Dispersive decay for the energy–critical nonlinear Schrödinger equation

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Linear Schrödinger Equation

We recall the linear Schrödinger equation,

$$\begin{cases} iu_t + \Delta u = 0 \\ u(0,x) = u_0 \in L^2(\mathbb{R}^d). \end{cases}$$

Taking a spatial Fourier transform, we quickly find

$$u(t,x) = e^{it\Delta} u_0(x).$$

Dispersive Decay

We recall

$$\|e^{it\Delta}f\|_{L^2} = \|f\|_{L^2} \qquad \text{(conservation of mass)}$$

$$\|e^{it\Delta}f\|_{L^p} \lesssim |t|^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|f\|_{L^{p'}} \qquad \text{for} \quad 2 < p < \infty$$

$$\|e^{it\Delta}f\|_{L^\infty} \lesssim |t|^{-\frac{d}{2}}\|f\|_{L^1} \qquad \text{(from fundamental solution)}$$

Energy–Critical Nonlinear Schrödinger Equation

We consider the asymptotic behavior of solutions to the *energy–critical nonlinear Schrödinger equation*:

$$\begin{cases} iu_t + \Delta u \pm |u|^{\frac{4}{d-2}} u = 0 \\ u(0,x) = u_0(x) \in \dot{H}^1(\mathbb{R}^d), \end{cases} \tag{1}$$

where $u(t,x)$ is a complex–valued function on spacetime $\mathbb{R}_t \times \mathbb{R}_x^d$. With this convention, $+$ represents the focusing equation and $-$ the defocusing.

We say that $u \in C_t \dot{H}_x^1$ is a solution if

$$u(t) = e^{it\Delta} u_0 \mp i \int_0^t e^{i(t-s)\Delta} \big[|u|^{\frac{4}{d-2}} u\big](s) ds. \tag{2}$$

Well-Posedness

Theorem (CKSTT '04, RV '05, KM '06 [1, 6, 8])

Fix $d = 3, 4$ and let $u_0 \in \dot{H}^1$. In the focusing case, assume that $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$ where

$$W(x) = \left(1 + \tfrac{1}{d(d+2)}|x|^2\right)^{\frac{2-d}{2}}.$$

In the $d = 3$ focusing case, further assume that u_0 is radial. Then there exists a unique global solution $u \in C_t \dot{H}^1_x$ to (1) which satisfies

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2(d+2)}{d-2}} dxdt \leq C(\|u_0\|_{\dot{H}^1}).$$

Scattering

Corollary (Scattering)

Suppose that $u_0 \in \dot{H}^1$ *and satisfies Theorem 1. Then there exists scattering states* $u_\pm \in \dot{H}^1$ *such that*

$$\|u(t) - e^{it\Delta}u_\pm\|_{\dot{H}^1} \to 0 \quad as \quad t \to \pm\infty.$$

Intuitively, this says that the nonlinear equation *parallels* the linear Schrödinger equation.

The Main Theorem

Theorem (K. in preparation)

Fix $d = 3, 4$ *and*

$$\begin{cases} 2 < p \leq \infty, & \text{if } d = 3 \\ 2 < p < \infty, & \text{if } d = 4. \end{cases}$$

Given $u_0 \in L^{p'} \cap \dot{H}^1(\mathbb{R}^d)$ *satisfying Theorem 1, let* $u(t)$ *denote the unique global solution to* (1)*. Then*

$$\|u(t)\|_{L^p} \leq C(\|u_0\|_{\dot{H}^1})|t|^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|u_0\|_{L^{p'}}$$

for all $t \in \mathbb{R}$ *for a constant depending only on* $\|u_0\|_{\dot{H}^1}$, d, *and* p.

