APPLICATION OF CONFORMAL MAPPINGS TO ELECTROSTATICS

MATTHEW KOWALSKI

Abstract. We study the application of complex analysis, particularly conformal mappings, to the problem of solving for an electric potential subject to Dirichlet boundary conditions. By conformally mapping a region of interest to a simpler region with exploitable symmetries, we are able to solve increasingly difficult electrostatic problems with minimal effort. Though a relatively standard technique in physics, this paper aims to provide a rigorously defined mathematical foundation for the method, which is often lacking in a physics-focused source.

1 Introduction

The purpose of this paper is to establish a method of using conformal maps to solve electrostatics problems in two dimensions. Questions of the existence of electric fields satisfying certain boundary conditions are not addressed, with a focus instead on taking existing solutions and transforming them to other domains.

The paper will begin with a brief overview of electrostatics, emphasizing the importance of Laplace's equation and its consequences. From there, the complex electric potential will be introduced and results following from complex analysis will be shown. Finally, the method of applying conformal maps to solve electrostatics will be shown and generalized to three dimensions. The paper will conclude with a simple example demonstrating the ability of conformal maps to reveal symmetries.

1.1 Note on Notation

The problem of solving for electric fields or potentials can be made more general by working in the one point compactification of $\mathbb{R}^n$ or $\mathbb{C}$. To this end, we denote the one point compactifications of $\mathbb{R}^n$ and $\mathbb{C}$, respectively, by $\mathbb{R}^2_\infty$ and $\mathbb{C}_\infty$.

2 Electric Potential

Given a fixed charge distribution, ρ, over a subset of Euclidean space, electrostatics is concerned with computing the electric field generated by said charge distribution. Given that the magnetic field and time-dependent charge distributions are considerations of electrostatics, Maxwell's equations [2] imply that electrostatics is governed by

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad\qquad \nabla \times \mathbf{E} = 0$$

Where ε_0 is the permitivity of free space.

The second condition, that $\mathbf{E}$ is curl-less, is equivalent to the condition that $\mathbf{E}$ can be expressed as the gradient of a scalar potential, $V : \mathbb{R}^2 \to \mathbb{R}$. This equivalence follows from Poincaré's lemma that states that all closed forms on $\mathbb{R}^2$ are exact, a result which extends to vector fields such as $\mathbf{E}$. Therefore, the problem of computing $\mathbf{E}$ given a charge distribution is reduced to computing the electric potential V, from which $\mathbf{E}$ can be easily found as $\mathbf{E} = -\nabla V$. The negative sign here is convention arising from physical considerations, as the electric potential additionally represents electric potential energy per charge and the electric field should follow the direction of maximal energy decrease.

By restricting our domain, we may further work with the assumption that there are no charges in our area of interest. Therefore, we will work entirely in the case where $\rho = 0$ on our domain. This restriction does not greatly limit practical applications, as electric fields are largely non-useful at points of charge. In this case, Maxwell's equations restrict to the forms $\nabla \cdot \mathbf{E} = 0$ and $\nabla \times \mathbf{E} = 0$. When the electric field is expressed as $\mathbf{E} = -\nabla V$, these equations restrict further to only Laplace's equation, which is as follows.

Definition 2.1 (Laplace's Equation)**.**

$$\nabla^2 V = 0$$

It should be noted that Laplace's equation is relevant to the study of heat conduction, gravitation, fluid dynamics, and many other areas of interest. Therefore, while this paper is intended for electrostatics, it can be expanded easily to many areas. The study of solutions to Laplace's equation in general is referred to as *potential theory*.

By definition, satisfying Laplace's equation is equivalent to being harmonic. With this in mind, we establish the following definition of an electric potential in $\mathbb{R}^n_\infty$ and $\mathbb{C}_\infty$,

Definition 2.2 (Electric Potential)**.** Let D be an open subset of $\mathbb{R}^n$ or $\mathbb{C}$ and let $\overline{D}$ be the closure of D in $\mathbb{R}^n_\infty$ or $\mathbb{C}_\infty$ respectively. A function $V : \overline{D} \to \mathbb{R}$ is said to be an *electric potential* on D if V is harmonic on D and continuous when approaching the boundary of D from the interior. That is to say that for all sequences $(x_n) \subset D$ such that $x_n \to x \in \overline{D}$, $\lim_{n\to\infty} f(x_n) = f(x)$.

