Proposed Plan for Completing the Dissertation

Matthew Kowalski

Abstract. Modern communication—whether through optical fibers or even a message in a bottle—must contend with *dispersion*: the phenomenon in which waves of different frequencies propagate at different velocities. While dispersion alone is well-understood, the interactions of waves with the medium introduce nonlinear effects, resulting in complex dynamics. In nonlinear dispersive models, dispersion and the nonlinearity compete, leading to three asymptotic behaviors, classified by which effect, if either, dominates.

In my dissertation, I investigate the precise regularity and size of initial conditions that give rise to these distinct scenarios. To ensure broad applicability, I analyze a variety of nonlinear dispersive models, culminating in the study of the dispersion-managed nonlinear Schrödinger equation. This equation is the primary model of pulse propagation in long-haul optical fibers. To maintain signal integrity and counteract pulse broadening, these fibers alternate between positive and negative dispersion materials.

When the nonlinearity dominates the dispersion, waves can exhibit turbulence, an unbounded growth in size and roughness over time. This behavior has sparked recent interest in the continuum Calogero–Moser model, the hydrodynamic limit of the respective particle system; see [1]. In my joint work [4], published in *Pure and Applied Analysis*, I demonstrated the existence of solutions that cascade energy to ever finer scales when exceeding a minimal mass. This result proved the sharp mass threshold for turbulence in this model.

In contrast, when dispersion is stronger than the nonlinearity, waves will spread out, decaying in height over time. In a series of works [5, 6, 7], I proved that solutions to three fundamental nonlinear dispersive models exhibit the strong quantitative decay of their linear counterparts. These studies assumed minimal hypotheses and developed broadly applicable tools that have since been applied to other equations; see, e.g., [8]. The paper [6] is forthcoming in the *Journal of Differential Equations*.

When dispersion balances the nonlinearity, solitons—localized waves that maintain their shape over time—emerge as a boundary between global existence and finite-time blowup. In the latter case, waves reach infinite height in finite time. A striking example is the dispersion-managed nonlinear Schrödinger equation (DM-NLS), which not only serves as a fundamental model of pulses in optical fibers but also introduces new mathematical challenges due to its rapidly varying dispersion. Unlike the classical (NLS), where the threshold for the existence of solutions is long-established, the theory for (DM-NLS) remains uncertain. Despite this, my research suggests that its well-posedness theory parallels that of (NLS). This is supported by preliminary investigations that confirm the local existence of solutions at the conjectured regularity and instability below it. Ongoing efforts seek to strengthen these results, showing global existence at the threshold and instantaneous norm inflation below. In doing so, I aim to clarify the limits of this widely-used model, determining the regularity and size for which it accurately describes pulse propagation.

Introduction. To quantify the regularity and size of waves, we use the Sobolev norms H^s, which measure the total mass of a function and its first s derivatives. For example, the H^1 norm is given by $\|u(x)\|_{H^1}^2 = \int_{\mathbb{R}^d} |u(x)|^2 + |\nabla u(x)|^2 dx$. For non-integer values of s, we define fractional derivatives on the frequency side.

The majority of my work focuses on variants of the (nonlinear) Schrödinger equation:

$$\text{(NLS)} \qquad i\partial_t u + \Delta u + g|u|^p u = 0, \quad u(0,x) = u_0(x) \in H^s(\mathbb{R}^d), \quad g = -1, 0, 1, \quad 0 < p < \infty.$$

Like most dispersive partial differential equations (PDEs), (NLS) admits a scaling symmetry, $u(t,x) \mapsto \lambda^{2/p} u(\lambda^2, \lambda x)$, that preserves the class of solutions. This symmetry defines a critical regularity: $s_c = \frac{d}{2} - \frac{2}{p}$. Specifically, H^{s_c} becomes a dimensionless measure of size for (NLS). It is widely conjectured that this scaling-critical regularity forms the threshold for well-posedness. Precisely, when $s > s_c$, we expect well-posedness. That is, there exists a unique, continuous map from initial data in H^s to solutions evolving continuously in H^s. If $s < s_c$, then we expect ill-posedness, indicating that one of these conditions fails.