The Main Theorem (Edge Case)

Theorem (K. in preparation)

Given $u_0 \in L^1 \cap \dot{B}^{1,2}_1(\mathbb{R}^4)$ *satisfying Theorem 1, let* $u(t)$ *denote the unique global solution to* (1)*. Then*

$$\|u(t)\|_{L^\infty} \le C(\|u_0\|_{\dot{B}^{1,2}_1})|t|^{-2}\|u_0\|_{L^1}$$

for all $t \in \mathbb{R}$*.*

Here $\dot{B}^{1,2}_1$ is a homogeneous Besov space defined by

$$\|u_0\|^2_{\dot{B}^{1,2}_1} = \sum_{N \in 2^{\mathbb{Z}}} N\|P_N u_0\|_{L^2_x},$$

where P_N is a *Littlewood–Paley projection* onto frequencies $\sim N$.

Note that $\dot{B}^{1,2}_1 \hookrightarrow \dot{H}^1$.

The Main Theorem (Edge Case)

Corollary (K. in preparation)

Fix some $\varepsilon > 0$. Given $u_0 \in L^1 \cap \dot{H}^{1+\varepsilon} \cap \dot{H}^{1-\varepsilon}(\mathbb{R}^4)$ satisfying Theorem 1, let $u(t)$ denote the unique global solution to (1). Then

$$\|u(t)\|_{L_x^\infty} \leq C(\|u_0\|_{\dot{H}^{1+\varepsilon}}\|u_0\|_{\dot{H}^{1-\varepsilon}})|t|^{-2}\|u_0\|_{L^1}.$$

Theorem (Lin–Strauss '78, [7])

For smooth initial data $u_0 \in H^\infty(\mathbb{R}^3)$*, satisfying additional decay properties, let* $u(t)$ *denote the maximal solution to* **cubic** *NLS. Then for all times of existence,*

$$|u(t,x)| \lesssim \frac{1}{1+|t|^{3/2}}$$

uniformly in x*.*

Similar estimates used to prove global well-posedness for such smooth initial data and to show scattering.

Prior Work

(Lin–Strauss '78, [7]) : L^∞ dispersive decay for cubic NLS in $\mathbb{R}^3$, used to prove scattering and well-posedness. Ideas from their proof used here.

(Hogan–K. '24, [5]) : Similar result for CM-DNLS in the upper half plane. Requires high-regularity and additional spatial decay.

(Fan–Zhao '20 [3], Guo–Huang–Song '22 [4]) : Same result for energy-critical NLS, requires $u_0 \in H^3 \cap L^1$ and does not recover the linear dependence on initial data.

(Fan–Killip–Vișan–Zhao '24, [2]) : Scaling-critical result for mass-critical NLS. Their proof forms a template for our base case.

Lorentz Spaces

Definition (Lorentz space)

Fix $d \geq 1$; $0 < p < \infty$; and $0 < q \leq \infty$. The Lorentz space $L^{p,q}$ is the space of measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ which have finite quasinorm

$$\|f\|_{L^{p,q}(\mathbb{R}^d)} = p^{1/q} \Big\| \lambda \big| \{x \in \mathbb{R}^d : |f(x)| > \lambda\} \big|^{1/p} \Big\|_{L^q((0,\infty), \frac{d\lambda}{\lambda})},$$

where $| * |$ denotes the Lebesgue measure on $\mathbb{R}^d$.

Remarks.

- $L^{p,p} = L^p$
- For any $p > 0$,

$$|x|^{-1/p} \in L^{p,\infty}(\mathbb{R}).$$

Lorentz Spaces

Hölder's inequality. *Given* $0 < p, p_1, p_2, q, q_1, q_2 \leq \infty$ *such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$,

$$\|fg\|_{L^{p,q}} \lesssim \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}.$$

Young's inequality. *Given* $1 < p, p_1, p_2 \leq \infty$ *and* $0 < q, q_1, q_2 \leq \infty$ *such that* $\frac{1}{p} + 1 = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$,

$$\|f * g\|_{L^{p,q}} \lesssim \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}.$$

Lorentz-Improved Dispersive Decay

Applying Hunt's interpolation inequality, for all $0 < \theta \leq \infty$,

$$\|e^{it\Delta} f\|_{L^2} = \|f\|_{L^2}$$

$$\|e^{it\Delta} f\|_{L^{p,\theta}} \lesssim |t|^{-d\left(\frac{1}{2}-\frac{1}{p}\right)} \|f\|_{L^{p',\theta}} \quad \text{for} \quad 2 < p < \infty$$

$$\|e^{it\Delta} f\|_{L^\infty} \lesssim |t|^{-\frac{d}{2}} \|f\|_{L^1}$$

Lorentz–Strichartz Estimates

Proposition (Nakanishi '01)

