Dispersive Decay for the Energy-Critical Nonlinear Schrödinger Equation

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Dispersive Decay for the Linear Schrödinger Equation

Consider the linear Schrödinger equation:

$$\begin{cases} iu_t + \Delta u = 0 \\ u(0,x) = u_0 \end{cases} \qquad \text{equivalently} \qquad u(t) = e^{it\Delta}u_0, \tag{1}$$

where $u(t,x)$ is a complex-valued function on spacetime $\mathbb{R}_t \times \mathbb{R}^d_x$.

This equation is *dispersive* in the sense that different frequencies travel at different velocities. Consequently, it exhibits dispersive decay of the form

$$\left\|e^{it\Delta}u_0\right\|_{L^p} \lesssim |t|^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|u_0\|_{L^{p'}} \tag{2}$$

for $2 \le p \le \infty$.

Energy-Critical Nonlinear Schrödinger Equation

We study the asymptotic behavior of solutions to the *energy-critical nonlinear Schrödinger equation*:

$$\begin{cases} iu_t + \Delta u \pm |u|^{\frac{4}{d-2}}u = 0 \\ u(0,x) = u_0(x) \in \dot{H}^1(\mathbb{R}^d), \end{cases} \tag{3}$$

where $u(t,x)$ is a complex-valued function on spacetime $\mathbb{R}_t \times \mathbb{R}^d_x$.

Global Well-Posedness

Theorem 1 (CKSTT, RV, KM [1, 4, 6]). *Fix $d \in \{3,4\}$ and suppose $u_0 \in \dot{H}^1$. In the focusing case, further assume that u_0 is radial and satisfies $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$ where*

$$W(x) = \left(1 + \tfrac{1}{d(d+2)}|x|^2\right)^{\frac{2-d}{2}}.$$

Then there exists a unique global solution $u \in C_t\dot{H}^1_x$ to (3) which satisfies

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+2)}{d-2}}\,dx\,dt \le C(E(u_0)).$$

Moreover, there exists scattering states $u_\pm \in \dot{H}^1$ such that

$$\|u(t) - e^{it\Delta}u_\pm\|_{\dot{H}^1_x} \to \quad \text{as} \quad t \to \pm\infty. \tag{4}$$

This theorem implies that solutions to (3) *scatter* and hence parallel the linear evolution (1) asymptotically. We then ask whether the decay property (2) of the linear evolution extends to these solutions to (3). We answer this in the affirmative for optimal-regularity solutions below.

The Main Results

Theorem (K. in preparation). *Fix $d \in \{3,4\}$ and*

$$\begin{cases} 2 < p \le \infty, & \text{if } d = 3 \\ 2 < p < \infty, & \text{if } d = 4. \end{cases}$$

Given $u_0 \in L^{p'} \cap \dot{H}^1(\mathbb{R}^d)$ satisfying Theorem 1, let $u(t)$ denote the unique global solution to (3). Then

$$\|u(t)\|_{L^p} \le C(\|u_0\|_{\dot{H}^1})|t|^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|u_0\|_{L^{p'}}$$

for all $t \in \mathbb{R}$ and for some constant depending only on $\|u_0\|_{\dot{H}^1}$, d, and p.

Theorem (K. in preparation). *Given $u_0 \in L^1 \cap \dot{B}^{1,2}_1(\mathbb{R}^4)$ satisfying Theorem 1 let $u(t)$ denote the unique global solution to (3). Then*

$$\|u(t)\|_{L^\infty_x} \lesssim \sum_{N \in 2^{\mathbb{Z}}} \|P_N u(t)\|_{L^\infty_x} \le C\left(\|u_0\|_{\dot{B}^{1,2}_1}\right)|t|^{-2}\|u_0\|_{L^1}$$

for all $t \in \mathbb{R}$ and for some constant depending only on $\|u_0\|_{\dot{B}^{1,2}_1}$, d, and p.

Here P_N is a Littlewood–Paley projection onto frequencies $\sim N$ and $\dot{B}^{1,2}_1$ is a scaling–critical homogeneous Besov space.

Lorentz Spacetime Bounds

Theorem. *Fix $d \in \{3,4\}$; $\phi, \theta \ge 2$; and $2 < p, q < \infty$ such that $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$. Suppose that $u_0 \in \dot{H}^1(\mathbb{R}^d)$ satisfies Theorem 1. Then the corresponding global solution $u(t)$ to (3) satisfies*

$$\|\nabla u\|_{L^{p,\theta}_t L^{q,\phi}_x} \le C(\|u_0\|_{\dot{H}^1}).$$

In the case of $\theta \ge p$ and $\phi \ge q$, this further holds for the endpoints $(p,q) = \left(2, \frac{2d}{d-2}\right), (\infty, 2)$.

Moreover, for $\phi, \theta \ge \frac{2(d-2)}{d+2}$,

$$\left\|\nabla \int_0^t e^{i(t-s)\Delta}\left[|u|^{\frac{4}{d-2}}u\right](s)ds\right\|_{L^{p,\theta}_t L^{q,\phi}_x(\mathbb{R}\times\mathbb{R}^d)} \le C(\|u_0\|_{\dot{H}^1}).$$

History of the Problem

Estimates of this form have historically been used to prove well–posedness and scattering for a wide variety of dispersive models. Here, we approach this problem in hindsight with well–posedness and scattering determined first.

