \quad \text{in} \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix}^m \begin{bmatrix} A & B \\ C & D \end{bmatrix}^n \begin{bmatrix} A & B \\ C & D \end{bmatrix}^k$$

where $A$ is a $k \times k$ invertible matrix. We first aim to show that $M_k$ is of rank $k$ iff $D - CA^{-1}B = 0$.

Consider the map

$$\Phi: \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \begin{bmatrix} A^{-1} & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{bmatrix}$$

By direct computation, $\det \begin{bmatrix} A^{-1} & 0 \\ -CA^{-1} & I \end{bmatrix} = \det A^{-1} \neq 0$ and so $\begin{bmatrix} A^{-1} & 0 \\ -CA^{-1} & I \end{bmatrix}$ is invertible. Thus $\begin{bmatrix} A^{-1} & 0 \\ -CA^{-1} & I \end{bmatrix}$ is invertible and hence a homeomorphism. Therefore $\Phi$ is invertible and hence a homeomorphism.

In particular, $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ has rank $k$ iff $\begin{bmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{bmatrix}$ has rank $k$. Since $I$ is invertible, if $D - CA^{-1}B \neq 0$ then we can find a larger invertible submatrix $D - CA^{-1}B$. Therefore $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is of rank $k$ iff $D - CA^{-1}B = 0$.

Define $f: M \to \mathbb{R}^{m \times n \times k}$ by $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto D - CA^{-1}B$. We claim that $0$ is a regular value of $f$. Fix some $E \in M\{m \times m \times k\}$ then $\forall \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in f^{-1}(0)$

$$df_{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \lim_{E \to 0} \frac{D_{E=CA^{-1}B} - D_{CA^{-1}B}}{E} = 0$$

and so $df_{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}$ is negative at $M \cap f^{-1}(0)$. Therefore $0$ is a regular value of $f$. 

We now aim to extend $f$. We note that the determinant of a submatrix is a smooth map and so we can partition $\mathcal{M}_{mn}$ into the union

\[ \mathcal{M}_{mn} = \{(no \text{ kxk invertible submatrices}) \cup \{\text{ one kxk invertible submatrix}\}\]

where $f$ is open and $f^c$ is closed. By $f$ is open, $f$ is a submanifold of $\mathcal{M}_{mn}$ of dimension $mn$. We now aim to extend $f$ to all of $f$.

We note that each $\mathcal{M}_{mn}$ matrix has $(m-k)(n-k)$ kxk submatrices. Let the positions of these submatrices be enumerated 1,...,(m-k)(n-k).

We can write $f$ as the union of open sets

\[ f = \bigcup_{i=1}^{(m-k)(n-k)} \{ \text{kxk submatrix in position } i \text{ is invertible}\} \]

On $f_i$, by permuting rows and columns, $f$ a diffeomorphism $\Psi_i f_i \to M$.

Define $f: M \to \mathbb{R}^{(m-k)(n-k)}$ by

\[ f = \sum_{i=1}^{(m-k)(n-k)} \Psi_i f_i \circ \Phi_i \]

By the earlier reasoning for $f$, 0 is a regular value of $f$ if $f^{-1}(0)$ is a regular value of $f$ and $f^{-1}(0) = \mathcal{T}$. Therefore $T$ is a submanifold of dimension

\[ mn - (m-k)(n-k) = mk + nk - k^2 = k(mn - k) \]
(a) Let \( \{S_n\} \) be a partition of \([0,1]\), n.t. \( c_{[t_n,t_{n+1}]} \) is smooth. We define
\[
\int_c w = \frac{1}{n} \sum \int_{t_n}^{t_{n+1}} c^* w
\]
Hence integration of a smooth \( w \) in \( \mathbb{R} \) is well-defined, this is as well.

(b) Suppose that \( w = df \) for some smooth \( f \). Then by direct computation, \( \forall \) path, piecewise smooth \( c \),
\[
\int_c w = \int_c df = \frac{1}{n} \sum \int_{t_n}^{t_{n+1}} c^* df = \frac{1}{n} \sum \int_{t_n}^{t_{n+1}} d(f(c)) = \frac{1}{n} \sum \int_{t_n}^{t_{n+1}} (f(c'))(t) \, dt
\]
(FTC)
\[
= \frac{1}{n} (f(c(t_{n+1})) - f(c(t_n))) = f(c(1)) - f(c(0))
\]
In particular, if \( c \) is closed then \( \int_c w = 0 \).

(=) Suppose instead that \( \int_c w = 0 \) \( \forall \) piecewise smooth closed curve \( c \). We aim to show that \( M \) or \( w \) is exact. WLOG repeat the argument on all path-connected component of \( M \).

Fix some \( p \in M \). Since \( M \) is path-connected, \( \forall p \in M \) \( \exists \) a smooth path \( \gamma(p) \) from \( p_0 \) to \( p \). Define \( f : M \to \mathbb{R} \) by
\[
f(p) = \int_{\gamma(p)} w
\]
we find claim that \( f \) is well-defined, i.e. that \( f(p) \) is independent of the choice of path \( \gamma(p) \), \( \forall p \in M \) and suppose \( f \) two...
Then $Y_t = \gamma_t : [0,1] \rightarrow M$ is a piecewise smooth 'closed' path.

Assumption and definition then implies

$$\mathcal{P} = \int_{Y_t} \omega = \int_{Y_0} \gamma_t^* \omega + \int_D (\gamma_t^* \omega) = \int_{\gamma_0} \omega - \int_{\gamma_1} \omega$$

and $\omega \int_{\gamma_0} \omega = \int_{\gamma_1} \omega$. Therefore $f$ is independent of the choice of path $Y_t(p)$ and hence is well-defined.

Since $\omega$ is smooth and $Y_t(p)$ can be locally chosen to be ray smoothly at $p$, it then follows that $f(p)$ is a smooth function.

Moreover, by definition, since $Y_t(p)$ is smooth

$$df_p = d\left( \int_{Y_t(p)} \omega \right) = d\left( \int_{0}^{1} (Y_t(p))^* \omega \right)$$

Fix some $p \in M$ and local coordinates $(x_1, \ldots, x_n)$ on a neighborhood $U$ of $p$.

By definition,

$$df_p = 2 \frac{\partial}{\partial x_i}(y_i)_p$$

and

$$\frac{\partial}{\partial x_i} \bigg|_p = \lim_{t \to 0} \frac{f(p + t \hat{x}_i) - f(p)}{t}$$

hence $f$ is independent of the path chosen, we may choose $Y(p)$, $Y(p + t \hat{x}_i)$ i.e. they agree up to $p$ and $Y(p + t \hat{x}_i)$ tends to the $x_i$ direction at unit speed.

$$\frac{\partial}{\partial x_i} \bigg|_p = \lim_{t \to 0} \frac{1}{t} \int_{0}^{t} g_i \, ds$$

$$= \lim_{t \to 0} \frac{1}{t} \int_{0}^{t} g_i \, ds = g_i(p)$$

Therefore $df_p = \xi g_i(p) dx_i = \omega_p$ and so $df = \omega$ as desired.
(a) Two smooth submanifolds are transversal, denoted $S_1 \pitchfork S_2$, if $\forall p \in S_1 \cap S_2$, $T_p S_1 + T_p S_2 = T_p M$.

(b) Suppose that $S_1 \pitchfork S_2$.

If $S_1 \cap S_2 = \emptyset$, then this claim fails. This can be seen by taking two disjoint open sets $U_1, U_2 \subset M$. Then $U_1 \cap U_2 = \emptyset$ but $\dim U_1 + \dim U_2 - \dim M = \dim M$. Therefore we assume $S_1 \cap S_2 \neq \emptyset$.

Let $i_1 : S_1 \hookrightarrow M$ and $i_2 : S_2 \hookrightarrow M$ be the two smooth embeddings. Fix $p \in S_1 \cap S_2$. Then by the implicit function theorem, I find a neighborhood $U_1 \subset S_1$ of $p$ that is the zero set of functions $\{f_1, \ldots, f_{n_1}\}$ where $n_1 = \dim S_1$. Similarly, I find a neighborhood $U_2 \subset S_2$ of $p$ that is the zero set of functions $\{g_1, \ldots, g_{n_2}\}$ where $n_2 = \dim S_2$.

Let $U = U_1 \cap U_2 \subset S_1 \cap S_2$. By construction, $U = \{f_1 = 0, \ldots, f_{n_1} = 0, g_1 = 0, \ldots, g_{n_2} = 0\}$.

We claim that $0$ is a regular value of $\Phi = (f_1, \ldots, f_{n_1}, g_1, \ldots, g_{n_2})$. By definition, $\Phi : S_1 \cap S_2 \to \mathbb{R}^{2n_1 + 2n_2}$. Therefore $0$ must be a regular value of $\Phi$. By definition, $\ker d\Phi_x = \mathbb{R}^{2n_1 + 2n_2}$.

By direct computation, $d\Phi_x = df_x \otimes dg_x$ and so

$$\dim \ker d\Phi_x = \dim (\ker df_x + \ker dg_x)$$

$$= \dim \ker df_x + \dim \ker dg_x - \dim (\ker df_x + \ker dg_x)$$

$$= n_1 + n_2 - \dim (T_x S_1 + T_x S_2)$$

$$= n_1 + n_2 - n$$

Since $S_1 \pitchfork S_2$. Therefore $0$ is a regular value of $\Phi$ and $0$ is a $n$-dimensional submanifold of $M$. Therefore $U = \Phi^{-1}(0)$ is a submanifold of $M$ of dimension $n = (n_1 - 1) + (n_2 - 1) = n_1 + n_2 - n$. As $n_1 + n_2$ can be found and $p \in S_1 \cap S_2$, this implies $S_1 \cap S_2$ is a regular value, and $0$ is a regular value. Therefore $S_1 \cap S_2$ is a submanifold of dimension $n_1 + n_2 - n$. 

* and zero is a regular value.
For some $p \in S$.

$(\Leftarrow)$ Suppose $f$ is constant on $S$.

\[ df_p = \sum \lambda_i dF_p^i \]

Let $\iota : S \to M$ be the inclusion of $S$ in $M$. Then $f|_S = f \circ \iota$, and so

\[ d(f|_S)_p = df_p \circ d\iota^*_p = \sum \lambda_i dF_p^i \circ d\iota^*_p = \sum \lambda_i d(F \circ \iota)_p \]

hence $F^i$ is constant on $S$, $d(F^i \circ \iota)_p = 0 \Rightarrow d(f|_S)_p = d(f \circ \iota)_p = 0$ as desired.

$(\Rightarrow)$ Suppose that $f|_S = f \circ \iota$ has a critical point at $p$.

Then $df_p \circ d\iota^*_p = 0$ and $\ker df_p \supset T_p S$. Hence, $\ker df_p$ is only nonzero in direction normal perpendicular to $S$. Since $S = F^{-1}(c)$, we know that $\ker df_p = T_p S^\perp \subset T_p M$. Therefore $\exists$ constants $\lambda, \ldots, \lambda_k \neq 0$.

\[ df_p = \sum \lambda_i dF_p^i \]

as desired.
Let $M$ be a smooth orientable manifold with boundary of dimension $n$.

Suppose for the sake of contradiction that there exists a smooth

map $r : M 	o DM$.

By Sard's theorem, if $p \in DM$ is a regular value of $r$, then $r^{-1}(p)$ is a submanifold of codimension $n-1$. Since $M$ is compact and $r^{-1}(p)$ is closed, it follows that $r^{-1}(p)$ is a compact 1-dimensional manifold.

Therefore, by the classification of compact 1-dimensional manifolds, $r^{-1}(p)$ is the disjoint union of closed intervals and copies of $S^1$.

In particular, $\#(\partial (r^{-1}(p)))$ is even if it is finite.

However, since $r = id$ on $\partial M$,

$$\partial (r^{-1}(p)) = (\partial M) \cap r^{-1}(p) = p$$

which is odd. Therefore, no such smooth map exists. $\square$.
For $A \in \text{GL}_{n+1}(\mathbb{C})$, 

(a) We note that $A$ is a smooth linear map $C^{n+1} \to C^{n+1}$. 

Since $A$ is linear and invertible, $\ker A = \{0\}$. Therefore 

$A: C^{n+1} \setminus \{0\} \to C^{n+1} \setminus \{0\}$ is smooth. To show that this 

descends to a smooth map $\mathbb{C}P^n \to \mathbb{C}P^n$, it suffices to show 

that $A$ factors through the quotient $C^{n+1} \setminus \{0\} \to C^{n+1} \setminus \{0\} / \sim = \mathbb{C}P^n$ 

where $\sim$ is $x \sim y$ if $x = y$. 

We define $\tilde{A}: \mathbb{C}P^n \to \mathbb{C}P^n$ by 

$[z_0: \ldots : z_n] \mapsto [A(z_0, \ldots, z_n)]$. 

By linearity, $A(x \cdot A: \ldots : y) = A(x_0, \ldots, z_n) = x \cdot A(z_0, \ldots, z_n)$. 

Therefore $\tilde{A}: \mathbb{C}P^n \to \mathbb{C}P^n$ is well-defined and smooth.

(b) Suppose that $[z_0: \ldots : z_n]$ is a fixed point of $A$. Then 

$[z_0: \ldots : z_n] = A(z_0, \ldots, z_n) = [A(z_0, \ldots, z_n)]$ 

and so $A(z_0, \ldots, z_n) = \lambda (z_0, \ldots, z_n)$ for some $\lambda \neq 0$. By definition, 

of $A$, 

(c) We recall that a map $f$ is finite if $f$ has finitely many fixed points $x$, 

and so that all eigenvalues of $A$ are distinct. Then by changing 

bases, we may assume that $\text{diag}(\lambda_1, \ldots, \lambda_n) = A$. 

By the previous part, the only possible fixed points of $A$ 

are of the form $[0: \ldots : 1: \ldots : 0]$. Moreover, we note that a fixed 

point $[0: \ldots : 1: \ldots : 0] = [0: \ldots : 1: \ldots : 0]$, 

and so the fixed points of $A$ are precisely $[0: \ldots : 1: \ldots : 0]$. 

and is the fixed point of $A$ on precisely $[0: \ldots : 1: \ldots : 0]$.
For $i$ and consider the neighborhood

$$U_i = \left\{ (z_0, \ldots, z_i, \ldots, z_n) : z_i \neq 0 \right\}$$

we note that $U_i$ is diffeomorphic to $\mathbb{C}^n$ via the map $\Psi : [z_0, \ldots, z_i, \ldots, z_n] = \left[ \frac{z_0}{z_i}, \ldots, 1, \ldots, \frac{z_n}{z_i} \right] \mapsto \left( \frac{z_0}{z_i}, \ldots, 1, \ldots, \frac{z_n}{z_i} \right)$.

With this, we may represent $A$ as a map $A : U_i \to U_i$.

$$A(z_0, \ldots, \frac{1}{z_i}, \ldots, z_n) = \Psi(A[z_0, \ldots, 1, \ldots, z_n]) = \Psi([\frac{\lambda_0 z_0}{z_i}, \ldots, \frac{\lambda_n z_n}{z_i}]) = \left( \frac{\lambda_0 z_0}{z_i}, \ldots, 1, \ldots, \frac{\lambda_n z_n}{z_i} \right)$$

Therefore $A = \text{diag} \left( \frac{\lambda_0}{z_i}, \ldots, \frac{\lambda_n}{z_i} \right)$ on $U_i \subseteq \mathbb{C}^n$.

In particular, $dA = \text{diag} \left( \frac{\lambda_0}{z_i}, \ldots, \frac{\lambda_n}{z_i} \right)$ since $TU_i = TU_i \subseteq U_i$.

$$\Rightarrow dA - I = \text{diag} \left( \frac{\lambda_0}{z_i} - 1, \ldots, 0, \ldots, \frac{\lambda_n}{z_i} - 1 \right)$$

$$\Rightarrow \det (dA - I) \neq 0 \text{ since } \frac{\lambda_i}{z_i} \neq 1 \forall j \neq i.$$ 

Therefore $[0, \ldots, 1, \ldots, 0]$ is a Teichmüller fixed point $V_i$ and so $A$ is a Teichmüller map if all eigenvalues are distinct.

In particular, we assume that $A$ has distinct eigenvalues again.

We recall that the Teichmüller number of $A$ is the sum of local Teichmüller numbers which are $\pm 1$ depending on the sign of whether $dA - I$ is orientation preserving or reversing.

As calculated in (c), a fixed point $z_i = [0, \ldots, 1, \ldots, 0]$.

$$\det (dA - I) = \prod_{j \neq i} \left( \frac{\lambda_j}{z_i} - 1 \right)$$

Therefore the local Teichmüller number $\ell_i(A)$ at $z_i$ is given by

$$\ell_i(A) = \text{sign} \prod_{j \neq i} \left( \frac{\lambda_j}{z_i} - 1 \right) = \text{sign} \left( z_i^{-n} \prod_{j \neq i} (\lambda_j - z_i) \right).$$
(d) We recall that fundamental number is homotopy
invariant and that $L(\text{id}) = \chi(M)$.
Because $\text{GL}_n(\mathbb{C})$ is connected, \textit{\(\exists\) a path and hence a}
homotopy, from $A$ to the identity matrix. This \textit{\(\Rightarrow\)}
ascending to
a homotopy from $A: \mathbb{C}P^n \to \mathbb{C}P^n$ to $\text{id}: \mathbb{C}P^n \to \mathbb{C}P^n$.

Therefore

$$L(A) = L(\text{id}) = \chi(\mathbb{C}P^n).$$

we recall that the CW structure of $\mathbb{C}P^n$ has \textit{\(\Rightarrow\)} a single
cell in every even dimension. Therefore

$$L(A) = \sum_{i=0}^{n} 1 = n+1$$
as desired. \(\square\)
Let $F : S^n \to S^n$ be a continuous map. Then $F$ induces a map $F_* : H_n(S^n) \to H_n(S^n)$ on top homology. We recall that $H_n(S^n) \cong \mathbb{Z}$ and so $F_*$ gives a homomorphism $\mathbb{Z} \to \mathbb{Z}$.

As the only homomorphism $\mathbb{Z} \to \mathbb{Z}$ are multiplication by an integer, $F_* : \mathbb{Z} \to \mathbb{Z}$ is multiplication by $k$. We define $\deg F = k$.

Additionally, we recall that the homomorphism $H_0(S^n) \to \mathbb{Z}$.

Therefore, for all $n$-forms $w$ on $S^n$

$$
\int_{S^n} F_* w = \int_{F_* S^n} w = \int_{S^n} w = (\deg F) \int_{S^n} w
$$

while $S^n$ is regarded as an $n$-cycle. Depending on whether

(b) Suppose $F$ has no fixed point. We claim that $F$ is homotopic to $-\text{id}$. Consider the straight line homotopy

$$
H_t(x) = \frac{(1-t)F(x) + t(-x)}{1 - t|F(x) + t(-x)|} = \frac{(1-t)F(x) - tx}{1 - t|F(x) - tx|}
$$

Thus, $H_t$ is well-defined, it must be shown that $(1-t)F(x) - tx \neq 0 \forall x, t$. Suppose $\exists x \in M, (1-t)F(x) - tx = 0$. We note that since $F(x) \neq 0$ and $-x \neq 0$, $t \neq 0, 1$. Therefore

$$
F(x) = \frac{tx}{1-t} \implies x = \frac{tx}{t-1}
$$

and so $t = 1/2$. However, then $F(x) = \frac{11/2}{1-1/2} x = x$ which contradicts the fact that $F$ has no fixed points. Therefore $(1-t)F(x) - tx \neq 0 \forall x, t$.

Thus $H_t$ is a homotopy from $H_0 = F$ to $H_1 = -\text{id}$.

Then $\deg F = \deg (-\text{id}) = (-1)^{n+1}$ as desired.
We define

$$D^n \cup D^n = \frac{D^n \times S^{n-1}}{\text{ev}}$$

where $$(x,0) \sim (f(x),1) \forall x \in D^n \approx \mathbb{S}^{n-1}$$.

This is to say that $D^n \cup D^n$ is two copies of $D^n$ with boundaries attached via $f$.

(6) Equivalently, for $n \geq 2$ and $\deg f = k$, we continue $D^n \cup D^n$ via

1. 0-cell: $\cdot p$
2. $n$-cells: $A,B$ with $\partial A = e$
   \[ \partial B = ke \quad \text{sold} \]

Deduce computation then gives the homology groups

$$H_0(X) = \frac{kr_{e_0}}{im_{d_0}} = \mathbb{Z} \langle p \rangle = \mathbb{Z}$$
$$H_k(X) \equiv \mathbb{Z} \quad \text{for } 1 \leq k \leq n-2$$
$$H_{n-1}(X) = \frac{kr_{e_{n-1}}}{im_{d_{n-1}}} = \mathbb{Z} \langle e \rangle = \mathbb{Z} \langle e, ke \rangle = \mathbb{Z}$$
$$H_n(X) = \frac{kr_{e_n}}{im_{d_{n+1}}} = \mathbb{Z} \langle B, kA \rangle = \mathbb{Z}$$

and $0$ for all higher order homologies.

For $n = 1$, we instead have the CW complex

2 0-cells: $a,b$
2 1-cells: $A,B$ with $\partial A = a-b$
$$\partial B = f(a) - f(b) = \begin{cases} 0 & k = \pm 2 \\ \pm (a-b) & k = \pm 1 \end{cases}$$

For both $k = \pm 2$, the deformation extends onto $S^1$ and so

$$H_n(D^n \cup D^n) = \begin{cases} \mathbb{Z} & k = 0,1 \\ 0 & \text{else} \end{cases}$$
(c) If \( f \) is a homeomorphism, then \( \text{deg} f = \pm 1 \) and so we have the CW complex

\[
\begin{align*}
\text{n} & \geq 2 \\
1 & \text{n-cell} : p \\
2 & (n-1)\text{-cell} : e \quad \text{with} \quad \partial e = p - 0 \\
3 & n\text{-cell} : \mathbb{A}, \mathbb{B} \quad \text{with} \quad \partial \mathbb{A} = e, \quad \partial \mathbb{B} = \pm e
\end{align*}
\]

By reversing the orientation of \( B \) by \( \text{deg} f = 1 \), this yields

\[
\begin{align*}
\text{n} & \geq 2 \\
1 & \text{n-cell} : p \\
2 & (n-1)\text{-cell} : e - 1 \quad \text{with} \quad \partial e = p - 0 \\
3 & n\text{-cell} : \mathbb{A}, \mathbb{B} \quad \text{with} \quad \partial \mathbb{A} = \partial \mathbb{B} = e
\end{align*}
\]

In both cases, this is precisely the CW complex for \( S^n \).

Therefore \( D^n \cup D^n \) is homeomorphic to \( S^n \) if \( f \) is a homeomorphism since all these CW complexes for all even \( n \) are homeomorphic. \( \square \)
(a) No, $M$ and $N$ need not have the same fundamental group. We recall that $S^2$ is a 2-sheeted cover of $\mathbb{RP}^2$. By considering their CW complexes,

\begin{align*}
S^2 & & \mathbb{RP}^2 \\
1 \text{ 0-cell} : p & & 1 \text{ 0-cell} : q \\
1 \text{ 2-cell} : A & & 1 \text{ 2-cell} : \alpha \text{ w/ } \int_A \alpha = 0 \\
1 \text{ 2-cell} : B & & 1 \text{ 2-cell} : \beta \text{ w/ } \int_B \beta = 2a
\end{align*}

we can compute their fundamental groups as being generated by their 1-cells with relations given by the boundaries of their 2-cells. Thus $\pi_1(S^2) \cong \langle 0, 1 \rangle \cong 0$ and $\pi_1(\mathbb{RP}^2) \cong \langle \alpha, 2a \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

Then $\pi_1(S^2) \neq \pi_1(\mathbb{RP}^2)$.

(c) No, $M$ and $N$ need not have the same singular homology. Consider $S^2 \to \mathbb{RP}^2$ as done in (a). It suffices to compute cellular homology over $\mathbb{Z}$ as that is equivalent. By direct computation we find the CW complexes above.

\begin{align*}
H_1(S^2) & = 0 \\
H_2(S^2) & \cong \mathbb{Z} \\
H_1(\mathbb{RP}^2) & = \frac{k \alpha}{\text{im} k} = \frac{\mathbb{Z} \langle 2a \rangle}{\mathbb{Z} \langle 2a \rangle \cong \mathbb{Z}/2\mathbb{Z}} \\
H_2(\mathbb{RP}^2) & = \frac{k \alpha \beta}{\text{im} k} = \frac{\mathbb{Z} \langle 2a \rangle \langle 2\beta \rangle}{\langle 2a \rangle \langle 2\beta \rangle} \cong 0
\end{align*}

and so these homologies do not agree.

(b) No, these de Rham cohomologies need not agree. Consider $S^2, \mathbb{RP}^2$ as above. By universal coefficients,

\begin{align*}
H^2_{\text{dR}}(S^2, \mathbb{R}) & = H^2(S^2; \mathbb{R}) \cong \mathbb{R} \\
H^2_{\text{dR}}(\mathbb{RP}^2, \mathbb{R}) & = H^2(\mathbb{RP}^2; \mathbb{R}) \cong \mathbb{R} \\
H^3_{\text{dR}}(\mathbb{RP}^2, \mathbb{R}) & = H^3(\mathbb{RP}^2; \mathbb{R}) \cong 0
\end{align*}

and so these de Rham cohomologies need not agree.
Let \( A, X \) be endowed with a simplicial (or cellular) structure.

and let \( C_n(X), C_n(A) \) denote.

Let \( C_n(X), C_n(A) \) denote the free abelian groups on the regular \( n \)-simplices in \( X,A \) respectively. Let \( i: A \to X \) denote the inclusion of \( A \) in \( X \). Then \( i \) induces an inclusion \( i_*: C_n(A) \to C_n(X) \)

and so we can consider the quotient \( C_n(X,A) = C_n(X)/C_n(A) \).

Hence the boundary map \( \partial_c: C_n(X,A) \to C_{n-1}(X,A) \) agrees with the boundary map \( \partial_c: C_n(A) \to C_{n-1}(A) \) on their overlap. We have a well-defined boundary map \( \partial_n: C_n(X,A) \to C_{n-1}(X,A) \).

Since \( i_*(C_n(X)) \subseteq C_n(X) \), and \( i_*(C_n(A)) \subseteq C_n(A) \), it follows that we have a well-defined chain complex

\[ \ldots \to C_n(X,A) \to C_{n-1}(X,A) \to \ldots \]

We thus define the relative homology \( H_n(X,A) = \ker \partial_n/\text{im} \partial_{n+1} \).

Setting \( \pi_n \) be the quotient map \( C_n(X) \to C_n(X,A) \), we get a SES

\[ 0 \to C_n(A) \to C_n(X) \to C_n(X,A) \to 0 \]

The standard snake lemma proof then yields a LES on homology.
GeoTop
Fall 2012
(a) Let \( A \in SL_2(\mathbb{R}) \). We recall the polar decomposition

\[ A = UP \]

where \( U \) is an orthogonal matrix and \( P \) is a symmetric positive semi-definite matrix. Then \( \det A = \det U \cdot \det P \). By \( \det U = 1 \) and \( \det P > 0 \), it follows: \( \det P, \det U = 1 \). Therefore \( U \in SO(2) \) and hence as a rotation \( U = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \) for \( \theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z} \).

Then

\[ A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \quad \text{with} \quad ac - b^2 = 1 \implies \cos \theta = \frac{b^2 + 1}{a} \quad \text{(note } a, c > 0 \text{ and } b^2 > 0) \]

and

\[ A = \begin{bmatrix} a & b \\ b & \frac{b^2 + 1}{a} \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \]

Since \( A \) is invertible, this decomposition is unique and smooth in \( A \). Therefore \( SL_2(\mathbb{R}) \to S^1 \times \mathbb{R}^2: A \mapsto (\theta, a, b) \) is a diffeomorphism.

(b) Now suppose \( A \in SL_2(\mathbb{C}) \). Again, by the polar decomposition \( A = UP \) where \( U \) is unitary and \( P \) is positive semi-definite Hermitian.

By the same reasoning as above, \( \det U = \det P = 1 \) and \( U \in SU_2(\mathbb{C}) \). Then \( U = \begin{bmatrix} \alpha & -\overline{\beta} \\ \overline{\beta} & \alpha \end{bmatrix} \) so \( \frac{\alpha^2 + \beta^2}{2} = 1 \). Therefore

\[ A = \begin{bmatrix} \alpha & -\overline{\beta} \\ \overline{\beta} & \alpha \end{bmatrix} \begin{bmatrix} x & y \\ y & z \end{bmatrix} \]

We note that \( a, c \in \mathbb{R} \) with \( c = \frac{b^2 + 1}{a} \) whenever \( \det P = 1 \). As before, \( U, P \) are unique and vary smoothly with \( A \).

\( SL_2(\mathbb{C}) \to S^2 \times \mathbb{R}^2 \cong S^1 \times \mathbb{R} \times \mathbb{C}: \lambda \mapsto (x, y, a, c) \) is a diffeomorphism. \( \square \)
We shall that \( \text{RIP}^{2n-1} \) can be viewed as

\[
\text{RIP}^{2n-1} = \mathbb{S}^{2n-1} / \varepsilon
\]

where \( \mathbb{S}^{2n-1} \subset \mathbb{R}^{2n} \) and \( \varepsilon \) is the antipodal identification \( x \sim -x \).

We aim to construct a non-vanishing smooth vector field on \( \mathbb{S}^{2n} \) and then transfer that to \( \text{RIP}^{2n-1} \).

Consider some \( x = (x_1, y_1, \ldots, x_n, y_n) \in \mathbb{S}^{2n-1} \subset \mathbb{R}^{2n} \). We shall that

\[
T_x \mathbb{S}^{2n-1} = \{ z \in \mathbb{R}^{2n} : z \perp x \}. \text{ In particular,}
\]

\[
(y_1, -x_1, \ldots, y_n, -x_n) \in T_x \mathbb{S}^{2n-1}
\]

since

\[
x \cdot (y_1, -x_1, \ldots, y_n, -x_n) = \sum_i x_i y_i - x_i y_i = 0.
\]

The map \( x \mapsto (y_1, -x_1, \ldots, y_n, -x_n) \) is smooth but is only a wording and scaling of coordinates. Therefore

\[
\mathbb{S}^{2n-1} \rightarrow T \mathbb{S}^{2n-1} : x \mapsto (y_1, x_1, \ldots, y_n, -x_n) \in T_x \mathbb{S}^{2n-1}
\]

is a smooth vector field on \( \mathbb{S}^{2n-1} \). Moreover, since \( 0 \notin \mathbb{S}^{2n-1} \), \( V \) is everywhere non-vanishing.

We now show that \( V \) factors through \( \varepsilon \) to a non-vanishing VF on \( \text{RIP}^{2n-1} \).

To show this, it suffices to show that

\[
[V_x] = [V_{-x}]
\]

By direct computation, for \( x = (x_1, y_1, \ldots, x_n, y_n) \)

\[
V_{-x} = ((-y_1), ((-x_1), \ldots, (-y_n), (-x_n)) = (-y_1, x_1, \ldots, y_n, -x_n) = V_x
\]

and so \( [V_x] = [V_{-x}] \).
we now show that $V$ factors through the quotient map $\Pi : S^{2n-1} \to \mathbb{R}P^{2n-1}$.

We recall that $\Pi$ induces a map $d\Pi : TS^{2n-1} \to T\mathbb{R}P^{2n-1}$.

It then suffices to show that

$$d\Pi_x V_x = d\Pi_{-x} V_x$$

Let $f$ be the antipodal map $x \mapsto -x$. Then $\Pi \circ f = -\Pi$.

Direct computation then implies, for $x = (x_1, y_1, \ldots, x_n, y_n)$

$$d\Pi_{-x} V_x = d(\Pi \circ f)_x ((-y_1), (-x_1), \ldots, (-y_n), (-x_n))$$

$$= d\Pi_{-x} ((-Y_1, \ldots, -Y_n, -X_n))$$

$$= d\Pi_{-x} V_x$$

as desired. Therefore $V$ defines a well-defined vector field on $\mathbb{R}P^{2n-1}$.

Since $V$ is non-vanishing, the vector field on $\mathbb{R}P^{2n-1}$ is non-vanishing.

$\square$
We recall the extension theorem:

**Theorem (extension theorem):** Let $\mathbb{C}^n$ be an open manifold and $\mathcal{C}^x$ a closed set. Let $f: x \mapsto y$ satisfy $f \in \mathcal{C}^x$. Then $\exists$ a homotopy map $g: x \mapsto y$ s.t. $g \in \mathcal{C}^x$ and $f \cdot h \in \mathcal{C}^x$ on $\mathcal{C}^x$.

Suppose $x, y \in \mathbb{R}^n \setminus M$. Let $f: [0,1] \to \mathbb{R}^n$ be the straight line

$$f(t) = (1-t)x + ty$$

where $M$ is a manifold of dimension $\leq n$, $M$ is closed. Suppose $C = [0,1]$. Then $C \subset [0,1]$ is closed, and $C \cap f^{-1}(M) = \emptyset$, and $C \cap (\partial f)^{-1}(M) = \emptyset$. So $f \in \mathcal{C}^x$ and $f \cdot h \in \mathcal{C}^x$. The extension theorem then implies that $\exists g: [0,1] \to \mathbb{R}^n$ s.t. $g(0) = x$, $g(1) = y$, and $g$ is homotopic to $f$. In particular, $g$ is a path from $x$ to $y$ and $g \in \mathcal{C}^x$.

Note that $[0,1]$ is one-dimensional, so $\forall p \in [0,1]$, $\dim (\text{im} dgp) = 1$.

Therefore, if $f: p \mapsto g(p) \in M$ then

$$n = \dim (\text{im} dgp + T_g(p)M) \leq 1 + n - 2 = n - 1 < n$$

which is a contradiction. Therefore $g([0,1]) \subset \mathbb{R}^n \setminus M$, and so $\mathbb{R}^n \setminus M$ is path-connected and hence connected.

We now claim that $\mathbb{R}^n \setminus M$ is simply connected.

Let $Y: [S^1] \to \mathbb{R}^n \setminus M$ be a closed path. Let $h: [0,1] \times S^1 \to \mathbb{R}^n$ be the straight line homotopy in $\mathbb{R}^n$,

$$h_s(t) = h(s, t) = (1-s)Y(t) + sY(0)$$

Then $C = \{0,1\} \times S^1 = \partial([0,1] \times S^1)$. Then $C$ is closed, and $C \cap h^{-1}(M) = \emptyset$ and $C \cap (\partial h)^{-1}(M) = \emptyset$. So $h \in \mathcal{C}^x$ and $\partial h \in \mathcal{C}^x$.

$\longrightarrow$
The extension theorem then implies that if \( \mathcal{Y} \) is a homotopy
\[
k : [0,1] \times S^1 \to \mathbb{R}^n
\]
s.t. \( k \circ M \) and \( k|_{0,1} \times S^1 = h|_{0,1} \times S^1 \). Since \( h \) is a homotopy from \( Y \) to \( \gamma(0) \), and \( k|_{0,1} \times S^1 = h|_{0,1} \times S^1 \), it follows that \( k \) is a homotopy from \( Y \) to \( \gamma(0) \). Moreover, since \([0,1] \times S^1\) is 2-dimensional,
\[
\dim (\text{im } dk_p) \leq 2 \quad \forall \ p \in [0,1] \times S^1
\]
Therefore, if \( p \in [0,1] \times S^1 \) s.t. \( k(p) \in M \), then
\[
n = \dim (\text{im } dk_p + T_{k(p)}M) < 2 + n - 2 \quad n
\]
which is a contradiction. Then \( \text{im}(k) \subset \mathbb{R}^n \setminus \mathcal{M} \) and so \( k \) is a homotopy in \( \mathbb{R}^n \setminus \mathcal{M} \) from \( Y \) to \( \gamma(0) \). Therefore \( \mathcal{Y} \) is contractible in \( \mathbb{R}^n \setminus \mathcal{M} \). As this holds for all closed loop paths in \( \mathbb{R}^n \setminus \mathcal{M} \), this implies that \( \mathbb{R}^n \setminus \mathcal{M} \) is simply connected. \( \square \)
(a) For \( n \geq 1 \) and \( k \in \mathbb{Z} \), we aim to construct a continuous map 
\( f: S^n \to S^n \) of degree \( k \).

If \( k = 0 \) then a constant map satisfies.

Now suppose \( k > 0 \). Let \( B_1, \ldots, B_k \) be disjoint \( n \)-disks in \( S^n \).

Consider the quotient map \( \pi: S^n \to S^n / \left( S^n \setminus \bigcup_{i=1}^{k} B_i \right) \).

Since each \( B_i / \partial B_i \cong S^n \), we note that \( \pi(B_i / \partial B_i) \cong S^n \).

Therefore \( S^n / \left( S^n \setminus \bigcup_{i=1}^{k} B_i \right) \cong \bigvee_{i=1}^{k} S^n \).

This \( \pi \) induces a continuous map \( \tilde{f}: S^n \to \bigvee_{i=1}^{k} S^n \).

Finally, define \( f: S^n \to S^n \) by \( f = \tilde{f} \cdot g \cdot \text{identity} \) on each \( S^n \), where \( g \) is the identity or a reflection on each \( S^n \), and maps into a single copy of \( S^n \), with the identity or reflection on each \( S^n \) chosen such that \( f \) is orientation-preserving.

To compute the degree of \( f \), it suffices to consider local degree.

Choose some \( q \in S^n \) s.t. \( f^{-1}(q) \) is finite (i.e., \( q \neq p \)). By construction

If \( |f^{-1}(q)| = k \) and \( V \ni f^{-1}(q) \), \( f \) is locally an orientation-preserving diffeomorphism.

Therefore \( \deg f = 1 \) and so \( \deg \tilde{f} = k \), as desired.

Finally conclude \( k < 0 \). Repeating the above above argument w/ \( k < 1 \) disks and \( g \) chosen to reverse orientation, we find a suitable \( f \).

(b) The same construction as above holds, hence \( \forall k \)

\[
X / \left( X \setminus \bigcup_{i=1}^{k} B_i \right) \cong \bigvee_{i=1}^{k} S^n
\]
(⇒) Suppose that $\Delta$ is integrable. Fix some $p \in \Omega$. Then by definition 3 local coordinates $(x_1, \ldots, x_n)$ defined on a neighborhood $V \subset \Omega \cap \mathbb{R}^n$, $\Delta = \mathbb{R} \langle d\Delta x, \ldots, d\Delta x_n \rangle$ on $V$.

Define $u_i = x_{k+i}$. Hence $\Delta = \mathbb{R} \langle x_1, \ldots, x_k \rangle = \mathbb{R} \langle d\Delta x_1, \ldots, d\Delta x_k \rangle$ on $V$, we can write locally write

$$x_j = \sum f_{ij} \frac{\partial \Delta x_i}{\partial x_j}$$

Then $X_j(u_k) = \sum f_{ij} \frac{\partial \Delta x_i}{\partial x_j}(u_k)$

$$= \sum f_{ij} \frac{\partial \Delta x_i}{\partial x_j}(x_{k+i}) = 0$$
on $V$. In particular, $u_k \notin \mathbb{R} \forall k = 1, \ldots, n-k$. Moreover, since $\{x_i\}$ are coordinates for $V$, it follows that $\{d\Delta x_k, 3x_k\}$ are linearly independent on $V$.

(⇐) Suppose statement (b) holds. Fix some $p \in \Omega$. Then 3 a neighborhood $V \subset \Omega \cap \mathbb{R}^n$ and $n-k$ functions $u_1, \ldots, u_{n-k} \in \mathbb{E}^n$ i.e. $du_1, \ldots, du_{n-k}$ are linearly independent on $V$.

Let $\Phi_{i}(t, q)$ be a local flow of $X_i$ defined on a neighborhood $U_i$ of $p$.

$$\Phi_{i}(t, q) = \frac{\partial}{\partial t} \big|_{t=0} \Phi_{i} \big(0, q\big)$$

Define $U = \cup U_i$, $\cup U_i$. Then $U$ is a neighborhood of $p$.

We recall that a distribution is integrable iff $\forall$ 1-forms $\omega$ that annihilate $\Delta$, $du_1$ also annihilates $\Delta$. Let $I(\omega) = \{1$-form $w| w$ annihilates $\Delta\}$. Have $\dim I(\omega) = k$, it follows that $\dim I(\Delta) = n-k$. Hence $du_1, \ldots, du_{n-k} \in I(\Delta)$ are linearly dependent, if follows that they span $I(\Delta)$ near $p$. In particular, any $\omega \in I(\Delta)$ can be locally written as near $p$ as $\omega = \sum f_i \frac{\partial \Delta x_i}{\partial x_j} \Rightarrow du = \sum \partial f_i \Delta x_i \Rightarrow$ that is, $\Delta$ annihilates all 1-forms.
Then \( \forall x, y \in \Delta, \)
\[
dw(x, y) = \sum_i d\xi_i \wedge du_i(x, y) \\
= \sum \left( d\xi_i(x) du_i(y) - d\xi_i(y) du_i(x) \right) \\
= \sum 0 \\
= 0
\]
and \( \omega \) is an annihilator \( \Delta \). Therefore \( \Delta \) is an integrable distribution.
It suffices to work with de Rham cohomology.

Suppose that $H^{2n+1}(M; \mathbb{R}) \cong \mathbb{R}^k$.

Bk $M$ is compact and orientable, we know that $H^{4n+2}(M; \mathbb{R}) \cong \mathbb{R}$.

Therefore, the wedge product $\wedge : H^{2n+1}(M; \mathbb{R}) \times H^{2n+1}(M; \mathbb{R}) \to H^{4n+2}(M; \mathbb{R})$ defines a map $A : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$, hence the wedge product is bilinear and alternating, it follows that $A$ is bilinear and alternating. In particular, $A$ can be represented by a $k \times k$ skew-symmetric matrix.

We claim that $A$ is invertible. To show this, it suffices to show that $k \cdot A = 0$. In particular, it suffices to show that if $\exists u \neq 0$, $A(u, u) = u^T A u = 0 \quad \forall v$, then $u = 0$.

Suppose that such a $u$ exists. Then $\exists$ a corresponding form $w \neq 0$. $w \wedge w = 0 \quad \forall \nu \in H^{2n+1}(M; \mathbb{R})$.

Let $x_1, \ldots, x_{2n+2}$ be local coordinates for $M$. Then locally,

$$w = \sum_{0 \leq i < \ldots < i_{2n+1} \leq 4n+2} f_{i_{\ldots}, i_{2n+1}} d x_i \wedge \ldots \wedge d x_{i_{2n+1}}$$

By the given property of $w$, this implies that locally, $\forall 0 \leq i_1 < \ldots < i_{2n+1} \leq 4n+2$

$$0 = d x_{i_1} \wedge \ldots \wedge d x_{i_{2n+1}} \wedge w = f_{i_1, i_{2n+1}} \, d \nu \Rightarrow f_{i_1, i_{2n+1}} = 0$$

and $w \wedge w = 0$ locally and hence identically. Therefore $A$ is an invertible matrix. Hence $A$ is skew-symmetric,

$$\det(A) = \det(A^T) = \det(-A) = (-1)^k \det(A)$$

Bk $\det A = 0$, this implies $k$ is constant, as desired.
Suppose on the contrary that \( M \) is a compact 3-dimensional manifold, \( \partial M = \mathbb{R}P^2 \).

Since \( \mathbb{R}P^2 \) is connected, \( M \) has a connected component \( M_1 \) of \( M \) such that \( \partial M_1 = \mathbb{R}P^2 \) and \( \partial M_2 \neq \emptyset \) for other connected components. Therefore, we may assume \( M_1 \) is connected and \( \partial M_1 \neq \emptyset \) by restricting to \( M_1 \).

We first establish a lemma.

**Lemma 1:** Suppose \( X \) is an odd-dimensional closed manifold, \( \mathbb{R} \)-oriented with boundary.

Then \( \chi(X) = 0 \).