Suppose that $2 < p, q < \infty$ satisfy

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}.$$

Then,

$$\left\| e^{it\Delta} f \right\|_{L_t^{p,2} L_x^{q,2}(\mathbb{R}\times\mathbb{R}^d)} \lesssim_{p,q} \|f\|_{L^2(\mathbb{R}^d)}.$$

Moreover for all $0 < \theta, \phi \leq \infty$,

$$\left\| \int_0^t e^{i(t-s)\Delta} F(s,x) ds \right\|_{L_t^{p,\theta} L_x^{q,\phi}(\mathbb{R}\times\mathbb{R}^d)} \lesssim_{p,q,\theta,\phi} \|F\|_{L_t^{p',\theta} L_x^{q',\phi}(\mathbb{R}\times\mathbb{R}^d)}.$$

Lorentz–Strichartz Estimates

Proof. We proceed via a TT^* argument. Define $T : L_x^2 \to L_t^{p,2} L_x^{q,2}$ by

$$[Tf](t,x) = [e^{it\Delta} f](x).$$

Therefore $TT^* : L_t^{p',2} L_x^{q',2} \to L_t^{p,2} L_x^{q,2}$ is given by

$$[TT^* F](t,x) = \int e^{i(t-s)\Delta} F(s,x) ds.$$

To prove the first claim, it then suffices to show that TT^* is bounded from $L_t^{p',2} L_x^{q',2} \to L_t^{p,2} L_x^{q,2}$.

Lorentz–Strichartz Estimates

Proof continued. To show that TT^* is bounded, we estimate directly. By definition,

$$\begin{aligned}
\left\| [TT^*F](t,x) \right\|_{L_t^{p,2} L_x^{q,2}(\mathbb{R}\times\mathbb{R}^d)} &= \left\| \int e^{i(t-s)\Delta} F(s,x) ds \right\|_{L_t^{p,2} L_x^{q,2}(\mathbb{R}\times\mathbb{R}^d)} \\
\text{(dispersive decay)} &\lesssim \left\| \int |t-s|^{-d(\frac{1}{2}-\frac{1}{q})} \|F(s,x)\|_{L_x^{q',2}} ds \right\|_{L_t^{p,2}} \\
\text{(Young's)} &\lesssim \left\| |t|^{-d(\frac{1}{2}-\frac{1}{q})} \right\|_{L_t^{\frac{2q}{d(q-2)},\infty}} \|F(s,x)\|_{L_t^{p',2} L_x^{q',2}} \\
&\lesssim \|F(s,x)\|_{L_t^{p',2} L_x^{q',2}}.
\end{aligned}$$

Which completes the first claim.

Lorentz–Strichartz Estimates

Proof continued. To show that TT^* is bounded, we estimate directly. By definition,

$$
\begin{aligned}
\left\|[TT^*F](t,x)\right\|_{L_t^{p,2}L_x^{q,2}(\mathbb{R}\times\mathbb{R}^d)} &= \left\|\int e^{i(t-s)\Delta}F(s,x)ds\right\|_{L_t^{p,2}L_x^{q,2}(\mathbb{R}\times\mathbb{R}^d)} \\
\text{(dispersive decay)} &\lesssim \left\|\int |t-s|^{-d(\frac{1}{2}-\frac{1}{q})}\|F(s,x)\|_{L_x^{q',2}}ds\right\|_{L_t^{p,2}} \\
\text{(Young's)} &\lesssim \left\||t|^{-d(\frac{1}{2}-\frac{1}{q})}\right\|_{L_t^{\frac{2q}{d(q-2)},\infty}}\|F(s,x)\|_{L_t^{p',2}L_x^{q',2}} \\
&\lesssim \|F(s,x)\|_{L_t^{p',2}L_x^{q',2}}.
\end{aligned}
$$

Which completes the first claim.