My goal is to analyze three asymptotic behaviors—turbulence, dispersive decay, and well-posedness—for initial data in the critical space H^{s_c}. Moreover, I seek to determine the precise size threshold at which they occur. In doing so, I ensure that my work is optimal within the practical limitations of these models.

Project 1: Turbulence (complete). In my joint publication [4], I investigated the continuum Calogero–Moser model (CCM): a mass-critical variant of (NLS) where $|u|^p u$ is replaced with $iu\Pi^+\partial_x|u|^2$. Here Π^+ is a projection onto positive frequencies. This model has gained recent attention due to the unexpected interplay between complete integrability, which normally suggests control over the H^s norms of solutions, and turbulent behavior, which gives solutions that exhibit unbounded H^s norm growth for all $s > 0$.

In [3], it was shown that solutions to (CCM) with mass less than 2π must remain bounded in H^s for all time, while solutions of mass 4π can grow turbulent. This gap raised the question of the true mass threshold for turbulence. In [4], I answered this question by constructing solutions $u(t,x)$ with mass $2\pi + \varepsilon$, for any $\varepsilon > 0$, for which $\|u(t,x)\|_{H^s} \gtrsim |t|^s$ for all $s > 0$. My proof rests on two main ingredients: an orbital stability for the ground state (stating that solutions which begin close in H^1 to the ground state must remain close) and dispersive decay (showing that the complex extension of $u(t,x)$ decays like $|u(t,z)| \lesssim \mathrm{Im}(z)^{-1}|t|^{-1/2}$). Combining these effects, I demonstrated that small perturbations of the ground state must concentrate to increasingly fine scales on the real line, leading to H^s norm growth.

Project 2: Dispersive decay (complete). In the recent series of papers [5, 6, 7], I investigated dispersive decay for the nonlinear wave equation (NLW), the nonlinear Schrödinger equation (NLS), and the generalized Korteweg-de Vries equation (gKdV) respectively. Together, these works cover the three most-studied dispersive models and offer a broad range of tools for proving dispersive decay, synthesizing methods from harmonic analysis and dispersive PDEs.

Before Strichartz inequalities, quantitative dispersive estimates were the primary tool for understanding long-time behavior. For linear models, this decay is a simple exercise in harmonic analysis which shows, for instance, that solutions $u(t,x)$ to the linear Schrödinger equation ((NLS) with $g=0$) disperse and decay like $|u(t,x)| \lesssim |t|^{-d/2}\int_{\mathbb{R}^d} |u(0,y)|dy$. For nonlinear models such as (NLS), (NLW), and (gKdV), the question then becomes whether the nonlinearity interferes with the time decay of the underlying linear equation.

In [5, 6], I successfully demonstrated that solutions to (NLS) and (NLW) decay at the same rate as the respective linear models, while assuming only that the initial data belonged to the scaling-critical regularity. This necessitated the development of detailed estimates on the fine structure of solutions, analyzing the distribution and evolution of the amplitudes and frequencies of a wave over time.

My contribution to [7] is ongoing. Though previously released, my suggestions allowed the author to strengthen their results to small initial data in $H^{1/4}$. In an appendix to be added this month, I extend their results to large initial data with an added negative regularity requirement. Following my work on (DM-NLS), I would like to return to (gKdV) and sharpen this result to the critical space H^0.

Project 3: Well-posedness (in progress). In long-haul optical fibers, pulse broadening poses a significant challenge, causing bits to smear together and limiting the rate of data transmission. This is mitigated by introducing alternating sections of positive and negative dispersion glass along the fiber. The resulting pulse propagation is then described by the dispersion-managed nonlinear Schrödinger equation (DM-NLS).

I am studying (DM-NLS) in the strong regime, where sections of extreme positive and negative dispersion glass are alternated rapidly. The canonical model of this regime is the *Gabitov–Turitsyn equation* [2]:

$$\text{(GT)} \qquad i\partial_t u + \Delta u \pm \int_0^1 e^{-i\sigma\Delta}\left[\left|e^{i\sigma\Delta}u\right|^p e^{i\sigma\Delta}u\right]d\sigma = 0,$$

where $e^{i\sigma\Delta}u$ is the linear Schrödinger evolution of u up to time σ. Paradoxically, the averaging over σ in (GT) both suppresses the effects of the nonlinearity and introduces novel mathematical challenges, invalidating many methods commonly used for (NLS). Together with the lack of scaling symmetry, this has led to major uncertainty over the precise threshold for the well-posedness of (GT).