**Proof:** We calculate \( \chi(X) \) via Poincaré duality. Since \( \mathbb{R} \) is a field,

\[
H^k(X; \mathbb{R}) = \text{Hom}(H_k(X; \mathbb{R}), \mathbb{R}) \oplus \text{Ext}^k_{\mathbb{R}}(H_{n-k}(X; \mathbb{R}), \mathbb{R})
\]

By the universal coefficients theorem, since \( \mathbb{R} \) is a field,

\[
\Rightarrow H^k(X; \mathbb{R}) = \text{Hom}(H_k(X; \mathbb{R}), \mathbb{R})
\]

Moreover, since \( X \) is compact, Poincaré duality implies

\[
H^k(X; \mathbb{R}) = H_{n-k}(X; \mathbb{R}). \Rightarrow H^k(X; \mathbb{R}) \cong H_{n-k}(X; \mathbb{R})
\]

Therefore since \( X \) is \( \mathbb{R} \)-oriented, since \( n \) is odd

\[
\chi(X) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \text{rank}(H_i(X; \mathbb{R}))
\]

\[
= \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \text{rank}(H_i(X; \mathbb{R})) + (-1)^{n-i} \text{rank}(H_{n-i}(X; \mathbb{R}))
\]

\[
= \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i + (-1)^{n-i} \text{rank}(H_i(X; \mathbb{R}))
\]

\[
= 0
\]

which is what was to be shown. \( \Box \)

Replace all \( \mathbb{R} \)'s with \( \mathbb{Z}/2\mathbb{Z} \).
Lemma 2: Suppose $X$ is an odd-dimensional manifold with boundary.

Then $2\chi(M) = \chi(\partial M)$.

Proof: Let $M_1, M_2$ be 2 copies of $M$. Let $\tilde{M}$ be the doubling of $M$ defined by $\tilde{M} = M_1 \sqcup (\partial M \times [0,1]) \sqcup M_2 / \sim$ where $m_1 \sim (m_0,0)$ for all $m_0 \in \partial M$, $(m_0,0) \sim (m_0,1)$, and $m_2 \sim (m_2,1)$ for all $m_2 \in M_2$, $(m_2,1) \sim (m_2,0)$, which is to say that $\tilde{M}$ is two copies of $M$ attached via a thickening of their boundary. We now proceed via Mayer-Vietoris.

Let $U = M_1 \sqcup \partial M \times [0,1] / \sim$ and $V = \partial M \times (0,1) \sqcup M_2 / \sim$.

Then $U$, $V$, and $U \cap V$ are open subsets of $\tilde{M}$, $U \cap V$ and $V \cap U$ are open subsets of $\partial M$, and $U \cap V \cap U \cap V = \emptyset$.

We then acquire the LES

$$\cdots \rightarrow H_k(U \cap V) \rightarrow H_k(U) \oplus H_k(V) \rightarrow H_k(U \cap V) \rightarrow \cdots$$

$$\cdots \rightarrow H_k(\partial M) \rightarrow H_k(\tilde{M}) \rightarrow \cdots$$

Taking an alternating sum and rearranging, we find that

$$\chi(\tilde{M}) = \sum_{k=0}^{\infty} (-1)^k \text{rank}(H_k(\tilde{M})) - 2 \sum_{k=1}^{\infty} (-1)^k \text{rank}(H_k(\partial M)) + \sum_{k=1}^{\infty} (-1)^k \text{rank}(H_k(\partial M))$$

$$= \chi(\tilde{M}) - 2 \chi(M) + \chi(\partial M)$$

Since $\tilde{M}$ is odd-dimensional, $\partial M$ boundary, and compact, the previous lemma implies $\chi(\tilde{M}) = 0$. Then $\chi(\partial M) = 2\chi(M)$.

Based on Lemma 2, $2\chi(M) = \chi(\partial M) = \chi(\partial \Pi^2)$. Hence we recall that $\chi(\Pi^2) = 1 \neq 0$. 

$\blacksquare$
We note that

\[ X = \mathbb{R}^n \setminus \{L_1, \ldots, L_n\}. \]
If \( n = 0, 1 \), then \( X = \emptyset \) and the homology is trivial. We just take \( n > 2 \).

We note that \( X \) deformation retracts onto \( S^{2n-3} \) via the from straight line homotopy equivalence

\[ X \mapsto (1-t)X + t\frac{X}{|X|}. \]

Therefore it suffices to compute the \( 2n-1 \)-homology of a 2n-punctured sphere.

We note that the 2n-punctured sphere is diffeomorphic to \( \mathbb{R}^{n-1} \) \( \setminus \) the 2n-1-punctured \( \mathbb{R}^{n-1} \) via a stereographic projection.

Via a diffeomorphism, we may arrange all the punctures on a straight line. From here, the space deformation retracts onto a 2n-1-punctured cylinder.

Finally, since a 1-punctured cylinder deformation retracts onto \( S^{n-2} \), one more deformation retracts onto the wedge of 2n-1 copies of \( S^{n-2} \).

Therefore

\[ H_k(X) = H_k\left(\bigvee_{j=1}^{2n-2} S^{n-2}\right) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}^{2n-1} & k = n-2 \\ 0 & \text{else} \end{cases} \]

which is what was to be found.

For \( n = 2 \), \( X \) is the disjoint union of 4 contractible spaces.

Therefore

\[ H_k(X) = \begin{cases} \mathbb{Z}^4 & k = 0 \\ 0 & \text{else} \end{cases} \]
We recall the Künneth formula, which states

\[ H_k(A \times B) = \bigoplus_{j=0}^k H_j(A) \otimes H_{k-j}(B) \]

(a) We construct \( S' \) as one 0-cell \( p \) and \( 1 \)-cell \( e = p \cdot p \cdot e \). Let \( \mathcal{X} \) have \( k \)-cells \( e_1^k, \ldots, e_n^k, \ldots, n \). Then \( \mathcal{X} \times S' \) has \( k \)-cells \( e_i^k \times p \) and \( e_{i-1}^k \times e \). By the toolbox rule, \( A_{k, i,j} \)

\[ \partial_k(e_i^k \times p) = \partial_k e_i^k \times p + (-1)^k e_{i-1}^k \times \partial_k p = \partial_k e_i^k \times p \]

\[ \partial_k(e_{i-1}^k \times e) = \partial_k e_{i-1}^k \times e + (-1)^k e_{i-1}^k \times \partial_k e = \partial_k e_{i-1}^k \times e \]

Then \( \text{Im} \partial_k = (\text{Im} \partial_k) \times p \oplus (\text{Im} \partial_{k-1}) \times e \)

\( \ker \partial_k = (\ker \partial_k) \times p \oplus (\ker \partial_{k-1}) \times e \)

In particular,

\[ H_k(\mathcal{X} \times S') = \frac{\ker \partial_k}{\text{Im} \partial_{k+1}} = \frac{(\ker \partial_{k}) \times p \oplus (\ker \partial_{k-1}) \times e}{(\text{Im} \partial_{k+1}) \times p \oplus (\text{Im} \partial_{k}) \times e} \]

have \( (\text{Im} \partial_{k+1}) \times p \subset (\ker \partial_{k+1}) \times p \) and \( (\text{Im} \partial_k) \times e \subset (\ker \partial_k) \times e \)

\( \cap \) \( (\ker \partial_k) \times p \cap (\ker \partial_{k-1}) \times e = \emptyset \), it follows that

\[ H_k(\mathcal{X} \times S') = \frac{(\ker \partial_{k}) \times p}{(\text{Im} \partial_{k+1}) \times p} \oplus \frac{(\ker \partial_{k-1}) \times e}{(\text{Im} \partial_k) \times e} \]

\[ = \frac{\ker \partial_{k}}{\text{Im} \partial_{k+1}} \oplus \frac{\ker \partial_{k-1}}{\text{Im} \partial_k} = H_k(\mathcal{X}) \oplus H_{k-1}(\mathcal{X}). \]

(b) For \( n > 0 \), we recall that \( \mathbb{C}P^n \) is contractible in all even dimensions. Therefore

\[ H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & \text{k even} \\ 0 & \text{else} \end{cases} \]

By part a, \( H_k(\mathbb{C}P^n \times S') = H_k(\mathbb{C}P^n) \oplus H_{k-1}(\mathbb{C}P^n) \oplus \mathbb{Z} \forall k, 0, \ldots, 2n+1 \).

This implies

\[ \text{as desired.} \]
GeoTop

FALL 2011
Let $M$ be a compact smooth manifold. We claim that a smooth embedding

$$\Psi: M \to \mathbb{R}^k$$

for some $k \geq 1$.

By definition of a smooth manifold, there is an atlas $(U_i, \psi_i)$ of $M$. Since $M$ is compact, there is a finite subcover $\{U_1, \ldots, U_m\}$ of $M$.

Let $\psi_1, \ldots, \psi_m$ be a partition of unity subordinate to $\{U_1, \ldots, U_m\}$.

Define $\Psi: M \to \mathbb{R}^{m+n}$

$$\Psi = (\psi_1, \psi_2, \ldots, \psi_m, \psi_i, \ldots, \psi_m)$$

where $\psi_i, \psi_j$ are smooth. $\Psi$ is smooth. It then suffices to show $\Psi$ is an injective immersion.

We first show injectivity. Suppose $\Psi(x) = \Psi(y)$ for some $x, y \in M$.

Then $\psi_i(x) = \psi_i(y)$ for all $i$ and $\psi_i(x) = \psi_i(y)$ for all $i$.

Since $\psi_i$ is supported on $U_i$, and $\{U_i\}$ covers $M$, this implies that $x = y$.

It follows that $\psi_i(x) = \psi_i(y)$. Since $\psi_i(x) = \psi_i(y)$ and $\psi_i(x) = \psi_i(y)$, it follows that $\psi_i(x) = \psi_i(y)$. Hence, $\psi_i(x) = \psi_i(y)$.

It remains to show $d\Psi$ is injective. Let $x, y \in M$ such that $d\Psi_x(v) = d\Psi_y(w)$ for some $v \in U_i \cap M$. Then $d\Psi_x(v) = d\Psi_y(w)$, and $d_x(\psi_i, \psi_j)(v) = d_x(\psi_i, \psi_j)(w)$.

By the product rule,

$$d\Psi_x(v) = d\Psi_s(w)$$

where $\psi_i$ is a diffeomorphism, $d\psi_i$ is smooth injective. Hence $v = w$ and $w \in d\Psi_x$ is injective. Therefore $\Psi$ is an immersion.
we recall that \( \mathbb{RP}^n = (\mathbb{R}^{n+1})^* / \sim \), where \( \sim \) is the
identification \( x \sim cx \). We denote elements of \( \mathbb{RP}^n \) via
\[
[x_0 : \ldots : x_n] = [(x_0, \ldots, x_n)]
\]
for any \((x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}\). To show \( \mathbb{RP}^n \) is a smooth manifold
of dimension \( n \), we construct coordinate charts on \( \mathbb{RP}^n \) and
check that the transitions are smooth.

Let \( U_i = \{ [x_0 : \ldots : x_n] \in \mathbb{RP}^n : x_i \neq 0 \} \). Then none \( x_i \neq 0 \).

We map \( \mathbb{RP}^n \) via the map \( \psi_i : U_i \rightarrow \mathbb{R}^n \)
\[
[\frac{x_0}{x_i} : \ldots : \frac{x_n}{x_i}] \mapsto (\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}).
\]
By construction, since \( \mathbb{RP}^n \) is the quotient of \( \mathbb{R}^{n+1} \setminus \{0\}\),
\( \forall [x_0 : \ldots : x_n] \in \mathbb{RP}^n \exists \) some \( j \) s.t. \( x_j \neq 0 \). Therefore \( \mathbb{RP}^n = \bigcup U_i \).

Moreover, since \( \mathbb{R}^n \) is open and the above map is a homeomorphism,
It remains to check that the transition functions are smooth.

By construction, \( \forall x, y \in \mathbb{R}^n \), \( \psi_j(\psi_i^{-1}(x)) = \mathbb{R}^n \setminus \{j\text{-th axis}\} \).

Up to reordering coordinates, we may assume that \( i = n-1 \), \( j = n \) and
check the transition function \( \psi_{n-1} \circ \psi^{-1} \).

By construction, \( \psi_{n}(U_{n} \cap U_{n-1}) = \mathbb{R}^n \setminus \{n\text{-th axis}\} \)
\( \psi_{n-1}(U_{n} \cap U_{n-1}) = \mathbb{R}^n \setminus \{n\text{-th axis}\} \).

Thus
\( \varphi_n \circ \varphi_n^{-1} : \mathbb{R}^n \setminus \{n^{th} \text{ axis}\} \rightarrow \mathbb{R}^n \setminus \{(n-1)^{th} \text{ axis}\} \)

\( (x_0, \ldots, x_{n-2}, x_n \neq 0) = (x_0, \ldots, x_{n-2}, 1, x_n \neq 0) \)

\( \varphi_n^{-1} \rightarrow \left[ x_0 : \ldots : x_{n-2} : 1 : x_n \neq 0 \right] \)

\( = \left[ \frac{x_0}{x_n} : \ldots : \frac{1}{x_n} : 1 \right] \)

\( \varphi_n \rightarrow \left( \frac{x_0}{x_n}, \ldots, \frac{x_{n-2}}{x_n}, \frac{1}{x_n} \right) \)

Re-embedding to standard coordinates,

\( \varphi_n \circ \varphi_n^{-1} : (x_1, \ldots, x_n \neq 0) \rightarrow (\frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}, \frac{1}{x_n} \neq 0) \)

which is smooth a diffeomorphism near \( x_n \neq 0 \). \( \Box \)
Support for the rule of contradiction that an immersion

\[ f : M \rightarrow T^n \]

equiv. We need to recall that \( \mathbb{R}^n \) is the universal cover of \( T^n \) via the covering quotient map \( p : \mathbb{R}^n \rightarrow \mathbb{R}^n/\mathbb{Z}^n = T^n \).

Since \( \pi_1(M) = 0 \), \( f^*(\pi_1(M)) = 0 \subset p^*(\pi_1(\mathbb{R}^n)) \). Therefore, \( f \) is a lift \( \tilde{f} : M \rightarrow \mathbb{R}^n \) s.t. \( p \circ \tilde{f} = f \). Since \( \tilde{f} \) is an immersion, \( \tilde{f} \) is an immersion. Since \( M, \mathbb{R}^n \) are both dimension \( n \), \( \tilde{f} \) is an embedding. The inverse function theorem then implies that \( \tilde{f} \) is a local diffeomorphism and hence an open map.

Therefore, \( \tilde{f}(M) \subset \mathbb{R}^n \) is open. Since \( M \) is compact, \( \tilde{f}(M) \) is compact. But \( f \) is continuous.

Therefore, \( f(M) \) is an open compact subset of \( \mathbb{R}^n \), which is a contradiction. \( \ast \)

Alternate

Since \( M, T^n \) are compact and of the same dimension, the

stark of arcwise theorem implies that \( M \) is a covering space of \( T^n \) with covering map \( f : M \rightarrow T^n \). Hence \( M \) is simply connected, \( M \) is the universal cover of \( T^n \). We recall that \( \mathbb{R}^n \) is a

universal cover of \( T^n \) and so \( M \cong \mathbb{R}^n \). However \( \mathbb{R}^n \) is not compact \( \ast \).
Let \( p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0 \) be a non-constant complex polynomial. Consider the map
\[
\gamma_R : S^1 \to \mathbb{C} : \Theta \mapsto p(Re^{i\Theta})
\]
for \( R > 0 \) and argue via the winding number about 0.

Conclude \( R = 0 \). Then \( Y_0(0) = a_0 \neq 0 \). If \( a_0 = 0 \) then \( p \) has a zero and we are done. Otherwise suppose \( a_0 \neq 0 \), then \( Y_0 \) winds around 0 \( n > 0 \) times.

As \( R \to 0 \), we know that \( |z|^n \) dominates \( |a_{n-1}z^{n-1} + \ldots + a_0| \).

Choose \( R \) sufficiently large so that \( |a_{n-1}z^{n-1} + \ldots + a_0| < |z|^n \) \( \forall |z| = R \). Then by the standard "dog on a leash" theorem,
\[
\gamma_R \text{ and } Y : S^1 \to \mathbb{C} : \Theta \mapsto p(Re^{i\Theta}) \text{ have the same winding number about 0. Since } Y \text{ winds around 0 } n > 0 \text{ times, this implies that } \gamma_R \text{ winds around 0 } n > 0 \text{ times.}
\]

We note that \( \gamma_R \) is homotopic to \( Y_0 \) via the homotopy
\[
Y_t : \Theta \mapsto p(tRe^{i\Theta})
\]
around 0. However, we should recall that the winding number is invariant under homotopies that avoid 0. Therefore, \( \exists \ t \in \mathbb{R} \) s.t.
\[
p(tRe^{i\Theta}) = Y_t(\Theta) = 0
\]
and so \( p \) has a \( \Theta \) zero at \( tRe^{i\Theta} \).
It suffices to work locally since $d$ is defined locally.

Given local coordinates $x_1, \ldots, x_n$ around $q \in U$, we can express $\omega$ as

$$\omega = \sum_{i_1, \ldots, i_p} f_{i_1, \ldots, i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

By linearity, it then suffices to consider $p$-forms $\omega = g dx_{i_1} \wedge \cdots \wedge dx_{i_p}$.

By definition $\varphi$ the pullback,

$$(\varphi^* \omega)_q = (g \circ \varphi)(d_{i_1} x_i \circ \varphi_t f) \wedge \cdots \wedge (d_{i_p} x_i \circ \varphi_t f)$$

Then

$$d_{\varphi(q)}(\varphi^* \omega) = d_{\varphi(q)}(g \circ \varphi)(d_{i_1} x_i \circ \varphi_t f) \wedge \cdots \wedge (d_{i_p} x_i \circ \varphi_t f)$$

we proved by induction on $p$.

Suppose that $\omega$ is a 0-form. Then the chain rule implies that $\forall \varphi \forall \omega$

$$d_{\varphi(q)}(\varphi^* \omega) = d_{\varphi(q)}(g \circ \varphi) (d_{i_1} x_i \circ \varphi_t f) \wedge \cdots \wedge (d_{i_p} x_i \circ \varphi_t f) = \varphi^*(dx_i)$$

$$= \varphi^*(dx)$$

Now suppose that the claim holds for all $k < p$. Since $d$ is defined locally, it suffices to work locally. Moreover, by linearity of $\varphi^* d$, it suffices to consider forms of the form $\omega = g dx_{i_1} \wedge \cdots \wedge dx_{i_p}$.

Let $\omega = g dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. Then $\omega$ is a $p-1$-form and $\omega = dx_i \wedge \omega$.

Dually, computation and the inductive assumption imply

$$d(\varphi^* \omega) = d(\varphi^*(dx_i \wedge \omega)) = d((\varphi^* dx_i) \wedge (\varphi^* \omega))$$

$$= d(\varphi^* dx_i) \wedge (\varphi^* \omega) = \cdots$$

$$(\cdots \text{induction hypothesis})$$

$$= d(dx_i) \wedge (f^* dx_i) \wedge f^* (dx_i) \wedge f^* (dx_i) \wedge f^* (dx_i)$$

$$= f^*(dx_i \wedge \omega)$$

As desired.
(a) Let $B^3$ be the closed unit ball in $\mathbb{R}^3$. Then $\partial B^3 = S^2 \subset \mathbb{R}^3$ and so Stokes' theorem implies that
\[
\int_{S^2} w = \int_{\partial B^3} i^* w = \int_{S^2} d(w|_{S^2}) = \int_{S^2} (2x+1) \, dx \wedge dy \wedge dz
\]
\[
= 2 \int_{B^3} x \, dV + \int_{S^2} dV
\]
By symmetry, $\int_{S^2} dV = 0$. So
\[
\int_{S^2} w = \int_{B^3} dV = \text{vol}(B^3)
\]
the value of $\text{vol}(B^3)$ differs depending on normalization but is usually $\text{vol}(B^3) = \frac{4}{3} \pi$.

(b) Suppose that such an $x$ exists. Then by Stokes,
\[
\int_{S^2} w = \int_{S^2} i^* x = \int_{S^2} x = \int_{B^3} dV = \int_{B^3} 0 = 0
\]
which contradicts part (a). Therefore no such $x$ exists.

(b) de Rham's theorem
Suppose $M$ is a smooth manifold. Let $H^k(M; \mathbb{R})$ be the singular cohomology of $M$ with real coefficients, given by
\[
H^k(M; \mathbb{R}) \cong \text{Hom}(H_k(M; \mathbb{R}); \mathbb{R})
\]
by the UCT. Then $\exists$ an isomorphism $H^k_{dR}(M) \to H^k(M; \mathbb{R})$ given by
\[
H^k_{dR}(M) \to \text{Hom}(H_k(M; \mathbb{R}), \mathbb{R}); \omega \to \int_{\omega}
\]
Based on the figure, we can express $X$ via the CW complex

\begin{align*}
1 & \text{-cell: } \rho \\
1 & \text{-cell: } c \quad \mathrm{im} \, \partial_1 = \mathbb{Z}(e) \\
1 & \text{-cell: } f \quad \mathrm{im} \, \partial_2 = 5\mathbb{Z}(e) \\
\end{align*}

Then \( \mathrm{im} \, \partial_1 = 0 \), \( \ker \partial_1 = \mathbb{Z}(e) \), \( \mathrm{im} \, \partial_2 = \mathbb{Z}(5e) = 5\mathbb{Z}(e) \) and \( \ker \partial_2 = 0 \).

This gives the chain complex

\[ \mathbb{Z} \xrightarrow{\partial_2} C_1 (X) = \mathbb{Z}(e) \xrightarrow{\partial_1} C_0 (X) = \mathbb{Z}(p) \xrightarrow{\partial_0} 0 \]

and the corresponding homology groups

\begin{align*}
H_0 (X) &= \frac{\ker \partial_0}{\mathrm{im} \, \partial_1} = \frac{\mathbb{Z}(p)}{0} \cong \mathbb{Z} \\
H_1 (X) &= \frac{\ker \partial_1}{\mathrm{im} \, \partial_2} = \frac{\mathbb{Z}(e)}{5\mathbb{Z}(e)} \cong \mathbb{Z}/5\mathbb{Z} \\
H_2 (X) &= \frac{\ker \partial_2}{\mathrm{im} \, \partial_3} = \ker \partial_2 = 0 \\
\end{align*}

\( \forall \, H_k (X) = 0 \quad \forall \, k > 2 \).

This can be shortened to:

This gives the chain complex

\[ \mathbb{Z} \xrightarrow{\partial_2} C_1 (X) \xrightarrow{\partial_1} C_0 (X) \xrightarrow{\partial_0} 0 \]

and therefore gives homology groups

\( H_2 (X) = 0 \), \( H_1 (X) = \mathbb{Z}/5\mathbb{Z} \), \( H_0 (X) = \mathbb{Z} \)

and \( H_k (X) = 0 \quad \forall \, k > 2 \).

To obtain the cohomology, we dualize the chain complex to get the cochain complex. Dualizing, we find

\[ C^*_i = \text{Hom} (C_i, \mathbb{Z}) \leq \text{Hom} (\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z} \]

\( \delta_2 = 5 \), \( \delta_1 = 0 \), \( \delta_0 = 0 \), \( \delta = 5 \cdot 5 \)

which gives the cochain complex

\[ \mathbb{Z} \xleftarrow{0} C^*_1 \xleftarrow{\delta} C^*_0 \xleftarrow{0} \mathbb{Z} \xleftarrow{0} 0 \]

and cohomology groups

\[ H^2 (X) = \frac{\ker \delta_2}{\mathrm{im} \, \delta_1} = \mathbb{Z}/5\mathbb{Z}, \quad H^1 (X) = \frac{\ker \delta_1}{\mathrm{im} \, \delta_0} = 0, \quad H^0 (X) = \mathbb{Z} \]

and \( H^k (X) = 0 \quad \forall \, k > 2 \).
Alternatively, the universal coefficient theorem implies that
\[ 0 \to \text{Ext}(H_0(x), \mathbb{Z}) \to H^0(x ; \mathbb{Z}) \to \text{Hom}(H_0(x), \mathbb{Z}) \to 0 \]
is a split exact sequence. We then compute
\[
\begin{align*}
\text{Ext}(H_0(x), \mathbb{Z}) & \cong \text{Ext}(\mathbb{Z}, \mathbb{Z}) = 0 \\
\text{Ext}(H_1(x), \mathbb{Z}) & \cong \text{Ext}(\mathbb{Z}/5\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/5\mathbb{Z} \\
\text{Ext}(H_2(x), \mathbb{Z}) & \cong \text{Ext}(0, \mathbb{Z}) = 0 \\
\text{Hom}(H_0(x), \mathbb{Z}) & \cong \text{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z} \\
\text{Hom}(H_1(x), \mathbb{Z}) & \cong \text{Hom}(\mathbb{Z}/5\mathbb{Z}, \mathbb{Z}) = 0 \\
\text{Hom}(H_2(x), \mathbb{Z}) & \cong \text{Hom}(0, \mathbb{Z}) = 0
\end{align*}
\]
with all other entries being 0. Then
\[
\begin{align*}
H^0(x) & \cong \text{Ext}(H_0(x), \mathbb{Z}) \oplus \text{Hom}(H_0(x), \mathbb{Z}) = \mathbb{Z} \\
H^1(x) & \cong \text{Ext}(H_0(x), \mathbb{Z}) \oplus \text{Hom}(H_1(x), \mathbb{Z}) = 0 \\
H^2(x) & \cong \text{Ext}(H_1(x), \mathbb{Z}) \oplus \text{Hom}(H_2(x), \mathbb{Z}) = \mathbb{Z}/5\mathbb{Z}
\end{align*}
\]
and all other cohomologies being 0.
GEOTop
Spring 2011
Suppose $V$ is a smooth vector field on a smooth manifold of dimension $n$. Suppose $\exists$ $p \in V(p) \neq 0$. We claim that there exists a coordinate system $(x_1, \ldots, x_n)$ s.t. $V = \partial_1 x_1$ in a neighborhood of $p$.

As we are working locally at $p$, we may assume WLOG that we are working on $\mathbb{R}^n$ at $p=0$. Further, by rotating and scaling, we may assume that $V_0 = \partial_1 x_1$.

Let $\Phi_t$ be the flow of $V$ near 0, defined on some neighborhood $U$ of 0. Define $\chi: U \rightarrow \mathbb{R}^n$ by

$$\chi(a_1, \ldots, a_n) = \Phi_{a_1}(0, a_2, \ldots, a_n)$$

where $U$ is a sufficiently small neighborhood of the origin.

We claim that $\chi$ is our desired coordinate system.

Express $V$ as $V = f_1 \partial_1 x_1 + \cdots + f_n \partial_n x_n \Rightarrow f = (f_1, \ldots, f_n)$.

Then

$$\frac{\partial}{\partial x_i} \chi(a_1, \ldots, a_n) = \partial_i x_i \frac{\partial}{\partial x_i} \phi_{a_1}(0, a_2, \ldots, a_n)$$

$$= \frac{\partial}{\partial x_i} V \phi_{a_1}(0, a_2, \ldots, a_n)$$

$$= \frac{\partial}{\partial x_i} V \chi(a_1, \ldots, a_n)$$

$$= f(\chi(a_1, \ldots, a_n))$$

We claim that $(d\chi)_0$ is non-singular. Calculating $\frac{\partial}{\partial x_i} V |_{0}$ in general yields

and evaluating at 0 this gives $\frac{\partial}{\partial x_1} V = (1, 0, \ldots, 0)$.

For $i = 2, \ldots, n$, we find that

$$\frac{\partial}{\partial x_1} \chi |_{0} = \frac{\partial}{\partial x_1} \phi_{x_1}(0, x_2, \ldots, x_n) = \frac{\partial}{\partial x_1} \phi_{0}(0, 0, \ldots, x_i, \ldots, 0)$$

$$= \frac{\partial}{\partial x_1} (0, \ldots, x_i, \ldots, 0) = (0, \ldots, 1, \ldots, 0)$$

Therefore $(d\chi)_0$ is non-singular. The inverse function theorem
a) We aim to demonstrate Cartan's formula

\[ L_X = doix + ix \circ ad \]

To do so, we work locally. By linearity, it suffices to consider k-forms of the form \( f dx_1 \wedge \ldots \wedge dx_k \). We proceed by induction.

First, we consider a 0-form \( f \). Hence, by definition,

\[ (L_X(f))_p = \lim_{h \to 0} \frac{f \circ \phi_h(p) - f(p)}{h} = df(p)_p \]

where \( \phi_h \) is the flow of \( X \) (local to \( p \) if needed).

Additionally,

\[ (doix(f) + ix \circ ad(f))_p = (ix \circ df)_p = df(p)_p \]

Thus, \( \phi_h \) is a curve through \( p \) with \( \frac{d}{dt} \phi_t(p)|_{t=0} = X_p \).

\[ (Xf)_p = (f \circ \phi_t(p))'(0) = (L_X(f))_p \] as desired.

Now suppose the result holds for m-forms. For \( m < k \), consider a k-form \( f dx_1 \wedge \ldots \wedge dx_k = dx_1 \wedge (fdx_2 \wedge \ldots \wedge dx_k) \).

Note \( \eta \) is a \((k-1)\)-form. Then by the inductive hypothesis,

\[ L_X(dx_1 \wedge \eta) = L_X(dx_1) \wedge \eta + dx_1 \wedge L_X(\eta) \]

\[ = (ix \circ ad(dx_1) + dioix(dx_1)) \wedge \eta + dx_1 \wedge (ix \circ ad(\eta) + dioix(\eta)) \]

\[ = d \left( X(x_1) \wedge \eta + dx_1 \wedge (ix \circ ad(\eta) + dioix(\eta)) \right) \]

Additionally, by the Fubini rule

\[ (dioix + ix \circ ad)(dx_1 \wedge \eta) = d(ix(dx_1 \wedge \eta)) - ix(dx_1 \wedge d\eta) \]

\[ = d \left( X(x_1) \wedge \eta - dx_1 \wedge ix(\eta) \right) - ix(dx_1 \wedge d\eta) \]

\[ = d \left( X(x_1) \wedge \eta + x(x_1) \wedge d\eta + dx_1 \wedge dioix(\eta) \right) \]

\[ - x(x_1) \wedge d\eta + dx_1 \wedge ix(d\eta) \]

\[ = L_X(dx_1 \wedge \eta) \]
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a) We note that $\mathbb{R}^3 \setminus \{0\}$ deformation retracts onto $S^2$ via the map $x \mapsto \frac{x}{\|x\|}$. Therefore
\[ H^2(\mathbb{R}^3 \setminus \{0\}) \cong H^2(S^2) \cong \mathbb{R}. \]
In particular, this implies that if a closed 2-form $\psi$ on $\mathbb{R}^3 \setminus \{0\}$ is not exact, as desired.

b) Let $\psi$ be such a form. We recall that since $S^2$ is compact, the map $H^2(S^2) \to \mathbb{R}: \psi \mapsto \int_{S^2} \psi$ is an isomorphism. Therefore since $[\psi] \neq [0] \in H^2(S^2)$, it follows that $\int_{S^2} \psi \neq 0$.

Let $f: S^2 \to S^2$ be smooth. We recall that the degree is defined as follows. The map $f: S^2 \to S^2$ induces a homomorphism $f_*: H_2(S^2) \to H_2(S^2)$, and we know that $\beta_{\text{c}} \in H_2(S^2)$ is a generator of $H_2(S^2)$, it follows that $f$ induces a homomorphism $f_*: \mathbb{Z} \to \mathbb{Z}$ which must be multiplication by an integer, which we call $\deg f$.

Then
\[ \int_{S^2} f^* \psi = \int_{f_* S^2} \psi \]

The space $f_* S^2$ is a $\deg f$-fold cover of $S^2$, and so
\[ \int_{S^2} f^* \psi = \int_{f_* S^2} \psi = \deg f \int_{S^2} \psi \]

Therefore
\[ \deg f = \frac{\int_{S^2} f^* \psi}{\int_{S^2} \psi} \neq 0. \]
Suppose that $\Omega$ is a 2-form on $S^2$ s.t. $\int_{S^2} \Omega = 0$. We claim that $\Omega$ is exact.

Let $N = S^2 \setminus \{north\ \text{pole}\}$ and $S = S^2 \setminus \{south\ \text{pole}\}$.

Then $N \cup S \equiv \mathbb{R}^2$. Since $\mathbb{R}^2$ is contractible, the Poincaré lemma then implies that $\Omega|_N$, $\Omega|_S$ are exact and $\int_N \theta_1 = \int_S \theta_2 = 0$.

We now aim to glue $\theta_1$ and $\theta_2$ together to find $\eta$ s.t. $d\eta = \Omega$.

Let $U$ be the upper hemisphere of $S^2$ and $L$ the lower hemisphere. Since $U \cap L = \emptyset$ and $U \cap L$ has measure 0, it follows that

$$0 = \int_{S^2} \Omega = \int_U \theta_1 + \int_U \theta_2 + \int_U d\theta_1 + \int_L d\theta_2$$

Taking the standard orientation on $S^2$, Stokes' theorem implies

$$0 = \int_U d\theta_1 + \int_L d\theta_2 = \int_U \theta_1|_U - \int_L \theta_2|_L$$

$$\Rightarrow 0 = \int_{S^1} (\theta_1 - \theta_2)|_S$$

In particular, by the $S^1$ version of this result, $(\theta_1 - \theta_2)|_S$ is exact on $S^1$.

By $N$, $S$ deformation retracts onto $U \cap L \equiv S^1$ via $i^*$, it follows that $i^*$ is an isomorphism on cohomology. Therefore $(\theta_1|_{\partial N} - \theta_2|_{\partial N})$ is exact by $i^*(\theta_1|_{\partial N} - \theta_2|_{\partial N}) = (\theta_1 - \theta_2)|_S$.

Let $\psi_1, \psi_2$ be a partition of unity subordinate to $N, S$.

Define

$$\eta = \begin{cases} 
\theta_1 - d(\psi_1) & \text{on } N \\
\theta_2 + d(\psi_2) & \text{on } S 
\end{cases}$$
Suppose that $V: U \to S^2$ is a smooth map, considered as a $\text{VF}$ of $S^2$ and that $U = \mathbb{R}^3 \setminus \{p_1, \ldots, p_n\}$ where $p_1, \ldots, p_n \in \mathbb{B}^3$. We aim to explain why $\deg V|_{S^2} = \sum_i \text{ind}_{p_i} V$.

For each $p_i$, choose an open neighborhood $U_i$ of $p_i$ s.t. $U_i \cap \mathbb{B}^3$ and $U_i \cup U_j = \emptyset$ for $i \neq j$. This is possible as $p_1, \ldots, p_n$ are distinct and strictly inside $\mathbb{B}^3$ which is open.

Consider $X = \mathbb{B}^3 \setminus (\cup U_i)$. Then $\partial X = S^2 \sqcup (\cup U_i)$.

Since $V: \partial X \to S^2$ is a map between 2-manifolds, $S^2$ is connected, the extension theorem implies that and $V$ extends to all of $X$, the extension theorem then implies that $\deg V|_{\partial X} = 0$. Since $\partial X = S^2 \setminus (\cup U_i)$, when orientation is taken into account, this implies that

$$\text{O} = \deg V|_{S^2} - \deg V|_{\cup U_i}$$

$$\Rightarrow \deg V|_{S^2} = \sum_i \deg V|_{U_i}$$

$$= \sum_i \text{ind}_{p_i} V$$

As desired. \qed
Let $A^\bullet, B^\bullet, C^\bullet$ be chain complexes. Then $0 \to A^\bullet \overset{f^\bullet}{\to} B^\bullet \overset{g^\bullet}{\to} C^\bullet \to 0$

is a short exact sequence of chain complexes if for all $n,$

$$0 \to A_n \overset{f_n}{\to} B_n \overset{g_n}{\to} C_n \to 0$$

is a short exact sequence. To be clear, $0 \to A_n \overset{f_n}{\to} B_n \overset{g_n}{\to} C_n \to 0$

is a short exact sequence if $\text{im } f_n = \ker g_n,$ $\ker f_n = 0,$ $\text{im } g_n = C_n.$
a) We define $\mathbb{C}P^n$ as $S^{2n+1}/\sim$, where $S^{2n+1}\subseteq \mathbb{C}P^n$ in the usual way and $z\sim w$ if $z = \lambda w$ for some $\lambda \in \mathbb{C}^\times$.

We denote the equivalence class of $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ as $[z_0, \ldots, z_n]$.

b) We define the cell structure of $\mathbb{C}P^n$ inductively as follows.

For $\mathbb{C}P^0$, we note that $S^1/\sim \cong \mathbb{C}P^1$ since $\mathbb{C}P^0 \cong \mathbb{C}P^1 \cong S^1$.

Therefore $\mathbb{C}P^0$ has the structure of a single 0-cell.

Consider $\mathbb{C}P^n$ for $n > 1$. We claim that $\mathbb{C}P^n$ can be expressed as an $2n$-cell $C^{2n} = D^{2n}$ attached to $\mathbb{C}P^{n-1}$ via the map

$$D^{2n} \cong S^{2n-1} \rightarrow \mathbb{C}P^{n-1}: (z_0, \ldots, z_{n-1}) \mapsto [z_0, \ldots, z_{n-1}]$$

To see this, we decompose $\mathbb{C}P^n$ into $U \sqcup V$ where $U \cong \mathbb{C}P^{n-1}$, $V \cong D^{2n}$ and $U \sqcup V$ are attached as above.

Define $U = \{ [z_0, \ldots, z_{n-1}, 0] \in \mathbb{C}P^n \}$. Then $U \cong \mathbb{C}P^{n-1}$ by the map $[z_0, \ldots, z_{n-1}, 0] \mapsto [z_0, \ldots, z_{n-1}]$.

Consider $\mathbb{C}P^n \setminus U = V$. Then $V = \{ [z_0, \ldots, z_n] : z_n \neq 0 \}$.

By $|z_n| = \frac{2n}{|z_{n-1}|}$ for $z_n \neq 0$ where $|\frac{z_n}{|z_{n-1}|}| = 1$, it follows that $V = \{ [z_0, \ldots, z_{n-1}, x] : x > 0 \}$. Since $(z_0, \ldots, z_{n-1}, x) \in S^{2n+1}$, it follows that $V = \{ [z_0, \ldots, z_{n-1}, \sqrt{|1 - \frac{z_{n-1}^2}{z_n}|}] \}$. Then $V \cong D^n$ via the map $V \rightarrow D^n: [z_0, \ldots, z_{n-1}, \sqrt{|1 - \frac{z_{n-1}^2}{z_n}|}] \mapsto (z_0, \ldots, z_{n-1})$.

We note this is well-defined since $|z_0|^2 + \cdots + |z_{n-1}|^2 < 1$.

Since $\mathbb{C}P^n = U \sqcup V$ and $V = \mathbb{C}P^{n-1}$, the above attaching map gives the claim.

Therefore, by the above recursion, $\mathbb{C}P^n$ can be expressed as a CW complex with one cell in each even dimension.
(b) By the same reasoning as the previous problem, we can express $\mathbb{RP}^2$ as the chain complex

1. $\mathbb{Z}^2$ with $p_0$.
2. $\mathbb{Z}$ with $p_1$.
3. $\mathbb{Z}$ with $p_2$.

Then we assume the chain complex

$0 \rightarrow \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \rightarrow 0$

Therefore

$H_2(\mathbb{RP}^2) \cong \ker \partial_2 \cong 0$,

$H_1(\mathbb{RP}^2) \cong \frac{\ker \partial_1}{\text{Im} \partial_2} \cong \mathbb{Z}/2\mathbb{Z}$,

$H_0(\mathbb{RP}^2) \cong \frac{\ker 0}{\text{Im} \partial_1} \cong \mathbb{Z}/\mathbb{Z} \cong \mathbb{Z}$,

$H_k(\mathbb{RP}^2) \cong 0 \quad \forall k > 2$.

Alternate: We recall that $\mathbb{RP}^2$ can be represented implicitly by

Then $\partial_2 U = c + (b-a)$, $\partial_2 L = (b-a)c$,

$\partial_1 a = q - p \quad \partial_1 b = q - p \quad \partial_1 c = p - p = 0$

Then

$H_0(\mathbb{RP}^2) \cong \frac{\ker \partial_1}{\text{Im} \partial_2} \cong \mathbb{Z}(b-a,c)$,

$H_1(\mathbb{RP}^2) \cong \frac{\mathbb{Z}b-a,c}{\mathbb{Z}(b-a,c)} \cong \mathbb{Z}/2\mathbb{Z}$,

$H_2(\mathbb{RP}^2) \cong \ker \partial_2 = 0$.

As desired.
a) **Theorem (Infinitesimal Fixed Point)**

If $f: X \to X$ is a smooth map on a compact orientable manifold with $L(f) \neq 0$, then $f$ has a fixed point.

Here $L(f)$ is the **infinitesimal number** and is defined as

$$L(f) = \sum_{k \geq 0} (-1)^k \tr(f^*|_{H_k(X)} \to H_k(X)).$$

b) **Suppose that $f$ a smooth map $f : \mathbb{C}P^n \to \mathbb{C}P^n$.**

We recall that $H^2(\mathbb{C}P^n) \cong \mathbb{Z}$ (this can be obtained nearly immediately from earlier via the universal covering theorem).

Let $\omega$ denote the generator of $H^2(\mathbb{C}P^n)$. Then $\omega \wedge \ldots \wedge \omega$ generates $H^{2k}(\mathbb{C}P^n)$ for $k \geq 0$.

Consider $f^* : H^2(\mathbb{C}P^n) \cong \mathbb{Z} \to H^2(\mathbb{C}P^n) \cong \mathbb{Z}$. Then $f^*$ is an automorphism $\mathbb{Z} \to \mathbb{Z}$ and $\omega$ is multiplication by some $m \in \mathbb{Z}$. Then $f^*(\omega) = m \cdot \omega$ and $f^*(\omega \wedge \ldots \wedge \omega) = m^k \cdot \omega \wedge \ldots \wedge \omega$.

Thus, for $k \geq 0$, $f^*|_{H^{2k}(\mathbb{C}P^n)}$ is multiplication by $m^k$.