Lorentz–Strichartz Estimates

Proof continued. For the second claim, we argue similarly:

$$\begin{aligned}\left\| \int_0^t e^{i(t-s)\Delta} F(s,x) ds \right\|_{L_t^{p,\theta} L_x^{q,\phi}(\mathbb{R}\times\mathbb{R}^d)} &\lesssim \left\| \int |t-s|^{-d(\frac{1}{2}-\frac{1}{q})} \|F(s,x)\|_{L_x^{q',\phi}} ds \right\|_{L_t^{p,\theta}} \\ &\lesssim \left\| |t|^{-d(\frac{1}{2}-\frac{1}{q})} \right\|_{L_t^{\frac{2q}{d(q-2)},\infty}} \|F(s,x)\|_{L_t^{p',\theta} L_x^{q',\phi}} \\ &\lesssim \|F(s,x)\|_{L_t^{p',\theta} L_x^{q',\phi}}.\end{aligned}$$

Which completes the proof of the proposition. □

Spacetime Bounds

Theorem

Fix $d = 3, 4$. Suppose that $2 < p, q < \infty$ satisfy

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2},$$

and that $u_0 \in \dot{H}^1(\mathbb{R}^d)$ satisfies Theorem 1.

Then for all $\theta, \phi \geq 2$, the corresponding global solution $u(t)$ satisfies

$$\|\nabla u\|_{L_t^{p,\theta} L_x^{q,\phi}} \leq C(\|u_0\|_{\dot{H}^1}).$$

Moreover, for all $\theta, \phi \geq \frac{2(d-2)}{d+2}$,

$$\left\| \nabla \int_0^t e^{i(t-s)\Delta} \left[|u|^{\frac{4}{d-2}} u \right](s) ds \right\|_{L_t^{p,\theta} L_x^{q,\phi}(\mathbb{R} \times \mathbb{R}^d)} \leq C(\|u_0\|_{\dot{H}^1}).$$

Theorem (K. in preparation)

Fix $d = 3, 4$ and

$$\begin{cases} 2 < p \leq \infty, & \text{if } d = 3 \\ 2 < p < \infty, & \text{if } d = 4. \end{cases}$$

Given $u_0 \in L^{p'} \cap \dot{H}^1(\mathbb{R}^d)$ satisfying Theorem 1, let $u(t)$ denote the unique global solution to (1). Then

$$\|u(t)\|_{L^p} \leq C(\|u_0\|_{\dot{H}^1})|t|^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|u_0\|_{L^{p'}}$$

for all $t \in \mathbb{R}$ for a constant depending only on $\|u_0\|_{\dot{H}^1}$, d, and p.

Slow Decay Case : $\left(2 < p < \frac{2d}{d-2}\right)$

Proof. We fix $p = 3, d = 3$ for concreteness.

For simplicity, we work with the additional assumption that

$$\|u\|_{L_t^{8,4} L_x^{12}} \lesssim \|\nabla u\|_{L_t^{8,4} L_x^{12/5}} \leq C(\|u_0\|_{\dot{H}^1}) < \eta$$

for η sufficiently small.

We define

$$\|u\|_X = \sup_{t \in \mathbb{R}} |t|^{\frac{1}{2}} \|u(t)\|_{L_x^3}.$$

We seek a bootstrap of the form*

$$\|u\|_X \lesssim C(\|u_0\|_{\dot{H}^1}) \Big[\|u_0\|_{L^{3/2}} + \eta^4 \|u\|_X \Big]. \tag{3}$$

Proof continued. Recall the Duhamel formula:

$$u(t) = e^{it\Delta}u_0 \mp i\int_0^t e^{i(t-s)\Delta}\big[|u|^4 u\big](s)ds. \tag{4}$$

By linear dispersive decay,

$$\|e^{it\Delta}u_0\|_X \lesssim \|u_0\|_{L^{3/2}}$$

and so the first term is acceptable.