My working hypothesis has been that the well-posedness theory of (GT) aligns with the scaling-critical threshold H^{s_c} of the classical (NLS). As evidence, I have established local well-posedness in H^{s_c}, proving that solutions exist for short times and depend smoothly on the initial data. Moreover, I have shown that for $s < s_c$, (GT) is mildly ill-posed, in the sense that solutions fail to depend smoothly on initial data.

In my ongoing project, I am working to strengthen the well-posedness at H^{s_c} by demonstrating the global existence of solutions. To do so, I am separating the evolution of low and high frequencies, using energy conservation to control low frequencies and forcing high frequencies to evolve linearly. The remaining task is to control the interactions between these regimes. I am also working to demonstrate a stronger form of ill-posedness in the case $s < s_c$, namely instantaneous norm inflation. This demonstrates a failure of even continuous dependence on initial conditions by constructing solutions with arbitrarily small initial H^s norm that grow arbitrarily large, arbitrarily quickly. To accomplish this, I am considering the small-frequency limit of the model (where $e^{i\sigma\Delta}u \approx u$) in which (GT) has explicit solutions that grow like $\|\phi(t)\|_{H^s} \sim |t|^s$. The remaining challenge is to commute this growth through the small-frequency limit.

Future work. Having worked in all three main themes in dispersive PDEs, I am well-prepared to continue investigating the rich dynamics of these models. Looking forward, I am interested in studying how dispersion and the nonlinearity can be balanced to benefit communications, namely in the production of solitons in (DM-NLS) and (GT). My foray into this topic begins with the question of scattering for (GT), which compares the asymptotic behavior of solutions to (GT) to those of the linear Schrödinger model. In much later projects, I aim to incorporate additional methods used in optical fiber communication, such as optical amplification, where pulses are periodically amplified to combat signal attenuation.

Dissertation Timeline.

March 2025 – April 2025	Strengthen well-posedness threshold for (GT)
May 2025	Write and submit paper on sharp well-posedness of (GT) Begin work on scattering for (GT)
June 2025 – August 2025	Draft application materials for postdoctoral grants and positions
September 2025	Write and submit paper on scattering for (GT).
October 2025	**Activate dissertation year award** Write introductory material for CCM section of dissertation
November 2025	Interview for postdoctoral positions. Revisit dispersive decay for (gKdV).
Dec. 2025 – Jan. 2026	Write introductory material for dispersive decay and (GT) sections
Feb. 2026 – March 2026	Edit and finalize dissertation. Begin work on amplified (DM-NLS).
April 2026	**Write and submit midpoint report for DYA**
June 2026	Defend and submit my dissertation.

References

[1] A. G. Abanov, E. Bettelheim, and P. Wiegmann. Integrable hydrodynamics of Calogero–Sutherland model: bidirectional Benjamin–Ono equation. *J. Phys. A*, 42(13):135201, Mar 2009.
[2] I. R. Gabitov and S. K. Turitsyn. Averaged pulse dynamics in a cascaded transmission system with passive dispersion compensation. *Opt. Lett.*, 21(5):327–329, Mar 1996.
[3] P. Gérard and E. Lenzmann. The Calogero–Moser derivative nonlinear Schrödinger equation. *Communications on Pure and Applied Mathematics*, 2024.
[4] J. Hogan and M. Kowalski. Turbulent threshold for continuum Calogero–Moser models. *Pure and Applied Analysis*, 6(4):941–954, 2024.
[5] M. Kowalski. Dispersive decay for the energy-critical nonlinear wave equation. Preprint `arXiv:2501.06387`.
[6] M. Kowalski. Dispersive decay for the energy-critical nonlinear Schrödinger equation. *Journal of Differential Equations*, to appear. Preprint `arXiv:2411.01466`.
[7] M. Shan. Dispersive decay for the mass-critical generalized Korteweg-de Vries equation and generalized Zakharov–Kuznetsov equations. Preprint `arXiv:2409.05550`.
[8] J. Wang, C. Xu, and F. Zhang. Decay estimates for nonlinear Schrödinger equation with the inverse-square potential, 2024. Preprint `arXiv:2412.11424`.