We now compute $L(f)$. By definition and Poincaré duality, since $\mathbb{C}P^n$ is closed and orientable,

$$L(f) = \sum_{k \geq 0} (-1)^k \tr(f^*|_{H^k(\mathbb{C}P^n)} \to H^k(\mathbb{C}P^n))$$

for odd $k$,

$$H_k(\mathbb{C}P^n) \cong 0,$$

for odd $k$,

$$= \sum_{j \geq 0} \tr(f^*|_{H^{2j}(\mathbb{C}P^n)})$$

$$= \sum_{j \geq 0} \tr(m^j) = \sum_{j \geq 0} m^j = \begin{cases} \frac{m^{2n+1} - 1}{m-1} & (m \neq 1) \\ n+1 & m=1 \end{cases}.$$
(b) We can decompose $\mathbb{RP}^2 \times \mathbb{RP}^2$ into cells according to the cell decomposition of $\mathbb{RP}^2$. Let $p_i, q_i, a_i, b_i, A_i$ for $i = 1, 2$ be the cells of two copies of $\mathbb{RP}^2$, as before. Then $\mathbb{RP}^2 \times \mathbb{RP}^2$ has the cellular structure

1. **0-cells**: $(p_1, p_2), (p_1, q_2), (q_1, p_2), (q_1, q_2)$
2. **1-cells**: $(a_1, p_2), (a_1, q_2), (b_1, p_2), (b_1, q_2), (p_1, a_2), (p_1, b_2), (q_1, a_2), (q_1, b_2)$
3. **2-cells**: $(A_1, p_2), (A_1, q_2), (p_1, A_2), (q_1, A_2), (a_1, a_2), (a_1, b_2), (b_1, a_2), (b_1, b_2)$
4. **3-cells**: $(A_1, A_2), (A_1, b_2), (a_1, A_2), (b_1, A_2)$
5. **4-cell**: $(A_1, A_2)$

Moreover, $\partial_4 (A_1, A_2) = (A_1, 2(b_2-a_2)) + 2(b_1-a_2), A_2) = 2((A_1, b_2) - (A_1, a_2) + (b_1, A_2) - (a_1, A_2))$

Decompose $\mathbb{RP}^2$ as done in the first solution, we give the same decomposition to 2 copies of $\mathbb{RP}^2$ in cells $\Phi_i, e_i, f_i$ $i = 1, 2$. This yields a decomposition of $\mathbb{RP}^2 \times \mathbb{RP}^2$ in cells:

1. **0-cells**: $(p_1, p_2)$
2. **1-cells**: $(e_1, p_2), (p_1, e_2)$
3. **2-cells**: $(f_1, p_2), (p_1, f_2), (e_1, e_2)$
4. **3-cells**: $(f_1, e_2), (e_1, f_2)$
5. **4-cell**: $(f_1, f_2)$
Moreover,
\[\partial_3(f_1, f_2) = (\partial_2 f_1, f_2) + (f_1, \partial_2 f_2)\]
\[= (2e_1, f_2) + (f_1, 2e_2) = 2(\langle e_1, f_2 \rangle, (f_1, e_2))\]
\[= (\partial_2 e_1, e_2) + (f_1, \partial_1 e_2)\]
\[= (2e_1, e_2) - 2(e_1, e_2) \quad \text{(ignore X)}\]
\[= -2(e_1, e_2)\]

Then
\[H_3(\mathbb{R}P^2 \times \mathbb{R}P^2) = \frac{k \cdot \partial_3}{m \cdot \partial_4}\]
\[= \frac{\mathbb{Z}(\langle f_1, e_2 \rangle + (e_1, f_2))}{\mathbb{Z}(2(e_1, f_2) + 2(f_1, e_2))}\]
\[= \frac{\mathbb{Z}}{2\mathbb{Z}}\]

which in particular has a nonzero element \((f_1, e_2) + (e_1, f_2)\).
We decompose the tetrahedron into simplicial complexes as follows:

\[
X = [v_0, v_1, v_2] \cup [v_1, v_2, v_3] \cup [v_0, v_2, v_3] \cup [v_0, v_1, v_3]
\]

Then we arrange the chain complex

\[
0 \rightarrow \mathbb{Z}(A, B, C, D) \xrightarrow{\partial_2} \mathbb{Z}(a, b, c, d, e, f) \xrightarrow{\partial_1} \mathbb{Z}(v_0, v_1, v_2, v_3) \rightarrow 0
\]

with

\[
\partial_2 A = a + d - b, \quad \partial_2 B = a + e - c, \quad \partial_2 C = b + f - c, \quad \partial_2 D = c + f - e
\]

\[
\partial_1 a = v_1 - v_0, \quad \partial_1 b = v_2 - v_0, \quad \partial_1 c = v_3 - v_0, \quad \partial_1 d = v_2 - v_1, \quad \partial_1 e = v_3 - v_1, \quad \partial_1 f = v_3 - v_2
\]

Taking \(a, b, c, d, e, f\) and \(A, B, C, D\) and \(v_0, v_1, v_2, v_3\) as bases, we can then represent \(\partial_2\) and \(\partial_1\) as the matrices,

\[
\partial_2 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\partial_1 = \begin{bmatrix}
-1 & -1 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & -1 & 0 \\
0 & 1 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}
\]

Consider \([x, y, z, w]^T \in \text{ker} \partial_2\). By the above matrix, we have

\[
x + y = 0 \quad \Rightarrow \quad x = -y
\]

\[
-x + z = 0 \quad \Rightarrow \quad x = z
\]

\[
x + w = 0 \quad \Rightarrow \quad x = -w
\]

Thus, \([1, -1, 1, -1]^T \in \text{ker} \partial_2\), this implies that \(\text{ker} \partial_2 = \mathbb{Z}(A - B + C - D) = \mathbb{Z}\).
1. $a = b$  
2. ?  
3. $\sqrt{\_\_\_\_\_\_}$  
4. $a = b$  
5. $a \sqrt{b} = $  
6. $a \sqrt{b} \ c$  
7. $\sqrt{\_\_\_\_\_\_}$  
8. $=$  
9. $a \sqrt{b} = $  
10. $\sqrt{\_\_\_\_\_\_}$  
11. $a \sqrt{b} \sqrt{\_\_\_\_\_\_} \sqrt{\_\_\_\_\_\_}$  
12. ?  
13. $\sqrt{\_\_\_\_\_\_}$
GEO Top

Spring 2010
a) To show 0 is a regular value of \( F \), it suffices to show that \( dF_A \) is surjective \( \forall A \in F^{-1}(0) \). Fix some \( A \in F^{-1}(0) \). We recall that \( T_0 \mathbb{M}_n = \mathbb{M}_n \) and \( T_{\text{Tr} A \mathbb{S}_n} = \mathbb{S}_n \). Consider some \( C \in \mathbb{S}_n \). Then by direct computation,

\[
dF_A \left( \frac{1}{2} CA \right) = \lim_{t \to 0} \frac{A + \frac{1}{2} CA - \frac{1}{2} (CA)^T I - AA^T + I}{t} \]

\[
= \lim_{t \to 0} \frac{AA^T + \frac{1}{2} CAA^T + \frac{1}{2} A(CA)^T + \frac{1}{2} (CA)(CA)^T - AA^T}{t} \]

\[
= \lim_{t \to 0} \frac{\frac{1}{2} (C(AA^T + AA^T C^T) + \frac{1}{2} (CA)(CA)^T}{t} \]

\[
= \frac{1}{2} (C(AA^T + AA^T C^T) + C

Which the final inequality follows b/c \( F(A) = 0 \iff AA^T = I \) and \( C \in \mathbb{S}_n \). Therefore \( dF_A : T_0 \mathbb{M}_n \to T_{\text{Tr} A \mathbb{S}_n} \) is surjective \( \forall A \in F^{-1}(0) \) and so \( F \) has 0 as a regular value.

b) By construction \( F(A) = 0 \iff AA^T = I \) and so \( F(A) = 0 \iff A^* = A^T \). Then \( F^{-1}(0) = O(n) \). The regular value theorem and pad a then imply that \( O(n) \) is a smooth submanifold of \( \mathbb{M}_n \).

c) B/c \( F : \mathbb{M}_n \to \mathbb{S}_n \) and \( \dim \mathbb{M}_n = n^2 \), \( \dim \mathbb{S}_n = n + (n-1) + \ldots + 1 = \frac{n(n+1)}{2} \), the regular value theorem implies \( \dim O(n) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2} \).

Since \( O(n) : F^{-1}(0) \), it follows that

\[
T_{\text{Tr} O(n)} = \ker \left( dF_{\text{Tr}} \right) = 1 \quad dF_{\text{Tr}} : T_0 \mathbb{M}_n \to T_{\text{Tr} \mathbb{S}_n} \]

By direct calculation, \( \forall B \in \mathbb{M}_n \)

\[
dF_{\text{Tr}}(B) = \lim_{t \to 0} \frac{(I + tB)(I + tB)^T - I}{t} \]

\[
= \lim_{t \to 0} \frac{I - I + tB + tB^T + t^2 BB^T}{t} \]

\[
= B + B^T

hence \( \ker \left( dF_{\text{Tr}} \right) = \{ B \in \mathbb{M}_n : B + B^T = 0 \} = \{ B \in \mathbb{M}_n : B^T = -B \} \) which is the skew-symmetric matrices.
Lemma: the product of parallelizable manifolds is parallelizable.

Proof: take $M \times N$ parallelizable, $\dim M = m$, $\dim N = n$.

Then $\exists$ vector fields $V_1, \ldots, V_m$ on $M$ and $U_1, \ldots, U_n$ on $N$ s.t. $V_i(p)$, $U_j(q)$ are linearly independent $V_i(p)$, and $U_j(q)$ are lin. ind. $V_i(p), U_j(q)$.

We can extend $V_i, U_i$ to vector fields $\tilde{V}_i, \tilde{U}_i$ on $M \times N$ by taking $\tilde{V}_i(p,q) = (V_i(p), 0) \in T_{(p,q)}(M \times N) \cong T_p^* M \oplus T_q^* N$ and $\tilde{U}_i(p,q) = (0, U_i(q)) \in T_{(p,q)}(M \times N) \cong T_p^* M \oplus T_q^* N$.

where $T_p M, T_q N$ are the projections $M \times N \to M$ and $M \times N \to N$.

Then $V_i(p,q) \in T_{(p,q)} M \times N$, $\tilde{V}_i(p,q), \ldots, \tilde{V}_m(p,q), \tilde{U}_i(p,q), \ldots, \tilde{U}_n(p,q)$ are linearly independent. Then $M \times N$ is parallelizable.

We recall that parallelizability is equivalent to the trivializability of the tangent bundle. Additionally, we recall that

$$TS^1 = S^1 \times \mathbb{R}, \quad R \oplus TS^n = \mathbb{R}^{n+1}$$

since the tangent bundle $S^1$ has a nowhere vanishing vector field.

Moreover, since it follows that $TS^n \oplus R \cong S^n \times \mathbb{R}^{n+1}$ since the additional $\mathbb{R}$ term can be realized as the normal bundle to $S^n \subset \mathbb{R}^{n+1}$.

Thus

$$T(T^2 \times S^n) \cong T^* S^1 \oplus \pi_{S^1}^* TS^1 \oplus \pi_{S^n}^* TS^n$$

$$\cong \pi_{S^1}^* TS^1 \oplus S^1 \times R \oplus \pi_{S^n}^* TS^n$$

$$\cong \pi_{S^1}^* TS^1 \oplus S^1 \times S^n \times \mathbb{R}^{n+1}$$

$$\cong S^1 \times S^1 \times S^n \times \mathbb{R}^{n+2}$$

$$\cong (T^2 \times S^n) \times \mathbb{R}^{n+2}$$

so $T(T^2 \times S^n)$ is trivializable and hence parallelizable.
Suppose $M_1$ is compact. To show it is a covering map, it suffices to show

that $\forall p \in M_1$, $\exists$ an open neighborhood $U$ of $p$ s.t. $\pi^{-1}(U) = \bigcup_{i=1}^{n} V_i$ for open $V_i$ s.t. $\pi|_{V_i}: V_i \to U$ is a homeomorphism.

Consider $\pi^{-1}(p) = \{ q \in \pi^{-1}(p) \mid$ the differential $d\pi_q = T_q H_2 : T_q M_1 \to T_q H_2$ is an isomorphism and hence surjective. Therefore $p$ is a regular value of $\pi$ and so $\pi^{-1}(p)$ is an $0$-dimensional submanifold of $M_1$. In particular, this implies that the pre-image $\pi^{-1}(p)$ is discrete and countable. Hence $M_1$ is compact, any discrete set is finite. Therefore $\pi^{-1}(p) = q_1, \ldots, q_n$.

Since $d\pi_q$ is an isomorphism $\forall i$, the inverse function theorem implies that $\pi$ is a local diffeomorphism. Therefore $\forall i$ open neighborhood $V_i$ of $q_i$ s.t. $\pi|_{V_i}$ is a diffeomorphism. Since $\{ q_i \}$ is discrete, we may extract $V_i$ s.t. $\{ V_i \}$ is pairwise disjoint. Since $\pi|_{V_i}$ is a diffeomorphism, it is open. Therefore $\tilde{U}_i = \pi(V_i)$ is an open neighborhood of $p$. Let $U' = \bigcap_{i=1}^{n} \tilde{U}_i$. Then $U'$ is an open neighborhood of $p$.

Let $V' = V_i \cap \pi^{-1}(U)$. Then $V'$ is an open neighborhood of $q_i$ and $\pi^{-1}(U) = V'_1 \cup \cdots \cup V'_n$.

Let $V = V'_1 \cup \cdots \cup V'_n$ and $\pi(U) = V$. Then $\pi(U)$ is closed and hence compact since $M_1$ is compact. Then $\pi(U)$ is closed.

By construction, $\pi(U) \supseteq U' \setminus \pi(U_1)$ is an open neighborhood of $p$. Take $U = U' \setminus \pi(U_1)$ and $V = V_i \setminus \pi^{-1}(U)$. Then since
we claim that
\[ H_j(R^n \setminus \{x_1, \ldots, x_k\}) = \begin{cases} \mathbb{Z} & j = 0 \\ \mathbb{Z}^{k-j} & j = n-1 \\ 0 & \text{else} \end{cases} \]

To show this, we proceed by induction on \( k \), the # of points removed.

Suppose \( k = 0 \). Then we recall
\[ H_j(R^n) = \begin{cases} \mathbb{Z} & j = 0 \\ 0 & \text{else} \end{cases} \]
which is consistent with our claim.

Suppose \( k = 1 \). Then \( R^n \setminus \{x_1\} \) deformation retracts onto \( S^{n-1} \)
and so
\[ H_j(R^n \setminus \{x_1\}) = \begin{cases} \mathbb{Z} & j = 0, 1 \\ 0 & \text{else} \end{cases} \]
which aligns with our claim.

We claim that \( R^n \setminus \{x_1, \ldots, x_k\} \) deformation retracts onto the wedge of \( k \) copies of \( S^{n-1} \). There exists a diffeomorphism \( R^n \to R^n \) i.e. \( x_i \to (i, 0, \ldots, 0) \). In the standard way, we can then deformation retract \( R^n \setminus \{x_1, \ldots, x_k\} \) onto the solid cylinder of length \( k+1 \) and radius \( 1 \) centered on the 1st axis, excluding the points \( x_1, \ldots, x_k \).

From there we deformation retract radially inward, towards the \( k \) solid spheres, excluding their centers. Finally, we deformation retract radially outward, the inside of each solid sphere onto its boundary. Composing these, we find that \( R^n \setminus \{x_1, \ldots, x_k\} \) deformation retracts onto \( V_j \subset S^{n-1} \).
We recall that the singular homology of $S^{n-1}$ is given by

$$H_i(S^{n-1}) = \begin{cases} \mathbb{Z} & i = 0, n-1 \\ 0 & \text{else} \end{cases}$$

We additionally recall that the homology of a wedge sum is the direct sum of homologies, excluding the 0th homology. Therefore

$$H_i\left(\mathbb{R}^n \vee \mathbb{R}^n \vee \cdots \vee \mathbb{R}^n\right) \cong H_i\left(V_i^k S^{n-1}\right) = \begin{cases} \mathbb{Z} & j = 0 \\ \mathbb{Z}^k & j = n-1 \\ 0 & \text{else} \end{cases}$$

where the 0th homology follows from the fact that $V_i^k S^{n-1}$ is connected.
a) Let \( U, V \) be two disjoint open subsets of \( M \). i.e.
\( \overline{U} \cap \overline{V} = \emptyset \) and \( U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^n \). Since \( M \) is orientable, there exist an orientation on \( M \setminus U \cup V \) and an induced orientation on \( \partial U, \partial V \). We then glue one end of a cylinder \( C \) of fixed length to \( \partial U \) and the other to \( \partial V \) such that orientation is preserved.

b) Consider \( S^2 \) with \( g \) handles some attached. Denote this by \( S^2_g \).
If \( g = 1 \), then it is clear that \( S^2_g = M_g \), the orientable genus \( g \) surface. Repeating this at increasing \( g \), we find that \( S^2_g = M_g \cup g \).

We claim \( \pi_1(M_g) \neq 0 \) \( \forall g > 0 \). We recall that \( M_g \) can be constructed via:

1. 0-cell: \( p \)
2. \( 2g \) 1-cells: \( a_1, b_1, a_2, b_2, \ldots, a_g, b_g \) with \( \partial a_i = \pm b_i \) for \( i = 1, 2, \ldots, g \)
1. 2-cell: \( f \) with \( \partial f = a_1 b_1 - a_1 b_1 + \cdots + a_g b_g - a_g b_g \)

Then \( H^1(M_g) = \frac{\ker d_1}{\text{Im} d_2} \cong \mathbb{Z}^{2g} \) since \( d_1 = 0 \) and \( d_2 = 0 \).

Since \( H^1 \) is the abelianization of \( \pi_1(M_g) \), this implies that \( \pi_1(M_g) \neq 0 \) and \( \pi_1 \) \( S^2_g \) is not simply connected.
a) Let $\tilde{F}: S^2 \to S^2$. Then $\tilde{F}$ induces a map $F_*: H_2(S^2) \to H_2(S^2)$. Since $H_2(S^2) = \mathbb{Z}$, $F_*$ can be viewed as a homomorphism $F^*: \mathbb{Z} \to \mathbb{Z}$. As the only such homomorphisms are multiplication by an integer, it follows that $F_*(a) = ka$ for some $k \in \mathbb{Z}$. We let $k = \deg \tilde{F}$. No choices were necessary in this construction and the above homomorphism guarantees well-defined.

5) Let $\overline{B}_1, \ldots, \overline{B}_n$ be disjoint disks on $S^3$.

For each $\overline{B}_i$, find a $C^0$ map $\overline{B}_i \to S^3$ that extends to a map $\overline{\varphi}_i: \overline{B}_i \to S^n$ with $d\overline{\varphi}_i \to \mathbb{N}$. Define $F: S^2 \to S^2$ by $F|_{\overline{B}_i} = \overline{\varphi}_i$ and $F|_{S^2 \setminus \bigcup \overline{B}_i} = \mathbb{N}$. Then $F$ is $C^0$ and
Let \( V = \mathbb{R}^3 \), \( \mathbb{R}^3 \), \( \mathbb{R}^3 \)
. Then as usual,

\[
d\omega(V) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}
\]

This implies that

\[
d\omega(V) = \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx dy dz
\]

Take \( \omega = -P dx dz - Q dy dz + R dx dy \). Then

\[
d\omega = \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dy dz dx + dP \wedge dx dz + dR \wedge dx dy
\]

\[
= \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx dy dz
\]

Mohr's theorem for manifolds with boundaries then gives

\[
\int_M d\omega(V) d(vol) = \int_M d\omega = \int_{\partial M} \omega
\]

\[
= \int_{\partial M} (P dx dz - Q dy dz + R dx dy)
\]

\[
= \int_{\partial M} V
\]

Let \( \nu \) be the normal vector field to \( M \). We test with that

the divergence theorem states

\[
\int_M d\omega(V) d(vol) = \int_{\partial M} \langle V, \nu \rangle ds
\]

where \( ds = \omega d(vol) \) is the embedding of \( \partial M \) into \( \mathbb{R}^n \).

We first claim \( \omega \) by \( d(vol) = \langle \nu, w \rangle ds \). Let \( T = V \cdot \langle V, \nu \rangle \nu \). Then

\( T \) is precisely the portion of \( V \) tangent \( T \partial M \) and to \( T \)

is a vector field on \( \partial M \). We claim \( \omega \cdot T \partial d(vol) = 0 \). For any \( p \in \partial M \) and

for any vector fields \( V_1, \ldots, V_{n-1} \in T_p(\partial M) \), we have that

\[
(\omega \cdot T \partial d(vol))_p (V_1, \ldots, V_{n-1}) = d(vol)(T, d_{V_1} V_1, \ldots, d_{V_{n-1}} V_{n-1})
\]

Since \( \partial M \) is \( n-1 \)-dimensional and \( T, d_{V_1} V_1, \ldots, d_{V_{n-1}} V_{n-1} \) is \( n \) elements

\( T \partial \partial M \), it follows they are linearly dependent and so

\[
(\omega \cdot T \partial d(vol))_p (V_1, \ldots, V_{n-1}) = 0
\]

As this holds for vector fields, it follows that \( \omega \cdot T \partial d(vol) = 0 \) and

\[
\omega \cdot T \partial d(vol) \Rightarrow \omega \cdot d(vol) = \langle V, \nu \rangle ds.
\]
Let \( V = P \partial_x + Q \partial_y + R \partial_z \). Then
\[
\mathbf{i} \cdot \nabla d\omega = P \partial_y \omega - Q \partial_x \omega + R \partial_z \omega.
\]
Mohr's theorem then implies
\[
\int_M \langle \mathbf{v}, \mathbf{w} \rangle d\omega = \int_M \mathbf{v} \cdot \nabla d\omega
\]
\[
= \int_M d(\mathbf{v} \cdot d\omega).
\]
By construction,
\[
d(\mathbf{v} \cdot d\omega) = d(P \partial_y \omega - Q \partial_x \omega + R \partial_z \omega) \\
= \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) d\omega
\]
\[
= \text{div}(\mathbf{v}) \ d(\omega).
\]
And \( \omega \) is
dual.

\[\square\]
We assume \( n > 1 \).

1. We recall that the universal cover of \( S^1 \times \cdots \times S^1 = T^n \) is \( \mathbb{R}^n \). Let \( \pi : \mathbb{R}^n \to T^n \) be the canonical projection.

Since \( n > 1 \), \( \pi'(S^n) = 0 \). Then \( \pi'(\mathbb{R}^n) = 0 \). It follows that \( F^* (\pi'(S^n)) \subseteq \pi^* (\pi'(\mathbb{R}^n)) \)

and so \( F \) a lift \( \tilde{F} : S^n \to \mathbb{R}^n \) of \( F \). B/c \( \mathbb{R}^n \) is contractible, \( \tilde{F} \) a homotopy \( \tilde{h} : S^n \to \mathbb{R}^n \) from \( \tilde{F} \) to a constant map.

This demands it a homotopy \( h : S^n \to T^n \) from \( F \) to a constant map. Therefore \( F \) is nullhomotopic.

2. Pick some open \( U \subseteq T^n \) s.t. \( U \cap \mathbb{R}^n = S^n \setminus \{pU\} \). Since \( T^n \) is a smooth manifold, this is possible. Let \( \varphi : U \to S^n \setminus \{pU\} \) be this diffeomorphism and define

\[
F : T^n \to S^n \text{ by } F|_U = \varphi \text{ and } F|_{T^n \setminus U} = \pi.
\]

Then \( F \) is cont. by construction and can be made smooth if necessary.

Consider a regular value \( c \) of \( F \). Since a regular value must have finite pre-images, \( x + \mathbb{N} \). Since \( F(p) + N \) iff \( p \in U \), and \( F|_U = \varphi \), a diffeomorphism, it follows that \( F^{-1}(c) \) consists of one point in \( U \). Since \( F|_U \) is a homomorphism, its degree at this follows that \( \deg F = \pm 1 \). Hence degree is homotopy invariant, this implies that \( F \) is not nullhomotopic.
c) We now use degree defined via differential forms. For each $i$, let $w_i$ be a nowhere vanishing volume form on $S^{n_i}$. Then $w_1 \wedge \ldots \wedge w_k$ is a nowhere vanishing volume form on $S^{n_1} \times \ldots \times S^{n_k}$. Normalizing, we may assume
\[
\int_{S^{n_1} \times \ldots \times S^{n_k}} w_1 \wedge \ldots \wedge w_k = 1
\]
so that
\[
\deg F = \int_{S^n} F^*(w_1 \wedge \ldots \wedge w_k) = \int_{S^n} F^*w_1 \wedge \ldots \wedge F^*w_k
\]
Consider $F^*w_1$. Since $w_1$ is a closed form on $S^{n_1}$, it follows that $w_1$ extended to $S^{n_1} \times \ldots \times S^{n_k}$ is also closed. Then $F^*w_1$ is a closed $n_1$-form on $S^n$. Hence, if $n_1 \leq n$, it then follows that $F^*w_1$ is exact and so $\exists \Psi \in C^\infty(S^n)$ such that $d\Psi = F^*w_1$. Then $d(\Psi \wedge F^*w_2 \wedge \ldots \wedge F^*w_k) = F^*(w_1 \wedge \ldots \wedge w_k)$. Mōri's theorem then implies
\[
\deg F = \int_{S^n} \Psi \wedge F^*w_2 \wedge \ldots \wedge F^*w_k = \int_{S^n} \Psi = 0
\]
As desired.
1. 
2. 
3. a \lor b \lor c \lor 
4. a \land b \land 
5. = 
6. 
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Since $M$ is smooth and connected, $\exists$ a path $Y: (0,1) \rightarrow M$
from $x_1$ to $x_2$. Here, we confine $Y$ to its image $Y((0,1))$.

Since $(0,1)$ is compact, $Y((0,1))$ is compact. Since $M$ is
a smooth manifold, $\forall x \in Y$ $\exists$ a neighborhood $U_x$ of $x$
$s.t. U_x$ is diffeomorphic to a subset of $\mathbb{R}^n$. Let $n = \text{dim}(M)$.

Since $Y$ is compact, $\exists$ a finite subcover $U_0, \ldots , U_m$ of $Y$.

By choosing sufficiently small initial sets $U_0$ and rearranging
our sets, we may assume that $x_1 \in U_0$, $x_2 \in U_m$.

The index of $U_i$ increases as it goes
from $0$ to $1$, and that $U_i \cap U_j = \emptyset$ if $|i-j| \leq 1$. This is
illustrated to the right. For each $i = 1, \ldots , m$, choose $z_i \in U_i \cap U_{i+1}$.$\forall y_i \in U_i$, and let $z_0 = x_1$ and $z_m = x_2$. **

To construct $\phi$, we aim to construct $\phi_i$ for $i = 0, \ldots , m$
s.t. $\phi_i(z_i) = z_{i+1}$ and $d(x_i)z_i(y_i) = U_{i+1}$. Then $\phi = \phi_0 \circ \ldots \circ \phi_m$
will satisfy $\phi(x) = x_2$ w.l. $d(x_i)(\phi_i(y_i)) = U_i$.

The case of $M = V \subseteq \mathbb{R}^n$ and we define a diffeomorphism $\phi$
which sends $x \mapsto y$ and $d(x)(v) = w$ for specified $x, y, v, w$
and is equal to the identity outside of a compact $\phi$ neighborhood
of $x$ and $y$.

Since we are working in $V \subseteq \mathbb{R}^n$, $\exists$ a smooth
path $Y: (0,1) \rightarrow V$ s.t. $Y(0) = x$, $Y(1) = y$, and
$Y'(0) = v$ and $Y'(1) = w$.

** Additionally choose nonzero $v \in T_{z_i}M$
$s.t. U_0 = V_1$ and $U_m = V_2$. 
we aim to use the transversality theorem.

Define \( F: X \times \mathbb{R}^n \to \mathbb{R}^n \) by \( (x, a) \mapsto x + a \).

we aim to show \( F \mid_Y \).

Suppose \( \exists (x, a) s.t. F(x, a) \in Y \). Then,

\[
\begin{align*}
\text{d}F(x, a): T_{(x, a)}(X \times \mathbb{R}^n) &\cong (T_x X) \oplus (T_a \mathbb{R}^n) \to T_{F(x, a)} \mathbb{R}^n \\
\end{align*}
\]

In local coordinates, \( (x_1, \ldots, x_n, a_1, \ldots, a_n) \in X \times \mathbb{R}^n \) we may write \( \text{d}F(x, a) \) as a \((n+m) \times n\) matrix. By construction we see that \( \text{d}F(x, a) \) has an \( n \times n \) identity matrix in the furthest right position. Then \( \text{rank}(\text{d}F(x, a)) = n \implies \text{rank}(\text{d}F(x, a)) = n \) and \( \text{d}F(x, a): T_{(x, a)}(X \times \mathbb{R}^n) \to T_{F(x, a)} \mathbb{R}^n \) is surjective.

In particular, this implies

\[
\begin{align*}
\text{d}F(x, a)(T_{(x, a)}(X \times \mathbb{R}^n)) + T_{F(x, a)} Y &\cong T_{F(x, a)} \mathbb{R}^n \\
\end{align*}
\]

As thus holds \( \forall (x, a) \in F^{-1}(Y) \), this implies that \( F \mid_Y \).

The transversality theorem then implies that

\[
f_a = F(-, a): X \to \mathbb{R}^n
\]

is transversal to \( Y \) for \( a \in a \). By construction, since \( f_a \) is translation by \( a \), \( \text{d}(f_a) = \text{id} \) and so \( \forall x \in f_a^{-1}(Y) \),

\[
\begin{align*}
\text{d}(f_a)_x(T_x X) + T_{f_a(x)} Y &\cong T_{f_a(x)} \mathbb{R}^n \\
\implies T_{f_a(x)}(x + a) + T_{f_a(x)} Y &\cong T_{f_a(x)} \mathbb{R}^n
\end{align*}
\]

and \( x + a \not\in Y \) for \( a \in a \).
3) \[ 408 \ 846 \ 515 \]

a) We use the regular value theorem.

Consider \( F: \text{Mat}_n(\mathbb{R}) \to \mathbb{R} \) defined by \( F(M) = \det M \).

Then \( F^{-1}(1) = \text{SL}(n, \mathbb{R}) \). To show \( \text{SL}(n, \mathbb{R}) \) is a smooth submanifold, it then suffices to show \( 1 \) is a regular value of \( F \). Let \( \mathbf{M} \) some \( M \in \text{SL}(n, \mathbb{R}) \). Then \( \forall \mathbf{T} \in \text{Mat}_n(\mathbb{R}) = \text{Tr} \text{Mat}_n(\mathbb{R}) \)

\[
d F_M(T) = \lim_{t \to 0} \frac{\det(M + tT) - \det(M)}{t}.
\]

\[
= \lim_{t \to 0} \frac{\det(M)(\det(I + tM^{-1}T) - 1)}{t}
\]

\[
= \lim_{t \to 0} \frac{\det(I + tM^{-1}T) - 1}{t}
\]

Conclude some \( k \in \mathbb{R} \) \( \mathbf{T} = k \mathbf{M} \). Then \( \mathbf{W} \mapsto \mathbf{T} = \frac{k}{n} \mathbf{W} \).

\[
d F_M(k\mathbf{M}) = \lim_{t \to 0} \frac{\det((I + \frac{k}{n}t)\mathbf{M}) - 1}{t}
\]

\[
= \lim_{t \to 0} \frac{(1 + \frac{k}{n}t)^n - 1}{t}
\]

\[
= \lim_{t \to 0} \frac{k(\frac{2}{2}k(\frac{k}{n})^2 \ldots + t^{n-1}(\frac{k}{n})^{n-1})}{t}
\]

and so \( 1 \) is a regular value of \( F \). Thus \( F^{-1}(1) \) is a smooth submanifold of \( \text{Mat}_n(\mathbb{R}) \).

b) By the previous part, it follows that

\[
T_{\mathbf{x}} \text{SL}(n, \mathbb{R}) = \ker (d F_{\mathbf{x}})
\]

By direct calculation,

\[
d F_{\mathbf{x}}(A) = \lim_{t \to 0} \frac{\det(I + tA) - 1}{t}
\]

since \( \det(I + tA) = \det(I + t \text{Tr}(A)\mathbf{I}) \) for sufficiently small \( t \).

It follows that

\[
d F_{\mathbf{x}}(A) = \text{Tr}(A)
\]

so

\[
T_{\mathbf{x}} \text{SL}(n, \mathbb{R}) = \{ A : \text{Tr}(A) = 0 \}.
\]
a) A cochain homotopy between $f_0^*$ and $f_1^*$ is a collection of linear maps $h_n: \Omega^n(N) \to \Omega^{n-1}(M)$ s.t.

$$f_1^* - f_0^* = d_h h + h d$$

If such a cochain homotopy exists, then $f_0^* = f_1^*$ on the level of cohomology since $\omega$ closed implies $(f_1^* - f_0^*) \omega = d(h(\omega)) = \text{exact}$. 

b) Let $I$ some $u \in \Omega^*(N)$. Then by definition,

$$\phi_1^* u - \phi_0^* u = \int_0^1 \left( \frac{d}{dt} \phi_t^* u \right) |_{t=0} ds$$

By definition of theLie derivative and Cartan's formula,

$$\phi_1^* u - \phi_0^* u = \int_0^1 d \left( \phi_t^* (\omega) \right) ds$$

$$= \int_0^1 \left( d \phi_t^* (\omega) + i_{\phi_t} d \right) (\omega) ds$$

Define $h: \Omega^*(N) \to \Omega^{n-1}(M)$ by $h(\omega) = \int_0^1 i_{\phi_t}(\omega) ds$. Then

$$\phi_1^* - \phi_0^* = d_h h + h d$$

as desired. \qed
a) By definition, for $p = (x_1, \ldots, x_{2n})$,

$$-X_p(x_{2i-1}) = \frac{d}{dt} q(t) |_{t=0}(x_{2i}) = \frac{d}{dt} \left( \cos(t)x_{2i}(p) - \sin(t)x_{2i-1}(p) \right) |_{t=0} = -x_{2i}(p) \quad (i = 1, 2, \ldots, n)$$

$$-X_p(x_{2i}) = \frac{d}{dt} q(t) |_{t=0}(x_{2i}) = \frac{d}{dt} \left( \sin(t)x_{2i-1}(p) - \cos(t)x_{2i}(p) \right) |_{t=0} = -x_{2i-1}(p) \quad (i = 1, 2, \ldots, n)$$

$$X = \sum_{j=1}^{n} \left( -x_{2j} \frac{d}{dx_{2j-1}} + x_{2j-1} \frac{d}{dx_{2j}} \right)$$

Additionally, $\omega \equiv 0$ by definition. Then by linearity,

$$\mathcal{L}X(\omega) = d(\omega) i_X(\omega)$$

$$= d \left( \sum_{j=1}^{n} \left( -x_{2j} \frac{d}{dx_{2j-1}} + x_{2j-1} \frac{d}{dx_{2j}} \right) \right) = d \left( \sum_{k=1}^{2n} x_k^2 \frac{d}{dx_k} \right)$$

$$= \sum_{k=1}^{2n} \left( -x_k \frac{d}{dx_k} \right)$$

$$= 0$$

Finally, we find $f = i_X(\omega) = \sum_{k=1}^{2n} x_k^2 \frac{d}{dx_k}$.

It then follows that $f = \frac{\sum_{k=1}^{2n} x_k^2}{2}$ satisfies this. \( \Box \)
Hence on the contrary, define
\[ H: [0,1] \times \mathbb{S}^n \to \mathbb{S}^n : (t, x) \mapsto \frac{(1-t)f(x) + t(-x)}{|(1-t)f(x) + t(-x)|} \]

To show \( H \) is smooth it suffices to show the denominator is non-vanishing. Suppose on the contrary that \((1-t)f(x) + t(-x) = 0\) for some \( t, x \). Then \(|(1-t)f(x)| = |tx| \Rightarrow (1-t)|f(x)| = t|x| \Rightarrow (1-t)x = t\frac{|x|}{|f(x)|} \Rightarrow x = \frac{1}{2} f(x) - \frac{1}{2} x = 0 \Rightarrow f(x) = x \neq x \).

Therefore the denominator is non-vanishing and so \( H \) is smooth.

By construction, \( H \) is a homotopy from \( f \) to \( -id \). Since degree is homotopy invariant, this implies \( \deg f = (-1)^{n+1} \) and \( \deg (id) = (-1)^{n+1} \). However, this contradicts the given fact that \( \deg f + (-1)^{n+1} = 0 \). Hence \( f \) must have a fixed point. \( \square \)
a) Let $G$ be a finitely presented group given by

$$G = \langle a_1, \ldots, a_n \mid f_1, \ldots, f_m \rangle$$

Define a CW complex $X$ given by

1. 0-cell: $p$
2. $n$ 1-cells: $a_1, \ldots, a_n$ with $\partial a_i = p - p$
3. $m$ 2-cells: $A_1, \ldots, A_m$ with $\partial A_i = f_i$

Since the fundamental group of a CW complex is equivalent to the fundamental group of its 1-skeleton modulo the attaching of its 2-cells, we find that since $X^n \cong S^1 \vee \cdots \vee S^1$ ($n$ times)

$$\pi_1(X) = \langle a_1, \ldots, a_n \mid f_1, \ldots, f_m \rangle$$

b) Consider $X = S^1 \vee S^1$. Then $\pi_1(X) \cong \mathbb{Z} \times \mathbb{Z}$ as desired.

c) Since $X$ is realized as a connected graph with 4 vertices and 2 edges, it follows that any 2-sheeted covering spaces of $X$ can be viewed as a connected graph with 2 vertices and 4 edges. Given

we then see all 2-sheeted cover connected coverings are

$$\exists \, \mapsto \begin{array}{c}
| \begin{array}{c}
\text{2-sheeted covering spaces of } X, \text{ w/ 3 connected}
\end{array}
\end{array}$$
To show that it.
Suppose $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic. Then $f$ a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^m$. Then $f^* : H^k(\mathbb{R}^n) \to H^k(\mathbb{R}^m)$ is an isomorphism.

Hence $H_c^k(\mathbb{R}^n) \cong H_c^k(\mathbb{R}^m)$. Thus $f^* : H_c^k(\mathbb{R}^n) \to H_c^k(\mathbb{R}^m)$ is an isomorphism.

We recall that by Poincaré duality,

$$H_c^k(\mathbb{R}^n) \cong \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

and $H_c^n(\mathbb{R}^n) \cong H_c^n(\mathbb{R}^m)$. Thus $\mathbb{R} \cong H_c^k(\mathbb{R}^m)$. Hence $n = m$ as desired. \qed

If $n = 0$ or $m = 0$, the result is immediate as $\mathbb{R}^n$ or $\mathbb{R}^m$ is compact.

We also remove a point and deformation retract to a sphere.

Need to deal with $n = 0$ case.
we recall that $N_3$ can be constructed via the cell complex

1. 1-cell: $p$

2. 2-cells: $a_1, \ldots, a_g$ with $\partial a_i = p-p$

3. 2-cells: $f$ with $\partial f = 2a_1 + 2a_2 + \ldots + 2a_g$

Thus, the fundamental group is given by

$$\pi_1(N_3) = \langle a_1, \ldots, a_g \mid a_1^2 a_2^2 \ldots a_g^2 \rangle$$

To compute the homology, we get the following chain complex

$$0 \rightarrow \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^g \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0$$

where $\partial_i = 0$ since $\partial_i a_i = p - p = 0$ for all $i$ and

$$\partial_2 f = 2a_1 + \ldots + 2a_g.$$  

So, $\partial_2 : \mathbb{Z} \rightarrow \mathbb{Z}^g : a \mapsto (2a, 2a, \ldots, 2a)$.

Then

$$H_2(N_3) = \frac{\ker(\partial_2)}{\operatorname{im}(\partial_3)} = \ker(\partial_2) = \{a : 2a = 0\} = 0$$

$$H_1(N_3) = \frac{\ker(\partial_1)}{\operatorname{im}(\partial_2)} = \mathbb{Z}^g / \{2x : x \in \mathbb{Z}^g\} \cong \mathbb{Z}^{g-1} \times \mathbb{Z}/2\mathbb{Z}$$

$$H_0(N_3) = \frac{\ker(\partial_0)}{\operatorname{im}(\partial_1)} = \mathbb{Z}/0 = \mathbb{Z}$$

all other homologies are trivial.
GEO Top

Spring 2020
Suppose that \( g \) is a locally constant function s.t. \( f \) is homotopic to \( g \). Let \( M_i \) denote the connected components of \( M \). We claim that \( g \) is constant on \( M_i \) for each \( i \).

Fix \( i \) and some \( p \in M_i \). Let \( \Omega = \{ q \in M_i : g(q) = g(p) \} \).

Since \( p \in \Omega \), \( \Omega \) is non-empty.

By \( g \) is locally constant, \( \forall q \in \Omega \) there exists an open neighborhood \( U \) of \( q \) s.t. \( \forall x \in U \). \( g(x) = g(q) = g(p) \). Then \( U \cap \Omega \) is open and \( \omega \in U \) is open.

Finally, by the continuity of \( g \), \( \Omega \) is closed.

Therefore since \( M_i \) is connected, \( \Omega \) is non-empty open and closed, each \( M_i \). Let \( g = c_i \) on \( M_i \). Therefore, \( \forall i \), \( \forall p \in M_i \).

\[
\left( (g|_{M_i})^*(w|_{M_i}) \right)_p = (w|_{M_i})_{c_i \circ d_p (g|_{M_i})}^*
\]

and \( \omega^* = 0 \) on \( M_i \). By this holds \( \forall i \).

In particular, on the level of\( c_i \) exact cohomology, \( g^* = 0 \).

Then since \( f \) is homotopic to \( g \), \( f^* = 0 \) on the level of cohomology. In particular, \( f^* \omega \) is closed.

Then \( f^* \omega \) is exact.
(a) Define
\[ w = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \]
Then \( dw = 0 \) so \( w \) is a closed and
\[ w \wedge w = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + dx_3 \wedge dx_4 \wedge dx_1 \wedge dx_2 \]
\[ = dV + dV = 2dV \]
which is non-vanishing.

(b) Suppose that such an \( w \) exists on \( S^4 \) exists.
We recall that
\[ H^k_{dR}(S^4) \equiv \begin{cases} \mathbb{R} & k = 0, 4 \\ 0 & \text{else} \end{cases} \]
Therefore since \( w \) is closed, \( w = d\eta \) for some 1-form \( \eta \).
Then \( w \wedge w = d(\eta \wedge w) \) and so \( w \wedge w \) is exact.
Hence \( H^4_{dR}(S^4) \to \mathbb{R}: \Theta \mapsto \int_{S^4} \Theta \) is an isomorphism.
This implies that \( \int_{S^4} w \wedge w = 0 \).

Alternatively, Nakao gives \( \int_{S^4} w \wedge w = \int_{S^4} \theta \eta \wedge w = \int \eta \wedge w = 0 \).

Since \( w \wedge w \) is non-vanishing volume form and \( S^4 \) is connected, \( w \wedge w \) is strictly positive definite or strictly negative.
In either case, \( \int_{S^4} w \wedge w \leq 0 \) which is a contradiction.
Therefore no such \( w \) exists.
Suppose that $\Omega$ is $1$-closed. Then

$$
\Omega = d(f w) = df \wedge w + f \wedge dw = \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right) \wedge w + f (2 dx \wedge dy)$$

$$= \frac{\partial f}{\partial x} dx \wedge dy - \frac{\partial f}{\partial x} dx \wedge dz + y \frac{\partial f}{\partial y} dx \wedge dy - \frac{\partial f}{\partial y} dy \wedge dz$$

$$+ x \frac{\partial f}{\partial z} dx \wedge dz + y \frac{\partial f}{\partial z} dy \wedge dz + 2 f dx \wedge dy$$

and $w$

$$\left(2 f + x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}\right) dx \wedge dy = 0$$

$$\left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial x}\right) dx \wedge dz = 0$$

$$-\left(\frac{\partial f}{\partial y} + x \frac{\partial f}{\partial z}\right) dy \wedge dz = 0$$

which implies $\frac{\partial f}{\partial x} = y \frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial y} = -x \frac{\partial f}{\partial z}$.

Thus

$$2 f + x \left(y \frac{\partial f}{\partial z}\right) + y \left(-x \frac{\partial f}{\partial z}\right) = 0$$

$$\Rightarrow 2 f = 0$$

$$\Rightarrow f = 0$$

Therefore if $\Omega$ is $1$-closed then $f=0$, which is not non-vanishing.
For vector fields $X, Y$, we aim to show that
\[
[L_X, i_Y] = i_{[X,Y]}.
\]
Let $w$ be a $k$-form and let $Z_1, \ldots, Z_{k-1}$ be vector fields. Then by direct computation,
\[
L_X(i_Y(w(Z_1, \ldots, Z_{k-1}))) = L_X(w(Y, Z_1, \ldots, Z_{k-1})) \quad \Rightarrow \quad \frac{\partial}{\partial X} w = (L_X w)(Y, Z_1, \ldots, Z_{k-1}) - w([X,Y], Z_1, \ldots, Z_{k-1})
\]

Similarly,
\[
i_Y \circ L_X(w(Z_1, \ldots, Z_{k-1})) = i_Y((L_X w)(Z_1, \ldots, Z_{k-1}) - \frac{\partial}{\partial X} w(Z_1, \ldots, [X,Z_2]\ldots,Z_{k-1})) \quad \Rightarrow \quad \frac{\partial}{\partial Y} w = (L_Y w)(Y, Z_1, \ldots, Z_{k-1}) - w(Y, Z_1, \ldots, [X,Z_i]\ldots,Z_{k-1})
\]

Therefore,
\[
[L_X, i_Y](w(Z_1, \ldots, Z_{k-1})) = (L_X w)(Z_1, \ldots, Z_{k-1}) - w([X,Y], Z_1, \ldots, Z_{k-1}) = (i_{[X,Y]} w)(Z_1, \ldots, Z_{k-1})
\]

As this holds for $w$ and $\forall Z_1, \ldots, Z_{k-1}$, the concludes
\[
[L_X, i_Y] = i_{[X,Y]}.
\]

as desired.
Suppose that $\mathbb{C}P^{2n}$ covers $X$.