Proof continued. For the nonlinear correction, we estimate

$$\left\| \int_0^t e^{i(t-s)\Delta}\left[|u|^4 u\right](s)ds \right\|_{L^3_x} \lesssim \int_0^t |t-s|^{-\frac{1}{2}} \left\| u^5(s) \right\|_{L^{3/2}_x} ds$$

$$\lesssim \int_0^t |t-s|^{-\frac{1}{2}} \|u(s)\|_{L^3_x} \|u(s)\|^4_{L^{12}_x} ds$$

Slow Decay Case : $d = 3, p = 3$

Proof continued. For the nonlinear correction, we estimate

$$\left\| \int_0^t e^{i(t-s)\Delta} \left[|u|^4 u \right](s) ds \right\|_{L_x^3} \lesssim \int_0^t |t-s|^{-\frac{1}{2}} \left\| u^5(s) \right\|_{L_x^{3/2}} ds$$

$$\lesssim \|u\|_X \int_0^t |t-s|^{-\frac{1}{2}} |s|^{-\frac{1}{2}} \|u(s)\|_{L_x^{12}}^4 ds$$

$$= \|u\|_X \left[\int_0^{t/2} + \int_{t/2}^t \right] |t-s|^{-\frac{1}{2}} |s|^{-\frac{1}{2}} \|u(s)\|_{L_x^{12}}^4 ds$$

Slow Decay Case : $d = 3, p = 3$

Proof continued. For the nonlinear correction, we estimate

$$
\begin{aligned}
\left\| \int_0^t e^{i(t-s)\Delta} \left[|u|^4 u\right](s) ds \right\|_{L_x^3} &\lesssim \int_0^t |t-s|^{-\frac{1}{2}} \left\| u^5(s) \right\|_{L_x^{3/2}} ds \\
&\lesssim \|u\|_X \int_0^t |t-s|^{-\frac{1}{2}} |s|^{-\frac{1}{2}} \|u(s)\|_{L_x^{12}}^4 ds \\
&\sim \|u\|_X |t|^{-1/2} \left[\int_0^{t/2} |s|^{-\frac{1}{2}} + \int_{t/2}^t |t-s|^{-\frac{1}{2}} \right] \|u(s)\|_{L_x^{12}}^4 ds \\
&\lesssim \|u\|_X |t|^{-1/2} \left\| |s|^{-1/2} \right\|_{L_s^{2,\infty}} \|u\|_{L_t^{8,4} L_x^{12}}^4 .
\end{aligned}
$$

Overall, this implies

$$\|u\|_X \lesssim \|u_0\|_{L^{3/2}} + \eta^4 \|u\|_X .$$

$\square$

Slow Decay Case : $d = 3, p = 3$

Extension to large data. We decompose $\mathbb{R}$ into $J = J(\|u_0\|_{\dot{H}^1}, \eta)$ many intervals $I_j = [T_{j-1}, T_j)$ such that

$$\|u\|_{L_t^{8,4} L_x^{12}(I_j \times \mathbb{R}^3)} < \eta.$$

We introduce a new bootstrap norm

$$\|u\|_{X(T)} = \sup_{t \in (0,T]} |t|^{1/2} \|u(t)\|_{L_x^3}$$

Then for $t \in I_j$, the previous argument shows

$$\|u\|_{X(T_j)} \leq C(\|u_0\|_{\dot{H}^1}) \big[\|u_0\|_{L^{3/2}} + \|u\|_{X(T_{j-1})} + \eta^4 \|u\|_{X(T_j)} \big].$$

Choosing η sufficiently small and the iterating over j concludes the proof the theorem.

Dimension 3 - Fast Decay Case : $6 \leq p \leq \infty$

Proof. We fix $p = \infty, d = 3$ for concreteness.

For simplicity, we work with the additional assumption that

$$\|u\|_{L_t^{6,3} L_x^{18,6}} \lesssim \|\nabla u\|_{L_t^{6,3} L_x^{18/7,6}} \leq C(\|u_0\|_{\dot{H}^1}) < \eta$$

for η sufficiently small.