These are (a few of) the relevant prior results:

- **Lin–Strauss '78 [5]** : L^∞ dispersive decay for cubic NLS in three-dimensions, used to prove scattering and well-posedness. Ideas from their proof used here.
- **Guo–Huang–Song '22 [3]** : Previous best result for the energy-critical NLS (3). Required initial data in H^3 and did not recover the linear dependence on the initial data.
- **Fan–Killip–Vişan–Zhao '24 [2]** : Optimal (scaling–critical) regularity result for mass–critical NLS. Their argument is a template for the slow-decay case presented below.

Proof Idea : Slow–Decay Case $\left(2 < p < \frac{2d}{d-2}\right)$

In this case, the decay (2) is integrable near $t = 0$. Due to this, we may use a relatively simple argument adapted from [2]. We fix $d = 3, p = 3$ and $\|u_0\|_{\dot{H}^1} \ll 1$ for simplicity.

Proof. We define our bootstrap norm

$$\|u\|_X = \sup_{t\in\mathbb{R}} t^{d\left(\frac{1}{2}-\frac{1}{p}\right)}\|u(t)\|_{L^p_x} = \sup_{t\in\mathbb{R}} t^{1/2}\|u(t)\|_{L^3_x}.$$

Fix $\eta > 0$ and suppose $\|u_0\|_{\dot{H}^1} \ll 1$ such that $\|u\|_{L^{8,4}_t L^{12}_x} < \eta$. We seek a bootstrap of the form

$$\|u\|_X \lesssim C(\|u_0\|_{\dot{H}^1})\left[\|u_0\|_{L^{3/2}} + \eta^4\|u\|_X\right]. \tag{5}$$

To show (5), we recall the Duhamel formula :

$$u(t) = e^{it\Delta}u_0 \mp i\int_0^t e^{i(t-s)\Delta}\left[|u|^4u\right](s)ds. \tag{6}$$

By linear dispersive decay, the linear term is acceptable :

$$\left\|e^{it\Delta}u_0\right\|_X \lesssim \|u_0\|_{L^{3/2}}.$$

For the nonlinear correction, we estimate

$$\begin{aligned} \left\|\int_0^t e^{i(t-s)\Delta}\left[|u|^4u\right](s)ds\right\|_{L^3_x} &\lesssim \int_0^t |t-s|^{-1/2}\|u^5(s)\|_{L^{3/2}_x}ds \\ &\lesssim \int_0^t |t-s|^{-1/2}|s|^{-1/2}\|u\|_X\|u(s)\|^4_{L^{12}_x}ds \\ &\lesssim \left\||t-s|^{-1/2}|s|^{-1/2}\right\|_{L^{2,\infty}_s([0,t))}\|u\|^4_{L^{8,4}_t L^{12}_x}\|u\|_X \end{aligned}$$

For the first term, we decompose $[0,t)$ into $[0,t/2)$ and $[t/2,t)$. We note that $|t-s| \sim |t|$ on $[0,t/2)$ and $|s| \sim |t|$ on $[t/2,t)$. Because $\||s|^{-1/2}\|_{L^{2,\infty}} \sim 1$ and $\||t-s|^{-1/2}\|_{L^{2,\infty}} \sim 1$, this implies

$$\left\|\int_0^t e^{i(t-s)\Delta}\left[|u|^4u\right](s)ds\right\|_{L^3_x} \lesssim |t|^{-1/2}\eta^4\|u\|_X,$$

which yields the bootstrap statement (5). Taking η sufficiently small then completes the proof.

Proof Idea : Fast–Decay Case $(d = 3,\ 6 \le p \le \infty)$

In this case, the linear dispersive decay (2) give a decay which is no longer integrable near 0. We fix $p = \infty$ for simplicity and decompose $[0,t)$ into $[0,t/2)$ and $[t/2,t)$ as done before.

For $[0,t/2)$, we carefully apply the slow–decay case. We then estimate

$$\begin{aligned} \left\|\int_0^{t/2} e^{i(t-s)\Delta}\left[|u|^4u\right](s)ds\right\|_{L^\infty_x} &\lesssim |t|^{-3/2}\int_0^{t/2}\|u(s)\|^{5/3}_{L^3_x}\left\|u(s)\right\|^{10/3}_{L^{15/2}_x}ds \\ &\lesssim |t|^{-3/2}\|u_0\|_{L^1}\left\|u(s)\right\|^{10/3}_{L^{20,10/3}_t L^{15/2}_x}. \end{aligned}$$

For $[t/2,t)$, we Sobolev embed before applying (2) which makes the decay integrable. Due to the failure of endpoint Sobolev embedding, we work informally here. We then (informally) estimate

$$\begin{aligned} \left\|\int_{t/2}^t e^{i(t-s)\Delta}\left[|u|^4u\right](s)ds\right\|_{L^\infty_x} &\lesssim \int_0^{t/2}\left\|\nabla e^{i(t-s)\Delta}\left[|u|^4u\right](s)\right\|_{L^3_x}ds \\ &\lesssim \int_0^{t/2}|t-s|^{-1/2}\left\|\nabla u^5(s)\right\|_{L^{3/2}_x}ds \lesssim |t|^{-3/2}\|\nabla u\|_{L^\infty_t L^2_x}\|u\|^3_{L^{6,3}_t L^{18,6}_x}\|u\|_X. \end{aligned}$$

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