Then $\pi_1(X)$ acts on $\mathbb{C}P^{2n}$ via deck transformations.

Suppose that $g \in \pi_1(X)$ is one such deck transformation.

We aim to show that $g = \text{id}$. To do so, it suffices to show that $g$ has a fixed point and hence that $L(g) = 0$.

We recall that, by the infinity trace formula,

$$L(g) = \sum_{i=0}^{2n} (-1)^i \text{tr} (g^* : H^i_c(\mathbb{C}P^{2n}) \to H^i_c(\mathbb{C}P^{2n}))$$

We recall that

$$H^i_c(\mathbb{C}P^{2n}) = \begin{cases} \mathbb{Z} & \text{odd} \\ 0 & \text{even} \end{cases}$$

Therefore

$$L(g) = \sum_{k=0}^{n} \text{tr} (g^* : H^k_c(\mathbb{C}P^{2n}) \to H^k_c(\mathbb{C}P^{2n}))$$

$$= \sum_{k=0}^{n} \text{tr} (g^* : \Omega \to \Omega)$$

Consider $g^* : H^2_c(\mathbb{C}P^{2n}, \Omega) \to H^2(\mathbb{C}P^{2n}, \Omega)$. If $H^2(\mathbb{C}P^{2n}) = 0$ for all $k$, it follows that $\Omega$ generates $H^2(\mathbb{C}P^{2n})$ and since $\text{ker} g^*$ generates $H^k_c(\mathbb{C}P^{2n})$ for all $k$, then $\text{ker} g^*$ generates $H^k(\mathbb{C}P^{2n})$ for all $k$.

Therefore if $g^* : H^2(\mathbb{C}P^{2n}) \to H^2(\mathbb{C}P^{2n})$ is multiplication by $x$, then $g^* : H^k_c(\mathbb{C}P^{2n}) \to H^k(\mathbb{C}P^{2n})$ is multiplication by $x^k$.

In particular,

$$L(g) = \sum_{k=0}^{n} x^k$$

But $x^k > 0$ for all $k > 0$ and $x^0 = 1$, this implies that

$$L(g) = 0.$$
Therefore $g$ has a fixed point and hence is the identity by its orbit decomposition.

Therefore $\pi_i(x)$ is the identity on $\mathbb{CP}^n$, and hence $x \in \mathbb{CP}^n$.  \[\square\]
Hence that \( f : S^2 \times S^2 \to \mathbb{C}P^2 \) is continuous.
Then \( f \) induces a map on the cohomology rings
\[
f^* : H^*(\mathbb{C}P^2) \to H^*(S^2 \times S^2)
\]
We recall that
\[
H^k(\mathbb{C}P^2) = \begin{cases} \mathbb{Z} & k = 0, 2, 4 \\ 0 & \text{otherwise} \end{cases}
\]
\[
H^k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & \text{otherwise} \end{cases}
\]
Then
\[
H^*(\mathbb{C}P^2) = \mathbb{Z}[x^2]/x^6 \quad \text{and} \quad H^*(S^2 \times S^2) = \mathbb{Z}[a^2, b^2]/(a^4, b^4)
\]
hence \( f^* \) must preserve degree,
\[
f^*(x^2) = \kappa a^2 + \beta b^2
\]
for some \( \kappa, \beta \in \mathbb{Z} \). Then
\[
f^*(x^4) = f^*(x^2) \cdot f^*(x^2)
\]
\[
= \kappa^2 a^4 + \beta^2 b^4 + 2\kappa \beta a^2 b^2
\]
\[
= 2\kappa \beta a^2 b^2
\]
Therefore on top cohomology, \( f^* \) is multiplication by \( 2\kappa \beta \), which preserves some \( a^2 b^2 \) generator \( H^4(S^2 \times S^2) \).
hence \( 2\kappa \beta \) is even, thus concludes that \( \text{deg } f = 2\kappa \beta \) is even. \( \square \)
We recall the LES on relative homology.

\[ \cdots \to \tilde{H}_k(x) \to \tilde{H}_k(x) \to H_k(x, x) \to \cdots \]

For all \( k \), we have \( \tilde{H}_k(x) = 0 \) and so we acquire the SES

\[ 0 \to H_k(X) \to H_k(X, x) \to 0 \]

which implies that \( \forall k, \forall x \in X \exists \) an isomorphism

\[ \phi^X_x : H_k(x) \to H_k(x, x) \]

Define \( \eta^X_{xy} = \phi^X_y \circ (\phi^X_x)^{-1} : H_k(x, x) \to H_k(x) \to H_k(x, y) \).

Then \( \eta^X_{xy} \) is an isomorphism, \( \eta^X_{xx} = \text{id} \), and

\[
\begin{align*}
M^X_{xy} \circ M^X_{yx} &= \phi^X_x \circ (\phi^X_y)^{-1} \circ (\phi^X_x)^{-1} \\
&= \phi^X_x \circ (\phi^X_y)^{-1} \\
&= \eta^X_{xy}
\end{align*}
\]

Finally, let \( f : X \to Y \) be continuous. Then \( f \) induces maps

\[ f^X : H_k(X) \to H_k(Y, f(x)) \text{ and } f^X : H_k(x, x) \to H_k(Y, f(x)). \]

Moreover, since \( f \) induces a map on the LES,

\[ f^X \circ \phi^X_x = \phi^Y_{f(x)} \circ f^X \Rightarrow f_*(\phi^X_x)^{-1} = (\phi^Y_{f(x)})^{-1} \circ f_* \]

Then

\[ f^X \circ \eta^X_{xy} = f^X \circ \phi^X_y \circ (\phi^X_x)^{-1} = \phi^Y_{f(x)} \circ \phi^Y_{f(x)} \circ f^X = \eta^Y_{f(x), f(x)} \circ f \]

as desired.
we recall the universal coefficient theorem, which states

\[ 0 \rightarrow \text{Ext}(H^n(G), G) \rightarrow H^n(G) \rightarrow \text{Hom}(H^n(G), G) \rightarrow 0 \]

is a split exact sequence. Thus

\[ H^n(G) \cong \text{Hom}(H^n(G), G) \oplus \text{Ext}(H^n(G), G) \]

As given,

\[ H_k(X) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k = 1 \\ \mathbb{Z}/2\mathbb{Z} & k = 2 \\ \mathbb{Z}/3\mathbb{Z} & k = 3 \\ 0 & \text{else} \end{cases} \]

To compute \( H_k(X; \mathbb{Z}/6\mathbb{Z}) \), we compute

\[
\begin{align*}
\text{Hom}(H_0(X; \mathbb{Z}/6\mathbb{Z})) &= \text{Hom}(\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}) = \mathbb{Z}/6\mathbb{Z} \\
\text{Hom}(H_1(X; \mathbb{Z}/6\mathbb{Z})) &= \text{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \\
\text{Hom}(H_2(X; \mathbb{Z}/6\mathbb{Z})) &= \text{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}) = 0 \\
\text{Hom}(H_3(X; \mathbb{Z}/6\mathbb{Z})) &= \mathbb{Z}/2\mathbb{Z} \\
\text{Hom}(H_4(X; \mathbb{Z}/6\mathbb{Z})) &= \mathbb{Z}/3\mathbb{Z}
\end{align*}
\]

\[
\begin{align*}
\text{Ext}(H_0(X; \mathbb{Z}/6\mathbb{Z})) &= \text{Ext}(\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}) = 0 \\
\text{Ext}(H_1(X; \mathbb{Z}/6\mathbb{Z})) &= \text{Ext}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \\
\text{Ext}(H_2(X; \mathbb{Z}/6\mathbb{Z})) &= \text{Ext}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}) = 0 \\
\text{Ext}(H_3(X; \mathbb{Z}/6\mathbb{Z})) &= \text{Ext}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \\
\text{Ext}(H_4(X; \mathbb{Z}/6\mathbb{Z})) &= \mathbb{Z}/3\mathbb{Z}
\end{align*}
\]

Then

\[ H_k(X; \mathbb{Z}/6\mathbb{Z}) = \begin{cases} \mathbb{Z}/6\mathbb{Z} & k = 0 \\ 0 & k = 1 \\ \mathbb{Z}/2\mathbb{Z} & k = 2 \\ \mathbb{Z}/3\mathbb{Z} & k = 3 \\ \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & k = 3 \\ 0 & \text{else} \end{cases} \]
To construct such a space, define $X$ as

1. 0-cell: $P$

2. 1-cells:
   1. $f: P \to O$
   2. $A, B, C$ attached to $P$ via $O$

3. 2-cells:
   1. $\alpha: P \to O$
   2. $\beta: P \to O$

Then

- $H_0(X) = \frac{\ker D_0}{\img D_0} = \mathbb{Z}\langle P \rangle = \mathbb{Z}$
- $H_1(X) = 0$
- $H_2(X) = \frac{\ker D_2}{\img D_2} = \mathbb{Z}\langle f \rangle = \mathbb{Z}/2\mathbb{Z}$
- $H_3(X) = \frac{\ker D_3}{\img D_3} = \mathbb{Z}\langle B \rangle = \mathbb{Z}/3\mathbb{Z}$
- $H_4(X) = \frac{\ker D_4}{\img D_4} = 0$

as desired.
we recall that \( \mathcal{M}_n \), the compact orientable genus \( n \) surface
was constructed via

\[
\begin{align*}
1 \text{ cell: } & \varnothing \\
2n \text{ cells: } & a_1, b_1, \ldots, a_n, b_n \quad \text{such that } a_1 = b_1 = p, p \cdot 0 \\
1 \text{ cell: } & f \quad \text{if } a_1 + b_1 - a_2 - b_2 + \ldots + a_n + b_n = a_n - b_n
\end{align*}
\]

Therefore, \( \pi_1(\mathcal{M}_n) = \langle a_1, b_1, \ldots, a_n, b_n \mid a_1b_1a_1^{-1}b_1^{-1} \ldots a_nb_n = 1 \rangle \cong \mathbb{Z}_n \).

Additionally, we recall that the fundamental subgroups of \( \pi_1(\mathcal{M}_n) \) correspond to the finite covering spaces of \( \mathcal{M}_n \) via

\[ \pi_1(\mathcal{X}, p) \cong \pi_1(\mathcal{M}_n) \]

where the \# of sheets is the index of \( \pi_1(\mathcal{X}, p) \). I.e.,

\[ \left( \pi_1(\mathcal{M}_n) : \pi_1(\mathcal{X}, p) \right) = \# \pi_1(\mathcal{X}, p) : \forall \mathcal{X} \in \mathcal{M}_n \]

Therefore, in search for finite index subgroups isomorphic to \( \mathbb{Z}_n \),
we are looking for finite sheeted covers \( \mathcal{M}_m \) of \( \mathcal{M}_n \).

By the CW complex above, we note that \( \chi(\mathcal{M}_n) = 2 - 2n \).

Therefore, if \( \mathcal{M}_m \) is a \( k \)-sheeted covering of \( \mathcal{M}_n \), then

\[
\chi(\mathcal{M}_m) = k \chi(\mathcal{M}_n) \quad \iff \quad 2 - 2m = k(2 - 2n)
\]

\[
\iff \quad 1 - m = \frac{k}{k - 1} n
\]

\[
\iff \quad m = kn - k + 1
\]

We claim that \( m = kn - k + 1 \) is a sufficient condition.
Suppose $m = k(n-1) + 1$. To cover $M_n$ by $M_m$, we first draw $M_n$ as

and identifying a loop through a hole of $M_n$ as $Y$, and write the

two curves of $Y$, denoted $Y^+$ and $Y^-$ above.

We then take $k$ copies of $M_n$, $C_1, ..., C_k$, $C_n = Y_1, ..., Y_k$

and attach $C_1$ to $C_2$ by identifying $Y_1^+ = Y_2^-$.

$C_2$ to $C_3$ by identifying $Y_2^+ = Y_3^-$.

$C_k$ to $C_1$ by identifying $Y_k^+ = Y_1^-$.

When repeated at $Y$, each $C_i$ has $n-1$ holes. Attaching all copies of $Y$

then creates a hole, so the resulting resulting $k$-core has

$k(n-1) + 1 = m$ holes and hence is isomorphic to $M_m$. This

is pictured on the following page.
$M_n$

$M_m = 1 \quad m = k(n-1) + 1$

$C_1 \quad C_2 \quad C_3 \quad \ldots \quad C_k$

$C_4$ - $C_m$ here

$\text{Example: } \quad n = 3, \quad k = 3, \quad m = 7$

$M_n$

$C_1 \quad C_2 \quad C_3$
Let $U$ be an $\varepsilon$-neighborhood of $D^2 \times S^1 \times S^0 / \varepsilon$ that deformation retracts onto $D^2 \times S^1 \times S^0 / \varepsilon$ and similarly for $U = D^2 \times S^1 \times \{1\} / \varepsilon$.

Then $U \cap V$ deformation retracts onto $\partial D^2 \times S^1 \times S^0 \cap \{1\} \cong T^2$

and $U \cup V = X$. We note that $D^2 \times S^1 \times S^0 / \varepsilon$ deformation retracts onto $S^1$. Mayer-Vietoris then gives the LES

$$\ldots \rightarrow H_k(T^2) \rightarrow H_k(S^1) \rightarrow H_k(X) \rightarrow \ldots$$

(1)

We recall

$$H_k(T^2) = \begin{cases} \mathbb{Z}, & k = 0, 2 \\ \mathbb{Z}^2, & k = 1 \end{cases}$$

$$H_k(S^1) = \begin{cases} \mathbb{Z}, & k = 0, 1 \\ 0, & \text{else} \end{cases}$$

From (1), we obtain the SES.

$$0 \rightarrow H_3(X) \rightarrow \mathbb{Z} \rightarrow 0$$

so $H_3(X) \cong \mathbb{Z}$. 

From (1) we also have

$$0 \rightarrow H_2(X) \rightarrow H_2(T^2) \rightarrow H_2(S^1)^{\oplus 2} \rightarrow H_2(X) \rightarrow H_0(T^2) \rightarrow H_0(S^1)^{\oplus 2} \rightarrow H_0(X) \cong 0$$

where $H_0(X) = \mathbb{Z}$ since $X$ is path connected.

Since $H_0(S^1)^{\oplus 2} \rightarrow H_0(X)$ is a retraction,

$H_0(T^2) \rightarrow H_0(S^1)^{\oplus 2}$ has kernel $\cong \mathbb{Z}$ and so $H_2(X) \rightarrow H_0(T^2)$ is the $0$ map. Then we have the SES

$$0 \rightarrow H_2(X) \rightarrow H_2(T^2) \rightarrow H_2(S^1)^{\oplus 2} \rightarrow H_2(X) \rightarrow 0$$

(2)
The map \( H_i(T^2) \to H_i(S') \oplus H_i(S') \) is given by the inclusions

\[
T^2 \to D^2 \times S' \times \{0\}/
\]
\[
T^2 \to D^2 \times S' \times \{1\}/
\]
where \( T^2 \) is the shared boundary in \( X \).

By construction, these are

\[
T^2 \to D^2 \times S' \times \{0\} : (x,y) \mapsto (x,y,0)
\]
\[
T^2 \to D^2 \times S' \times \{1\} : (x,y) \mapsto (xy^5, y, 1)
\]
when \( D^2 \) is contracted to a point then both become

\[
T^2 \to S' : (x,y) \mapsto y
\]
and so

\[
H_i(T^2) \to H_i(S') \oplus H_i(S') : (a,b) \mapsto (b,b)
\]
In particular, the map has image \( = \mathbb{Z} \) and kernel \( = \mathbb{Z} \).

By equation \((2)\), this gives \( H_2(X) = \mathbb{Z} \) and \( H_1(X) = \mathbb{Z} \).

To conclude,

\[
H_k(X) = \begin{cases} 
\mathbb{Z} & k=0,1,2,3 \\
0 & \text{else}
\end{cases}
\]
which is what was to be found.

GeoTop
Fall 2019
Let $M$ be a compact submanifold of $\mathbb{R}^3$ of dimension 3 with smooth boundary $\partial M$. The classical divergence theorem states that for a smooth vector field $X$,

$$\int_{\partial M} \langle X, N \rangle \, dA = \int_M \text{div}(X) \, dV$$

where $dV$ is the volume form on $M$ given by $dV$ on $\mathbb{R}^3$,

$$\text{div}(X) = \frac{\partial X}{\partial x} + \frac{\partial X}{\partial y} + \frac{\partial X}{\partial z}, \quad \langle X, N \rangle \in \text{the inner product of } X \text{ with } N,$$

$N$ the unit normal of $\partial M$, and $dA$ is the induced surface area form on $\partial M$ given by $dA = \iota^* \text{ind} dV$, $\iota : \partial M \to M$ is the inclusion.

Define $T = X - \langle X, N \rangle N$ on $\partial M$ to be the tangent component of $X$ to $\partial M$. We claim that $\iota^* \iota^* \text{ind} dV = 0$. Let $Y_1, Y_2, \ldots$ be vector fields on $\partial M$.

Then \{ $T, Y_1, Y_2 \}$ is linearly independent and so $\partial M$ is a 2-dimensional manifold, and we have $\iota^* \iota^* \text{ind} dV(Y_1, Y_2) = \iota^* \text{ind} dV(T, Y_1, Y_2) = 0 \Rightarrow \iota^* \iota^* \text{ind} dV = 0$.

Linearity then implies that $\iota^* \iota^* \text{ind} dV = \langle X, N \rangle \iota^* \iota^* \text{ind} dV = \langle X, N \rangle dA$.

Therefore, by Stokes's theorem,

$$\int_{\partial M} \langle X, N \rangle \, dA = \int_M \iota^* \iota^* \text{ind} dV = \int_M d \text{ind} dV$$

Let $X = f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} + h \frac{\partial}{\partial z}$. Then

$$d \text{ind} dV = d \left( \text{ind} dV \right) = d \left( f dydz - g dxdz + h dxdy \right)$$

$$= \left( \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \right) dxdydz$$

$$= \text{D} \langle X, N \rangle dV$$

and we have

$$\int_{\partial M} \langle X, N \rangle \, dA = \int_M \text{D} \langle X, N \rangle \, dV$$

as desired.
Let $X = \mathbb{RP}^{nm}/\mathbb{RP}^n$.

Consider $\mathbb{RP}^{nm} \subset \mathbb{RP}^n$. We recall that $\mathbb{RP}^k$ can be constructed from $\mathbb{RP}^{k-1}$ by attaching a $k$-disk $B^k$ to $\mathbb{RP}^{k-1}$ via the map

$$(x_0, \ldots, x_k) \mapsto [x_0 : \ldots : x_k].$$

Therefore, $\mathbb{RP}^{nm}/\mathbb{RP}^n$ has the structure

1. $0$-cell $p$
2. $n+1$-cell $e_{n+1}$ with $\partial e_{n+1} = p = 0$
3. $n+2$-cell $e_{n+2}$ with $\partial e_{n+2}$ attached via the quotient map $\pi_{n+1}$ (degree $1 + (-1)^n$)
4. $nm$-cell $e_{nm}$ with $\partial e_{nm}$ attached via a map $1 + (-1)^{nm-1}$.

We note that the quotient only collapses the 0, 1, \ldots, $n$-cells of $\mathbb{RP}^{nm}$ to a point. Therefore, it only affects the boundary maps of the $k \leq n+1$ cells and leaves all higher cells unaffected.

Consider $H_{n+1}(\mathbb{RP}^{nm}/\mathbb{RP}^n)$. By definition

$$H_{n+1}(X) = \frac{ker \partial_{n+2}}{im \partial_{n+1}} = \frac{\mathbb{Z}}{(1 + (-1)^{n+1}) \mathbb{Z}} = \begin{cases} \mathbb{Z}/2^{n+1} & \text{if } n \text{ even} \\ \mathbb{Z} & \text{if } n \text{ odd} \end{cases}.$$

As all cells and boundary maps are the same for $k > n+1$ and all higher cells are trivial for $k \leq n$, it follows that

$$H_k(X) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k = 1, \ldots, n \\ \mathbb{Z} & k = n+1 \text{ if } n \text{ even} \\ \mathbb{Z}/2^{n+1} & k = n+1 \text{ if } n \text{ odd} \\ H_k(\mathbb{RP}^{nm}) & k > n+1 \end{cases}.$$

which is what was to be computed.
we proceed by the LES on relative $\mathrm{H}_k$. If $m \geq 0$, then this interval, we
assumption $m \geq 1$.

We note that $\forall m \geq 0$, $\mathrm{RP}^m \times \mathbb{R}^m$ has a neighborhood which deformation
contract onto $\mathrm{RP}^m$. Therefore $(\mathrm{RP}^m, \mathrm{RP}^n)$ is a good pair and

$$\mathrm{H}_k(\mathrm{RP}^m/\mathrm{RP}^n) \cong \mathrm{H}_k(\mathrm{RP}^m, \mathrm{RP}^n).$$

For $k > 0$, this implies that it suffices to find $\mathrm{H}_k(\mathrm{RP}^m, \mathrm{RP}^n)$.

For $k = 0$, we note that $\mathrm{RP}^m/\mathrm{RP}^n$ is the quotient of a connected
space and hence connected, so $\mathrm{H}_0(\mathrm{RP}^m/\mathrm{RP}^n) \cong \mathbb{Z}$.

We recall the LES for relative homology, which is

$$\cdots \to \mathrm{H}_k(\mathrm{RP}^n) \to \mathrm{H}_k(\mathrm{RP}^{m+n}) \to \mathrm{H}_k(\mathrm{RP}^m, \mathrm{RP}^n) \to \cdots \ (*)$$

We recall that the homology of $\mathrm{RP}^1$ in general is given by

$$\mathrm{H}_k(\mathrm{RP}^1) = \begin{cases} \mathbb{Z} & k = 0 \text{ or } k = 1 \text{ is odd} \\ \mathbb{Z}/2\mathbb{Z} & 0 \leq k \leq 1 \text{ is odd} \\ 0 & \text{else} \end{cases}$$

Suppose first that $n$ is even. Then $\forall k \geq n$, we acquire the SES

$$\mathrm{H}_k(\mathrm{RP}^n) = 0 \to \mathrm{H}_k(\mathrm{RP}^{m+n}) \to \mathrm{H}_k(\mathrm{RP}^m, \mathrm{RP}^n) \to 0 = \mathrm{H}_{k-1}(\mathrm{RP}^m)$$

$$\Rightarrow \mathrm{H}_k(\mathrm{RP}^m, \mathrm{RP}^n) \cong \mathrm{H}_k(\mathrm{RP}^{m+n})$$

Now consider $1 \leq k < n$. Since $\mathrm{RP}^{n+m}$ and $\mathrm{RP}^n$ have the same
simplex $\forall k = 0, \ldots, n$, it follows that the map $\mathrm{H}_k(\mathrm{RP}^n) \to \mathrm{H}_k(\mathrm{RP}^{n+m})$
in $(\ast)$ is an isomorphism. Therefore, by induction it is exact sequence

$$\mathrm{H}_k(\mathrm{RP}^n) \leftrightarrow \mathrm{H}_k(\mathrm{RP}^{m+n}) \leftrightarrow \mathrm{H}_k(\mathrm{RP}^m, \mathrm{RP}^n) \leftrightarrow \mathrm{H}_{k-1}(\mathrm{RP}^n) \leftrightarrow \mathrm{H}_k(\mathrm{RP})$$

In the $k=n$ case, since $n$ is even, $\mathrm{H}_n(\mathrm{RP}^n) = \mathrm{H}_n(\mathrm{RP}^{m+n}) = 0$ so the map is still an
isomorphism.
and so for \( n \neq 0 \),

\[
H_k(\mathbb{RP}^m, \mathbb{RP}^n) = 0 \quad \text{if} \quad m - n \neq k \\
H_k(\mathbb{RP}^m, \mathbb{RP}^n) = H_k(\mathbb{RP}^m)
\]

This yields, for \( n \) even,

\[
H_k(\mathbb{RP}^m/\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k \neq 0, k \neq n \\ H_k(\mathbb{RP}^m) & k > n \end{cases}
\]

Now consider \( n \) odd. For \( k < n \) and \( k > n+1 \), the same reasoning holds.

To find the \( k = n, n+1 \) cases, we have the exact sequence

\[
0 \rightarrow H_{n+1}(\mathbb{RP}^m, \mathbb{RP}^n) \rightarrow H_n(\mathbb{RP}^n) \rightarrow H_n(\mathbb{RP}^m) \rightarrow H_n(\mathbb{RP}^m, \mathbb{RP}^n) \rightarrow 0
\]

Since \( n+1 \) is even,

\[
\mathbb{Z} \quad \mathbb{Z}/2\mathbb{Z}
\]

Consider \( \mathbb{RP}^n \cap \mathbb{RP}^m \)

\[
\mathbb{RP}^n \cap \mathbb{RP}^{m+1} = \{(x, ..., x, 0) : x_i \in \mathbb{R}, i < m\}
\]

Consider \( \mathbb{RP}^n \cup \mathbb{RP}^{m+1} \)

\[
\mathbb{RP}^n \cup \mathbb{RP}^{m+1} = \{(x, ..., x, 0, 1) : x_i \in \mathbb{R}, i < m\}
\]

Consider \( \mathbb{RP}^m/\mathbb{RP}^n \)

\[
\text{all } 2^n \text{ attached van}
\]
we recall that $\mathbb{RP}^n$ is compact as the quotient of $S^n$ which is compact. Additionally, since $\mathbb{RP}^n$ can be constructed with a k-cell for all $k = 0, 1, \ldots, n$, it follows that $\mathbb{RP}^n$

\[ X(\mathbb{RP}^n) = \begin{cases} 0 & k \text{ odd} \\ 1 & k \text{ even} \end{cases} \]

Therefore if $n$ is even, then $\mathbb{RP}^n$ does not admit a non-vanishing vector field.

Suppose that $n$ is odd. Express $\mathbb{RP}^n$ as $S^n/\sim$ where $\sim$ is the anti-podal identification and $S^n \subset \mathbb{R}^{n+1}$. We recall that for $p \in S^n$, $T_p S^n \cong \mathbb{R}^n_+$ smoothly on $p$.

We can therefore construct a non-vanishing vector field $V$ on $S^n$ via the map $V: (x_1, y_1, x_2, y_2, \ldots, x_n, y_n, x_{n+1}, y_{n+1}) \mapsto (-y_1, x_1, -y_2, x_2, \ldots, -y_n, x_n, x_{n+1}, y_{n+1})$

Since $p \cdot V_p = 0 \Rightarrow p \perp V_p$.

Let $\pi: S^n \to \mathbb{RP}^n$ be the anti-podal quotient map. To show that $(\pi \circ V)(p) = d\pi_p V_p$ is a well-defined vector field on $\mathbb{RP}^n$, it must be shown that $d\pi_p V_p = d\pi_{p'} V_{p'}$. Let $f: p \mapsto -p$ be the anti-podal map. Then by construction,

\[ d\pi_p = d(\pi \circ f)_p = d\pi_{f(p)} df_p = -d\pi_{-p}, \]

Moreover, by construction,

\[ V_p = (y_1, x_1, \ldots) = -(y_1, x_1, \ldots) = -V_p \]

and so $d\pi_p V_p = d\pi_{p'} V_{p'}$. Then $V$ factors through $\pi$ to a vector field $\pi_* V$ on $\mathbb{RP}^n$. Since $V$ is non-vanishing, so is $\pi_* V$, as desired.
Let \( X = S^1 \times S^1 \cong T^2 \) and \( Y = X \times \mathbb{D}(0,1) \times \{y\}, 0 \times \{y\} \rightarrow (y, y, 1) \). The standard mapping cone LES then yields a LES

\[ \cdots \rightarrow H_k(X) \rightarrow H_k(X) \rightarrow H_k(Y) \rightarrow H_{k-1}(X) \rightarrow \cdots \]

where \( f: X \rightarrow Y, (x, y) \mapsto (y, x) \).

We recall that by the Kunneth formula,

\[ H_k(X) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2 \\ \mathbb{Z}^2 & \text{if } k = 1 \\ 0 & \text{else} \end{cases} \]

which yields the exact sequence

\[ 0 \rightarrow H_3(Y) \rightarrow \mathbb{Z} \rightarrow H_2(Y) \rightarrow \mathbb{Z} \rightarrow H_1(Y) \rightarrow \mathbb{Z} \rightarrow H_0(Y) \rightarrow 0 \]

Since \( Y \) is contractible, \( H_3(Y) = 0 \).

Thus, \( H_2(Y) \cong \mathbb{Z} \times \mathbb{Z} \). For all \( p \),

\[ \int_{S^1 \times S^1} d\sigma \times d\tau = \int_{S^1} d\sigma \int_{S^1} d\tau \]

and \( \int_{S^1} \frac{1}{(x-a)(x-b)} \) is singular.

By f maps int. \( f \), we note that \( f_\# : H_2(X) \rightarrow H_2(X) = \mathbb{Z} \) is multiplication by \(-1\), and \( f_\#: H_1(X) \cong \mathbb{Z}^2 \rightarrow H_1(X) \cong \mathbb{Z}^2 : (a, b) \mapsto (b, a) \).

Then

\[ \frac{\partial}{\partial x} = \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \]

\[ \frac{\partial}{\partial y} = \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \]

hence \( H_2(Y) \cong \mathbb{Z} \) is injective and \( \ker 2 = 0 \), it follows that \( H_2(Y) = 0 \).
Bk $\mathbb{Z} \to H_2(y)$ has kernel $2\mathbb{Z}$ and $\mathbb{Z}^2 \to \mathbb{Z}^2$ has kernel $\mathbb{Z}$, it follows that

$$0 \to \mathbb{Z}/2\mathbb{Z} \to H_2(y) \to \mathbb{Z} \to 0$$

is a short exact sequence and hence $H_2(y) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Bk $\mathbb{Z}^2 \to H_1(y)$ has kernel $\{(a,-a) : a \in \mathbb{Z}\} \cong \mathbb{Z}$ and $H_1(y) \to \mathbb{Z}$ has image $\mathbb{Z}$,

$$0 \to \mathbb{Z} \to H_1(y) \to \mathbb{Z} \to 0$$

is a short exact sequence so $H_1(y) \cong \mathbb{Z}^2$. Therefore

$$H_n(y) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}^2 & k = 1 \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & k = 2 \\ 0 & \text{else} \end{cases}$$

as desired.
Since multiplication by \( g \) is a diffeomorphism,

\[
(Lg)_* (X)(f) = X(f \circ Lg)
\]

\( \forall \ g, f \). Therefore, by direct computation, \( \forall \) left-invariant \( X, Y \),

\[
(Lg)_* [X, Y](f) = (Lg)_* (X \circ Y - Y \circ X)(f)
\]

\[
= (X \circ Y \circ Lg(f) - Y \circ X \circ Lg(f))
\]

\[
= X(Lg_* Y(f)) - Y(Lg_* X(f))
\]

\[
= X(Y(f)) - Y(X(f))
\]

\[
= [X, Y](f)
\]

As this holds \( \forall f \), \([X, Y]\) is left-invariant.
We use the convention $E_0 = E_n$ and $X_i = X_i$ for ease of notation.

Define $X_n = \{ N_i, S_i, E_i, W_i : i = 1, 2, \ldots, n \}$ with topology generated by $\{ E_i, S_i, E_i, W_i, E_i, N_i, W_i, E_i, W_i, S_i, E_i, W_i \}$.

Define $f_n : X_n \to X$ by $N_i \mapsto N_i$, $E_i \mapsto E_i$, $S_i \mapsto S_i$, $W_i \mapsto W_i$.

Then $f_n$ is continuous since

$$f_n^{-1}(S_{i+1}) = \bigcup_{i=1}^{n} S_{i+1} = \text{open}$$

$$f_n^{-1}(W_{i+1}) = \bigcup_{i=1}^{n} W_{i+1} = \text{open}$$

$$f_n^{-1}(E_{i+1}) = \bigcup_{i=1}^{n} E_{i+1} = \text{open}$$

We first claim that $X_n$ is path connected. Pick $X_n$ is countable, it suffices to construct paths: $N_i \mapsto W_i \mapsto S_i \mapsto E_i \mapsto N_i$.

Define

$$Y_{N_i \mapsto W_i} : [0,1] \to X_n : t \mapsto \begin{cases} N_i & t = 0 \\ W_i & t < 1 \end{cases}$$

$$Y_{W_i \mapsto S_i} : [0,1] \to X_n : t \mapsto \begin{cases} W_i & t < 1 \\ S_i & t = 1 \end{cases}$$

$$Y_{S_i \mapsto E_i} : [0,1] \to X_n : t \mapsto \begin{cases} S_i & t = 0 \\ E_i & t < 1 \end{cases}$$

$$Y_{E_i \mapsto N_i} : [0,1] \to X_n : t \mapsto \begin{cases} E_i & t < 1 \\ N_i & t = 1 \end{cases}$$

Each of these are continuous paths since

$$Y_{N_i \mapsto W_i}^{-1} : \begin{cases} W_i & \mapsto (0,1) \text{ open } \\ E_i, W_i, S_i & \mapsto (0,1) \text{ open } \\ E_i, W_i & \mapsto (0,1) \text{ open } \\ E_i, W_i & \mapsto (0,1) \text{ open } \\ S_i, E_i, W_i & \mapsto (0,1) \text{ open } \\ \text{else} & \mapsto \emptyset \text{ open } \end{cases}$$

similar computation holds for other paths. Therefore $X_n$ is path connected.
Consider $N$. By definition, $\pi^{-1}(E \times N) = \bigcup_{i=1}^{n} (U_i \times N_i)$. Define $f_n^{-1}(E \times N) = \bigcup_{i=1}^{n} f_n^{-1}(E \times N_i)$, where $f_n|_{E_i} = \text{id}$. Therefore, $f_n$ satisfies the covering map definition at $E$. A symmetric argument holds for $W$.

Consider $N$. We have that $\pi^{-1}(E \times N) = \bigcup_{i=1}^{n} (U_i \times N_i)$. By construction, $f_n|_{E \times N_i} = \text{id}$ and so $f_n$ satisfies the definition at $N$.

The same reasoning holds for $S$.

Therefore, $f_n$ is an $n$-fold cover.

With this construction, we find that the universal cover will be

$$X_\infty = \{ N_i \times W_i \times S_i \times E_i : i \in \mathbb{Z} \}$$

with the topology generated by the same kinds of sets as before.

The covering map is then given by

$$f : \begin{cases} 
E_i \to E \\
W_i \to W \\
S_i \to S 
\end{cases}$$

as before.
For some $n > 1$, consider the quotient $[0,n]/\sim$. Define $f_n : [0,n]/\sim \to X$ by

$$f_n(x) = \begin{cases} 
N & x = 0, 1, \ldots, n-1 \\
S & x = \frac{1}{2}, \frac{3}{2}, \ldots, n-\frac{1}{2} \\
W & x = 0, \frac{1}{2}, \frac{3}{2}, \ldots, n-\frac{1}{2} \\
E & x = \frac{1}{2}, \frac{3}{2}, \ldots, n \end{cases}$$

This can be visualized by

```
W E N W S E ...
0 \frac{1}{2} 1 \frac{3}{2} ...
```

here we have given $[0,n]/\sim$ the usual topology of $S^1$.

First we claim that $f_n$ is continuous. By construction

$$f_n^{-1}(E^3) = \left(\frac{1}{2}, 1\right) \cup \cdots \cup \left(n-\frac{1}{2}, n\right) = \text{open}$$

$$f_n^{-1}(N) = \left(0, \frac{1}{2}\right) \cup \cdots \cup \left(n-1, n-\frac{1}{2}\right) = \text{open}$$

$$f_n^{-1}(E, W) = [0,n] \setminus \left(0, \frac{1}{2}, \frac{1}{2}, \ldots, n-\frac{1}{2}\right) = \text{open}$$

$$f_n^{-1}(N, E, W) = [0,n] \setminus \left(\frac{1}{2}, \frac{3}{2}, \cdots, n\right) = \text{open}$$

$$f_n^{-1}(S, E, W) = [0,n] \setminus \{0, 1, \ldots, n\} = \text{open}$$

and so $f_n$ is continuous.

We now show that $f_n$ is an $n$-fold cover. We only consider $N$.

As the remaining points will follow symmetry arguments,
(a) Let $0 \to A \to B \to C \to 0$ be a SES of chain complexes.

To get the boundary map in the associated LES, we wish to define the map $\mathcal{H}_n(C) \to \mathcal{H}_n(A)$.

Let $i : A \to B$ and $j : B \to C$ be the maps between the chain complexes.

Consider some $[b] \in \mathcal{H}_n(C)$. Since $j : B_n \to C_n$ is surjective, there is some $b \in B_n$ such that $j_n(b) = [b]$. Then since $j$ is a chain map, $0 = d_n(c) = d_n(j_n(bn)) = j_n(d_B bn)$.

So $d_B bn \in im(j_n)$. Then we may by exactness of the SES, use $i_m \in Hom$, s.t. $i_m(an) = d_B bn$. Define $S_n : \mathcal{H}_n(C) \to \mathcal{H}_{n-1}(A) : [c] \mapsto [an]$. To show that $S_n$ is well-defined, it must be shown that $S_n$ is independent of the choice of representative of $[c_n]$ and the choice of pre-image $bn$.

Suppose $I \in \mathcal{C}_n$, and consider $c_n + d_B cn \in [c_n]$. Then $I \in B_{n-1} \in B_0 \in B_n$, s.t. $j_n(bn + d_B cn) = c_n + d_B cn$.

Applying $S_n$ to $bn + d_B cn$, we get $d_B bn$. Therefore, this alternate representation of $[c_n]$ will yield the same $[an]$, since it yields the same $d_B bn \in B_{n-1}$.

We first show that $S_n$ is independent of the choice of $bn \in B_n$.

\[ (*) \]
Suppose \( f(b_n, b'_n) \in B_n \) i.e. \( j_n(b_n) = j_n(b'_n) = C_n \).

Then \( j_n(b_n - b'_n) = 0 \Rightarrow b_n - b'_n \in \ker j_n \). By contraposition, if \( a_n \in A_n \) i.e. \( i_n(a_n) = b_n - b'_n \), we have

As before, choose \( a_{n-1}, a_{n-1}' \in A_{n-1} \) i.e. \( i_{n-1}(a_{n-1}) = \partial b_{n-1} \) and \( i_{n-1}(a_{n-1}') = \partial b_{n-1}' \). We claim that \([a_{n-1}] = [a_{n-1}']\).

By construction,

\[
\begin{align*}
\partial i_{n-1}(a_{n-1} - a_{n-1}') &= \partial b_{n-1} - \partial b_{n-1}' \\
&= \partial b(i_n(a_n)) \\
&= i_n(\partial a(a_n))
\end{align*}
\]

Since \( i_n \) is injective, this implies \( a_{n-1} - a_{n-1}' = \partial a(a_n) \) and \([a_{n-1}] = [a_{n-1}']\). Therefore the choice of \( b_n \) does not affect the map \( j_n \).

We now show that the choice of representative of \([C_n]\) does not affect the map.

(continues here)

Therefore \( j_n \) is well-defined on homology.

Finally, we note that

\[
\begin{align*}
\text{im}(\partial_n) &= \ker(i_{n-1}). \\
\text{morphism} \quad b_n \in B_n, \\
\text{im} j_n(b_n) &= \text{im} j_n[b_n] = 0 \\
\text{im} j_{n-1}(b_n) &= \text{im} j_{n-1}[b_n] = 0
\end{align*}
\]

We do not show exactness of this LES.
(b) Tensoring the chain complex with the long exact sequence yields the diagram

\[
\begin{array}{ccc}
\circ & \circ & \circ \\
\downarrow 5 \circ & \downarrow 5 & \downarrow 5 \\
\circ \to \mathbb{Z}/5\mathbb{Z} \to \mathbb{Z}/25\mathbb{Z} \to \mathbb{Z}/5\mathbb{Z} \to \circ & (i) \\
\downarrow 5=0 & \downarrow 5 & \downarrow 5=0 \\
\circ \to \mathbb{Z}/5\mathbb{Z} \to \mathbb{Z}/25\mathbb{Z} \to \mathbb{Z}/5\mathbb{Z} \to \circ & (i) \\
\downarrow 0 & \downarrow 0 & \downarrow 0 \\
\circ & \circ & \circ \\
\end{array}
\]

In all the instances of $\mathbb{Z}/5\mathbb{Z}$ above, the homology is $\mathbb{Z}/5\mathbb{Z}$ since the maps are trivial, let $A, B, C, A_0, B_0, C_0$ denote the maps, reading from left to right. We then drop $[\cdot]$ when considering the homology.

Following the construction from part a, consider some $c \in H_1(C).$ Then $\exists b \in \mathbb{Z}/25\mathbb{Z}$ s.t. $b_1 \to b$. By construction of the map $\mathbb{Z}/25\mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}$ we may choose $b_1 = "c"$ in the sense of $\mathbb{Z}$. Then $\exists b = 5b \neq 0$ and we may choose $a_0 \in A_0 \subseteq \mathbb{Z}/5\mathbb{Z}$ s.t. $a_0 = b_1 \to 5a_0 = 2b = 5b_1$. Then $\delta : c_1 \to a_0 = b_1 = c_1$ is the identity map on $\mathbb{Z}/5\mathbb{Z}$. \qed
For each \( p \in M \) a neighborhood \( U_p \) of \( p \) is compact.

Corollary: This can be constructed explicitly for any \( p \in M \) by taking a neighborhood that diffeomorphic to a subset of \( \mathbb{R}^n \) and then taking the pre-image of a ball.

Corollary: the open cover \( \{ U_p^3 \}_{p \in M} \) and let \( \{ \psi_p \}_{p \in M} \) be a partition of unity subordinate to \( \{ U_p^3 \} \).

Since \( M \) is second-countable, we may restrict each \( U_p \) to a small neighborhood of \( p \) such that \( U_p \) remains compact and \( \{ U_p^3 \}_{p \in M} \) is a countable open cover of \( M \). *Enumerate these sets \( \{ U_1, U_2, \ldots \} \) and let \( \{ \psi_1, \psi_2, \ldots \} \) be a partition of unity subordinate to \( \{ U_1 \} \).

Define \( f : M \to \mathbb{R} \) by \( f(p) = \sum \psi_i(p) \). Since \( \{ U_i \} \) is locally finite, \( f \) is a finite sum at all \( p \) and \( \psi_1 \) is well-defined. Since \( \psi_i \) is smooth, this similarly implies that \( f \) is smooth.

We aim to show \( f \) is proper. Let \( K \) be a compact subset of \( \mathbb{R} \).

Since \( f \) is continuous, \( f^{-1}(K) \) is closed. To show \( f^{-1}(K) \) is compact, it then suffices to show \( f^{-1}(K) \) is contained in a compact set.

Since \( K \subseteq \mathbb{R} \) is compact, \( f^{-1}(\mathbb{R}) \subseteq [-n,n] \). Conclude \( f^{-1}([-n,n]) \).

Since \( f \) is non-negative, \( f^{-1}([-n,n]) = f^{-1}([0,n]) \).