We define

$$\|u\|_X = \sup_{t \in \mathbb{R}} |t|^{\frac{3}{2}} \|u(t)\|_{L_x^\infty}.$$

We seek a bootstrap of the form

$$\|u\|_X \lesssim C(\|u_0\|_{\dot{H}^1}) \Big[\|u_0\|_{L^1} + \eta^3 \|u\|_X \Big]. \tag{5}$$

Proof continued. Recall the Duhamel formula:

$$u(t) = e^{it\Delta}u_0 \mp i \int_0^t e^{i(t-s)\Delta}\big[|u|^4 u\big](s)ds. \tag{6}$$

By linear dispersive decay,

$$\|e^{it\Delta}u_0\|_X \lesssim \|u_0\|_{L^1}$$

and so the first term is acceptable.

For the nonlinear correction, we decompose $[0, t) = [0, t/2) \cup [t/2, t)$.

Proof continued. Consider the early-time interval $[0, t/2)$.

$$
\begin{aligned}
\left\| \int_0^{t/2} e^{i(t-s)\Delta} \left[|u|^4 u \right](s) ds \right\|_{L_x^\infty} &\lesssim \int_0^{t/2} |t-s|^{-3/2} \left\| u^5(s) \right\|_{L_x^1} ds \\
&\lesssim |t|^{-3/2} \int_0^{t/2} \|u(s)\|_{L_x^3}^{5/3} \left\| u(s) \right\|_{L_x^{15/2}}^{10/3} ds \\
&\lesssim |t|^{-3/2} \int_0^{t/2} |s|^{-5/6} \|u_0\|_{L_x^{3/2}}^{5/3} \left\| u(s) \right\|_{L_x^{15/2}}^{10/3} ds \\
&\lesssim |t|^{-3/2} \|u_0\|_{L^1} \|u_0\|_{\dot{H}^1}^{2/3} \int_0^{t/2} |s|^{-5/6} \left\| u(s) \right\|_{L_x^{15/2}}^{10/3} ds \\
&\lesssim C(\|u_0\|_{\dot{H}^1}) |t|^{-3/2} \|u_0\|_{L^1} \left\| u \right\|_{L_t^{20,10/3} L_x^{15/2}}^{10/3} \\
&\lesssim C(\|u_0\|_{\dot{H}^1}) |t|^{-3/2} \|u_0\|_{L^1}.
\end{aligned}
$$

Dimension 3 - Fast Decay Case : $p = \infty$

Proof continued. Consider the late-time interval $[t/2, t)$.

$$\begin{aligned}\left\| \int_{t/2}^{t} e^{i(t-s)\Delta}\left[|u|^4 u\right](s)ds \right\|_{L_x^\infty} &\lesssim \int_0^{t/2} \left\| \nabla e^{i(t-s)\Delta}\left[|u|^4 u\right](s) \right\|_{L_x^3} ds \\ &\lesssim \int_{t/2}^{t} |t-s|^{-1/2} \left\| \nabla u^5(s) \right\|_{L_x^{3/2}} ds\end{aligned}$$

Dimension 3 - Fast Decay Case : $p = \infty$

Proof continued. Consider the late-time interval $[t/2, t)$.

$$\left\| \int_{t/2}^{t} e^{i(t-s)\Delta}\big[|u|^4 u\big](s)ds \right\|_{L_x^\infty} \lesssim \int_{t/2}^{t} |t-s|^{-1/2} \|\nabla u^5(s)\|_{L_x^{3/2,1}} ds$$

$$\lesssim \|u\|_X |t|^{-3/2} \int_{t/2}^{t} |t-s|^{-1/2} \|\nabla u\|_{L^2} \|u\|_{L_x^{18,6}}^3 ds$$

$$\lesssim \|u\|_X |t|^{-3/2} \left\| \|\nabla u\|_{L^2} \|u\|_{L_x^{18,6}}^3 \right\|_{L_s^{2,1}[0,t/2)}$$

$$\leq C(\|u_0\|_{\dot{H}^1}) \|u\|_X |t|^{-3/2} \|u\|_{L_t^{6,3} L_x^{18,6}}^3 .$$

Along with the linear evolution, this yields

$$\|u\|_X \lesssim C(\|u_0\|_{\dot{H}^1}) \Big[\|u_0\|_{L^1} + \eta^3 \|u\|_X \Big].$$

$\square$

Dimension 4 - Fast Decay Case : $4 \leq p < \infty$

Ideas. Fix $p = \infty$ for example only. Consider the early-time interval $[0, t/2)$.

$$\left\| \int_0^{t/2} e^{i(t-s)\Delta}\left[|u|^2 u\right](s)ds \right\|_{L_x^\infty} \lesssim |t|^{-2} \int_0^{t/2} \left\| u^3(s) \right\|_{L_x^1} ds$$
$$\lesssim |t|^{-2} \int_0^{t/2} \|u(s)\|_{L_x^{5/2}}^{15/7} \|u(s)\|_{L_x^6}^{6/7} ds$$
$$\leq C(\|u_0\|_{\dot{H}^1}) |t|^{-2} \|u_0\|_{L^1} \int_0^{t/2} |s|^{-6/7} \|u(s)\|_{L_x^6}^{6/7} ds$$
$$\leq C(\|u_0\|_{\dot{H}^1}) |t|^{-2} \|u_0\|_{L^1} \|u\|_{L_t^{7,6/7} L_x^6}^{6/7}.$$

Fix : $$u^3(s) = \left[e^{is\Delta} u_0 \mp i \int_0^s e^{i(s-\tau)\Delta}\left[|u|^2 u\right](\tau) d\tau \right]^3$$

Lemma. $\|e^{it\Delta} f\|_{L_{t,x}^3}^3 \lesssim \|f\|_{L^1} \|f\|_{\dot{H}^1}^2$.

Dimension 4 - Fast Decay Case : $4 \leq p < \infty$

Ideas continued. Consider the late-time interval $[t/2, t)$.

$$\left\| \int_{t/2}^{t} e^{i(t-s)\Delta} \left[|u|^2 u\right](s) ds \right\|_{L_x^\infty} \lesssim \int_{t/2}^{t} \left\| \nabla e^{i(t-s)\Delta} \left[|u|^2 u\right](s) \right\|_{L_x^4} ds$$
$$\lesssim \int_{t/2}^{t} |t-s|^{-1} \left\| \nabla u^3(s) \right\|_{L_x^{4/3}} ds.$$

For $p < \infty$, this argument works and completes the proof for $d = 4$.

The Main Theorem (Edge Case)

Theorem (K. in preparation)

Given $u_0 \in L^1 \cap \dot{B}^{1,2}_1(\mathbb{R}^4)$ *satisfying Theorem 1, let* $u(t)$ *denote the unique global solution to* (1). *Then*

$$\|u(t)\|_{L^\infty} \leq C(\|u_0\|_{\dot{B}^{1,2}_1})|t|^{-2}\|u_0\|_{L^1}$$

for all $t \in \mathbb{R}$.

Here $\dot{B}^{1,2}_1$ is a homogeneous Besov space defined by

$$\|u_0\|^2_{\dot{B}^{1,2}_1} = \sum_{N \in 2^{\mathbb{Z}}} N\|P_N u_0\|_{L^2_x},$$

where P_N is a *Littlewood–Paley projection* onto frequencies $\sim N$.

Note that $\dot{B}^{1,2}_1 \hookrightarrow \dot{H}^1$.

Ideas. It remains to consider the $[t/2, t)$ piece. We localize in frequency and introduce a cutoff $B > 0$:

$$\Big\| \int_{t/2}^{t} e^{i(t-s)\Delta} P_N\big[|u|^2 u\big](s) ds \Big\|_{L_x^\infty}$$

$$\leq \Big(\int_{t/2}^{t-B} + \int_{t-B}^{t} \Big) \big\| e^{i(t-s)\Delta} P_N\big[|u|^2 u\big](s) \big\|_{L_x^\infty} ds$$

$$\lesssim \int_{t/2}^{t-B} |t-s|^{-2} \big\| P_N\big[|u|^2 u\big](s) \big\|_{L_x^1} ds$$

$$+ \int_{t-B}^{t} N^2 \big\| P_N\big[|u|^2 u\big](s) \big\|_{L_x^2} ds.$$

Optimizing in B and summing over N, we find that

$$\Big\| \int_{t/2}^{t} e^{i(t-s)\Delta} \big[|u|^2 u\big](s) ds \Big\|_{L_x^\infty} \lesssim \|u\|_X \Big(\sum_{N \in 2^{\mathbb{Z}}} N \|P_N u\|_{L_t^\infty L_x^2} \Big)^2$$