By construction, we note that

* Additionally, by paracompactness, we may restrict \( \{ \psi_i \} \) to a locally finite...
By construction, \( \forall \rho \in M \),
\[
\left| f(\rho) \right| = \left| \sum_{i=1}^{\infty} i \psi_i(\rho) \right| \geq \min \{ i : \psi_i(\rho) > 0 \} \sum_{i=1}^{\infty} \psi_i(\rho) \geq \min \{ i : \psi_i(\rho) > 0 \}
\]

Therefore \( f^{-1}[0,1] \subset \overline{U_1 \cup \ldots \cup U_n} \) which is compact by construction. Therefore \( f^{-1}(K) \) is contained in a compact subset of \( M \) and hence is compact. Thus \( f \) is proper as desired. \( \square \)
Consider the anti-podal map \( f : S^n \to S^n : x \mapsto -x \). Since \( 0 \neq S^n \), 
\( f \) does not have any fixed points and hence is a nontrivial map.

We aim to calculate the Lefschetz number \( \chi f \).

By definition,
\[
\chi f = \sum_{j=0}^{n} (-1)^j \operatorname{tr}(f_* : H_j(S^n) \to H_j(S^n)).
\]

We recall that \( H_j(S^n) = \begin{cases} \mathbb{Z} & j = 0, n \text{ or } j = 0 \text{ even} \\ 0 & \text{else} \end{cases} \).

Thus,
\[
\chi f = \operatorname{tr}(f_* : H_0(S^n) \to H_0(S^n)) + (-1)^n \operatorname{tr}(f_* : H_n(S^n) \to H_n(S^n)).
\]

Recalling the CW structure of \( S^n \), we know that \( f_* \) fixes the 0-cell, inverts the n-cell if \( n \) is odd, and fixes the n-cell if \( n \) is even.

Then,
\[
\chi f = 1 + (-1)^n (-1)^n = 2.
\]

For the identity, we recall the Lefschetz number as
\[
\chi \text{id} = \chi(S^n) = 1 + (-1)^n.
\]

Moreover, since \( f \) does not have any fixed points, \( \chi f = 0 \).

For the identity, we recall that \( \chi \text{id} = \chi(S^n) = 1 + (-1)^n \).

Therefore if \( n \) is even, \( \chi f = 0 \neq 2 = \chi \text{id} \) and \( f \) and \( \text{id} \) are not homotopic. It then remains to show that they are in the case of \( n \) odd.

Suppose \( n \) is odd. Then we view \( S^n \subset \mathbb{R}^{n+1} = (\mathbb{R}^2)^{n+1} \).

Let \( R_0 \) be the rotation by \( \theta \) on \( \mathbb{R}^2 \). Then
\[
\phi_\theta = R_0 \circ \theta \circ R_0 : S^n \to S^n
\]

is a homotopy from \( \phi_0 = \text{id} \) to \( \phi_\pi = -\text{id} \) as desired.
We recall Cartan's formula which states

\[ L_x = d_0 i_x + i_x d_0 \]

From this, it follows that \( L_x \) commutes with \( d_0 \) since

\[ L_x d_0 = (d_0 i_x + i_x d_0) d_0 = d_0 i_x d_0 = d_0 (d_0 i_x + i_x d_0) = d_0 L_x \]

Additionally, as will be shown in a lemma if time permits,

\[ [L_x, i_y] = i_{[x,y]} \]

Combining these facts, direct computation yields

\[ [L_x, L_y] = [L_x, d_0 i_y] + [L_x, i_y d_0] \]

\[ = (L_x d_0 i_y - d_0 i_y L_x) + (L_x i_y d_0 - i_y d_0 L_x) \]

\[ = (d_0 L_x i_y - d_0 i_y L_x) + (L_x i_y d_0 - i_y d_0 L_x) \]

\[ = d_0[L_x, i_y] + [L_x, i_y] d_0 \]

\[ = d_o [L_x, i_y] + i_{[x,y]} d_0 \]

\[ = L_{[x,y]} \]

as desired.

\[ \text{Lemma} \]
Lemma \[ [L_x, i_y] = i_{[x,y]} \]

Proof: Fix a 1-form \( \omega \) and vector fields \( X_1, \ldots, X_k \).

Then by direct computation

\[
L_x \circ i_y (\omega(X_1, \ldots, X_k)) = L_x (\omega(Y, X_1, \ldots, X_k)) \\
= (L_x \omega)(Y, X_1, \ldots, X_k) + \omega(L_x Y, X_1, \ldots, X_k) + \sum_j \omega(Y, X_1, \ldots, L_x X_j, \ldots, X_k)
\]

and

\[
i_y \circ L_x (\omega(X_1, \ldots, X_k)) = i_y (L_x \omega)(X_1, \ldots, X_k) + \sum_j \omega(Y, X_1, \ldots, L_x X_j, \ldots, X_k) \\
= (L_x \omega)(Y, X_1, \ldots, X_k) + \sum_j \omega(Y, X_1, \ldots, L_x X_j, \ldots, X_k)
\]

Therefore

\[
[L_x, i_y] (\omega(X_1, \ldots, X_k)) = \omega(L_x Y, X_1, \ldots, X_k) \\
= \omega ([x,y], X_1, \ldots, X_k) \\
= i_{[x,y]} \omega(X_1, \ldots, X_k)
\]

and so \( \omega [L_x, i_y] = i_{[x,y]} \omega \).
Suppose that $w$ is exact. Then

$$(\Rightarrow)$$

Suppose first that $w$ is exact, i.e. $w = df$ for some smooth $f$. Consider a smooth map $f: S' \to M$. By $f^*$ commuting with the exterior derivative, Stokes' law implies

$$\int_{S'} f^* w = \int_{S'} d(f^* g) = \int_{S'} f^* g = 0$$

as desired. Alternatively, we may note that $H^1_{dR}(S') \cong \mathbb{R}$ via the homomorphism $\tilde{\eta} \mapsto \int_{S'} \eta$. Therefore since $f^* w = d(f^* g)$ is 0 on the level of cohomology, $\int_{S'} f^* w = 0$.

$$(\Leftarrow)$$

Suppose instead that $\int_{S'} f^* w = 0$ for all smooth $f: S' \to M$.

Let $M_1, M_2, \ldots$ denote the connected components of $M$ and fix a point $p \in M_i$. Since $M$ is a smooth manifold, $M_i$ is path connected to $p$.

Therefore $\forall x \in M_i$, $\exists$ a path $Y(x)$ from $p$ to $x$.

Define $g: M \to \mathbb{R}$ by

$$g(x) = \int_{Y(x)} w$$

To show that $g$ is well-defined, it must be shown that $g$ is independent of the choice of path $Y(x)$. Suppose $\exists$ 2 paths $Y_i, Y_e$ from $p$ to $x$. Consider the path $Y_i - Y_e$. By construction, $Y_i - Y_e$ is smooth except potentially at $p$, $x$. Working locally at $p$, $x$ we can construct a homotopy $\tilde{Y}_i$ to $Y_i$. Let $f: S^1 \to M$ be the smooth map defined by $\tilde{Y}_i$. Then by homotopy invariance and assumption

$$\int_{Y_i} w - \int_{Y_e} w = \int_{Y_i - Y_e} w = \int_S f^* w = \int_{S'} f^* w = 0$$
Therefore $g$ is independent of the choice of path $Y(t)$.

Since $w$ is smooth, this implies that $g : M \to \mathbb{R}$ is smooth.

$Y(x)$ can be chosen smoothly in $x$.

We finally claim that $dg = w$. To show this, it suffices to show it pointwise.

By the FTC, \( \forall x \in M \),

\[
dg_x = \frac{d}{dt} \bigg|_{t=0} \int_{Y(t)} w = w_{Y(0)}(x(t)) = w_x
\]

and so as an exact form $dg = w$.

\( \Box \)
Suppose \( f: \mathcal{M} \to \mathcal{N} \) is \( n \)-form diffeomorphism. Then \( f^* \omega - f^* \eta = d\phi \) for some \( \phi \) on \( \mathcal{N} \) on \( \mathcal{M} \).

If \( \omega - \eta \) is exact, we claim that \( f^* \omega - f^* \eta \) is exact.

\[
(f^* \omega)_p = df_p \omega_p, \quad (f^* \eta)_p = \omega_p \circ df_p
\]

\[
\text{df: } T \mathcal{M} \to T \mathcal{N}
\]

\[
\omega = \eta \quad f^* \omega - f^* \eta = \phi.
\]

If \( f \) is local diffeomorphism, then \( f^* \omega - f^* \eta = d\phi \Rightarrow (f^* \omega - f^* \eta)_p = df_p \phi \).

To show that \( f^* \omega - f^* \eta \) is exact, it suffices to construct a \((n-1)\)-form \( \Psi \) locally such that \( d\Psi = f^* \omega - f^* \eta \). Fix some \( p \in \mathcal{M} \). Then \( f \) is a covering map \( f \) on a neighborhood of \( f(p) \) and disjoint set \( P_1, \ldots, P_k \) of the inverse images of \( p \) under \( f \), which are finite since \( f \) is a finite covering map. There exists a chart \( \phi_i \) on \( \mathcal{N} \) and \( \mathcal{U}_i \) of \( f^* \mathcal{U}_i \) such that \( \phi_i \circ f \) is a diffeomorphism

\[
\psi_i = \frac{1}{k} \sum_{l=1}^{k} (\phi_i)_* (\delta_{l})_i = \phi_i \Psi
\]

since \( \phi_i \circ f \) is a diffeomorphism \( \mathcal{V}_i \), this is well-defined and smooth on \( \mathcal{U}_i \). Moreover, on \( \mathcal{U}_i \),

\[
d\psi_i = \frac{1}{k} \sum_{l=1}^{k} (\phi_i)_* d\psi_i((\phi_i)_i^{-1}(q))
\]

\[
= \frac{1}{k} \sum_{l=1}^{k} (\phi_i)_* (f^* \omega - f^* \eta)((\phi_i)_i^{-1}(q))
\]

\[
= \frac{1}{k} \sum_{l=1}^{k} (\omega - \eta)_i
\]

\[
= (\omega - \eta)_i
\]

Therefore \( f^* \omega - f^* \eta \) is exact. This implies that \( f \) is injective on cohomology. \( \Box \)
we first calculate \( H_1(x,A) \). Let \( C_k(x) \) denote the \( k \)-chains of \( X \) and \( C_k(A) \) denote the \( k \)-chains of \( A \). Since \( A \) is a union of points, we have

\[
C_2(x,A) = \mathbb{Z}, \quad C_2(x) = 0, \quad C_1(x,A) = C_1(x) = \mathbb{Z}
\]

we first calculate \( H_1(x,A) \). we recall the LES

\[
0 \rightarrow H_1(A) \rightarrow H_1(x) \rightarrow H_1(x,A) \rightarrow H_0(A) \rightarrow H_0(x) \rightarrow H_0(A) \rightarrow H_0(x,A) \rightarrow 0
\]

Hence \( X^0 \cup \{0,1\} \), \( X \) is contractible and hence \( H_1(x) = 0 \). Therefore \( H_1(x,A) \rightarrow H_0(A) \). Since \( A \) is the countable union of 0-cells, \( H_0(A) \) is countably generated. Therefore \( H_1(x,A) \) is countably generated.

we claim that \( H_1(x,A) \) is not countably generated. we recall that \( H_1(x,A) \) is the abelianization of \( T^1(x,A) \) and as it suffices to show \( T^1(x,A) \) is not countably generated. To do so, it suffices to construct a surjection \( \overline{T^1(x,A)} \rightarrow \bigoplus_{\mathbb{Z}} \mathbb{Z} \).

For some \( x \in \bigoplus_{\mathbb{Z}} \mathbb{Z} \), let binary expansion \( x = (x_1, x_2, \ldots) \).

Define \( Y: \{0,1\} \rightarrow xA \) s.t. \( Y(0) = [1] \) and s.t.

\( Y \) traverses \([1/n, 1/n+1]/A \) \( x_n \) times during time \([1/n, 1/n+1]/A \).

Therefore \( T^1(x,A) \rightarrow \bigoplus_{\mathbb{Z}} \mathbb{Z} \) is a surjection and \( H_1(x,A) \) is uncountably generated.

* Define \( \overline{T^1(x,A)} \rightarrow \bigoplus_{\mathbb{Z}} \mathbb{Z} \) by \( Y \rightarrow (x_1, x_2, \ldots) \) where \( x_n \) is the \# of times \( Y \) traverses \([1/n, 1/n+1]/A \). Since each loop of \( [1/n, 1/n+1]/A \) is non-contractible, this is well-defined on \( T^1(x,A) \).
(a) Let $f: \mathbb{RP}^2 \to T^2 = S^1 \times S^1$ be continuous. We recall that $\mathbb{RP}^2$ is the universal cover of $T^2$ and claim that $f$ lifts to a continuous map $\tilde{f}: \mathbb{RP}^2 \to \mathbb{R}^2$.

We recall that $\mathbb{RP}^2$ has CW structure

1. 0-cell: $p$
2. 1-cells: $c$ with $de = p - p = 0$
3. 2-cells: $f$ with $df = 2c$

Therefore $\pi_1(\mathbb{RP}^2) \cong \langle c \mid 2c \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

Similarly, since $S^1 \times S^1 \subseteq T^2$ has cell CW structure

1. 0-cell: $p$
2. 1-cells: $a, b$ with $da = db = p - p = 0$
3. 2-cells: $f$ with $df = a + b - a - b = 0$

It follows that $\pi_1(S^1 \times S^1) \cong \langle a, b \mid a + b - a - b \rangle = (a, b) \cong \mathbb{Z}^2$.

Consider $f_\ast(\pi_1(\mathbb{RP}^2)) \subseteq \pi_1(S^1 \times S^1)$. Since the only finite subgroup of $\mathbb{Z}^2$ is 0, it follows that $f_\ast(\pi_1(\mathbb{RP}^2)) = 0$. Therefore $f_\ast(\pi_1(\mathbb{RP}^2)) \subseteq q_\ast\pi_1(\mathbb{R}^2)$

and so $f$ is a lift $\tilde{f}: \mathbb{RP}^2 \to \mathbb{R}^2$ of $f$. By the straight line homotopy $\tilde{f}_t = (1-t)\tilde{f}$, we note that $\tilde{f}$ is nullhomotopic.

Then $f_t = q_\ast\tilde{f}_t$ is a homotopy from $q_\ast(f_0) = q_\ast\tilde{f} = f$ to $q_\ast(f_1) = q_\ast(0)$ and so $f$ is nullhomotopic as desired.

(b) As calculated above, $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}$. Therefore $f$ is not nullhomotopic.

Define $Y: S^1 \to \mathbb{RP}^2$ that is not nullhomotopic.

Define $f: S^1 \times S^1 \to \mathbb{RP}^2: (x, y) \mapsto Y(x)$.

Then $f$ is continuous since $(x, y) \mapsto x$ and $Y$ are continuous and $f$ is not nullhomotopic since $Y$ is not nullhomotopic. \qed
(a) As given, \( W \) is constucted as
\[
\begin{align*}
1 \text{ 0-cell:} & \quad \mathcal{P} \\
1 \text{ 1-cell:} & \quad \mathcal{E} \quad \partial \mathcal{E} = \mathcal{P} - \mathcal{P} = 0 \\
2 \text{ 2-cells:} & \quad \mathcal{A}, \mathcal{B} \quad \partial_2 \mathcal{A} = 4 \mathcal{E}, \quad \partial_2 \mathcal{B} = 7 \mathcal{E}
\end{align*}
\]

This gives the chain complex
\[
\begin{align*}
0 \rightarrow & \quad \mathbb{Z} \langle \mathcal{A}, \mathcal{B} \rangle \xrightarrow{\partial_2} \mathbb{Z} \langle \mathcal{E} \rangle \xrightarrow{\partial_1} \mathbb{Z} \langle \mathcal{P} \rangle \rightarrow 0
\end{align*}
\]

and hence the homology groups
\[
\begin{align*}
H_0(W) &= \frac{\ker(\partial_1)}{\text{im} \partial_2} = \mathbb{Z} \langle \mathcal{P} \rangle / 0 = \mathbb{Z} \\
H_1(W) &= \frac{\ker(\partial_2)}{\text{im} \partial_3} = \mathbb{Z} \langle \mathcal{E} \rangle / \mathbb{Z} \langle 4 \mathcal{E}, 7 \mathcal{E} \rangle = \mathbb{Z} \langle \mathcal{E} \rangle / 0 = \mathbb{Z} \\
H_2(W) &= \frac{\ker(\partial_3)}{\text{im} \partial_4} = \mathbb{Z} \langle \mathcal{A}, \mathcal{B} \rangle / 0 = \mathbb{Z}
\end{align*}
\]

which is what was to be found.

(b) We claim that \( W \) is not homotopy equivalent to \( S^2 \).

To show this, it suffices to show that \( \pi_1(S^2) \neq \pi_1(W) \).

We recall that \( \pi_1(S^2) = 0 \). This can be computed explicitly by noting that for any closed loop \( \gamma \subset S^2 \), we may find \( p \in S^2 \) i.e. \( \pi(\gamma) \), i.e. a straight line homotopy of \( \pi(\gamma) \) to \( 0 \in \mathbb{R}^2 \), which pulls back to a homotopy from \( \gamma \) to a point. Therefore, every closed loop on \( S^2 \) is homotopic to a point and \( \pi_1(S^2) = 0 \).

To compute \( \pi_1(W) \), we recall that for any CW complex, the fundamental group has a presentation where the generators are the 1-cells, and the relations are the boundaries of the 2-cells. This yields
\[
\pi_1(W) = \langle \mathcal{E}, 4 \mathcal{E}, 7 \mathcal{E} \rangle = 0
\]
which doesn't help.
GeoTop
Spring 2014
(a) Let \( \psi \) be a smooth bump function s.t.
\[
\psi = 1 \text{ on } (-1,1) \text{ and } \psi = 0 \text{ outside } (-2,2), \text{ and } \psi; \mathbb{R}^2 \rightarrow [0,1]
\]
Define \( f; \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) by
\[
f(x) = (x(1-\psi(x)), 1x((1-\psi(x)))
\]
Then \( \forall x \in \mathbb{R}, f(x) \in \mathbb{T}^1 \) since \( |x(1-\psi(x))| = |1x(1-\psi(x))| \).
Further, since \( 1-\psi(x) \) is 0 near 0 and 1 away from 0,
\[
x(1-\psi(x)) \rightarrow 0 \text{ as } x \rightarrow 0
\]
\[
x(1-\psi(x)) \rightarrow \infty \text{ as } x \rightarrow \infty
\]
\[
x(1-\psi(x)) \rightarrow -\infty \text{ as } x \rightarrow -\infty
\]
The intermediate value theorem then implies that \( x(1-\psi(x)) \) is negative on \( \mathbb{R} \). Therefore \( \text{im} f = \mathbb{T}^1 \).
Finally, since \( x, \psi(x) \) are smooth and \( 1x1 \) is smooth except at 0,
it is shown that \( f \) is smooth at 0 suffices to show that \( f \) is smooth at 0. By construction, \( 1-\psi(x) = 0 \) on \((-1,1)\). Therefore \( f \) is identically 0 on a neighborhood of 0 and hence is smooth.

(b) No, \( f \) cannot be an immersion.

Heuristically, if \( f \) were an immersion then \( \mathbb{T}^1 \) would be a smooth submanifold of \( \mathbb{R}^2 \). However, because of the corner at \((0,0)\), \( \mathbb{T}^1 \) is not smooth.

Negate on the contrary that \( f \) is an immersion. Then \( \mathbb{T}^1 \) is a smooth submanifold of \( \mathbb{R}^2 \) with the usual smooth structure. Then \( f \) maps some subset of \( \mathbb{R}^2 \) onto \( \mathbb{T}^1 \) and hence has the smooth structure of \( \mathbb{R}^2 \). In the identity, the implies that \( f \) is a neighborhood of 0 and
Option 1:

Suppose that such an immersion exists. Then locally it is locally the canonical immersion, i.e. in coordinates $x_i$, $x_j$ s.t.

$$f(t) = (t, 0).$$

From looking locally at the origin, this would imply that $f$ a diffeomorphism from the a corner to a line, which is impossible.

Option 2:

By definition, any immersion $f$ would be of the form

$$f(t) = (g(t), 1|g(t)|)$$

WLOG suppose that $f(0) = (0, 0)$.

To be an immersion, immersion,

$$df_0 = g'(0)dx + 1g'(0)dy$$

must be injective.
By definition of a manifold with boundary, \( \forall p \in \partial W \) there is an open neighborhood \( U_p \) of \( p \) and a diffeomorphism \( \psi_p : U_p \rightarrow V_p \subset H \)\(^{36} \)

where \( H = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0 \} \) with \( n = \dim W \), i.e., \( \psi_p \) takes \( \partial W \) to \( \partial H = \{ (x_1, \ldots, x_{n-1}, 0) \mid x_i \in \mathbb{R} \} \).

Let \( U = W \setminus \partial W \). Then \( \{ U \cup U_p \} \) is an open cover of \( W \).

By paracompactness, we may extract \( \{ U \cup U_p \} \) as a countable locally finite cover \( \{ U, U_1, \ldots \} \). Let \( \{ \psi_1, \psi_2, \ldots \} \) be a partition of unity subordinate to \( \{ U, U_1, \ldots \} \).

Define \( \pi : \mathbb{R}^n \rightarrow \mathbb{R}^n \) by \( (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, 0) \). With this define \( F_i \) on \( U_i \) by

\[
(\psi_i) = f \circ \psi_i ^{-1} \circ \pi \circ \psi_i^{-1}
\]

Then for \( p \in U_i \cap \partial W \), \( \pi(\psi_i(p)) = \psi_i(p) \) so \( F_i = f \) on \( \partial W \cap U_i \).

Define \( F : W \rightarrow \mathbb{R}^n \) by

\[
F = \sum_1 \psi_i F_i = \sum_1 \psi_i f \circ \psi_i ^{-1} \circ \pi \circ \psi_i^{-1}
\]

By local finiteness, this sum is locally finite and so \( F \) is smooth.

Additionally, by construction \( \sum_1 \psi_i = 1 \) on \( \partial W \) since \( U \cup \partial W = \partial W \).

Therefore on \( \partial W \), \( F = \sum_1 \psi_i f = f \), as desired.

Alternatively, this can be done via the tubular neighborhood theorem. \( \square \)
Suppose that \( n \) is even. Then \( \text{id} \) has degree \( 1 \) and \( -\text{id} \) has degree \( -1 \). This can be seen either by remembering that \( \deg(-\text{id}) = (-1)^{n+1} \) on \( S^n \) or by noticing that by restricting \( \mathbb{S}^n \subset \mathbb{R}^{n+1} \), \( -\text{id} \) flips the coordinates and hence has degree \((-1)^{n+1}\).

Since degree is homotopy invariant, this implies that \( -\text{id} \) and \( \text{id} \) are homotopic to \( \text{id} \) if \( n \) is even.

Now suppose that \( n \) is odd. Then \( n = 2k-1 \) and we can view \( S^n \subset \mathbb{R}^{n+1} \). Consider the map \( H_t : S^n \to S^n : z \mapsto e^{\pi i t} z \).

Then \( H_0 = \text{id} \) and \( H_1 = -\text{id} \) and \( H_t \) is continuous.

Therefore \( H_t \) is a homotopy from \( \text{id} \) to \( -\text{id} \).  \( \square \)
(⇒) Suppose \( w_1, \ldots, w_k \) are linearly independent 1-forms on \( M \).

Then \( \forall p \in M, (w_i)_p, (w_k)_p \in T^*_p M \) are linearly independent.

Therefore \( \exists v_1, \ldots, v_k \in T_p M \) s.t. dual vectors \( v_i, v_k \in T_p M \) s.t.

\[
(w_i)_p(v_j) = \delta_{ij}.
\]

Then \( (w_1, \ldots, w_k)_p(v_1, \ldots, v_k) = 1 \neq 0. \)

Then \( w_1 \wedge \ldots \wedge w_k \neq 0. \)

(⇐) Suppose \( w_1, \ldots, w_k \) are linearly dependent. Then \( \exists \alpha_1, \ldots, \alpha_k \) s.t.

\[
w_k = \alpha_1 w_1 + \cdots + \alpha_k w_k.
\]

Then

\[
w_1 \wedge \ldots \wedge w_{k-1} \wedge w_k = w_1 \wedge \ldots \wedge w_{k-1} \wedge \left( \sum_{j \neq k} \alpha_j w_j \right)
\]

\[
= \sum_{j \neq k} \alpha_j w_1 \wedge \ldots \wedge w_{k-1} \wedge w_j \cdot w_j = 0,
\]

as desired. \qed
we recall that the Poincare dual of a submanifold \( W \subset M \) is a differential form \( \eta \) on \( M \) such \( \int_M \omega \wedge \eta = -\int_W \omega \) \( \forall \omega \).

Consider \( 0 \leq t \leq 1 \). We note that, viewing \( M \) as the unit square \([-1,1] \times [-1,1]\) and identifying opposite edges, \( S \) loops 7 times in the \( x \)-direction and 3 times in the \( y \)-direction before returning to the origin. We recall that \( \text{H}_1(M) \cong \mathbb{R}^2 \), and so all 1-forms \( \omega \) on \( M \) can be written as \( adx + bdy \). For \( a,b \in \mathbb{R} \), up to \( \mathbb{R} \)-addition,\[
\int_S \omega = \int_{S_1} adx + \int_{S_2} bdy = 7a + 3b
\]

Now let \( \eta = 7dy - 3dx \). Then \( \forall 1 \)-form \( \omega = adx + bdy \) on \( M \),\[
\int_M \omega \wedge \eta = \int_M (7adx \wedge dy + 3bdx \wedge dy) = 7a + 3b = \int_S \omega.
\]

and therefore \( \omega \wedge 7dy - 3dx \) is a Poincare dual of \( S \).
By Van Kampen's, we recall that $\pi_1(S'\times S') = \langle a, b \rangle \cong \mathbb{Z} \times \mathbb{Z}$.

Since $S'\times S'$ can be visualized as a graph with two vertices and two edges,

Viewing $S'\times S'$ as a graph, we see that all covering spaces of $S'\times S'$ consist of graphs where each vertex has an outgoing and ingoing $a$ and $b$ identified edges. Consider in particular the infinite-sheeted cover $\tilde{X}$.

For any vertex $p$ in $\tilde{X}$, it follows that $\tilde{\pi}_1(\tilde{X}, p) = \langle a \rangle \ast \pi_1(X, p)$.

Consider the subgroup $\langle a \rangle \ast \langle a, b \rangle$, since $bab^{-1} \notin \langle a \rangle$,

$\langle a \rangle$ is not a normal subgroup. Therefore $\tilde{X}$ is an irregular cover.
(a) The cell decomposition produced depends on the parity of \( n \).

Suppose that \( n \) is even. Then since we identify an even \# of pairs of sides, there is only one remaining vertex. The cellular decomposition is then

\[
\begin{align*}
0 & \text{- cell: } p \\
1 & \text{- cells: } e_1, \ldots, e_n \text{ w/ } \delta e_i = p - p - 0 \\
2 & \text{- cell: } f \text{ w/ } \delta f = e_1 + e_2 + \ldots + e_n - e_1 - \ldots - e_n
\end{align*}
\]

The cellular chain complex is then

\[
0 \to C_2 \cong \mathbb{Z} \xrightarrow{\partial_2} C_1 \cong \mathbb{Z}^n \xrightarrow{\partial_1} C_0 \cong \mathbb{Z} \to 0
\]

Suppose instead that \( n \) is odd. Then there are 2 remaining vertices after identification. Therefore we have the cellular decomposition

\[
\begin{align*}
2 & \text{- cells: } p, q \\
1 & \text{- cells: } e_1, \ldots, e_n \text{ w/ } \delta e_i = (-1)^i (q - p) \\
2 & \text{- cell: } f \text{ w/ } \delta f = e_1 + \ldots + e_n - e_1 - \ldots - e_n
\end{align*}
\]

The cellular chain complex is then

\[
0 \to C_2 \cong \mathbb{Z} \xrightarrow{\partial_2} C_1 \cong \mathbb{Z}^n \xrightarrow{\partial_1} C_0 \cong \mathbb{Z}^2 \to 0
\]

(b) By construction, every 0-cell in \( X_n \) has an even \# of added attachments, so there are no shrinking.
(b) Consider a point \( x \in X_n \). We aim to show that there is an open neighborhood \( U \) of \( x \) which is diffeomorphic to a subset of \( \mathbb{R}^2 \). Let \( x \notin f \backslash A \). Then this diffeomorphism is close. Therefore, suppose that \( x \in e_i \) or \( x \) is a vertex.

In order to show that \( x \in e_i \) for some \( i \). By construction, part of \( Df \) is identified at \( e_i \) and part of \( Df \) is identified at \( -e_i \).

Therefore at \( x \), \( X_n \) locally looks like

\[
\begin{array}{c}
\text{even } n \\
\begin{array}{c}
e_i \\

\vdots \\
\end{array} \\
\end{array}
\]

from which it is clear that a neighborhood of \( x \) is diffeomorphic to an open subset of \( \mathbb{R}^2 \).

Similarly, if \( x \) is a vertex, \( X_n \) locally looks like

\[
\begin{array}{c}
\text{odd } n \\
\begin{array}{c}
e_i \\

\vdots \\
\end{array} \\
\end{array}
\]

in all cases, thus is a neighborhood diffeomorphic to a subset of \( \mathbb{R}^2 \).

Additionally, it follows from the diagrams that these transition maps are smooth. Therefore \( X_n \) is a smooth 2-manifold w/o boundary.

We also note that \( X_n \) is compact as the genus quaternion of a compact graph.

Therefore, if \( g \) is the genus of \( X_n \),

\[
2 - 2g = \chi(X_n) = \begin{cases} 1 - n + 1 & \text{n even} \\ 2 - n + 1 & \text{n odd} \end{cases} = \begin{cases} 2 & \text{n odd, } n \text{ odd} \\ 1 & \text{otherwise} \end{cases}
\]

\[
g = \begin{cases} \frac{n-1}{2} & \text{n odd} \\ \frac{n}{2} & \text{n even} \end{cases}
\]
(a) We recall that $\mathbb{RP}^3$ can be constructed as a 3-dimensional ball $B^3$ attached to $\mathbb{RP}^2$ via the map $\delta: \partial B^3 \rightarrow \mathbb{RP}^2$ given by $([x_0 : x_1 : x_2]) \mapsto ([x_0 : x_1 : x_2])$. This implies that $\mathbb{RP}^3 \setminus \mathbb{RP}^2$ can be constructed as an open 3-ball $U$, i.e., $\mathbb{RP}^3 \setminus \mathbb{RP}^2$ can be constructed as an open 3-dimensional annulus $\mathbb{A} = \mathbb{B} \setminus \mathbb{B}$ inner boundary attached to $\mathbb{RP}^2$ as done above. Since $S^2 \times [0, 1/2]$ is diffeomorphic to a 3-dimensional annulus and $\mathbb{RP}^2 = S^2 / \mathbb{Z}_2$, this implies that $\mathbb{RP}^3 \setminus \mathbb{RP}^2 = S^2 \times [0, 1/2] / \{(x_0, 0) \sim (-x_0, 0)\}$.

Similarly, $\mathbb{RP}^3 \setminus \mathbb{RP}^2 = S^2 \times [1/2, 1] / \{(x_1, 1) \sim (-x_1, 1)\}$.

Thus, $\mathbb{RP}^3 \# \mathbb{RP}^3 = (\mathbb{RP}^3 \setminus \mathbb{RP}^2) \cup (\mathbb{RP}^3 \setminus \mathbb{RP}^2) / \sim$, where $\sim$ identifies the boundaries of the two $\mathbb{RP}^3 \setminus \mathbb{RP}^2$'s.

Alternatively, $\mathbb{RP}^3 \setminus \mathbb{S}^3 = \mathbb{RP}^2 \cong \mathbb{RP}^3 \# \mathbb{RP}^3 = \mathbb{RP}^2 \sqcup \mathbb{RP}^2$.

(b) We recall that $S^2$ is a double cover of $\mathbb{RP}^2$ via a covering map given by the quotient map $\pi: S^2 \rightarrow S^2 / \mathbb{Z}_2 \cong \mathbb{RP}^2$.

Heuristically, $S^2$ can be seen as $\mathbb{RP}^2$ can be regarded as half sphere since $\mathbb{RP}^3 \# \mathbb{RP}^3$ is homeomorphic to $S^2 \times Y$, it suffices to show $S^2 \times S^1$ is a double cover of $Y$. We view $S^2 \times S^1$ as $S^2 \times [0, 1/2] / \{(x_0, 0) \sim (-x_0, 0)\}$. Then define $P: S^2 \times S^1 \rightarrow Y$ by

\[
\begin{align*}
S^2 \times \{0, 1\} &\rightarrow \mathbb{RP}^2 \times \{0, 1\} \subset Y \text{ via the 2-cover } \pi: S^2 \rightarrow \mathbb{RP}^2 \\
S^2 \times \{0, 1\} &\rightarrow S^2 \cup \{0, 1\} \subset Y \text{ via the identity (1-cover)} \\
S^2 \times S^1 &\rightarrow \mathbb{RP}^2 \times S^1 \subset Y \text{ via the 2-cover } \pi: S^1 \rightarrow \mathbb{RP}^2
\end{align*}
\]
we note this map is constructed locally as an identity and each point in \( Y \) has \( 2 \) pre-images. Therefore \( p : S^1 \times S^1 \to Y \) is a double cover as claimed.
we proceed by Mayer–Vietoris.

Let $U = X \setminus \{p_1\}$ and $V = X \setminus \{p_2\}$ where $\sim$ is the contraction of $X \setminus \{p_3\}$ and $X \setminus \{p_1\}$ to points. Then $U, V$ are both contractible, $UV$ deformation retracts to $X$ via the straight line homotopy along $\{p_1\}$, and $UV = S(X)$.

Thus yields the LES

\[ \cdots \to H_k(\{(p_1)\}) \to H_k(U) \oplus H_k(V) \to H_k(S(X)) \to \cdots \]

\[ H_k(X) \quad H_k(spt) \]

we note that by construction, $S(X)$ is connected so $H_0(S(X)) \cong \mathbb{Z}$.

For $k > 2$ some $H_k(spt) = 0$ and $k \neq 0$, the above LES yields a SES

\[ 0 \to H_k(spt) \to H_k(X) \to 0 \]

and $H_k(S(X)) \cong H_{k-1}(X) \forall k > 2$. Finally, for $k = 1$, we have the SES

\[ 0 \to H_1(spt) \to H_1(X) \to (H_0(spt))' \to H_0(S(X)) \to 0 \]

letting $r$ denote the \# of connected components of $X$, this yields

\[ 0 \to H_1(spt) \to \mathbb{Z}^r \to \mathbb{Z}^2 \to \mathbb{Z} \to 0 \]

Taking the alternating sum then implies that

\[ \text{rank } H_1(S(X)) = r - 2 + 1 = r - 1 \]

and $H_1(S(X)) \cong \mathbb{Z}^{r-1}$.

Summarizing,

\[ H_k(S(X)) \cong \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}^{r-1} & k = 1 \\ H_{k-1}(X) & k > 2 \end{cases} \]
GEO Top
Fall 2013
(a) No. $f$ need not be injective or surjective.

Consider $f: \mathbb{C} \to \mathbb{C}: z \mapsto e^z$. Then $df = e^z dz$ which is nonvanishing and hence nowhere nonsingular everywhere. However, if $0 \Rightarrow f$ is not injective and $f(0) = f(2\pi i) = 1$, $f$ is not surjective.

(b) Yes. This is the classic example of a closed theorem, which we repeat below.

(c) Yes.

Proof: Suppose that $U \subseteq M$ is open. Since $f$ is nonsingular, and $\dim_{\mathbb{R}} Y = \dim_{\mathbb{R}} M$.

A open $U$ is a neighborhood of $f(U)$, i.e., $f(U) \subseteq U$.

Let $f$ be a diffeomorphism by the inverse function theorem.

Therefore $f(U) \subseteq f(U)$ is open and $f(p) \in f(U) \subseteq f(U)$.

Then $V = \overline{f(U)}$ is an open neighborhood of $f(U)$.

If $p \in V = \overline{f(U)}$. Therefore $f(U)$ is open in open $U$.

and so $f$ is open.

(d) I was going to say yes, but then I fact.

No. Consider $f: (0,1) \to (0,1): f(x) = \frac{x}{1-x}$.

Then $df = \text{id} \not\Rightarrow f$ is nonsingular. However, $(0,1)$ is closed in itself but not in $(0,2)$, so $f$ is not a closed map.

No. Consider $f: (0,1) \to S^1 = \mathbb{C}/\mathbb{Z}$, $f(x) = e^{2\pi i x}$.

Then $S^1$ is compact, $df = \text{id}$, so $f$ is nonsingular, but $f$ is not injective and hence not a covering map.

No.
Sard's theorem solution

Suppose for the sake of contradiction that a situation \( p : M \to \mathbb{M} \) exists. Then by Sard's theorem, \( p \) has a regular value \( p_0 \in \mathbb{M} \).

Since \( p_0 \) is a regular value, \( r^{-1}(p)_0 \mathbb{M} \) is a codimension \( \dim \mathbb{M} - 1 \) submanifold and hence a 1-dimensional manifold.

Since \( r_0 \mathbb{M} \) is closed, \( r^{-1}(p)_0 \mathbb{M} \) is closed. Moreover since \( \mathbb{M} \) is compact, \( r^{-1}(p)_0 \mathbb{M} \) is compact. By the only compact 1-dimensional manifold are disjoint unions of \( S^1 \) and finite segments, it follows that

\[ \partial r^{-1}(p)_0 \mathbb{M} \] has even cardinality. Hence, since \( \partial r = r|_{\partial \mathbb{M}} = \text{id} \),

\[ \partial r^{-1}(p)_0 \mathbb{M} = \partial p_0 \mathbb{M} \] which is odd. Therefore our supposition led to a contradiction and as such situation exists.

Homology proof

Suppose \( i \) a retract \( r : M \to \mathbb{M} \).

The relative homology one \( \mathbb{Z}/2 \mathbb{Z} \) then gives a LES

\[ 0 \to H_n(\mathbb{M}; \mathbb{Z}/2 \mathbb{Z}) \to H_n(M; \mathbb{Z}/2 \mathbb{Z}) \to H_n(M, \mathbb{M}; \mathbb{Z}/2 \mathbb{Z}) \to H_{n-1}(\mathbb{M}; \mathbb{Z}/2 \mathbb{Z}) \to ... \]

\[ H_0(\mathbb{M}, \mathbb{M}; \mathbb{Z}/2 \mathbb{Z}) \]

Hence \( M \) is compact and orientable over \( \mathbb{Z}/2 \mathbb{Z} \), Poincaré duality implies

\[ H_n(M; \mathbb{Z}/2 \mathbb{Z}) \cong H_0(M, \mathbb{M}; \mathbb{Z}/2 \mathbb{Z}) = 0 \].

Therefore we have the exact sequence

\[ 0 \to H_n(M, \mathbb{M}; \mathbb{Z}/2 \mathbb{Z}) \to H_{n-1}(\mathbb{M}; \mathbb{Z}/2 \mathbb{Z}) \to H_n(M; \mathbb{Z}/2 \mathbb{Z}) \to ... \]

Hence we have a situation \( M \to \mathbb{M} \), \( i \) is a retraction. Therefore

\[ H_n(M, \mathbb{M}; \mathbb{Z}/2 \mathbb{Z}) = 0. \] However, by Poincaré duality,

\[ H_n(M, \mathbb{M}; \mathbb{Z}/2 \mathbb{Z}) = H_0(\mathbb{M}; \mathbb{Z}/2 \mathbb{Z}) \neq 0, \]

which is a contradiction.
(a) Define

\[ \mu: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{S}^p: (x, y) \mapsto \frac{y - x}{|y - x|} \]

and note that \( \lambda (N, M) = \deg \mu \) by construction.

Let \( T: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n: (x, y) \mapsto (y, x) \) be the swapping map

and \( \phi: \mathbb{S}^p \to \mathbb{S}^p: x \mapsto -x \) be the anti-podal map. Then

\[ \mu = \phi \circ \lambda \circ T: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{S}^p \to \mathbb{S}^p \]

Therefore

\[ \lambda (N, M) = \deg \mu = \deg (\phi) \deg (\lambda) \deg (T) = \deg (\phi) \deg (T) \lambda (M, N). \]

We note that the anti-podal map \( \phi: \mathbb{S}^p \to \mathbb{S}^p \) has

\[ \deg (\phi) = (-1)^{p+1} \]

since \( p+1 \) coordinates are being inverted.

To compute \( \deg T \), we note that locally,

\[ T: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^n: (x_1, \ldots, x_n, y_1, \ldots, y_m) \mapsto (y_1, \ldots, y_m, x_1, \ldots, x_n) \]

which is a proven that requires \( nm \) adjacent index changes.

Therefore \( \deg T = (-1)^{nm} \) and

\[ \lambda (N, M) = (-1)^{p+1 + nm} \lambda (M, N) = (-1)^{m+1 + m+1} \lambda (M, N) \]

as desired.

(b) We additionally assume \( N \) is boundaryless.

We extend \( \lambda \) to \( \tilde{\lambda}: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{S}^p: (x, y) \mapsto \frac{y - x}{|y - x|} \) since \( \mathbb{S}^p \cap N = \emptyset \).

Moreover, \( \tilde{\lambda} (N \times N) = \partial (N \times N) \cap \mathbb{S}^p \neq \emptyset \).

Therefore \( \mathbb{M} \times \mathbb{N} \) is the boundary of a manifold \( \mathbb{M} \times \mathbb{N} \) and \( \lambda \) is a map on the boundary which can be extended to the whole manifold.

The extension theorem then implies \( \tilde{\lambda}(M \times N) = \deg = 0 \). \( \square \)
For some p.e.M. Since $M$ is connected, $\forall x \in M \exists$ a path $\gamma$.

(\Rightarrow) Suppose that $w$ is exact. Let $I$ be a piecewise smooth curve $c: S^1 \to M$. If $I$ is constant, then $\int_c w = 0$ trivially.

So we assume $c$ is not constant.

Let $0 = t_0 < t_1 < \ldots$ be a partition of $[0, 1]$ with $S^1 = [0, 1] / \sim$.

Since $c$ is smooth on $[t_n, t_{n+1}] \forall n$. Then $c^{-1}([t_n, t_{n+1}])$ is a smooth submanifold of $M \forall N$. Stokes' theorem then implies

$$\int_c w = \int_{c([t_n, t_{n+1}])} df = \sum_n \int_{\partial c([t_n, t_{n+1}])} = \sum_n f(c([t_{n+1}])) - f(c([t_n]))$$

This means

$$\int_c w = f(c(1)) - f(c(0)) = 0$$

as desired.

(\Leftarrow) Suppose instead that $\int_c w = 0 \forall$ piecewise smooth $c$.

For some p.e.M. Since $M$ is connected, $\forall x \in M \exists$ a smooth path $\gamma(x)$ from $p$ to $x$. Define $f: M \to \mathbb{R}$ by

$$f(x) = \int_{\gamma(x)} w$$

We first claim that $f$ is independent of $\gamma(x)$.

Suppose there are two smooth paths $\gamma_1, \gamma_2$ from $p$ to $x$.

Then $\gamma_1 - \gamma_2$ is a piecewise smooth closed path, and so

$$\int_{\gamma_1} w - \int_{\gamma_2} w = \int_{\gamma_1 - \gamma_2} w = 0 \Rightarrow \int_{\gamma_1} w = \int_{\gamma_2} w$$

Therefore $f$ is independent of the choice of path, and $w$ is well-defined.

By $w$ is smooth, it follows that $f$ is smooth.

We claim that $df = w$. To show this, it suffices to show

$$df(v) = w_v(v) \forall \Phi \in \mathfrak{g} M$$

and $v \in T_x M$. 


It suffices to work locally. Let \((x_1, \ldots, x_n)\) be coordinates at \(q \in M\) on \(U\), and let \(V\) be the other chart. \(v \in \mathcal{C}^1, q = (0, \ldots, 0)\).

Then \(w\) can be written

\[
w = \sum_{i=1}^{n} q_i dx_i.
\]

For \(V \in T_q M\), by definition, \(V_i, V_j\),

\[
\frac{\partial f}{\partial x_i} = \lim_{t \to 0} \frac{f(q + tx_j) - f(q)}{t}
\]

\[
= \lim_{t \to 0} \frac{1}{t} \int_{[q, q + tx_j]} w.
\]

Since we are working locally at \(q\), we can regard \(w\) as an integration on \(\mathbb{R}^n\). Then

\[
\frac{\partial f}{\partial x_i} = \lim_{t \to 0} \frac{1}{t} \int_{0}^{t} q_i(s) ds = q_i(0) = q_i(q).
\]

Therefore \(df = w\) locally and hence \(df\) is globally. \(\square\)
(a) Suppose $\omega$ is an integrable distribution. Then locally $\exists$ coordinate $x,y,z$ s.t. $\omega = \mathbb{R} \langle \partial / \partial x, \partial / \partial y \rangle$. This implies that $\omega = df$ and $\omega$ is closed: $\omega \wedge d\omega = f d\omega \wedge df = 0$ as desired.

(b) Locally, $\omega = -ydx + xdy + dz$. Then $d\omega = -dy \wedge dx + dx \wedge dy = 2dx \wedge dy$.

$\omega \wedge d\omega = (-ydx + xdy + dz) \wedge (2dx \wedge dy) = 2dx \wedge dy \wedge dz$

so $\omega$ is smooth and non-vanishing but $\omega \wedge d\omega = 0$.

Therefore $\ker \omega$ is a 2-dimensional (codimension 1) co-distribution that is not integrable.

We now return to part (a).

(\Rightarrow) Suppose $\omega \wedge d\omega = 0$. Let $\xi, \eta$ be vector fields $X, Y \in \ker \omega$.

We aim to show $[X, Y] \in \ker \omega$. By direct computation:

$\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$

$b) X, Y \in \ker \omega$. Choose $Z \notin \ker \omega$. Then $\omega = \omega \wedge d\omega$.

$\Rightarrow \omega = \omega \wedge d\omega (X, Y, Z)$

$= -\omega(Z) d\omega(X, Y)$

$= +\omega(Z) \omega(X, Y)$

locally

since $\omega$ is non-vanishing, this implies $\omega([X, Y]) = 0$ and $[X, Y] \notin \ker \omega$ as desired. $\square$
(a) In the usual way, we define the gradient of \( f \) as

\[
\nabla f = \frac{\partial f}{\partial x_1} \, dx_1 + \cdots + \frac{\partial f}{\partial x_n} \, dx_n
\]

We note that this is dual to

\[
d f = \frac{\partial f}{\partial x_1} \, dx_1 + \cdots + \frac{\partial f}{\partial x_n} \, dx_n
\]

(b) We recall that the Hessian of \( f \) is given as

\[
H_f = \begin{bmatrix}
\frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\
\frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2}
\end{bmatrix}
\]

We note that since \( f \) is smooth, \( H_f \) is symmetric.

We thus define \( \text{Hess}(f) \) to be the \((n,2)\) symmetric \((n,n)\) tensor

\[
\text{Hess}(f)(x,y) = x^T H_f y^T
\]

or equivalently

\[
\text{Hess}(f) = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} \, dx_i \otimes dx_j
\]

(c) We write \( g = \sum_{i=1}^n dx_i \otimes dx_i \) as annilated matrix \( I \). Then by linearity and

\[
L_{dx_i} g = \sum_{i=1}^n L_{dx_i} (dx_i \otimes dx_i)
\]

By Cauchy's formula,

\[
L_{dx_i} dx_i = (d \circ \pi_i + \pi_i \circ d) \, dx_i = d \circ i_{dx_i} \, dx_i = d \frac{\partial f}{\partial x_i}
\]

\[
\sum_{i=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} \, dx_i \otimes dx_j
\]

Then

\[
L_{dx_i} g = \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} \, (dx_i \otimes dx_i + dx_i \otimes dx_j)
\]

which is what we set out to show.
Viewing $T^2$ as $[0,1] \times [0,1] / \sim$ where $(x,0) \sim (x,1)$ and $(0,y) \sim (1,y)$, we find that $T^2 - D^2$ deformation retracts onto $S'VS'$.

We find that $S'VS'$ consists of one vertex and 2 edges. The $3$-fold covers of $S'VS'$ therefore consist of connected graphs with 3 vertices and 6 edges. Running through these, we find

and all relevant permutations of rules and orientation.

There are all possible covers since every vertex can either have one edge connected to itself and two edges connected to other vertices or 4 edges connected to other vertices. Running through all possible combinations, we find three $3$-fold options, 4 permutations.

The $3$-fold covers of $T^2 - D^2$ are therefore all surfaces which deformation retract onto these graphs.
Suppose that $A$ has finite presentation

$$A = \langle a_1, \ldots, a_k \mid b_1, \ldots, b_k \rangle$$

Define a $\mathbb{C}W^X$ complex as follows.

1. 0-cell: $p$
2. $k$ $n$-cells: $a_1, \ldots, a_k \cup \partial^n a_i \cup \partial^n a_i$ attached via the constant map $p$
3. $l$ $n+1$-cells: $c_1, \ldots, c_l \cup \partial^{n+1} c_i = b_i$

Then

$$H_n(X) = \frac{\ker \partial_n}{\text{im} \partial_{n+1}} = \frac{\langle a_1, \ldots, a_k \rangle}{\langle b_1, \ldots, b_k \rangle} = A$$

as claimed.

Alternatively, consider the bouquet of $k$ $n$-spheres

$$\bigvee_{j=1}^k S^n$$

with the $j$th $S^n$ being attached $\cup b_j$.

Attach $l$ $n+1$-cells to $\bigvee_{j=1}^k S^n$ via $b_1, \ldots, b_l$. Then as above,

$$H_n(X) = \langle a_1, \ldots, a_k \mid b_1, \ldots, b_l \rangle = A$$

as claimed. \qed
Fix some \( p \in \mathbb{H} \). By identifying \( p \) as the north pole, we see that \( S^3 \setminus \mathbb{H} \) is equivalent to \( \mathbb{R}^3 \setminus \{ \text{axis of unknot in xy} \} \) via a stereographic projection.

We claim that this deformation retracts onto \( T^2 \).

Let \( z \) denote the \( z \)-axis and let \( C \) be the unit circle in the \( xy \) plane.

Then \( A \times C = \mathbb{R}^3 \setminus (z \cup C) \) is homeomorphic to \( \mathbb{R}^3 \setminus (z \cup C) \).

Define \( f_t : \mathbb{R}^3 \setminus (z \cup C) \to \mathbb{R}^3 \setminus (z \cup C) \) by

\[
f_t(x) = (1-t)x + t(y + \frac{x-y}{2|x-y|})
\]

By definition \( f_t \) is continuous for all \( t \), \( f_0 = \text{id}, \ldots, f_1 = \text{id} \) is the torus around \( C \) of radius \( 1/2 \), and \( f_1 = \text{id} \). \( f_1 \) is the torus.

Therefore \( \mathbb{R}^3 \setminus (z \cup C) \) deformation retracts onto \( T^2 \).

Combining these, it suffices to compute the fundamental group and homology of \( \mathbb{R}^3 \setminus (z \cup C) \). We recall that

\[
\pi_1(T^2) \cong \mathbb{Z}^2 \quad \text{and} \quad H_n(T^2) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z}^2 & n = 1 \\ 0 & \text{else} \end{cases}
\]

which is what we set to be shown.
We mince the CW construction of $\mathbb{HP}^n$ or $\mathbb{CP}^n$ to construct $\mathbb{HP}^n = \bigoplus_{k=0}^n V_k$ for all $k = 0, \ldots, n$.

To do so, we construct $\mathbb{HP}^{n+1}$ from $\mathbb{HP}^n$ structurally.

We start with $\mathbb{HP}^0$. We note that $\mathbb{HP}^0 = (\mathbb{H} - \{0\}) \cap (\mathbb{H} - \{0\}) \cong \mathbb{S}^0$ and so $\mathbb{HP}^0$ has the structure of a single point.

Suppose that we have constructed $\mathbb{HP}^n$ as described. We claim that we can construct $\mathbb{HP}^{n+1}$ by attaching a $U(n+1)$-cell to $\mathbb{HP}^n$.

For consistency in notation, we denote elements of $\mathbb{HP}^n$ by

$$\begin{bmatrix} k_0 : \ldots : k_n \end{bmatrix}$$

for $(k_0, \ldots, k_n) \in \mathbb{HP}^n \setminus \{0\}$.

We first claim that $\mathbb{HP}^n \to \mathbb{HP}^{n+1}$ via the map

$$\begin{bmatrix} k_0 : \ldots : k_n \end{bmatrix} \mapsto \begin{bmatrix} k_0 : \ldots : k_n : 0 \end{bmatrix}$$

where the smoothness is obvious.

Let $U = \{ (k_0 : \ldots : k_n : 0) \in \mathbb{HP}^{n+1} \}$. We claim that $\mathbb{HP}^{n+1} \setminus U \cong B^{U(n+1)}$.

Consider some $\begin{bmatrix} k_0 : \ldots : k_{n+1} \end{bmatrix} \in \mathbb{HP}^{n+1} \setminus U$. By scaling, we may assume that $|k_0| + \cdots + |k_{n+1}| = 1$. Then in particular, some $k_{n+1} \neq 0$.

$$\begin{bmatrix} k_0 : \ldots : k_{n+1} \end{bmatrix} = \begin{bmatrix} k_0 : \ldots : k_n : \sqrt{1 - \sum_{i=0}^{n} k_i^2} a_{n+1} \end{bmatrix}$$

Therefore $\mathbb{HP}^{n+1} \setminus U \cong B^{U(n+1)}$ via the map

$$\begin{bmatrix} k_0 : \ldots : k_n : \sqrt{1 - \sum_{i=0}^{n} k_i^2} a_{n+1} \end{bmatrix} \mapsto (k_0, \ldots, k_n) \in B^{U(n+1)}$$

where we note that $\sum_{i=0}^{n} k_i^2 < 1$ since $|k_{n+1}| > 0$ and $k_0$ can be viewed as an element of $\mathbb{R}^n$ in the obvious way.

Finally, we observe that $\partial(\mathbb{HP}^{n+1} \setminus U) = U \cong \mathbb{HP}^n$.

Therefore, we can construct $\mathbb{HP}^{n+1}$ as $B^{U(n+1)}$ attached to $\mathbb{HP}^n$ via the map $\partial B^{U(n+1)} \cong S^{U(n+1)} \to \mathbb{HP}^n : (k_0, \ldots, k_n) \to \begin{bmatrix} k_0 : \ldots : k_n \end{bmatrix}$

with $S^{U(n+1)} \cong (\mathbb{R}^n)$.
Iterating this construction we find that $H^{n}\cap P^n$ consists of \( 1 \cdot u_k \) all for each \( k = 0, \ldots, n \). In particular, this implies

\[
H_k(H^{n}\cap P^n) = \bigcap_{k = 0,1,\ldots, n} u_k
\]

which is what was to be found.
Algebraic Topology Prep
From Yan's Notes
If $n > 0$ then $\mathbb{R}^n$ is compact $\Rightarrow \mathbb{R}^m$ is compact $\Rightarrow m = 0$ or $n$.

We recall that the one-point compactification of $\mathbb{R}^n$ is diffeomorphic to $S^n$.

Suppose $f$ a homomorphism $f: \mathbb{R}^n \to \mathbb{R}^m$. Then by the one point compactification, $f$ extends to a homomorphism $\tilde{f}: S^n \to S^m$.

In particular, $\tilde{f}$ induces an isomorphism $\tilde{f}^*: H_n(S^n) \to H_n(S^m)$.

We recall that

$$H_n(S^k) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & \text{else} \end{cases}$$

Therefore $\tilde{f}^*: \mathbb{Z} \to H_n(S^m)$ is an isomorphism, so $m = n$. \qed
we note that $M$ is the Mobius strip and $X$ is the Klein bottle.

To calculate $\pi_1(X)$, we contract $X$ with a CW complex.

We proceed by Van Kampen.

First, we note that $M$ deformation retracts onto $S^1$ via the straight line homotopy $(x,y) \mapsto (x, (1-t)y, t_2)$.

Therefore $\pi_1(M) = \pi_1(S^1) \cong \mathbb{Z}$.

Now let $U$ be an open neighborhood of $M \times 0$3 that deformation retracts onto $M \times 0$3 and similarly let $V$ be an open neighborhood of $M \times 13$. Then $\pi_1(U) \cong \pi_1(U) \cong \pi_1(M) \cong \mathbb{Z}$, $U \cap V = X$, and $UV$ deformation retracts onto $2M$.

Let $i_U: UV \to U$ and $i_V: UV \to V$ be the inclusion maps.

Then Van Kampen theorem implies that

$$\pi_1(X) \cong \pi_1(U) \ast \pi_1(V)/N,$$

where $N$ is generated by $i_u^*([Y])i_v^*([Y])$ for $Y \in \pi_1(UV)$.

As found, $\pi_1(U) \cong \mathbb{Z}$, so let $a$ be the generator of $\pi_1(U)$.

Then $UV$ deformation retracts onto $2M$, any element of $\pi_1(UV)$ is homotopic to a product of $a$.

Then $i_u^*([a]) = a$ and $i_v^*([a]) = b^2$, since $2M$ maps twice around $M$. Therefore, by Van Kampen,

$$\pi_1(X) \cong \langle a, b \mid a^2b^2 \rangle$$
we first calculate $H_n(\mathcal{M})$ in terms of $H_n(M)$.

Let $M_1, M_2$ be copies of $M$. Define $\tilde{M}$ by

$$\tilde{M} = M_1 \sqcup \delta M \times \{0,1\} \sqcup M_2 / \nu$$

where $\nu = (y,0) \cup (y,1)$ and $y \in \partial M_1$. Let $U = M_1 \sqcup \delta M \times \{0,1\} / \nu$ and $V = \delta M \times \{0,1\} \sqcup M \times \nu$. Then $U, V$ are open, deformation retract onto $M_1, M_2$ respectively, $U \cap V = \tilde{M}$ and $U \cap V$ deformation retracts onto $\partial M$.

Mayer-Vietoris then yields a LES

$$\cdots \to H_k(U \cap V) \to H_k(U) \oplus H_k(V) \to H_k(U \cap V) \to \cdots$$

which is equivalent to the LES

$$\cdots \to H_k(\partial M) \to H_k(M)^{\oplus 2} \to H_k(\tilde{M}) \to \cdots$$

Taking an alternating sum then yields

$$C = \sum_{k=0}^{n} (-1)^k \left( \text{rank} \ H_k(\partial M) - \text{rank} \ H_k(M) + \text{rank} \ H_k(\tilde{M}) \right)$$

$$= \chi(\partial M) - 2 \chi(M) + \chi(\tilde{M})$$

It then remains to show that $\chi(\tilde{M}) = 0$. By construction, $\tilde{M}$ is a compact odd-dimensional submanifold with boundary.

By Poincaré duality and the universal coefficient theorem,

$$H_k(\tilde{M}; \mathbb{Z}/2) \cong H^{n-k}(\tilde{M}; \mathbb{Z}/2)$$

But $\tilde{M}$ is orientable, over $\mathbb{Z}/2$, this implies

$$\chi(\tilde{M}) = \sum_{k=0}^{n} (-1)^k \text{rank} \ H_k(\tilde{M}) = \sum_{k=0}^{n} (-1)^{n-k} \text{rank} \ H^{n-k}(\tilde{M}) = -\chi(M)$$

and so $\chi(\tilde{M}) = 0$.

Then $\chi(\partial M) = 2 \chi(M)$

as desired.
(a) In the standard way, we may visualize the punctured torus as

From this, it can be seen that a radially straight line homology outward from \( p \) with deformation retract \( T^2 \backslash \{ p \} \) onto \( S^1 \cup S^1 \).

Then by de Rham's theorem,

\[
H^k_{\text{dR}}(T^2 \backslash \{ p \}) = H_k(T^2 \backslash \{ p \}; \mathbb{R}) = \begin{cases} 
\mathbb{R}^2 & k = 1 \\
\mathbb{R} & k = 0 \\
0 & \text{else}
\end{cases}
\]

(b) By part (a), \( H^k_{\text{dR}}(T^2 \backslash \{ p \}) \cong 0 \). Therefore since d\omega is closed, it is exact.

\[\square\]
We recall that the fundamental group of a CW complex $X$

1. 0-cell, $k$ 1-cells $e_1, \ldots, e_k$, and $l$ 2-cells attached via $a_1, \ldots, a_l$

has fundamental group $\langle e_1, \ldots, e_k, a_1, \ldots, a_l \rangle$.

Therefore, we construct $X$ as

1. 0-cell: $p$
2. 1-cells: $a, b$ with $da = 0b - p - p = 0$
3. 2-cells: $f, g, h$ with $df = ma, dg = nb, Dh = a \cdot b - a \cdot b$

Then $\pi_1(X) = \langle a, b | a^m, b^n \rangle = \mathbb{Z}/m \times \mathbb{Z}/n$

Similarly, we construct $Y$ as

1. 0-cell: $p$
2. 1-cells: $a, b$ with $da = 0b - p - p = 0$
3. 2-cells: $f, g, h$ with $df = ma, dg = nb, Dh = a \cdot b - a \cdot b$

Then $\pi_1(Y) = \langle a, b | a^m, b^n, aba^{-1}b^{-1} \rangle = \mathbb{Z}/m \times \mathbb{Z}/n$

As desired.
\[ \mathcal{P}_{\kappa_1} (\frac{5}{\kappa_1}, s') \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} . \]
If \( n \) is even, this rotates degree.

If \( n \) is odd, construct homotopy via \( \frac{n+1}{2} \) rotations.

19F.8

(a) Universal cover \( S^1 \times S^1 \) in \( \mathbb{R}^2 \) which is simply connected.

Any map \( \text{TRP}^2 \rightarrow S^1 \times S^1 \) maps \( \pi_1 (\text{TRP}^2) \) to a finite subgroup of \( \mathbb{Z}^2 \rightarrow 0 \) subgroup.

Therefore any map lifts. These construct a homotopy in \( \mathbb{R}^2 \) the desired CED.

(b) \( \pi_1 (\text{TRP}^2) \cong \mathbb{Z}/2 \mathbb{Z} \) is \( 3 \) a non-nullhomotopic path \( Y : S^1 \rightarrow \text{TRP}^2 \).

Define \( f : S^1 \times S^1 \rightarrow \text{TRP}^2 : (\theta, \phi) \mapsto Y(\theta) \).

19F.7

Repeat \( \text{TRP}^2 \rightarrow S^1 \times S^1 \) argument.
As given, we construct $X$ as follows:

1. 0-cells: $P$
2. 2-cells: $a+b$ with $x_1 = x_2 = P$
3. 3-cells: $D$ with $D_3$ attached via $f$

Thus gives the chain complex

$$0 \rightarrow C_3(X) \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \rightarrow 0$$

By definition, this yields the homology groups

$$H_0(X) = \frac{ker \partial_0}{im \partial_1} = \mathbb{Z}<P> \cong \mathbb{Z}$$

$$H_1(X) = \frac{ker \partial_1}{im \partial_2} = 0$$

$$H_2(X) = \frac{ker \partial_2}{im \partial_3} = \mathbb{Z}<a,b> \cong \frac{\mathbb{Z} \times \mathbb{Z}}{gcd(d_1, d_2) \mathbb{Z}}$$

$$H_3(X) = \frac{ker \partial_3}{im \partial_4} = 0$$

and all higher homology groups are 0.
By the LES for reduced homology, since \((S^1, D^2)\) is a good pair,

\[\ldots \to H_k(\partial X) \to H_k(X) \to H_k(X, \partial X) \to \ldots\]

is a LES. We recall that

\[H_k(S^1 \times S^1) = H_k(T^2) = \begin{cases} \mathbb{Z} & k = 0, 1 \\ \mathbb{Z} \times \mathbb{Z} & k = 1 \\ 0 & \text{else} \end{cases}\]

To find the homology of \(X\), we note that \(D^2\) is contractible and hence \(X\) deformation retracts onto \(S^1\). Therefore

\[H_k(X) = H_k(S^1) = \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & \text{else} \end{cases}\]

Therefore we have the following LES

\[0 \to H_3(x, \partial X) \to \mathbb{Z} \to 0 \to H_2(x, \partial X) \to \]

\[\to \mathbb{Z} \to \mathbb{Z} \to H_1(x, \partial X) \to \]

\[\to \mathbb{Z} \to \mathbb{Z} \to H_0(x, \partial X) \to 0\]

Therefore \(H_3(x, \partial X) = \mathbb{Z}\). The map \(H_0(\partial X) \to H_0(x)\) is an isomorphism and so \(H_0(x, \partial X) = 0\) by excision. It then remains to calculate \(H_2(x, \partial X), H_1(x, \partial X)\) since \(H_k(X, \partial X) = 0\) \(\forall k \geq 2\).

For \(H_2(x, \partial X)\) and \(H_1(x, \partial X)\) we have the SES,

\[0 \to H_2(x, \partial X) \to \mathbb{Z}^2 \to \mathbb{Z} \to H_1(x, \partial X) \to 0\]

Consider the map \(\mathbb{Z}^2 \to \mathbb{Z}\). This was induced by the inclusion \(T^2 = \partial X\) in \(X = S^1 \times D^2\) and the deformation retraction \(S^1 \times D^2 \to S^1\).

Therefore, on first homology, \(\mathbb{Z}^2 \xrightarrow{f^*} \mathbb{Z}\) takes \((a, b) \mapsto a\) since the second 1-cycle is squashed by the retraction.
Therefore \( \text{im} f = \mathbb{Z} \) and \( \ker f^* = \mathbb{Z} \).

By contrapositive, this implies \( \text{im} e^* = \mathbb{Z} \).

Hence \( e^* \) is injective. This implies \( H_1(x, \partial x) = \mathbb{Z} \).

Similarly, \( \ker g^* = \mathbb{Z} \) and so \( H_0(y, \partial y) = 0 \) since \( g^* \) is surjective by contrapositive. \( Q.E.D. \)
(a) We construct $\text{RIP}^n_{\alpha}$ with:

1. $0$-all: $p$
2. $1$-all: $e_1$ with $|\sigma_1| = p - p = 0$
3. $2$-all: $e_2$ with $|\sigma_2| = 2e_1$
4. $k$-all: $e_k$ with $|\sigma_k| = e_{k-1} + (-1)^k e_{k-1}$
5. $n$-all: $e_n$ with $|\sigma_n| = e_{n-1} + (-1)^{n-1} e_{n-1}$

Then $H_0 (\text{RIP}^n) = \frac{k \alpha + \delta_0}{\text{Im} \delta_1} = \frac{2^{k-1} p^2}{0} = 0$

For $0 < k < n$, we have two cases. If $k$ is even then:

$$H_k (\text{RIP}^n) = \frac{k \alpha + \delta_k}{\text{Im} \delta_{k+1}} = 0$$

If $k$ is odd, then:

$$H_k (\text{RIP}^n) = \frac{k \alpha + \delta_k}{\text{Im} \delta_{k+1}} = \frac{2 \alpha (e_k)}{2 \alpha (2e_k)} = \mathbb{Z}/2\mathbb{Z}$$

Finally, for $k = n$:

$$H_n (\text{RIP}^n) = \frac{k \alpha + \delta_n}{\text{Im} \delta_{n+1}} - k \alpha \delta_n \in \begin{cases} \mathbb{Z} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

As usual, $H_k (\text{RIP}^n) = 0$ for $k > n$.

(b)
(b) As above, we construct $RP^2 \times RP^2$ via

1. $0 \text{-cell}: p$

2. $1 \text{-cells}: e, \; v \; \text{such that } 2e = p \cdot p = 0$

3. $2 \text{-cell}: f \; \text{such that } df = 2e$

Then we can give $RP^2 \times RP^2$ a corresponding CW complex via

1. $0 \text{-cell}: (p, p)$

2. $1 \text{-cells}: (e, p), \; (pe) \; \text{such that } \partial_1(e, p) = 0, \; \partial_1(\rho, e) = 0$

3. $2 \text{-cell}: (e, e), \; (f, p), \; (p, f) \; \text{such that } \partial_2(e, e) = 0, \; \partial_2(1, p) = 2(e, p), \; \partial_2(p, f) = 12(p, e)$

Hence, the homology groups of $X = RP^2 \times RP^2$

$$H_0(X) = \frac{\ker \partial_0}{\text{im} \partial_1} = \frac{\mathbb{Z}((p, p))}{0} = \mathbb{Z}$$

$$H_1(X) = \frac{\ker \partial_1}{\text{im} \partial_2} = \frac{\mathbb{Z}((e, p), (p, e))}{\mathbb{Z}(e, p), 12(p, e)} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z}$$

$$H_2(X) = \frac{\ker \partial_2}{\text{im} \partial_3} = \frac{\mathbb{Z}((e, e), (1, p) - (p, 1))}{\mathbb{Z}(2(e, e), -2(e, e))} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$$

$$H_3(X) = \frac{\ker \partial_3}{\text{im} \partial_4} = \frac{\mathbb{Z}((1, e) - (e, 1))}{\mathbb{Z}(2(1, e) - 2(e, 1))} = \mathbb{Z}/2\mathbb{Z}$$

$$H_n(X) = \ker \partial_n = 0$$

In particular, $H_3(y) \neq 0$.
Suppose that \( G = \langle a_1, \ldots, a_n \rangle \) is a finite rank free group. We would like to realize \( G \) as the fundamental group of the wedge of \( k \) copies of \( S^1 \), denoted \( X \).

Suppose for the sake of contradiction that \( G \) is a finite index subgroup of smaller rank; \( H \subseteq G \). Suppose \( H \) has index \( r \).

Then, corresponding to \( H \subseteq G \), \( \hat{X} \subseteq \tilde{X} \) is a \( r \)-sheeted covering space \( \hat{X} \subseteq \tilde{X} \).

Hence \( \hat{X} \) is the wedge of \( k \) \( r \)-copies of \( S^1 \).

\( \hat{X} \) can be realized as a graph with \( r \) vertices and \( rk \) edges.

Then \( \hat{X} \) has a spanning tree consisting of \( k-1 \) edges.

Contracting along this spanning tree, we see that \( \hat{X} \) deformation retracts to the wedge of \( nk-k+1 \) copies of \( S^1 \).

Therefore \( \pi_1(\hat{X}) \) is the free group on \( nk-k+1 \) generators.

In particular, this implies that the rank of \( H \) is

\[ nk-k+1 > n \]

Therefore \( H \) a finite index subgroup of smaller rank.
be proved by Mayer-Vietoris.

we recall that $\Sigma X$ is defined as

$$\Sigma X = \frac{X \times [0,1]}{\sim}$$

where $(x,0) \sim (y,0)$ and $(x,1) \sim (y,1)$ for $x, y \in X$.

Define $U = X \times \{0\}/\sim$ and $V = X \times \{1\}/\sim$. Then $U \cup V$ are contractible, $U \cup V$ deformation retracts onto $X$, and $U \cup V \cong X$.

By the Mayer-Vietoris sequence for reduced homology, this yields a LES

$$\cdots \to \tilde{H}_k(U \cap V) \to \tilde{H}_k(U) \oplus \tilde{H}_k(V) \to \tilde{H}_k(X) \to \tilde{H}_{k-1}(U \cap V) \to \cdots$$

Since $U, V$ are contractible, $\tilde{H}_k(U) \cong \tilde{H}_k(V) \cong 0$ for $k \geq 1$.

Therefore for $k \geq 1$, we argue the SES

$$0 \to \tilde{H}_k(X) \to \tilde{H}_k(U) \oplus \tilde{H}_k(V) \to 0$$

which implies $\tilde{H}_k(X) = \tilde{H}_{k-1}(V, U)$,

For $k = 0$, we have the SES

$$0 \to \tilde{H}_0(X) \to 0$$

which implies $\tilde{H}_0(X) = 0 = \tilde{H}_{-1}(V, U)$ by convention.

Thus should all be in terms of $\mathbb{Z}$.

Then universal coefficients theorem gives $M$.
(a) Let \( W \) be an neighborhood of \( \bar{Y} \) of \( \mathbb{Y} \) such that \( \text{image} \) \( \text{int} \) \( \bar{Y} \) is connected. Then \( 
abla \) is simply connected and so Van Kampen's implies that
\[
\Pi_1(\mathbb{Y}) = \Pi_1(Y) \times \Pi_1(S^1) = 0
\]
Since \( \Pi_1(S^1) = 0 \) and \( Y \) is simply connected.

(b) Since \( Y \) is simply connected, it is an own univalent cover.

We recall that the universal cover of \( S \) as \( R \) \( R \) each interval \( \text{int} \) \( [n, n+1] \) mapped to \( S \).

Then the universal cover of \( \mathbb{Y} \) will be \( R \) \( R \) a copy of \( S \) wedged at each integer.

Since the deck transformations of \( S \) are \( R \) \( R \) each integer translation, it follows that the deck transformations are integer translations for \( \mathbb{Y} \).

(c) As in part (b), since \( Y \) has no connected multi-covered covers, it suffices to consider the k-covered cover of \( S \).

We recall that the k-covered covers of \( S \) are
\[
[0, k]/\text{out}
\]
where each interval \( [n, n+1] \) is mapped to \( S \).

Then the k-covered covers of \( \mathbb{Y} \) are \([0,k]/\text{out} \) a copy of \( Y \) wedged at each integer.

\)
(d) Suppose now that \( V = \mathbb{R}P^2 \).

Then \( V \) is no longer simply connected, so \( \pi_1(V) \neq 0 \).

By recalling that \( \mathbb{R}P^2 \) is constructed via

\[
\begin{align*}
1 & \quad \text{cell: } p \\
1 & \quad \text{cell: } e &= 1 \quad \text{d}e = p - p = 0 \\
1 & \quad \text{cell: } f &= 1 \quad \text{d}f = 2e
\end{align*}
\]

it follows that \( \pi_1(\mathbb{R}P^2) \cong \langle e^1, e^2 \rangle \cong \mathbb{Z}/2\mathbb{Z} \)

Therefore \( \pi_1(Y \vee S^1) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).

Since \( \mathbb{R}P^2 \) is not simply connected, it is not its own universal cover. We recall that \( S^2 \) is the universal cover of \( \mathbb{R}P^2 \) via the quotient map \( S^2 \to S^2/\mathbb{Z}/2\mathbb{Z} \cong \mathbb{R}P^2 \).

Then, the universal cover of \( Y \vee S^1 \) will be the universal cover of \( S^1 \vee \mathbb{R} \), which is a universal cover of \( Y \vee S^1 \), amalgamated at each integer. Thus every copy of \( S^2 \) will have a copy of \( \mathbb{R} \) at the north and south pole. This process then iterates infinitely.

The resulting structure is then of the form

\[
\begin{array}{c}
\circ \quad \circ \quad \circ \\
S^2 & \quad \mathbb{R}
\end{array}
\]

though this is a rough sketch.

The duality transformation acts via translation on \( \mathbb{R} \) or reflection in \( S^2 \), corresponding to \( \mathbb{Z} \) and \( \mathbb{Z}/2\mathbb{Z} \) respectively.
(a) Suppose \( n \) is even and \( F : \mathbb{RP}^n \to \mathbb{RP}^n \) is smooth. To show \( F \) has a fixed point, it suffices to show that \( L(F) \neq 0 \).

By the Lefschetz trace formula,
\[
L(F) = \sum_{k=0}^{n} (-1)^k \text{tr}(F^*: H^k(\mathbb{RP}^n; \mathbb{Q}) \to H^k(\mathbb{RP}^n; \mathbb{Q}))
\]
we recall that for \( n \) even
\[
H^k(\mathbb{RP}^n) \cong \begin{cases} 
\mathbb{Q} & k \equiv 0, n \pmod{2} \\
0 & \text{otherwise}
\end{cases}
\]

Therefore by the universal coefficients theorem, since \( \mathbb{Q} \) is a field,
\[
H^k(\mathbb{RP}^n; \mathbb{Q}) \cong \begin{cases} 
\mathbb{Q} & k \equiv 0, n \pmod{2} \\
0 & \text{otherwise}
\end{cases}
\]

Then
\[
L(F) = \text{tr}(F^*: H^n(\mathbb{RP}^n; \mathbb{Q}) \to H^n(\mathbb{RP}^n; \mathbb{Q})) - \text{tr}(F^*: \mathbb{Q} \to \mathbb{Q})
\]

since \( F^* \) is a (cohomology) ring homomorphism, \( F^n(1) = 1 \), and \( F^*: H^0(\mathbb{RP}^n; \mathbb{Q}) \to H^0(\mathbb{RP}^n; \mathbb{Q}) \) is the identity. Thus \( L(F) \neq 0 \) \( \implies \) \( F \) has a fixed point.

(b) View \( \mathbb{RP}^{2k-1} \) as the quotient of \( S^{2k-1} \subset \mathbb{C}^k \) under the anti-podal identification. On \( S^{2k-1} \), we define \( \tilde{F} : S^{2k-1} \to S^{2k-1} \) via \( z \mapsto iz \). Then \( \tilde{F} \) has no fixed points. Moreover,
\[
\tilde{F}(iz) = -iz = \tilde{F}(z)
\]
and \( \tilde{F} \) factors to a map \( F : \mathbb{RP}^{2k-1} \to \mathbb{RP}^{2k-1} \).

If \( F[z] = [z] \) then \( [iz] = [z] \) \( \implies \) \( iz = z \), which is impossible.

Therefore \( F \) has no fixed point.
(a) \( \pi_1(x) = \langle a, b | a b^2, b a^2 \rangle \)

\( a b^2 = e \Rightarrow a^2 = b^{-2} \)

\( \Rightarrow \pi_1(x) = \langle b | b^{-3} \rangle \equiv U/3U \)

which is finite.

(b) 
\[ H_0(x) = \frac{k r \Delta_0}{1m \Delta_1} = \frac{\mathcal{Q} \langle p \rangle}{\alpha} \equiv \mathcal{Q} \]

\[ H_1(x) = \frac{k r \Delta_1}{1m \Delta_2} = \frac{\mathcal{Q} \langle a, b \rangle}{\mathcal{Q} \langle a + 2b, b + 2a \rangle} = \frac{\mathcal{Q} \langle a \rangle}{\mathcal{Q} \langle -3a \rangle} \equiv U/3U \]

\[ H_2(x) = \frac{k r \Delta_2}{1m \Delta_3} \equiv 0 \]

\[ H_k(x) = 0 \quad \forall \ k \geq 2. \]
GeoTop
Fall 2022
Let $M$ denote the space of non-symmetric matrices and let $S$ denote the space of non-symmetric matrices. We note that 
\[ \dim M = n^2, \ \dim S = n + (n-1) + \cdots + 1 = \frac{n(n+1)}{2}. \] Furthermore, we recall that $T_A M = M$ and $T_B S = S$ for all $A \in M$, $B \in S$.

Define $f : M \to S : A \mapsto A^T A$. Then $O(n) = f^{-1}(\text{Id})$ by definition. We claim that $\text{Id}$ is a regular value of $f$. To show this, it suffices to show that $\forall A \in O(n)$, $dF_A : T_A M \to T_{f(A)} S$ is negative. As $T_A M = M$ and $T_{f(A)} S = S$, it suffices to show $dF_A : M \to S$.

Consider some $C \in S$. Let $B = \frac{1}{2} AC$. Then by direct computation,
\[
dF_A(B) = \lim_{t \to 0} \frac{(A + tB)^T(A + tB) - A^T A}{t}
= \lim_{t \to 0} \frac{A^T A + tB^T A + tA^T B + t^2 B^T B - A^T A}{t}
= B^T A + A^T B
= \frac{1}{2} (C^T A^T A + A^T AC)
= \frac{1}{2} CT^T + \frac{1}{2} C = C \quad (\text{since } C \in S)
\]
Therefore $dF_A$ is everywhere $\forall A \in O(n)$. This implies that $O(n)$ is a smooth submanifold of dimension
\[ n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}\]
as claimed.
We recall that a tangent bundle is trivializable iff \( \exists \, M \) linearly independent non-vanishing vector fields. Therefore, we construct \( \frac{n(n-1)}{2} \) non-vanishing linearly independent vector fields on \( \text{SO}(n) \).

We note that \( \text{O}(n) \) is a Lie group. Therefore for each \( g \) notation we consider \( g \) be group \( G \) of dimension \( m = \frac{n(n-1)}{2} \).

Let \( e \) denote the identity in \( G \). Consider \( T_e G \). Since \( G \) is of dimension \( m \), \( G \) equals \( V_1, \ldots, V_m \in T_e G \) r.l.

\( v_1, \ldots, v_m \) are linearly independent and nonzero.

For each \( g \in G \), let \( m_g : G \to G; h \mapsto gh \). Since \( G \) is abelian,

\( m_g \) is a diffeomorphism and so \( \text{dm}_g : T_h G \to T_{gh} G \) is a diffeomorphism.

Define \( X_1, \ldots, X_m \) on \( G \) via

\[ X_j|_g = \text{dm}_g(v_j) \]

Then \( X_1, \ldots, X_m \) are linearly independent and non-vanishing.

\( \text{dm}_g \) is a diffeomorphism. Moreover, since \( m_g \) is smooth in \( g \),

\( X_1, \ldots, X_m \) are smooth.

Therefore \( TG \) is trivializable and so \( T\text{O}(n) \) is trivializable. \( \square \)
Fix suppose that $k > 0$.

Let $B_1, \ldots, B_k$ be $k$ disjoint balls in $M$. Then $M$ continuously maps to the wedge of $k$ copies of $S^n$ via the quotient map

$$\pi : M \to M/(M \setminus \bigcup_{i=1}^k B_i) \cong \bigvee_{i=1}^k S^n$$

This follows since $\overline{B_j}/\partial B_j = S^n \forall j$.

Define $g : \bigvee_{i=1}^k S^n \to S^n$ s.t. $g|_{S^n} = \text{id}$ for each copy of $S^n$, where $r$ is a reflection chosen that $f = g \circ \pi$ is orientation reversing on $B_i$ for all $i$.

Then $A \in S^n$ not at the wedge point, $\# f^{-1}(A) = k$, hence $f$ is orientation reversing on $f^{-1}(A)$ this implies that $\text{deg} f = k$.

Now suppose $k < 0$. Repeating the above argument $\omega | \Omega$ and then composing with an orientation reversing reflection yields a map $f \circ \text{id}_{\Omega} : -|k| = k$, as desired.
We note that since \( \omega \) is non-vanishing, \( \text{rank}(\omega_p) = 1 \) consistently and \( \dim \ker(\omega_p) = n-1 \) \( \forall p \). Therefore \( \ker(\omega) \) is consistently of dimension \( n-1 \) and hence is a smooth distribution.

(3\( \Rightarrow \)2) Suppose \( dw = \kappa \wedge \omega \). Then

\[
\omega \wedge dw = \omega \wedge \kappa \wedge \omega
\]

Hence \( \kappa, \omega \) are 1-forms,

\[
\omega \wedge \kappa \wedge \omega = - \kappa \wedge \omega \wedge \omega
\]

\[
= \kappa \wedge \omega \wedge \omega
\]

\[
= - \omega \wedge \kappa \wedge \omega
\]

and \( \omega \wedge dw = \omega \wedge \kappa \wedge \omega = 0 \).

(2\( \Rightarrow \)1) Suppose \( \omega \wedge dw = 0 \). We recall that by Frobenius' theorem, \( \ker(\omega) \) is an integrable distribution iff \( \forall \) vector fields \( X, Y \in \ker(\omega) \), \( \{X, Y\} \in \ker(\omega) \).

Suppose \( \exists X, Y \in \ker(\omega) \). Let \( \exists \, V_p \in T_p M \backslash \ker(\omega) \).

Then \( \omega(X) = \omega(Y) = 0 \).

\[
0 = \omega \wedge dw(X_p, Y_p, V_p) = \omega(V_p) dw(X_p, Y_p)
\]

and \( \omega dw(X_p, Y_p) = 0 \) \( \forall p \). Then \( dw(X, Y) = 0 \).

Hence \( X, Y \in \ker(\omega) \).

\[
0 = dw(X, Y) = X(Y(\omega)) - Y(X(\omega)) = \omega([Y, X]) = -\omega([X, Y])
\]

Therefore \( [X, Y] \in \ker(\omega) \). As the hutch \( \forall \, X, Y, V \in \ker(\omega) \), \( \ker(\omega) \) is an integrable distribution.
Suppose \( k_0(\omega) \) is an integrable distribution. We aim to define \( \mathbf{\kappa} \) s.t. \( d\omega = \mathbf{\kappa} \wedge d\mathbf{\omega} \). To do so, it suffices to work locally since we may then extend globally via a partition of unity.

Since \( k_0(\omega) \) is integrable, locally \( \exists \) coordinate s.t.

\[
k_0(\omega) = \mathbf{1}_K (\partial_1 \partial x_1, \ldots, \partial_1 \partial x_{n-1})
\]

Then \( d\omega = f \, d\mathbf{\omega} \) for some non-vanishing \( f \).

Then \( \mathbf{\kappa} = \frac{df}{f} \) is well-defined and satisfies

\[
\mathbf{\kappa} \wedge d\mathbf{\omega} = df \wedge d\mathbf{\omega} = d\omega
\]

as desired.
Lemma: Suppose that $M^n$ is a compact orientable manifold which admit a symplectic form. Then $H^{2k}_d(R)(M) = 0 \forall k = 1, \ldots, n$.

Proof: Let $\omega$ denote the a symplectic form on $M$. Then $\omega$ is closed and so $\omega \wedge \cdots \wedge \omega$ is closed $\forall k = 0, \ldots, n$.

Suppose $\exists k$ s.t. $\frac{\omega \wedge \cdots \wedge \omega}{n-k} = d\eta$ for some $\eta$. Then

$$\frac{\omega \wedge \cdots \wedge \omega}{n} \quad = \quad \frac{\omega \wedge \cdots \wedge \omega}{n-k} \quad \wedge \frac{\omega}{n-k}$$

$$\quad = \quad d(\eta \wedge \cdots \wedge \omega)$$

Therefore $\omega \wedge \cdots \wedge \omega$ is exact. By Stokes' theorem, this implies that $\int_M \frac{\omega \wedge \cdots \wedge \omega}{n} = \int_{\partial M} \frac{\eta \wedge \omega \wedge \cdots \wedge \omega}{n} = \int_{\phi} \omega \wedge \cdots \wedge \omega = 0$

However, this contradicts the fact that $\omega \wedge \cdots \wedge \omega$ is a non-vanishing volume form. Therefore no such $k$ can exist. Then $\frac{\omega \wedge \cdots \wedge \omega}{k}$ is closed but we recall that

$$H^k_d(R)(S^p) = \begin{cases} \mathbb{R} & k = 0, p \\ 0 & \text{else} \end{cases}$$

Therefore by Kuushiki formula

$$H^k_d(R)(S^p \times S^q) = \begin{cases} \mathbb{R} & k = 0 \\ \mathbb{R} & k = p+1 \\ \mathbb{R} & k = q+1 \\ \mathbb{R}^2 & k = p+q \\ \mathbb{R} & k = q+1 \\ \mathbb{R}^2 & k = q+1 \\ \mathbb{R}^2 & k = p+q \end{cases}$$

We now suppose $p > 1$. If $p+1$ is not even, then $S^p \times S^q$ cannot admit a symplectic form.
By the lemma, the implies that if $S^p \times S^q$ admits a symplectic form, then $\{2, 4, \ldots, p+2\} \subseteq \{p, 2, p+2\}$.

This implies that

$$(p, l) \in \{(1, 1), (2, 2), (4, 2)\}$$

We check each of these cases.

1. $(p, l) = (1, 1)$

   In this case $S^p \times S^q \cong T^2$. Let $\omega = dx \wedge dy$ be the standard canonical volume form on $T^2$. Then $\omega$ is a symplectic form on $S^p \times S^q$.

2. $(p, l) = (2, 2)$

   Let $\eta$ be the volume form on $S^2$.

   Let $\eta_1, \eta_2$ be the 2-forms on $S^3 \times S^2$ corresponding to $\eta$ in the first and second coordinate copy of $S^2$.