Besov Spacetime Bounds

Proposition

Suppose p, q *satisfy*

$$\frac{2}{p} + \frac{4}{q} = 2 \tag{7}$$

and that $u_0 \in \dot{B}^{1,2}_1(\mathbb{R}^4) \subset \dot{H}^1(\mathbb{R}^4)$ *satisfies Theorem 1. Then the corresponding global solution* $u(t)$ *to* (1) *satisfies*

$$\sum_{N \in 2^{\mathbb{Z}}} N \|u_N\|_{L^p_t L^q_x} \leq C(\|u_0\|_{\dot{B}^{1,2}_1}).$$

Dimension 4 - $p = \infty$

Ideas continued. As shown,

$$\left\| \int_{t/2}^{t} e^{i(t-s)\Delta} \big[|u|^2 u \big](s) ds \right\|_{L_x^\infty} \lesssim \|u\|_X C(\|u_0\|_{\dot{B}_1^{1,2}}) \lesssim \eta^2 \|u\|_X$$

Which yields the bootstrap statement

$$\|u\|_X \lesssim C(\|u_0\|_{\dot{B}_1^{1,2}}) \Big[\|u_0\|_{L^1} + \eta^2 \|u\|_X \Big],$$

and concludes the proof for small initial data $u_0 \in \dot{B}_1^{1,2}$.

Extension to Large Data

Problem : We have

$$\left\| \int_{t/2}^{t} e^{i(t-s)\Delta} P_N \left[|u|^2 u \right](s) ds \right\|_{L_x^\infty} \lesssim \|u\|_X \left(\sum_{N \in 2^{\mathbb{Z}}} N \|P_N u\|_{L_t^\infty L_x^2} \right)^2 .$$

L_t^∞ does not imply any decay in t. Therefore, we **cannot** decompose $\mathbb{R}$ into intervals I on which

$$\sum_{N \in 2^{\mathbb{Z}}} N \|P_N u\|_{L_t^\infty L_x^2 (I \times \mathbb{R}^4)} \ll 1.$$

Extension to Large Data

Solution : Induct on the size of u_0.

Suppose that the decay holds for $\|u_0\|_{\dot{B}^{1,2}_1} \leq R$. For some small $\varepsilon > 0$, we consider initial data of the form

$$u_0 = \underbrace{v_0}_{\|v_0\|_{\dot{B}^{1,2}_1} \leq R} + \underbrace{w_0}_{\|w_0\|_{\dot{B}^{1,2}_1} \leq \varepsilon} .$$

With corresponding solution

$$u(t) = v(t) + w(t)$$

where

$$iv_t + \Delta v \pm |v|^2 v = 0$$
$$iw_t + \Delta w \pm \underbrace{\left(|v + w|^2(v + w) - |v|^2 v\right)}_{\approx \text{cubic}} = 0.$$

Besov Stability

Proposition

Fix $2 \leq p, q \leq \infty$ *which satisfy*

$$\frac{2}{p} + \frac{4}{q} = 2.$$

Suppose that u_0, v_0 *satisfy Theorem 1 and* $\|u_0\|_{\dot{B}^{1,2}_1}, \|v_0\|_{\dot{B}^{1,2}_1} \leq R$. *Then the corresponding solutions* $u(t), v(t)$ *to* (1) *satisfy*

$$\sum_{N \in 2^{\mathbb{Z}}} \|\nabla(u_N - v_N)\|_{L^p_t L^q_x} \leq C(R) \|u_0 - v_0\|_{\dot{B}^{1,2}_1}.$$

Future Work

This method relies largely on scattering, Strichartz estimates, and the resulting spacetime control.

It is our hope that this can be generalized to many dispersive PDEs which exhibit global spacetime bounds strong enough to imply scattering.

Thank you!

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