   Then $\omega = \eta_1 + \eta_2$ satisfies $\omega \wedge \omega = 2\eta_1 \wedge \eta_2$. Since $\eta_1, \eta_2$ are linearly independent, $\omega$ is non-vanishing and hence $\omega$ is a symplectic form.

3. $(p, l) = (4, 2)$

   We claim that $S^4 \times S^2$ does not admit a symplectic form. 

   By the Kähler form, $H^2(\Sigma^4 \times S^2)$ consists of $\omega$ is a 0-form on $S^4$ and $\xi$ is a 2-form on $S^2$. Then $\omega \wedge \xi = \frac{1}{2} \Omega \wedge \xi = 0$, and in particular $\omega \wedge \xi = 0$. Therefore no symplectic form on $S^4 \times S^2$ exists.

To summarize, $S^p \times S^q$ admits a symplectic form only

$$(p, l) = (1, 1), (2, 2)$$

* We note that by the reasoning in the lemma, a symplectic form must be non-vanishing.
(a) For ease of notation, let
\[ 0 \to \mathbb{Z}/p \xrightarrow{i} \mathbb{Z}/p \xrightarrow{\partial} \mathbb{Z}/p \to 0 \]  
where \( i \) is multiplication by \( p \) and \( \partial \) is the quotient map. Then \( \mathcal{V} \cong \mathbb{Z}/p \), \( \partial(f(a)) = \mathcal{V}(pa) = 0 \). Then \( \mathcal{V} \circ \partial = 0 \) and \( \partial \) is a SES.

Given \( A_k = C_k \otimes \mathbb{Z}/p \), this gives a sequence of chain complexes:
\[ 0 \to A_k \xrightarrow{id \circ f} B_k \xrightarrow{id \circ g} A_k \to 0 \]
This is a SES since \( \text{id} \circ g \circ \text{id} \circ f = \text{id} \circ (\mathcal{V} \circ \partial) = 0 \).

(b) We note that this is precisely the snake lemma.

Let \( \partial_k, \partial_k^A \) denote the boundary maps on \( A_k, B_k \), respectively.

We aim to construct a map \( \ker \partial_k^A \to \ker \partial_k \) that is well-defined up to homology. This will then define a map \( B_k : H_k(A_k) \to H_{k-1}(A_k) \).

Define \( i_k, j_k \) s.t.
\[ 0 \to A_k \xrightarrow{i_k} B_k \xrightarrow{j_k} A_{k-1} \to 0 \]  
for ease of notation.

For some \( a_k \in A_k \) s.t. \( \partial_k^A(a_k) = 0 \), we have (2) in a SES, \( i_k(b_k) = a_k \).

We will show later that the resulting map is independent of this choice of \( b_k \).

Comute \( B_k(b_k) \in B_k \). By our diagram, commutes.
\[ j_{k-1}(\partial_k^B(b_k)) = \partial_k^A(j_k(b_k)) = \partial_k^A(a_k) = 0 \]

Therefore since (2) in exact, \( i_k \) a \( A_{k-1} \) s.t.
\[ i_{k-1}(a_{k-1}) = \partial_k^B(b_k) \]

We define \( B_k [a_k] = [b_k] \). To show that \( B_k \) is well-defined on homology, it must be shown that \( B_k \) is independent of the choice of \( b_k, a_{k-1} \) on the level of homology.
we first show the map is independent of the choice of

\(a_k-1\). Suppose \(a_k, a_k'\) s.t.

\[ i_{k-1}(a_k) = i_{k-1}(a_k') = \partial_k^B(b_k) \]

Then \(i_{k-1}(a_k - a_k') = 0\). Since (2) is a SES, this implies

that \(a_k - a_k' = 0 \Rightarrow a_k = a_k'\). Therefore \(a_k s.t. \ i_{k-1}(a_k) = \partial_k^B(b_k)\).

We now show the map is independent of the choice of

\(b_k\). Suppose \(b_k, b_k'\) s.t. \( j_k(b_k) = j_k(b_k') = a_k \).

Then \(j_k(b_k - b_k') = 0\) and so \(a_k \in A_k s.t. \ i_k(a_k) = b_k - b_k'\).

Since (2) is exact, therefore, since our diagram commutes

\[ \partial_k^B(b_k - b_k') = \partial_k^B(i_k(a_k)) \]

\[ = i_{k-1}(\partial_k^A(a_k)) \]

Let \(a_k, a_k' \in A_k s.t. \ i_{k-1}(a_k) = \partial_k^B(b_k)\) and \(i_{k-1}(a_k') = \partial_k^B(b_k')\).

Then

\[ i_{k-1}(a_k - a_k') = \partial_k^B(b_k - b_k') \]

\[ = i_{k-1}(\partial_k^A(a_k)) \]

Since \(i_{k-1}\) is injective, this implies \(a_k - a_k' = \partial_k^A(a_k)\).

Therefore \(a_k\) and \(a_k'\) are the same on the level of homology,

and so the choice of \(b_k\) is irrelevant.

Therefore \(B_k : H_k(A_\ast) \to H_k(A_\ast)\) is well-defined.
Suppose \( x, y \) s.t. \( d(x) = py \).

We view \( x \in C_k \otimes \mathbb{Z}/p \) as \( x \circ 1 \) and \( y \in C_{k-1} \otimes \mathbb{Z}/p \) as \( y \circ 1 \).

Let \( b_k, a_{k-1} \) be for \( x \) as they are in part (b).

We can show that \( a_{k-1} \) and \( y \) are the same on the level of homology.

By definition,
\[
i_{k-1}(a_{k-1} \cdot y) = \partial_k^B(b_k) - i_{k-1}(y) = \partial_k^B(b_k) - py = \partial_k^B(b_k - dx \mod p^2) = \partial_k^B(b_k - x \mod p^2)
\]

By definition, \( j_k(b_k - x \mod p^2) = x - x = 0 \). Therefore \( y \cdot a_{k-1} \) are 2 different choices for \( b_k \) in the definition of \( B^k \).

As shown before, this implies that \( a_{k-1} = y \) on the level of homology and so \( B(x) = [a_{k-1}] = [y] \) as claimed.
If \( n = 0 \) then \( H = \emptyset \) so, \( \pi_i (\mathbb{R}^3 \setminus H) = \pi_i (\mathbb{R}^3) \cong \mathbb{Z} \).

Assume \( n > 0 \) and let \( X = \mathbb{R}^3 \setminus H \).

We claim that \( X \) is homotopy equivalent to the wedge of \( 2n-1 \) circles.

First, since \( H \) is closed and contains 0, 
we may deformation retract \( X \) to the 2n-punctured sphere \( S^2 \). 
This can be done explicitly by the straight line homotopy

\[
X \xrightarrow{t \mapsto \frac{x}{1-t}} \left( 1-t \right) x + t \frac{x}{1-t}
\]

Second, the 2n-punctured \( S^2 \) is equivalent to the 2n-1-punctured \( \mathbb{R}^2 \) via a stereographic projection centered on or off the excluded point on \( S^2 \).

Third, we may apply a diffeomorphism so that the 2n-1 punctured plane is precisely

\[
\mathbb{R}^2 \setminus \{(1,0), (2,0), \ldots, (2n-1,0)\}
\]

Then we may apply a straight line homotopy to deformation retract \( \mathbb{R}^2 \setminus \{(1,0), \ldots, (2n-1,0)\} \) onto the wedge of 2n-1 closed punctured disks.

Finally, on each punctured disk we may apply a deformation retract onto the boundary.

Combining all this implies that \( X \) is homotopy equivalent to \( \bigvee_{i=1}^{2n-1} S^1 \). Then by Van Kampen,

\[
\pi_i(X) \cong \pi_i \left( \bigvee_{i=1}^{2n-1} S^1 \right) \cong \mathbb{Z} \oplus \mathbb{Z}^{2n-1}.
\]
By definition, the mapping cone $X$ is given by

$$X = \text{sp} 3 \cup S' x \{0\}, \text{ or } S' / u$$

where $p_t = (x, t) \forall x \in S'$ and $f_t(x) \in S' / u \forall x \in S'$ where $f$ is the commutator $[a, b]$. Let $c$ denote the constant map $S' \to sp 3$.

We recall the LES for the cone mapping cone, which is

$$\cdots \to H_0(X) \xrightarrow{f^*} H_0(S' / u) \to H_0(X) \to \cdots$$

This can be derived from the LES for Mayer-Vietoris by taking $U = sp 3 \cup S' x \{0, 1\}, \text{ or } S' / u$ and noting that $U$ deformation is contractible, $V$ deformation retracts onto $S' / u$, and $U \cap V$ deformation retracts onto $S$.

Recalling $H_0(S') = \{ 0 \}$ and $H_k(S' / u) = \{ 0 \text{ for } k \neq 0 \}

this yields the LES,

$$0 \to H_2(X) \xrightarrow{f} \mathbb{Z} \to \mathbb{Z} \to H_1(X) \to \mathbb{Z} \to \cdots$$

hence $X$ is connected, $H_0(X) = \mathbb{Z}$. Hence $X$ is 2-dimensional.

By construction, we know that $H_k(S') \rightleftharpoons H_k(S' / u)$ is the 0 map. Therefore we have 2 SES

$$0 \to H_2(X) \to \mathbb{Z} \to 0$$

$$0 \to \mathbb{Z} \to H_1(X) \to \mathbb{Z} \to 0$$

which imply $H_2(X) = \mathbb{Z}$ and $H_1(X) = \mathbb{Z}$ by taking alternating sums.
Geometry/Topology Qualifying Exam

Start each problem on a new sheet of paper.
Write your university identification number at the top of each sheet of paper.

DO NOT WRITE YOUR NAME!
Complete this sheet and staple to your answers.

Read the directions of the exam carefully.

STUDENT ID NUMBER

DATE:

EXAMINEES: DO NOT WRITE BELOW THIS LINE

1. ______________________ 6. ______________________
2. ______________________ 7. ______________________
3. ______________________ 8. ______________________
4. ______________________ 9. ______________________
5. ______________________ 10. ______________________

Pass/fail recommend on this form.

Total score: 

Form revised 3/08
Attempt all ten problems. Each problem is worth 10 points. Justify your answers carefully.

1. Without using homology groups or homotopy groups, directly derive Brouwer’s fixed point theorem (any continuous map \( f : D^2 \rightarrow D^2 \) has a fixed point, where \( D^2 \) is the closed 2-disk) from the hairy ball theorem (any continuous vector field on \( S^2 \) is somewhere 0).

2. Solve the following problems:
   
   (a) Let \( F : S^n \rightarrow S^n \) be a continuous map. Show that if \( F \) has no fixed point, then the degree of the map, \( \deg F = (-1)^{n+1} \).
   
   (b) Show that if \( X \) has \( S^{2n} \) as universal covering space, then \( \pi_1(X) = \{1\} \) or \( \mathbb{Z}_2 \).

3. Let \( n \) be an integer and let \( n \) be a distinct points in \( \mathbb{R}^3 \). Calculate the integral homology groups of \( \mathbb{R}^3 \setminus \{p_1, p_2, \ldots, p_n\} \).

4. Let \( \Delta_n^{(k)} \) be the \( k \)-dimensional skeleton of the \( n \)-simplex \( \Delta_n \). Calculate the reduced homology groups \( \tilde{H}_i(\Delta_n^{(k)}) \) for all values of \( i, k, n \).

5. Define the complex projective space \( CP^n \) to be the quotient of \( \mathbb{C}^{n+1} \setminus \{0\} \) by the relation \( z \sim \lambda z \) for all \( \lambda \in \mathbb{C} \setminus \{0\} \). Construct a CW complex structure on \( CP^n \) with no odd-dimensional cells and exactly 1 cell in each even dimension up to \( 2n \). Calculate the fundamental group and the integral homology groups of \( CP^n \).

6. Define the orientation double cover for any topological manifold. What is the orientation double cover of the real projective plane \( \mathbb{R}P^2 \)?

7. Show that \( S^1 \times S^2 \) and the connected sum \( CP^2 \# CP^2 \) are not homotopy equivalent.

8. Consider a differentiable map \( f : S^{2n-1} \rightarrow S^n \), with \( n \geq 2 \). If \( \alpha \in \Omega^n(S^n) \) is a differential form of degree \( n \) such that \( \int_{S^n} \alpha = 1 \), let \( f^*\alpha \in \Omega^n(S^{2n-1}) \) be its pull-back under \( f \).
   
   (a) Show that there exists \( \beta \in \Omega^{n-1}(S^{2n-1}) \) such that \( f^*\alpha = d\beta \).
   
   (b) Show that the integral \( I(f) = \int_{S^{2n-1}} \beta \wedge d\beta \) is independent of the choices of \( \beta \) and \( \alpha \).

9. Let \( f : M \rightarrow N \) be a smooth map between smooth manifolds, \( X \) and \( Y \) be smooth vector fields on \( M \) and \( N \), respectively, and suppose that \( f_*X = Y \) (i.e., \( f(X(x)) = Y(f(x)) \) for all \( x \in M \)). Then prove that
   
   \[ f^*(L_Y\omega) = L_X(f^*\omega) \]

   where \( \omega \) is a 1-form on \( N \). Here \( L \) denotes the Lie derivative.

10. Prove Cartan’s lemma: Let \( M \) be a smooth manifold of dimension \( n \). Fix \( 1 \leq k \leq n \). Let \( \omega^1, \ldots, \omega^k \) be 1-forms on \( M \). Suppose that the \( \{\omega^1, \ldots, \omega^k\} \) are linearly independent and that \( \sum_{i=1}^k \omega_i \wedge \omega^i = 0 \). Prove that there exist smooth functions \( h_{ij} : M \rightarrow \mathbb{R} \) such that for all \( i = 1, \ldots, k \), \( \omega_i = \sum_{j=1}^k h_{ij} \omega^j \).
Suppose for the sake of contradiction that $f : D^2 \to D^2$ is continuous s.t. $f(x) + x \forall x \in D^2$. 


(a) since degree is homotopy invariant and \( \deg(id) = (-1)^{n+1} \),
it suffices to construct a homotopy from \( f \) to \( -id \).

We note that \( \deg(id) = (-1)^{n+1} \) since \( S^n \cong \mathbb{R}^{n+1} \) and so we can
view \( -id \) as multiplying \( n+1 \) coordinates by \(-1\), where
each multiplication by \(-1\) defines orientation reversing injection
and hence has degree \(-1\). Then \( \deg(-id) = (-1)^{n+1} \).

Define \( H_t: S^n \to S^n \) by
\[
H_t(x) = \frac{(1-t)f(x) + t(-x)}{|(1-t)f(x) + t(-x)|}
\]

Then \( H_0 = f/\|f\| = f \) and \( H_1 = -x/\|x\| = -x \). To show that \( H_t \) is
a homotopy, it must be shown that \((1-t)f(x) - tx \neq 0 \) \( \forall t \in [0,1] \), \( x \in S^n \).

Suppose on the contrary that \((1-t)f(x) - tx = 0 \).
Then
\[
f(x) = \frac{tx}{1-t}
\]

Since \( \|f(x)\| = 1 \) \( \forall x \) and \( |x| = 1 \) \( \forall x \), this implies
\[
\frac{t}{1-t} = 1 \implies t = \frac{1}{2}
\]

Then \( f(x) = x \), contradicting the fact that \( f \) has no fixed
point. Therefore \((1-t)f(x) - tx \neq 0 \) \( \forall t, x \) and \( \deg f = -1 \).

Define \( H_t: S^n \to S^n \) by
\[
H_t(x) = \frac{(1-t)f(x) + t(-x)}{|(1-t)f(x) + t(-x)|}
\]

Then \( \deg(-id) = (-1)^{n+1} \).
(b) Suppose that $X$ has $S^{2n}$ as a universal covering space. Then this should follow from some lifting criterion stuff.

\[ \begin{array}{c}
S' \\
\downarrow \gamma \\
S \\
\downarrow \gamma \\
\ast \\
\end{array} \quad \begin{array}{c}
\rightarrow \\
\downarrow \pi \\
X \\
\downarrow \pi \\
\ast \\
\end{array} \]

Since $\pi_1(S^{2n}) = \mathbb{Z}$, any path can be lifted.
we claim that \(\mathbb{R}^3 \setminus p_1 \ldots p_n\) deformation retracts onto the wedge of a copies of \(S^2\). First, by applying a diffeomorphism, we may assume \(p_1 = (0,0,0), \ p_2 = (0,0,1), \ldots, \ p_n = (0,0,1,0)\). This is a standard result for smooth manifolds (shown on a lecture. \ref{npgr9}) and is even more standard on \(\mathbb{R}^3\).

Thus, we apply a deformation retraction from \(\mathbb{R}^3 \setminus p_1 \ldots p_n\) to the infinite rotated cylinder \(\mathbb{R}^2 \setminus \mathbb{S}^1\) in a point removed. This can be done explicitly via the straight line homotopy

\[
H_t(x, y, z) = \begin{cases} 
(x, y, z) & \text{if } x^2 + y^2 < 1 \\
(1-t)(x, y, z) + t \left(\frac{x, y, z}{x^2 + y^2}\right) & \text{if } x^2 + y^2 > 1
\end{cases}
\]

We then deformation retract to the finite cylinder containing \(p_1 \ldots p_n\) by sending \((x, y, z)\) to \((x, 1, z)\) if \(y \leq 1\) and \((x, y, z)\) to \((x, n, z)\) if \(y > n\). This again can be done via the straight line homotopy.

The deformation retraction to the wedge of \(n\) punctured balls by the straight line homotopy in the radial direction.

Finally, since each ball is punctured, we deformation retract to the wedge of \(n\) spheres by radially contracting each ball individually.

Thus

\[
H_k(\mathbb{R}^3 \setminus p_1 \ldots p_n) \cong H_k(\mathbb{R}^2 \setminus \mathbb{S}^1) \cong \begin{cases} \mathbb{Z} & k = 0 \\
0 & \text{otherwise}
\end{cases}
\]

as desired.
we proved by induction on \( n. \)

**Hypothesis:** \( n = 0. \) Then \( \mathbb{CP}^n \) is a single point and \( z \) can be contracted to \( 1 \) or \(-1\).

**Hypothesis:** for the sake of induction that \( \mathbb{CP}^n \) can be constricted as desired. We claim that then \( \mathbb{CP}^{n+1} \) can similarly be constricted as desired. To show this, it then suffices to show that \( \mathbb{CP}^{n+1} \) can be constricted by attaching a \( 2n+2 \)-cell to \( \mathbb{CP}^n \).

By definition, \( \mathbb{CP}^{n+1} \cong \mathbb{C}^{n+1}/\mathbb{Z}_{2n+2}. \) For any \( z \in \mathbb{C}^{n+1} \setminus \{0\} \), we can normalize \( |z| = 1 \) by \( z \mapsto \frac{z}{|z|} \). Hence \( z \sim \frac{z}{|z|} \), this implies

\[
\mathbb{CP}^{n+1} \cong S^{2n+1}/Z_{2n+2}
\]

where \( S^{2n+1} = \{ z \in \mathbb{C}^{n+1} : |z| = 1 \}. \)

Define \( U \subset \mathbb{CP}^{n+1} \) as \( U = \{ [z_0 : \ldots : z_{n+1}] : z_{n+1} \neq 0 \} \). Then \( U \cong \mathbb{CP}^n \) via the diffeomorphism \([z_0 : \ldots : z_{n+1}] : [z_{n+1}:0] \mapsto \frac{z_0}{z_{n+1}} \).

Consider \( \mathbb{CP}^{n+1} \setminus U = \{ [z_0 : \ldots : z_{n+1}] : z_{n+1} = 0 \} \). Since we have normalized \( |z_0 : \ldots : z_{n+1}|^2 = 1 \), we may write

\[
\mathbb{CP}^{n+1} \setminus U = \{ [z_0 : \ldots : z_n : \sqrt{1-|z_0|^2 - \cdots - |z_n|^2}] \}.
\]

Then \( |z_0 : \ldots : z_n| < 1 \) and so \( \mathbb{CP}^{n+1} \setminus U \cong B^{2n+2} = \{ z \in \mathbb{C}^{n+1} : |z| < 1 \} \), via the diffeomorphism

\[
[z_0 : \ldots : z_n : \sqrt{1-|z_0|^2 - \cdots - |z_n|^2}] \mapsto (z_0, \ldots, z_n)
\]

By construction, \( e(\mathbb{CP}^n \setminus U) = U \cup U \times \mathbb{C}^n \), where \( e \) is \( [z_0 : \ldots : z_{n+1}] \mapsto [z_0 : \ldots : z_n] \).

Therefore, viewing \( \mathbb{CP}^{n+1} \setminus U \cong B^{2n+2} \), we see that \( \mathbb{CP}^{n+1} \) can be constricted as \( B^{2n+2} \) attached to \( \mathbb{CP}^n \) by the map

\[
B^{2n+2} \times Z_{2n+2} \to \mathbb{CP}^n : (z_0 : \ldots : z_n) \mapsto [z_0 : \ldots : z_n]
\]

By induction, this completes the construction.
From this construction, we immediately obtain

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{C} & k = 0, 2, \ldots, 2n \\ 0 & \text{else} \end{cases}$$

Hence $\mathbb{C}P^n$ has no 1-nulls, we also find that $\pi_1(\mathbb{C}P^n) = 0$. \hfill \quad \Box
Let \( M \) be a topological manifold. We define the orientation cover \( \tilde{M} \) as follows.

Let \( \tilde{M} = \{ (p, O_p) : p \in M \} \), where \( O_p \) is an orientation of \( T_p M \). Using the fact that \( M \) is locally orientable, we define the topology on \( \tilde{M} \) to be generated by

sets of the form \( V_{u,0} \) where \( U \subset M \) is open and local orientation \( O \), i.e., \( V_{u,0} = \{ (p, O_p) : p \in U \} \).

From this, it immediately follows that \( \tilde{M} \) is locally diffeomorphic to \( M \) and \( \tilde{M} \) is a topological manifold.

We now show \( \tilde{M} \) is a double cover of \( M \). Let \( \pi : \tilde{M} \to M \). \( \pi \) is a diffeomorphism. Therefore \( \tilde{M} \) is a double cover.

For any orientable \( U \subset M \), \( \pi^{-1}(U) \) is 2 orientable \( O_1, -O_1 \). Therefore \( \pi^{-1} \) is a diffeomorphism. Therefore \( \tilde{M} \) is a double cover.

We recall that \( \mathbb{R}P^n \) is orientable if \( n \) is odd and non-orientable if \( n \) is even.

Therefore, if \( n \) is odd, the above definition implies that \( \mathbb{R}P^n \) consists of 2 disjoint open sets, each 1-orientable orientation. Hence \( \mathbb{R}P^n \) is a 2-fold cover, thus \( \mathbb{R}P^n \cong \mathbb{R}P^n \cup \mathbb{R}P^n \).

Now suppose \( n \) is even. Then \( \mathbb{R}P^n \) is non-orientable and \( \tilde{\mathbb{R}P^n} \) is so a connected 2-fold cover. We recall that \( S^n \) is a connected 2-fold cover of \( \mathbb{R}P^n \) via the quotient map \( \pi : S^n \to \mathbb{R}P^n \).

Then \( S^n \) can be viewed as \( S^n = \{ (p, \pm 1) : p \in \mathbb{R}P^n \} = \tilde{\mathbb{R}P^n} \). From the above definition
we recall that $\mathbb{C}P^2$ can be considered as a 4-ball attached to $\mathbb{C}P^1$. Therefore, since all connected sums are homotopy equivalent, $\mathbb{C}P^2 \# \mathbb{C}P^2$ is homotopy equivalent to $\mathbb{C}P^1$ attached to $\mathbb{C}P^1$ via the identity map. Therefore $\mathbb{C}P^2 \# \mathbb{C}P^2$ is homotopy equivalent to $\mathbb{C}P^1$.

By the Kunneth formula, we find that

$$H_k(S^3) = \begin{cases} \mathbb{Z} & k = 0,2 \\ 0 & \text{else} \end{cases} \quad \Rightarrow \quad H_k(S^2 \times S^2) = \begin{cases} \mathbb{Z} & k = 0,4 \\ 0 & \text{else} \end{cases}$$

Therefore $S^2 \times S^2$ and $\mathbb{C}P^1$ are not homotopy equivalent and so $S^2 \times S^2$ and $\mathbb{C}P^2 \# \mathbb{C}P^2$ are not homotopy equivalent. $\square$
(a) We recall that \( H_{dR}^n(S^{2n-1}) = 0 \). Therefore it suffices to show that \( \int^{\ast} \) is exact, it suffices to show that \( \int^{\ast} \) is closed. Since pullbacks commute with the exterior derivative, \( d(\int^{\ast}) = \int^{\ast}(d\alpha) \).

Since \( \alpha \in \Omega^{n-1}(S^{2n-1}) \), \( d\alpha = 0 \) and \( d(\int^{\ast}) = \int^{\ast}0 = 0 \).

Therefore \( \int^{\ast} \) is closed and \( \omega \) is exact.

Thus \( \exists \beta \in \Omega^{n-1}(S^{2n-1}) \) s.t. \( \int^{\ast} = d\beta \) as desired.

(b) Suppose \( \exists \beta, \beta' \) s.t. \( d\beta = d\beta' = \int^{\ast} \).

Then \( d\beta - d\beta = 0 \implies d(\beta - \beta') = 0 \). Hence \( \beta, \beta' \in \Omega^{n-1}(S^{2n-1}) \), closure implies existence and \( \beta - \beta' = d\psi \) for some \( \psi \).

Thus \( \beta' = \beta + d\psi \). It then suffices to show that

\[
\int_{S^{2n-1}} \beta \wedge d\beta = \int_{S^{2n-1}} (\beta + d\psi) \wedge (\beta + d\psi) - d\psi \wedge d\beta
\]

for all \( \psi \). By direct computation,

\[
\int_{S^{2n-1}} (\beta + d\psi) \wedge (\beta + d\psi) = \int_{S^{2n-1}} (\beta + d\psi) \wedge d\beta = \int_{S^{2n-1}} \beta \wedge d\beta + d\psi \wedge d\beta
\]

Since \( d\psi \wedge d\beta = d(\psi \wedge d\beta) \), Stokes' theorem implies \( \int_{S^{2n-1}} \beta \wedge d\beta = \int_{S^{2n-1}} \beta \wedge d\beta = 0 \).

Therefore \( \int_{S^{2n-1}} (\beta + d\psi) \wedge (\beta + d\psi) = \int_{S^{2n-1}} \beta \wedge d\beta \) as desired.

Now suppose \( \exists \xi, \xi' \in \Omega^n(S^n) \) s.t. \( \int_{S^n} = 1 = \int_{S^n} \).

Then \( \int_{S^n} \xi - \xi' = 0 \) and \( \xi - \xi' = d\Theta \) for some \( \Theta \in \Omega^{n-1}(S^n) \).

Since integration is an isomorphism \( H_{dR}^n(S^n) \to \mathbb{R} \).

Then \( \int^{\ast} \xi - \int^{\ast} \xi' = \int^{\ast}(\xi - \xi') = \int^{\ast}(d\Theta) = d\int^{\ast} \Theta \).

Taking \( \beta \) s.t. \( d\beta = \int^{\ast} \) and \( \beta' \) s.t. \( d\beta' = \int^{\ast} \xi' \), we see that \( d\beta \cdot d\beta' = d\int^{\ast} \Theta \).
By Cartan's formula, we find that
\[ L_x (f^* \sigma) = (i_x \circ d + d \circ i_x)(f^* \sigma) \]
\[ = i_x (f^* \sigma) + d([f^* \sigma](X)) \]
we now apply to some \( p \in M \). Repeating this,
\[ L_x (f^* \sigma)_p = X_p((f^* \sigma)_p) + d_p([f^* \sigma](X)) \]
\[ = X_p(f^* \sigma) \]

By Cartan's formula, \( \omega \) is a \( 1 \)-form,
\[ f^* (\omega(Y)) = f^* (d(\omega(Y)) + i_Y (d\omega)) \]
\[ = d(f^* \omega(Y)) + f^* (i_Y (d\omega)) \]

By definition, \( \omega(Y): N \to \mathbb{R}: q \mapsto \omega_q(Y(q)) \).
Therefore \( f^* (\omega(Y))(p) = \omega_Y(f(p)) = \omega_{f(p)}(Y(f(p))) \) for \( p \in M \).
As given, this implies \( f^* (\omega(Y))(p) = \omega_{f(p)}(f_x(x(Y)(f(p)))) = (f^* \omega)(X)(p) \).
Then \( f^* (\omega(Y)) = (f^* \omega)(X) = i_X (f^* \omega) \).

Similarly, \( (i_Y (d\omega))_q: T_q N \to \mathbb{R} \).
Therefore \( f^* (i_Y (d\omega))_p: T_{f(p)} M \to \mathbb{R} \) and so
\[ f^* (i_Y (d\omega))_p(v) = (i_Y (d\omega))_{f(p)}(f^* v) \]
\[ = d\omega_{f(p)}(Y_{f(p)}(f v)) \]
\[ = (f^* d\omega)_p((f^* X)_p f v) \]
and so \( f^* (i_Y (d\omega)) = i_X (f^* d\omega) \) i.e. \( i_X \circ d (f^* \omega) \).

Combining this,
\[ f^* (\omega(Y)) = d(i_X (f^* \omega)) + i_X (d(f^* \omega)) = L_x (f^* \omega) \]

as desired.
Idea: Extend \( \{w^1, \ldots, w^k\} \) to a basis \( w^1, \ldots, w^k, w^{k+1}, \ldots, w^n \) of \( T^*M \). Then \( \varphi_i = \sum_{j=1}^{n} h_{ij} w^j \).

It suffices to work locally as then we can extend \( h_{ij} \) to all of \( M \) via a partition of unity.

Locally, we may extend \( w^1, \ldots, w^k \) to a basis \( w^1, \ldots, w^k, w^{k+1}, \ldots, w^n \) of \( T^*M \). Then \( \forall i, j, \exists h_{ij} \in C \)

\[
\varphi_i = \sum_{j=1}^{n} h_{ij} w^j \quad (1)
\]

As given, \( \sum_{i=1}^{k} \varphi_i \wedge w^i = 0 \). Therefore

\[
\sum_{i=1}^{k} \sum_{j=1}^{n} h_{ij} w^j \wedge w^i = 0
\]

Expanding and associating like terms, we find that

\[
\sum_{i=1}^{k} \sum_{j=1}^{k} (h_{ij} - h_{ji}) w^j \wedge w^i + \sum_{j=k+1}^{n} \sum_{i=1}^{k} h_{ij} w^j \wedge w^i = 0
\]

Since \( w^1, \ldots, w^n \) are linearly independent, \( \{w^i \wedge w^j\} \) are linearly independent. Therefore

\[
h_{ij} = h_{ji} \quad \forall \ i, j \in 1, \ldots, k
\]

\[
h_{ij} = 0 \quad \forall \ j > k+1
\]

Therefore \( \varphi_i = \sum_{j=1}^{k} h_{ij} w^j \quad \forall i \)

\( \forall \ i \), \( h_{ij} = h_{ji} \), as desired.
(a) Since \( \ker(w) \) is 2-dimensional and integrable, \( \exists \) vector fields \( X, Y \) s.t. \( \ker(w) = \text{span}(X, Y) \).

About any \( p \), we can locally extend \((X, Y)\) to a basis \( X, Y, Z \).

Thus since \( X, Y \in \ker(w) \)
\[
\text{w} \text{ndes}(X, Y, Z) = \text{w}(Z) \text{d}w(X, Y) - \text{w}(Z) \text{d}w(Y, X) \\
= 2 \text{w}(Z) \text{d}w(X, Y)
\]

We recall that \( \text{d}w(X, Y) = X(\text{w}(Y)) - Y(\text{w}(X)) - \text{w}(X, Y) \).

Since \( \ker(w) \) is integrable, \((X, Y) \in \ker(w)\) and so
\[
\text{d}w(X, Y) = X(\text{w}(Y)) - Y(\text{w}(X)) = 0. \quad \text{Then}
\]
\[
\text{w} \text{ndes}(X, Y, Z) = 0. \quad \text{Thus}
\]

Hence \( X, Y, Z \) are a local basis, the above vanishes locally and hence everywhere.

(b), (c) follow as before
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(a) we find show onto.

Let \( n = \dim M = \dim N \). Since \( f \) is a submersion, \( \forall p \in M \),

\[
\text{rank } df_p = n
\]

Hence \( n = \dim M \), thus implies that \( df_p \) is an isomorphism \( \forall p \in M \). Therefore, by the inverse function theorem, \( f \) is a local diffeomorphism. In particular, \( f \) is an open map.

Since \( f \) is open, \( f(M) \subset N \) is open. Since \( f \) is continuous and \( M \) is compact, \( f(M) \) is compact. Therefore, \( f(M) \) is both open and closed. Because \( N \) is connected, this implies that \( f(M) = N \) and so \( f \) is surjective.

We now show \( f \) is a covering map. We note that this is essentially exactly the implicit function theorem.

Since \( f \) is smooth and onto, it suffices to show that \( \forall q \in N \) there exists a neighborhood \( V \) of \( q \) and disjoint neighborhoods \( U_0, \ldots, U_k \) of the pre-images of \( q \) and \( f \) s.t. \( \cup U_i = f^{-1}(V) = \bigcup_{i=1}^{k} U_i \).

For some \( q \in N \), since \( f \) is a local diffeomorphism, \( f^{-1}(q) \) is a 0-dimensional submanifold of \( M \). Since \( M \) is compact, the image of \( f^{-1}(q) \) is finite, we enumerate \( f^{-1}(q) \) as \( p_1, \ldots, p_k \).

For each \( i = 1, \ldots, k \), \( \exists \) a neighborhood \( \bar{U}_i \) of \( p_i \) s.t. \( f: \bar{U}_i \rightarrow f(\bar{U}_i) \) is a diffeomorphism. By shrinking \( \bar{U}_i \) if necessary, we may assume that \( \bar{U}_i \cap \bar{U}_j = \emptyset \) if \( i \neq j \). Let \( V = f(\bar{U}_1) \cap \cdots \cap f(\bar{U}_k) \). Then \( f(V) \) is a neighborhood of \( q \).
\[ f^{-1}(\bar{V}) = \cdot \cdot \cdot \]
\[ f^{-1}(V) = \|\|\|\]
Now define \( V = \tilde{V} \cap (M \setminus \{ \tilde{\omega}_i \}) \). Since \( M \) is compact, \( M \setminus \{ \tilde{\omega}_i \} \) is compact. Therefore \( \{ (M \setminus \{ \tilde{\omega}_i \}) \} \) is closed and \( \tilde{V} \) is open, hence all pre-images of \( q \) are contained in \( \tilde{\omega}_i \). 

\( V \) is an open neighborhood of \( q \). See Figure 1 for visual.

Finally, take \( U_i = \tilde{U}_i \cap f^{-1}(V) \). Then \( U_i \) is a neighborhood of \( \tilde{\omega}_i \), \( f(U_i) = V \), and \( U_i \cap U_j = \emptyset \forall i \neq j \). Moreover, since \( \tilde{U}_i \) is an image of \( \tilde{U}_i \), \( f: U_i \to V \) is a diffeomorphism. Finally, by construction, \( f^{-1}(V) = \bigcup_{i=1}^n U_i \).

Therefore \( f \) satisfies the covering property, and \( \tilde{f} \) is a covering map. To show that \( f \) is a covering map, it only remains to show that \( \# f^{-1}(q) \) is independent of \( q \).

Fix some \( q \in N \). Define \( k_0 = \# f^{-1}(q) \) and

\[ \Omega = \{ q \in N : \# f^{-1}(q) = k_0 \} \]

Hence \( q \in \Omega \). \( \Omega \) is non-empty.

We claim that \( \Omega \) is both open and closed.

Consider some \( q \in \Omega \). Then \( \exists \) a neighborhood \( V \) of \( q \) and \( \tilde{\omega}_0 \) disjoint neighborhoods \( U_1, \ldots, U_n \subset M \) s.t. \( f: U_i \to V \) is a diffeomorphism, and \( f^{-1}(V) = U_1 \cup \cdots \cup U_n \). Then \( \forall \tilde{\omega} \in V \), \( f^{-1}(\tilde{\omega}) \) is a subset of \( U_1 \cup \cdots \cup U_n \). Since \( f: U_i \to V \) is bijective, \( \# f^{-1}(q) = |V| \). Therefore \( \# f^{-1}(q) = k_0 \). Thus \( q \in \Omega \) and \( \Omega \) is open.

Repeating the argument for \( q \notin \Omega \) implies that \( M \setminus N \setminus \Omega \) is open and hence \( \Omega \) is closed.

Since \( N \) is connected, this implies \( \tilde{\omega}_0 \notin N \) and so \( \# f^{-1}(q) \) is constant in \( q \).

Therefore \( f \) is a covering map.
(b) Define
\[ N = S' = \mathbb{R}/\mathbb{Z} \]
\[ M = (0, 2) \circ \mathbb{R} \]
Then \( N, M \) are connected, of the same dimension, and \( N \) is compact.

Let \( \pi \) be the quotient map \( \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} = S' \)
and \( f = \pi | M \). Then \( f \) is onto, smooth, and a local diffeomorphism.

In particular, \( f \) is a smooth submersion.

However, \( f^{-1}[0] = 1 \) and \( f^{-1}[1/2] = \{1/2, 3/2, 5/2\} \).

Therefore \( f^{-1}(q) \) is not constant and so \( f \) is not a covering map. \( \square \)
(a) By construction $X$ is diffeomorphic to $\mathbb{C}P^n$ via the map

$$[z_0: \ldots : z_n : 0: \ldots : 0] \mapsto [z_0: \ldots : z_n] \in \mathbb{C}P^n$$

Therefore $X$ is a smooth submanifold.

(b) We claim that $J_2(x, x) = 1$.

We recall that the mod 2 intersection $\#$ is invariant under homotopies. We then aim to define a homotopy of $X$.

Define $H_t$ by

$$H_t [z_0: \ldots : z_n : 0: \ldots : 0] = [(1-t)z_0: \ldots : (1-t)z_n : t \cdot z_0: \ldots : t \cdot z_n]$$

Then $H_0$ is the identity on $X$ and $H_1 [z_0: \ldots : z_n : 0: \ldots : 0] = [0: \ldots : 0: z_0: \ldots : z_n]$. Therefore $H_t$ is a homotopy from $X$ to $\hat{X} = \{ [0: \ldots : 0: z_0: \ldots : z_n] \}$. By homotopy invariance, this implies that

$$J_2(x,x) = J_2(x, \hat{x})$$

By definition,

$$X \cap \hat{X} = \{ [0: \ldots : 0: z_0: \ldots : 0] \} = \{ [0: \ldots : 0: 1: 0: \ldots : 0] \}$$

and

$$J_2(x, \hat{x}) = 1.$$
(a) It suffices to show the result locally.

By the amphi-form function, since any smooth embedding is locally the canonical embedding, we may choose local coordinates \(x_1, \ldots, x_n \in M\) s.t. \(x_1, \ldots, x_k\) are local coordinates for \(N\).

Denote \(v\), a vector field \(V\) in tangent to \(N\) iff it can be written as \(V = V_1 \partial/\partial x_1 + \cdots + V_k \partial/\partial x_k\).

Suppose \(\exists\) vector fields \(X, Y\) tangent to \(N\). Then we may write, in the coordinates above,

\[
X = x_1 \partial/\partial x_1 + \cdots + x_k \partial/\partial x_k = \sum_{j=1}^k x_j \partial/\partial x_j
\]

\[
Y = y_1 \partial/\partial x_1 + \cdots + y_k \partial/\partial x_k = \sum_{j=1}^k y_j \partial/\partial x_j
\]

Thus

\[
XY = \sum_{j=1}^k \left( \sum_{i=1}^k x_i \frac{\partial y_j}{\partial x_i} \right) \partial/\partial x_i
\]

\[
YX = \sum_{j=1}^k \left( \sum_{i=1}^k y_i \frac{\partial x_j}{\partial x_i} \right) \partial/\partial x_i
\]

and so

\[
[X, Y] = \sum_{j=1}^k \left( \sum_{i=1}^k (X_i \frac{\partial y_j}{\partial x_i} - Y_i \frac{\partial x_j}{\partial x_i}) \right) \partial/\partial x_i
\]

Therefore \([X, Y]\) is tangent to \(N\).

(b) Define

\[
X = x \partial/\partial y - y \partial/\partial x
\]

\[
Y = x \partial/\partial z - z \partial/\partial x
\]

we note that these are continuous but not smooth.

If we denote smooth vector fields

To check that there are tangent to \(S^2\), it suffices to check that \(\langle X, N \rangle = \langle Y, N \rangle = 0\) where \(N\) is a normal vector field:

\[
N = x \partial/\partial x + y \partial/\partial y + z \partial/\partial z \text{ to } S^2
\]
By direct computation, on $S^2$

\[
\langle x, y \rangle = -xy + xy = 0
\]
\[
\langle y, x \rangle = -zx + xz = 0
\]
Therefore $x, y$ are tangent to $S^2$.

By direct computation,

\[
[x, y] = xy - yx = y_2 \frac{\partial}{\partial y_2} - y_1 \frac{\partial}{\partial y_1}
\]
\[
= y_2 \frac{\partial}{\partial y_2} - y_1 \frac{\partial}{\partial y_1}
\]
and similarly,

\[
\langle [x, y], y \rangle = 0 + 2y - 2y = 0
\]
Therefore $[x, y]$ is tangent to $S^2$. \qed
Lemma: Suppose that $\nu^2n$ admits a symplectic form $\omega$.

Then $\text{H}^k_{dR}(M^{2n}) = 0$ for $k = 0, \ldots, n$.

Proof: Suppose for the sake of contradiction that $\nu \wedge \ldots \wedge \nu = d\eta$, for some $\eta, k$. Then $\nu \wedge \ldots \wedge \nu = d(\nu \wedge \nu \wedge \ldots \wedge \nu)$, hence $M$ is compact, orientable, $\text{H}^{2n}_{dR}(M) = \mathbb{R}$ with isomorphism $\Theta \mapsto \int_M \Theta$.

Then $\int_M \nu \wedge \ldots \wedge \nu = 0$, contradicting the fact that $\nu \wedge \ldots \wedge \nu$ is non-vanishing. Therefore $\nu \wedge \ldots \wedge \nu$ is not exact for all $k = 1, \ldots, n$.

Hence $M$ is connected, $\text{H}^0_{dR}(M) = \mathbb{R} \neq 0$.

(a) We recall that

\[ \text{H}^k_{dR}(S^8) = \begin{cases} \mathbb{R} & k = 0, 8 \\ 0 & \text{else} \end{cases} \]

Therefore $S^8$ does not admit a symplectic form by the lemma.

(b) By the Künneth formula,

\[ \text{H}^k_{dR}(S^2 \times S^6) = \begin{cases} \mathbb{R} & k = 0, 2, 6, 8 \\ 0 & \text{else} \end{cases} \]

Therefore $\text{H}^8_{dR}(S^2 \times S^6) = 0$ and so $S^2 \times S^6$ does not admit a symplectic form.
(c) We claim that $S^2 \times S^2 \times S^2 \times S^2$ does admit a symplectic form.

Let $\Omega$ denote a volume form on $S^2$.

For each coordinate "copy" of $S^2$ in $S^2 \times S^2 \times S^2 \times S^2$, we let $
abla_i, \nabla_2, \nabla_3, \nabla_4$ denote these volume forms in $S^2 \times S^2 \times S^2 \times S^2$.

Thus $\nabla_i \wedge \nabla_j \neq 0$ for $i \neq j$.

Let $w = \nabla_1 + \nabla_2 + \nabla_3 + \nabla_4$. Then since $\nabla_i$ is a 2-form

$$w \wedge w = 2(\nabla_1 \wedge \nabla_2 + \nabla_1 \wedge \nabla_3 + \nabla_1 \wedge \nabla_4 + \nabla_2 \wedge \nabla_3 + \nabla_2 \wedge \nabla_4 + \nabla_3 \wedge \nabla_4)$$

$$\Rightarrow w \wedge w \wedge w = 4 \nabla_1 \wedge \nabla_2 \wedge \nabla_3 \wedge \nabla_4$$

which is non-vanishing. Therefore $w$ is a symplectic form on $S^2 \times S^2 \times S^2 \times S^2$. 

$\square$
The classical divergence theorem states
\[ \sum_{\partial \Omega} \left( \langle \nu, \mathbf{n} \rangle d\mathbf{n} \right) = \int_{\Omega} \mathbf{div}(\mathbf{v}) d\Omega \]
where \( \mathbf{n} \) is the unit normal at \( \partial \Omega \), \( \mathbf{v} \) is the boundary orientation, \( \langle \nu, \mathbf{n} \rangle \) is the standard inner product, and \( d\mathbf{n} \) is the induced surface measure form on \( \partial \Omega \) given by
\[ d\mathbf{n} = \nu^i d\mathbf{n}_i \]
where \( \nu : \partial \Omega \to \mathbb{R}^3 \) is the inclusion map.

Define \( T = \nu - \langle \nu, \mathbf{n} \rangle \mathbf{n} \) on \( \partial \Omega \). Then \( T \) is the tangential component of \( \mathbf{v} \) to \( \partial \Omega \). We claim that \( \nu^i d\mathbf{n}_i = 0 \) on \( \partial \Omega \).

Suppose \( \nu^i \) are linearly independent vector fields on \( \partial \Omega \).

Then \( \nu^i d\mathbf{n}_i (x, y) = \nu^i d\mathbf{n}_i (T, x, y) \). Since \( \partial \Omega \) is 2-dimensional and \( T \in T \partial \Omega \), \( [T, x, y] \) must be linearly dependent. Therefore
\[ d\mathbf{n}_i (T, x, y) = 0 \]
and so \( \nu^i d\mathbf{n}_i (x, y) = 0 \). As this holds \( \forall x, y \),
the implies \( \nu^i d\mathbf{n}_i = 0 \).

By linearity, this implies
\[ 0 = \nu^i d\mathbf{n}_i = \nu^i (i^* d\mathbf{v} - \langle \nu, \mathbf{n} \rangle i^* d\mathbf{v}) \]
\[ = \nu^i d\mathbf{v} - \langle \nu, \mathbf{n} \rangle d\mathbf{A} \]
\[ \Rightarrow \langle \nu, \mathbf{n} \rangle d\mathbf{A} = \nu^i d\mathbf{v} \]

By Stokes' theorem, this implies
\[ \sum_{\partial \Omega} \langle \nu, \mathbf{n} \rangle d\mathbf{A} = \int_{\Omega} \nu^i d\mathbf{v} = \int_{\partial \Omega} \nu^i d\mathbf{v} = \int_{\partial \Omega} \nu^i d\mathbf{v} \]
Let \( V = V_1 \frac{\partial}{\partial x} + V_2 \frac{\partial}{\partial y} + V_3 \frac{\partial}{\partial z} \), and with \( d\omega_1 = dx \wedge dy \wedge dz \).

Then by direct computation

\[
\iota_V d(\omega_1) = V_1 dy \wedge dz - V_2 dx \wedge dz + V_3 dx \wedge dy
\]

\[
\Rightarrow d(\iota_V d(\omega_1)) = \left( \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \right) dx \wedge dy \wedge dz
\]

\[
= \iota_V(\nabla \cdot V) d(\omega_1)
\]

Therefore

\[
\int_{\partial \Omega} \langle u, \omega \rangle dA = \int_{\Omega} \iota_u (\nabla \cdot V) d(\omega_1)
\]

as desired.

\( \Box \)
Part (a) is an immediate application of Van Kampen's lemma. Part (b) is an immediate application of Meyer-Vietoris.

(a) Let $B_1, B_2$ be open balls in $M, N$ respectively.

We define $M \# N$ to be $M \setminus B_1 \cup N \setminus B_2$ glued at the boundary of $B_1, B_2$.

Take $U$ to be an $\varepsilon$-neighborhood of $M \setminus B_1$ which deformation retracts to $M$ and $V$ similarly for $N \setminus B_2$.

Then $U \cup U$ is simply connected, $\pi_1(U) \cong \pi_1(M \setminus B_1)$, and $\pi_1(U) \cong \pi_1(N \setminus B_2)$. Therefore by Van Kampen, $\pi_1(M \# N) \cong \pi_1(M \setminus B_1) \ast \pi_1(N \setminus B_2)$.

Since $n \geq 3$, $\dim(B_i) \geq 3$ and so $\pi_1(B_i) = 0$.

Therefore $\pi_n(M \setminus B_1) \cong \pi_n(M)$ and $\pi_n(N \setminus B_2) \cong \pi_n(N)$.

Then $\pi_n(M \# N) \cong \pi_n(M) \ast \pi_n(N)$.

(b) Meyer-Vietoris.
(a) Let \( B_1 \) be an open ball in \( M \) and \( B_2 \) an open ball in \( N \). Then we define the connected sum \( M \# N \) as

\[
M \# N = \left( M \setminus B_1 \right) \cup \left( N \setminus B_2 \right) / \sim
\]

where \( \sim \) is defined in \( \overline{B}_1 \cup \overline{B}_2 \).

Let \( U \) be an open neighborhood of \( M \setminus B_1 \), that deformation retracts onto \( M \setminus B_1 \), and similarly \( V \) for \( N \setminus B_2 \). Then \( U \cup V = M \# N \) and \( U \cup V \) deformation retracts onto \( \overline{B}_1 \cup \overline{B}_2 \cong S^{n-1} \).

Since \( n \geq 3 \), \( S^{n-1} \) is simply connected. Therefore \( U \cup V \) is simply connected and Van Kampen's implies that, since the base point is in \( U \cup V \),

\[
\pi_1(M \# N) \cong \pi_1(M \setminus B_1) * \pi_1(N \setminus B_2)
\]

Now consider \( M \). Let \( U, B_1, V \) an open neighborhood of \( M \setminus B_1 \) and \( V = B_1 \). Then \( M = U \cup V \) and \( U \cup V \) deformation retracts onto \( S^{n-1} \). Then by Van Kampen's, \( \pi_1(M) \cong \pi_1(M \setminus B_1) * \pi_1(B_1) \cong \pi_1(M \setminus B_1) \)

since \( B_1 \) is simply connected. Therefore \( \pi_1(M) \cong \pi_1(M \setminus B_1) \).

Similarly \( \pi_1(N) \cong \pi_1(N \setminus B_2) \). Then

\[
\pi_1(M \# N) \cong \pi_1(M) * \pi_1(N)
\]

as desired.

(b) Since \( M, N \) are connected and orientable, \( M \# N \) is connected and orientable. Therefore

\[
H_0(M \# N) \cong H_0(M \# N) \cong \mathbb{Z}
\]
We first claim that all maps \( f : S^2 \to S^1 \times S^1 \) are nullhomotopic.

We recall that the universal cover of \( S^1 \times S^1 \) is \( \mathbb{R}^2 \) via the quotient map \( \pi : \mathbb{R}^2 \to \mathbb{R}^2 / \mathbb{Z}^2 \). We claim that \( f : S^2 \to S^1 \times S^1 \) lifts to a map \( \tilde{f} : S^2 \to \mathbb{R}^2 \).

Indeed, since \( S^2 \) is simply connected, \( \pi_1(S^2) \cong 0 \) and \( \pi_1(\mathbb{R}^2) \cong 0 \), and in particular \( \pi_k(S^2) \cong \pi_k(\mathbb{R}^2) \) for \( k \geq 0 \).

Therefore \( \tilde{f} \) lifts to a map \( \tilde{f} : S^2 \to \mathbb{R}^2 \).

Since \( \mathbb{R}^2 \) is contractible, \( \tilde{f} \) is nullhomotopic and \( f \) a homotopy \( \tilde{f} \cdot \tilde{f} \) from \( \tilde{f}_0 = \tilde{f} \) to \( \tilde{f}_1 = 0 \). Then \( \pi_1(f) \) is a homotopy from \( f = \pi_1(f) \) to the constant map \( \pi_1(0) \). Therefore \( f \) is nullhomotopic.

We recall that degree is homotopy invariant and is go defined by the induced map \( \tilde{f}_* : H_2(S^2) \to H_2(S^1 \times S^1) \).

Since the constant map induces the 0 map on top homology, this implies that \( \deg f = 0 \). A continuous \( f : S^2 \to S^1 \times S^1 \).
We recall that $S^2$ can be constructed via 10-cell and 12-cell attached to the 0-cell. Also, we recall that a cartesian product of CW complexes can be constructed via the cartesian product of the cellular complexes. Therefore, by giving one copy of $S^2$ the cell structure 10-cell: $p$, 12-cell: $f_i$, and the other $p_0, f_2$, we find that $S^2 \times S^2$ has the following cellular structure:

1. 0-cell: $p = (p_0, p_2)$

2. 2-cells: $a = (f_1, p_2), b = (p_1, f_2)$ 
   \[ \partial a = (\partial f_1, p_2) = (p_1, p_0) = p \]
   \[ \partial b = (p_1, \partial f_2) = (p_1, p_2) = p \]

3. 4-cell: $A = (f_1, f_2)$ 
   \[ \partial A = (\partial f_1, f_2) - (f_1, \partial f_2) \]
   \[ = b - a = 0 \]

By adjoining $p, p_0$ are the south poles of their respective copies of $S^2$, this implies that $S^2 \times S^2 \rightarrow S^2 \times S^2$ via $S^2 \times S^2 \cong a \vee b$ with the wedge at point $p$, as done in $S^2 \times S^2$.

(b) As given, we attach a 3-cell to $S^2 \times S^2$ via the map $S^2 \rightarrow S^2 \vee S^2$ which creates a great circle connecting the north and south poles. Therefore, using the structure from part (a), we now have the structure:

1. 0-cell: $p$

2. 2-cells: $a, b$ 
   \[ \partial a + \partial b = p = 0 \]

3. 3-cell: $g$ 
   \[ \partial g = a + b \]

4. 4-cell: $A$ 
   \[ \partial A = a - b = 0 \]
Our homology groups are thus

\[ H_0(X) = \frac{\ker d_0}{\text{im} \, d_1} = \mathbb{Z} \langle p \rangle \cong \mathbb{Z} \]

\[ H_1(X) = \frac{\ker d_2}{\text{im} \, d_3} = \mathbb{Z} \langle a, b \rangle = \mathbb{Z} \langle a + b \rangle = \mathbb{Z} \langle a, -a \rangle \cong \mathbb{Z} \]

\[ H_2(X) = \frac{\ker d_3}{\text{im} \, d_4} = 0 \]

\[ H_3(X) = \frac{\ker d_4}{\text{im} \, d_5} = \mathbb{Z} \langle A \rangle \cong \mathbb{Z} \]

\[ H_4(X) = \frac{\ker d_5}{\text{im} \, d_6} = 0 \]

and all other homology groups are 0.
(a) By definition, the universal cover \( \tilde{X} \) is simply connected.

Therefore \( \pi_1(\tilde{X}) = 0 \) and so if \( p: \tilde{X} \to X \) is the covering map then \( p_\ast(\pi_1(\tilde{X})) = 0 \subset \sigma_\ast(\pi_1(\Delta)) \).

(a) We recall that the n-sphere \( S^n \) is simply connected.

Therefore \( \pi_1(S^n) = 0 \) and in particular \( \sigma_\ast(\pi_1(S^n)) = 0 \).

Then \( \sigma_\ast(\pi_1(S^n)) \subset p_\ast(\pi_1(\tilde{X})) \) where \( p: \tilde{X} \to X \) is the covering map.

The lifting criterion then implies \( \tilde{f} \) a lift \( \tilde{f}: S^n \to \tilde{X} \) as claimed.

(b) We recall that

\[ \sigma = p \circ \tilde{f} = p \circ \tilde{f}_2. \]
Suppose that $\Theta = df$ for some $F$. We recall that

$\int_{S'} i^*\Theta = \int_{S'} d i^* F$.

Then since $d i^* F$ is exact on $S'$, it follows that $\int_{S'} i^* \Theta = \int_{S'} d i^* F = 0$ as desired.

Suppose instead that $\int_{S'} i^* \Theta = 0$. Let $Y$ be a closed path on $S' \times (-1,1)$. Then since $S' \times (1,1)$ contracts to $S' \times \{0\}$ either $Y$ is contractible or is homotopic to $k$ copies of $S' \times \{0\}$. In either case, since $\int_{S'} i^* \Theta = 0$ it follows that $\int_Y \Theta = 0$. We now define $F$ by

$\Theta = df$. Fix some $x_0 \in S' \times (-1,1)$.

Since $S' \times (-1,1)$ is path connected, $\forall x \in S' \times (-1,1)$, $\exists$ a path $Y_x$ from $x_0$ to $x$. Define $F(x) = \int_{Y_x} \Theta$.

Since $\int_Y \Theta = 0$ $\forall$ closed paths, if $Y_1, Y_2$ are two paths from $x_0$ to $x$, then $\int_{Y_1 - Y_2} \Theta = 0$.

Therefore $F$ is well-defined independent of the path chosen and $w$ is well-defined. The smoothness of the integral $\Theta$ and $\Theta$ then imply that $F$ is smooth.

It remains to show $df = w$. For any piecewise smooth $Y$ curve, the FTC implies

$$\int_Y dF = F(y(1)) - F(y(0)) = F(x) - F(x_0) = 0.$$
a) It suffices to construct \( f_k : F^k(U \cap V) \to F^k(U) \otimes F^k(V) \) and \( g_k : F^k(U) \otimes F^k(V) \to F^k(U \cap V) \), i.e., \( f_k, g_k \) commute with \( d \) in the sense that 
\[
 d \circ f_k = f_k \circ d, \quad \text{and} \quad 0 \to F^k(U \cap V) \xrightarrow{f_k} F^k(U) \otimes F^k(V) \xrightarrow{g_k} F^k(U \cap V) \to 0
\]
is exact.

Define \( f_k : F^k(U \cap V) \to F^k(U) \otimes F^k(V) \) by \( f_k(w) = (w_{LU}, w_{LV}) \).

Clearly \( f_k \) is \( C^0 \)-linear and 
\[
 d(f_k(w)) = d(w_{LU}, w_{LV}) = (dw_{LU}, dw_{LV}) = f_k(dw).
\]

Define \( g_k : F^k(U) \otimes F^k(V) \to F^k(U \cap V) \) by \( g_k(w, n) = w_{LU} - n_{LV} \).

Then \( g_k \) is \( C^0 \)-linear and 
\[
 d(g_k(w, n)) = dw_{LU} - dn_{LV} = g_k(dw, dn).
\]

We now show the sequence is exact. Consider \( w \in F^k(U \cap V) \)
then it suffices to show \( g_k(f_k(w)) = 0 \). By construction,
\[
g_k(f_k(w)) = g_k((w_{LU}, w_{LV})) = w_{LU} - w_{LV} = 0
\]
Therefore the sequence is exact.
a) Since $GL^+ \subset \mathbb{R}^n$ is open, it has the structure of a smooth manifold. Therefore, to show $S^1$ is a submanifold, it suffices to show it is the preimage under the regular value theorem. Define $F: GL^+ \to \mathbb{R}$ by $A \mapsto \det(A)$. Then $S^1 = F^{-1}(1)$. We want to show 1 is a regular value of $F$. 
Let \( x_1, \ldots, x_k \) denote the removed points. Around each \( x_i \), there exists a neighborhood which deformation retracts onto \( S^2 \) centered at \( x_i \). Then, since the usual two-holed torus deformation retracts onto \( S' \cup S' \cup (V_{k=1}^k S^2) \), it follows that \( X \) deformation retracts onto \( S' \cup S' \cup (V_{k=1}^k S^2) \).

Van Kampen's theorem implies that \( \pi_1(X) \cong \mathbb{Z} \times \mathbb{Z} \).

b) Since \( X \) is connected, \( H_0(X) \cong \mathbb{Z} \). Since \( H_1(X) \) is the abelianization of \( \pi_1(X) \), \( H_1(X) \cong \mathbb{Z}^2 \).

Since \( X \) deformation retracts onto a cell structure with only containing 0, 1, 2 cells, \( H_2(X) = 0 \).

Finally, since \( X \) deformation retracts onto \( S' \cup S' \cup (V_{k=1}^k S^2) \), it follows that \( H_2(X) \cong \mathbb{Z}^k \). For \( n \geq 3 \), \( H_n(X) = 0 \).

This follows more rigorously by constructing \( S' \cup S' \cup (V_{k=1}^k S^2) \) as:

1. 0-cells \( p \)
2. 1-cells \( a, b \) attached via \( da = db = p \cdot p \)
3. 2-cells \( f_0, \ldots, f_k \) attached via \( df_i = p \)

Thus \( \partial_2 = 0, \partial_1 = 0 \Rightarrow H_2(X) \cong \mathbb{Z}^k, H_1(X) \cong \mathbb{Z}^2, H_0(X) = \mathbb{Z} \).
(a) Fix two points \( p, q \in \mathbb{R}^n \setminus M^m \) and let \( \gamma : [0,1] \to \mathbb{R}^n \) be a smooth path from \( \gamma(0) = p \) to \( \gamma(1) = q \).

Then \( \gamma \) is a 1-dimensional submanifold of \( \mathbb{R}^n \).

Also \( p, q \notin M^m \), \( \gamma(0) \), \( \gamma(1) \) are transversal to \( M^m \).

Therefore since \( \gamma([0,1]) \) is closed, the extension theorem implies that \( \exists \tilde{\gamma} : [0,1] \to \mathbb{R}^n \) homotopic to \( \gamma \) s.t.

\( \tilde{\gamma}(0) = p \), \( \tilde{\gamma}(1) = q \) and \( \tilde{\gamma} \) is transversal to \( M^m \).

Suppose \( \exists \tilde{\gamma} \) s.t. \( \tilde{\gamma}(t) \in M^m \). Then

\[ \text{im} d\tilde{\gamma} + T_{\tilde{\gamma}(t)} M^m = T_{\tilde{\gamma}(t)} \mathbb{R}^n \equiv \mathbb{R}^n \]

Thus \( \dim (\text{im} d\tilde{\gamma}) + \dim T_{\tilde{\gamma}(t)} M^m \geq n \)

\( \Rightarrow 1 + m > n \quad \Rightarrow m > n - 1 \)

which contradicts the fact that \( m \leq n - 2 \).

Therefore \( \exists \tilde{\gamma} \) s.t. \( \tilde{\gamma}(t) \notin M^m \) and \( \tilde{\gamma} \in \mathbb{R}^n \setminus M^m \),

as such a path \( \tilde{\gamma} \) can be constructed \( \forall p, q \in \mathbb{R}^n \setminus M^m \), this implies

(b)
(a) By definition, \( O \) is a closed, connected, oriented, n-manifold. Therefore, \( H^*_\partial(O) \cong \mathbb{R} \) via isomorphism given by
\[
\eta \mapsto \int_{\partial M} \eta
\]
Let \( \omega \) be an \( n \)-form on \( M \). To show that it is exact, it then suffices to show that \( \int_{\partial} \pi^* \omega = 0 \).

Let \( f \) be the deck transformation on \( O \) corresponding to the covering map \( \pi \). Since \( \omega \) is an orientation cover, \( f \) has degree -1, and it takes positive orientation to negative orientation. Additionally, \( \pi \circ f = \pi \). Therefore
\[
\int_{\partial} \pi^* \omega = \int_{\partial} \pi^* (\pi \circ f)^* \omega = \int_{\partial} f^* \pi^* \omega
\]
and \( \omega \mid_{\partial} = 0 \). Thus \( \pi^* \omega \) is exact.

(b) To show that \( \omega \) is exact, we construct an \( n \)-form \( \eta \) on \( M \)

Assume \( \pi \) is a 2-fold cover, \( \forall \, p \in M \). \( \exists \, q_1, q_2 \in O \), \( \pi(\{q_1, q_2\}) = p \) and \( \forall \) neighborhoods \( U_p \) of \( p \), \( V_i \) of \( q_i \), \( \forall \) \( \pi \big|_{V_i} \) is a diffeomorphism by
\[
\eta \big|_{U_p} = \frac{1}{2 \epsilon_i} \left( \pi \big|_{V_i} \right)^{-1*} \Theta \big|_{V_i}
\]
Then, on $U_p$,
\[
\begin{align*}
\left. d\eta \right|_{U_p} &= \frac{1}{2} \sum_{i=1}^{2} (\pi_i v_i)^* \omega \\
&= \frac{1}{2} \sum_{i=1}^{2} (\pi_i v_i)^* \pi_i^* \omega \\
&= \frac{1}{2} \sum_{i=1}^{2} \omega \\
&= \omega
\end{align*}
\]

We then extend $\eta$ to all of $M$ via a standard locally finite cover and partition of unity.

Alternatively, we recall that since $\emptyset$ is a finite-echelon cover, $\pi$ induces an injection on deRham cohomology. Therefore, $\pi^* : H^\omega_{dR}(M) \hookrightarrow H^\omega_{dR}(\emptyset)$ is injective. As shown in part a, $\pi^* \omega = 0$ on the level of cohomology. Thus, $H^\omega_{dR}(M) = 0$.

Therefore, $H^\omega_{dR}(M) = 0$ and thus all $n$-forms are closed.
Let \( \Phi: \mathbb{R}^3 \rightarrow \mathbb{R} \) be smooth such that \( \Phi = 0 \) on \( [1,1/2] \) and \( \Phi = 1 \) on \( (1/2,1) \).

Define
\[
F: \mathbb{D}^n \rightarrow \mathbb{D}^n \quad \text{s.t.} \quad F(x) = \begin{cases} \Phi(\frac{x}{|x|}) f(\frac{x}{|x|}) & x \neq 0 \\ 0 & x = 0 \end{cases}
\]

For \( x \neq 0 \), \( x \mapsto \frac{x}{|x|} \) is smooth and \( x \mapsto f(x) \) is smooth. Therefore, by

By construction of \( \Phi \), \( F = 0 \) on a neighborhood of \( 0 \). Therefore \( F \) is smooth at \( 0 \), and hence smooth on \( \mathbb{D}^n \).

Finally, by construction, for \( x \in S^{n-1} \), \( \frac{x}{|x|} = x \) and \( \phi(\frac{x}{|x|}) = 1 \)

so \( F(x) = f(x) \) as desired.

($\Rightarrow$) Suppose that $\omega = \lambda df$ locally.

Then locally,

$$\omega \wedge dw = \lambda df \wedge d\lambda nf = -\lambda d\lambda (df \wedge dt)$$

hence $dt$ is a 1-form

$$df \wedge dt = -df \wedge dt$$

and so $df \wedge dt = 0 \Rightarrow \omega \wedge dw = 0$.

($\Leftarrow$) Suppose initial that $\omega \wedge dw = 0$.

For each point $p$ a neighborhood $U$ of $p$ and coordinates $x_1, \ldots, x_n$.

Then we claim that $\ker \omega$ is integrable.

It suffices to show that if $x, y \in \ker \omega$ then $[x, y] \in \ker \omega$. We recall that

$$d\omega(x, y) = x \omega(y) - y \omega(x) - \omega([x, y])$$

If $x, y$ are linearly dependent, then $d\omega(x, y) = -\omega([x, y]) = 0$ which concludes. Otherwise $\mathcal{C}$ is locally flat locally $\mathcal{C}$ is a $n$-form $x, y, z$ form a basis. Then are linearly independent and $\omega \wedge dw = 0$

$$O = \omega \wedge dw(x, y, z) = \omega(z) \wedge dw(x, y) = d\omega(x, y) = 0$$

hence $\ker \omega$ has codimension 3. Then $\ker \omega$ is integrable.

By definition, the implies a local coordinate so that

$$\ker \omega = R \langle \partial / \partial x_1, \ldots, \partial / \partial x_n \rangle \Rightarrow \omega = \lambda d\tilde{x}_n$$

for some $\lambda$. Taking $f = \tilde{x}_n$ and optimizing via a thin concludes.
(a) Let \( \{ U_k, \psi_k \} \) be an atlas for \( H \). We aim to use \( \{ U_k, \psi_k \} \) to construct an atlas \( \{ V_b, \psi_b \} \) for \( M \). Let \( H_1, H_2 \) denote the 2 handlebodies in \( M \) and let \( \{ U_k, \psi_k^1, \psi_k^2 \} \) be their associated atlases.

For all \( k \) and \( l \), let \( U_k \cap \partial H = \emptyset \).

Define
\[
A = \left\{ (U_k, \psi_k^1), (U_k, \psi_k^2) \quad \forall k \neq \ell, U_k \cap \partial H = \emptyset \right\} \\
\quad \quad \cup \left\{ (U_k \cup U_l, \psi_k) \quad \forall k \neq \ell, U_k \cap \partial H \neq \emptyset \right\}
\]

where \( \psi_k : U_k \cup U_l \to V_b \) is defined by
\[
\psi_k(p) = \begin{cases} 
\psi_k^1(p) & p \in U_k \\
-\psi_k^2(p) & p \in U_l
\end{cases}
\]

Since \( \psi_k^1, \psi_k^2 \) map diffeomorphically into \( \mathbb{H}^3 \), \( \psi_k : U_k \cup U_l \to V_b(U_k \cup U_l) \) is diffeomorphic (and symmetric across \( \partial \mathbb{H}^3 \)). We note that on \( A \), the only overlaps of \( U_k \cup U_l \) have the transition maps in \( H_1, H_2 \) are smooth,

this gives smooth transition maps on \( M \).

Therefore \( M \) is a smooth 3-manifold.

Since \( H_1, H_2 \) are compact, \( M \) is compact. Hence \( H_1, H_2 \) are identified at their boundary, \( M \) is boundaryless.

Therefore \( M \) is closed.
(b) We proceed by Mayer–Vietoris.

Let $U$ be an $\varepsilon$-neighborhood of $H,C,M$ that deformation retracts onto $H_1$ and similarly for $V \cup H_2$.

Then $UV = M$ and $UV$ deformation retracts onto $\partial H_1 = \partial H_2 \subset M$. Then we get a long exact sequence

$$\cdots \rightarrow H_k(M_3) \xrightarrow{(i^*)} H_k(H) \oplus H_k(H) \xrightarrow{k^* - l^*} H_k(M) \xrightarrow{j^*} \cdots$$

where $i : UV \rightarrow U$, $j : UV \rightarrow V$, $k : U \rightarrow M$, $l : V \rightarrow M$ and $d$ is the map boundary map from the snake lemma.

We recall that $M_3$ has homology

$$H_k(M_3) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ \mathbb{Z}^{2g} & k = 1 \\ 0 & \text{else} \end{cases}$$

For $H_3$, we note that $H$ is the union of a genus $g$ surface. Therefore $H$ deformation retracts onto the wedge of $g$ circles $V \sqcup S^1$. Then

$$H_k(H) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}^g & k = 1 \\ 0 & \text{else} \end{cases}$$

which gives an exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow H_2(M) \rightarrow$$

$$\rightarrow \mathbb{Z}^{2g} \rightarrow \mathbb{Z}^{2g} \rightarrow H_1(M) \rightarrow$$

$$\rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow H_0(M) \rightarrow 0$$

we note $H_0(M) \cong \mathbb{Z}$ since $M$ is connected.
(b) We can construct a genus $g$ surface by

1. 0-cells: $p$

2. 1-cells: $a_1, b_1, \ldots, a_g, b_g$ with $\partial a_i = \partial b_i = p - p$

3. 2-cells: $f_1, \ldots, f_g$ with $\partial_2 f_i = a_i + b_i - a_i - b_i + c_i$

4. 3-cells $A_1, \ldots, A_g$ with $\partial_3 A_i = f_i$

To construct a handlebody $H$, it is necessary to add additional 1-cells and 2-cells. We do this by

1. 0-cell $p$

2. 1-cells $a_1, b_1, c_1, \ldots, a_g, b_g, c_g$ with $\partial a_i = \partial b_i = \partial c_i = p - p$

3. 2-cells $f_1, \ldots, f_g$ with $\partial_2 f_i = a_i + b_i - a_i - b_i + c_i$

4. 3-cells $A_1, \ldots, A_g$ with $\partial_3 A_i = f_i$

This gets the notation/index "bull"
(b) We can construct a genus $g$ surface via

1. 0-cells: $p$

2. 1-cells: $a_1, b_1, \ldots, a_g, b_g$ s.t. $\partial a_i = \partial b_i = p - p$

3. 2-cells: $t_1, \ldots, t_g$ s.t. $\partial t_i = \partial$
(a) We recall that the Mobius strip has a cellular decomposition as follows.

1. 2 0-cells: p, q
2. 1-cell: a, b, c
3. 2-cell: f

\[ \partial_1 a = p - q, \quad \partial_1 b = q - p, \quad \partial_1 c = q - p \]

By the boundary of \( M \), we can express \( X \) via the cellular decomposition.

1. 2 0-cells: p, q
2. 1-cells: a, b, c as above (\( \partial_1 a = q - p \))
3. 2 1-cells: f, g

\[ \partial_2 f = a + c - b + c = a - b + 2c \]
\[ \partial_2 g = a - b - 2d \]

We note that in the above decomposition, \( f + g \) form a cylinder which is attached on one end to c-d and the other end to d-c. Therefore, we may simplify the cellular decomposition to:

1. 0-cell: p
2. 1-cell: a, b
3. 2-cell: f

\[ \partial_1 a = \partial_1 b = p - p = 0 \]
\[ \partial_2 f = a - b - a - b = -2b \]

Hence \( X \) then only has 1 0-cell. \( \pi_1(X) \) has a representation with generators a, b, and relations given by \( \partial_1 f \). Then

\[ \pi_1(X) = \langle a, b | aba^{-1}b^{-1} = 1 \rangle = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \]

which is what was to be found.
(b) From the result in (a), denote the compact orientable surface of genus 2. We recall that \( \chi(M_g) = 2 - 2g \).

From the above cellular decomposition, \( \chi(X) = 1 - 2 + 1 = 0 \).

Therefore, \( X \) is homotopy equivalent to \( M_g \) then \( g = 1 \).

Hence, Euler characteristic is homotopy equivalent invariant.

However, we can construct \( H_i \) as

\[
\begin{align*}
1 & \quad \text{all } x^i \\
2 & \quad \text{cells: } a, b \quad \text{with } da = 2b = p - P \\
1 & \quad 2\text{-cell: } f \quad \text{with } df = a + 6 - a - b = 0
\end{align*}
\]

Therefore, \( H_i(X) \cong \langle a, b | aba^{-1}b^{-1} \rangle \cong \mathbb{Z}^2 \).

However, \( \mathbb{Z}^2 \neq \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) and so \( H_i \) and \( X \) are not homology equivalent.

Since \( H_i \) is homotopy invariant.

\[\square\]

Alternatively, from (a),

\[ H_i(X) \cong \langle a, b | aba^{-1}b^{-1} \rangle \]

\[ \langle aib | aba^{-1}b^{-1}, aba^{-1}b^{-1} \rangle \]

so \( ab = ba \) and \( a = bab \Rightarrow b^2 = e \). Then

\[ H_i(X) \cong \langle aib | b^2, aba^{-1}b^{-1} \rangle \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \]

However \( H_i(M_g) \cong \mathbb{Z}^2 \) since homology is homotopy invariant.

Thus concludes. \[\square\]
We recall that the connected covers of $\mathbb{RP}^n$ are $S^n$, which is a double cover via the antipodal identification, and $\mathbb{RP}^n$, which is trivially a single cover via the identity. We note that there are the only covers of $\mathbb{RP}^n$ because the universal cover of $S^n$ is $S^n$, and so any additional covers of $\mathbb{RP}^n$ would contradict this.

Consider $\mathbb{RP}^n \times \mathbb{RP}^n$. A wedge point $p$. Any cover of $\mathbb{RP}^n \times \mathbb{RP}^n$ will consist of wedges of covers of $\mathbb{RP}^n, \mathbb{RP}^n$ respectively, where the wedge a region covering $\mathbb{RP}^n$ to a region covering $\mathbb{RP}^n$.

Each of which can be attached to a cover of $\mathbb{RP}^n$. On the other hand, since $\mathbb{RP}^n$ is a single cover of $\mathbb{RP}^n$, and so each copy of $\mathbb{RP}^n$ in the cover can only be attached to a single copy of $\mathbb{RP}^n$. The same reasoning holds for $m$ of $\mathbb{RP}^n, S^n, S^m, \mathbb{RP}^m$, of the form alternating in $n, m$. These are of the form

\[
\begin{align*}
\mathbb{RP}^n & \quad \cdots \quad S^n \quad \mathbb{RP}^m \\
S^m & \quad \cdots \quad S^m \quad \mathbb{RP}^n \\
\mathbb{RP}^n & \quad \cdots \quad S^n \quad \mathbb{RP}^m
\end{align*}
\]

\[
\begin{align*}
\mathbb{RP}^n & \quad S^m \quad \cdots \quad S^m \quad \mathbb{RP}^n \\
S^m & \quad \cdots \quad S^m \quad \mathbb{RP}^n \\
\mathbb{RP}^n & \quad S^n \quad \cdots \quad S^n \quad \mathbb{RP}^m
\end{align*}
\]

\[
\begin{align*}
\mathbb{RP}^n & \quad \cdots \quad S^n \quad \mathbb{RP}^m \\
S^m & \quad \cdots \quad S^m \quad \mathbb{RP}^n \\
\mathbb{RP}^m & \quad \cdots \quad S^n \quad \mathbb{RP}^m
\end{align*}
\]
we note that $D_1$ is attached to $T^2$ via a degree $p$ map along one coordinate.

$D_2$ is attached to $T^2$ via a degree $q$ map along another coordinate.

we can then construct $X$ as follows:

1. cell: $u$

2. 1-cell: $a, b$ such that $a = b = \alpha_1 u = 0$

3. 2-cell: $f_1 D_1, D_2$ such that $f_1 = a + b - a - b = 0$

which yields the chain complex:

$$
\cdots \xrightarrow{\partial_3} C_3(X) \xrightarrow{\partial_2} C_2(X) \xrightarrow{\partial_1} C_1(X) \xrightarrow{\partial_0} C_0(X) \xrightarrow{\partial} 0
$$

$$
\mathbb{Z}^3 \xrightarrow{\partial_3} \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z}
$$

This yields the following homology groups:

$$
H_0(X) = \frac{k \mathbb{Z} \partial_0}{\text{im } \partial_1} = \frac{\mathbb{Z}<u>}{\mathbb{Z}} = \mathbb{Z}
$$

$$
H_1(X) = \frac{k \mathbb{Z} \partial_1}{\text{im } \partial_2} = \frac{\mathbb{Z}<a, b>}{\mathbb{Z}<a, b> \oplus \mathbb{Z}<a, b> \oplus \mathbb{Z}<a, b> \oplus \mathbb{Z}<a, b> \oplus \mathbb{Z}<a, b> \oplus \mathbb{Z}<a, b> \oplus \mathbb{Z}<a, b> \oplus \mathbb{Z}<a, b>} = \mathbb{Z}/p^2 \oplus \mathbb{Z}/q^2
$$

$$
H_2(X) = \frac{k \mathbb{Z} \partial_2}{\text{im } \partial_3} = \frac{\mathbb{Z}<f>}{\mathbb{Z}} = \mathbb{Z}
$$

and 0 for all higher homology. Therefore

$$
H_k(X) = \begin{cases} 
\mathbb{Z} & k = 0, 1
\mathbb{Z}/p^2 \oplus \mathbb{Z}/q^2 & k = 2
0 & \text{else}
\end{cases}
$$
(Q-1) Let $M$ be a compact smooth $n$-manifold, and $f: M \to \mathbb{R}^n$ a smooth map. Let

$$S = \{ p \in M \mid \text{rank}(df_p) < n \}.$$

(a) Prove $S \neq \emptyset$.
(b) Prove $f(S) \subset \mathbb{R}^n$ has empty interior.

(Q-2) Let $M_n$ be the space of $n \times n$ real matrices, viewed as the smooth manifold $\mathbb{R}^{n^2}$. Let $M_n^k$ be the subset of matrices of rank $k$. Prove that $M_n^k$ is a smooth submanifold of $M_n$. (Hint: First prove the subset of $M_n^k$ where the top-left $k \times k$ minor is non-singular is a smooth submanifold $M_n^k$.)

(Q-3) Let $\theta$ be the restriction of

$$(x^2dx^1 - x^1dx^2) + (x^4dx^3 - x^3dx^4) + \cdots + (x^{2n}dx^{2n-1} - x^{2n-1}dx^{2n})$$

to the unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$. Prove $\ker(\theta)$ is a distribution on $S^{2n-1}$. Is it integrable?

(Q-4) Let $M$ be a compact smooth 3-manifold and $\omega \in \Omega^1(M)$ a nowhere zero 1-form, so that $\ker(\omega)$ is an integrable distribution. Prove the following.
(a) $\omega \wedge d\omega = 0$.
(b) There exists some 1-form $\alpha$ with $d\omega = \alpha \wedge \omega$.
(c) $d\alpha \wedge \omega = 0$.

(Q-5) Let $M \subset \mathbb{R}^n$ be a compact $(n-1)$-dimensional submanifold, let $\iota: M \to \mathbb{R}^n$ be the inclusion map, and let $D \subset \mathbb{R}^n$ be the $n$-dimensional compact region with $\partial D = M$. Let $dV = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \in \Omega^n(\mathbb{R}^n)$ be the standard volume form on $\mathbb{R}^n$.
(a) Define $dA \in \Omega^{n-1}(M)$, the standard volume form on $M$, induced by the embedding $\iota$.
(b) Prove $\iota^*(i_XdV) = (X,N)dA$, for any smooth vector field $X$ on $\mathbb{R}^n$. (Here, $N$ is the unit normal vector field along $M$, pointing outward from $D$.)
(c) Prove

$$\int_D L_X(dV) = \int_M (X,N)dA.$$

(d) Derive Gauss' Divergence Theorem from the case $n = 3$.

(Q-6) Can a finite rank free group have a finite index subgroup of smaller rank?

(Q-7) Prove that the covering map $S^n \to \mathbb{RP}^n$ induces an isomorphism on de Rham cohomology if and only if $n$ is odd. What is the orientable double cover of $\mathbb{RP}^n$?

(Q-8) Assume the integral homology of a space is $\mathbb{Z}$ in grading 0, $\mathbb{Z}$ in grading 2, $\mathbb{Z}/2$ in grading 3, and 0 in all other gradings.
(a) What is its integral cohomology group?
(b) Construct a simply connected CW complex $X$ with the given homology.
(c) Construct another simply connected CW complex $Y$ with the same homology, which is not homotopy equivalent to $X$.

(Q-9) Let $X$ be a connected CW-complex. Show that there is a natural isomorphism

$$\tilde{H}_k(\Sigma X; M) \cong \tilde{H}_{k-1}(X; M)$$

for all $k$ and for all abelian groups $M$.

(Q-10) Let $Y$ be a connected and simply connected CW-complex.
(a) Compute the fundamental group of $Y \vee S^1$.
(b) Describe the universal cover of $Y \vee S^1$, together with the action of the deck transformations.
(c) Describe all finite covers of $Y \vee S^1$, again with the action of the deck transformations.
(d) Describe what changes in the first two parts for $Y = \mathbb{RP}^2$. 
Attempt all ten problems. Each problem is worth 10 points. Justify your answers carefully.

1. Suppose that $M$ and $N$ are connected smooth manifolds of the same dimension and $f : M \to N$ is a smooth submersion.

(a) Prove that if $M$ is compact, then $f$ is onto and $f$ is a covering map.

(b) Give an example of a smooth submersion $f : M \to N$ such that $M$ and $N$ have the same dimension, $N$ is compact, and $f$ is onto, but $f$ is not a covering map.

2. Let $\Phi_N, \Phi_S : \mathbb{R} \times S^2 \to S^2$ be two global flows on the sphere $S^2$. Show that there exist $\varepsilon > 0$, a neighborhood $U$ of the North pole, a neighborhood $V$ of the South pole, and a global flow $\Phi : \mathbb{R} \times S^2 \to S^2$ such that $\Phi(t, q) = \Phi_N(t, q)$ for all $t \in (-\varepsilon, \varepsilon), q \in U$, and $\Phi(t, q) = \Phi_S(t, q)$ for all $t \in (-\varepsilon, \varepsilon), q \in V$.

3. For $n \geq 1$, consider the subset $X \subset \mathbb{C}P^n$ given by

$$X = \{[z_0 : z_1 : \cdots : z_{2n}] \in \mathbb{C}P^n | z_{n+1} = z_{n+2} = \cdots = z_{2n} = 0\}.$$

(a) Show that $X$ is a smooth submanifold.

(b) Calculate the mod 2 intersection number of $X$ with itself.

4. Suppose $N$ is a smoothly embedded submanifold of a smooth manifold $M$. A vector field $X$ on $M$ is called tangent to $N$ if $X_p \in T_pN \subset T_pM$ for all $p \in M$.

(a) Show that if $X$ and $Y$ are vector fields on $M$ both tangent to $N$, then $[X, Y]$ is also tangent to $N$.

(b) Illustrate this principle by choosing two vector fields $X, Y$ tangent to $S^2 \subset \mathbb{R}^3$ (such that $[X, Y]$ is not identically zero), computing $[X, Y]$ and checking that it is tangent to $S^2$.

5. A symplectic form on an eight-dimensional manifold is defined to be a closed two-form $\omega$ such that $\omega \wedge \omega \wedge \omega$ is a volume form (that is, everywhere nonvanishing). Determine which of the following manifolds admit symplectic forms: (a) $S^8$; (b) $S^2 \times S^6$; (c) $S^2 \times S^2 \times S^2 \times S^2$.

6. Let $U$ be a bounded open set in $\mathbb{R}^3$ with smooth boundary, and let $V$ be a smooth vector field on $\mathbb{R}^3$. The classical divergence theorem expresses the triple integral $\int_U \int_V \div V d(\text{vol})$ as a surface integral over the boundary of $V$. State this theorem, and show how it can be obtained as a particular case of Stokes' Theorem for differential forms.

7. Let $M$ and $N$ be smooth, connected, orientable $n$-manifolds for $n \geq 3$, and let $M \# N$ denote their connect sum.

(a) Compute the fundamental group of $M \# N$ in terms of that of $M$ and of $N$ (you may assume that the basepoint is on the boundary sphere along which we glue $M$ and $N$).
(b) Compute the homology groups of $M \# N$. (You may use without proof that $H_n(-; \mathbb{Z})$ of a connected orientable $n$-manifold is always isomorphic to $\mathbb{Z}$).

(c) For part (a), what changes if $n=2$? Use this to describe the fundamental groups of orientable surfaces.

8. Determine all of the possible degrees of maps $S^2 \to S^1 \times S^1$.

9. Point $S^2$ via the south pole, and consider the Cartesian product $S^2 \times S^2$.

(a) Describe a cell structure on $S^2 \times S^2$ that is compatible with the inclusion of $S^2 \vee S^2 \hookrightarrow S^2 \times S^2$ as those pairs where one coordinate is the south pole.

(b) Let $X$ be $(S^2 \times S^2) \cup_{S^2} D^3$, where we attach the 3-disk via the map $S^2 \to S^2 \vee S^2$ which crushes a great circle connecting the north and south poles. Compute the homology groups of $X$.

10. Let $X$ be a semi-locally simply connected space and let $\tilde{X} \to X$ be the universal cover.

(a) Show that any map $\sigma : \Delta^n \to X$ lifts to a map $\tilde{\sigma} : \Delta^n \to \tilde{X}$, where $\Delta^n$ is the standard $n$-simplex.

(b) Show that if $\tilde{\sigma}_1, \tilde{\sigma}_2 : \Delta^n \to \tilde{X}$ are two lifts of $\sigma$, then there is an element $g$ of the fundamental group of $X$ such that $g \circ \tilde{\sigma}_1 = \tilde{\sigma}_2$, where we view $g$ as an automorphism of $\tilde{X}$ via the deck transformations.