This definition is not concerned with whether an electric potential is physically realizable, as that question is far broader than the scope of this paper. Rather, it establishes electric potentials as a solely theoretical concept.

2.1 Boundary Conditions

Suppose there exists an open, simply connected region $D \subset \mathbb{R}^n$ that we wish to solve for an electric potential on. In order to have an interesting solution, there must be some boundary condition on $\partial D \subset \mathbb{R}^n_\infty$. Moreover, in order for this system to have physical significance, the boundary condition must be sufficiently strong so that it gives a unique solution on the interior.

Physical reasoning would imply that the specification of the electric potential or electric field on the boundary should give a unique solution on the interior. Therefore, in order to ensure that Laplace's equation has a unique and well-behaved solution, two types of boundary conditions are usually specified : *Dirichlet* or *Neumann* boundary conditions. Dirichlet boundary conditions correspond to specifying the value of the potential on the boundary and Neumann boundary conditions specify the normal derivative of the potential, which is equivalent to specifying the electric field. For our purposes, we will restrict our focus to Dirichlet boundary conditions as they are the most natural. However, the theory developed can easily be extended to Neumann boundary conditions.

3 Complex Electric Potential

Electrostatics can be done entirely working with real spaces and real functions. However, expanding to complex functions and converting to complex spaces can reveal elegant and simple proofs that are more enlightening than their real counterparts. To that end, we will work on subsets of $\mathbb{C}$ rather than the classic subsets of $\mathbb{R}^n$. Note that the definition of a harmonic function and the electric potential remain the same. In order to access the elegance of complex analysis, we further wish to expand the real, harmonic electric potential to a holomorphic function. To do so, we establish the following theorem.

Theorem 3.1 (Extension from Harmonic to Holomorphic)**.** *Let there exist an open, simply connected $O \subset \mathbb{C}$. Let $V : O \to \mathbb{R}$ be a harmonic function. Then there exists a holomorphic function $\varphi : O \to \mathbb{R}$ such that $\operatorname{Re}\varphi = V$ where φ is unique up to an imaginary constant.*

We call φ the holomorphic extension *of V and we call the imaginary part of φ a* harmonic conjugate *of V.*

Proof. This proof is shown in Donald Sarason's "Notes on Complex Function Theory" [3]. For completion, we present a version here.

Define $f : O \to \mathbb{C}$ as $f = \frac{\partial V}{\partial x} - i\frac{\partial V}{\partial y}$. We first aim to show that f is holomorphic. By the harmonicity of V, $\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}$ are continuously differentiable. Therefore, to show that f is holomorphic, it suffices to show that the Cauchy Riemann equations are satisfied. By the harmonicity of V and the symmetry of mixed partials,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \implies \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial V}{\partial y}\right)$$

$$\frac{\partial^2 V}{\partial y \partial x} = \frac{\partial^2 V}{\partial x \partial y} \implies \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial x}\right) = -\frac{\partial}{\partial x}\left(-\frac{\partial V}{\partial y}\right)$$

Therefore f satisfies the Cauchy-Riemann equations and thus is holormorphic.

Using f, we will now construct a holomorphic extension of V. Because f is holomorphic on a simply connected domain, there exists a holomorphic function φ on O such that $\varphi' = f$. By adding a constant, we may assume that $\varphi(z_0) = V(z_0)$ for some $z_0 \in O$. We claim that $\operatorname{Re}\varphi = V$. To show this, we split φ into real and imaginary parts, α and β respectively. Because φ is holomorphic, $\frac{d\varphi}{dz} = 2\frac{\partial\alpha}{\partial z}$ where $\frac{\partial}{\partial z}$ is the Wirtinger derivative. Manipulating this with the Cauchy-Riemann equations, we find the following two equalities.

$$\frac{d\varphi}{dz} = \frac{\partial\alpha}{\partial x} + i\frac{\partial\beta}{\partial x} = \frac{\partial\alpha}{\partial y} - i\frac{\partial\alpha}{\partial y}$$

Comparing this with the fact that $\varphi' = f$, we find that $\frac{\partial\alpha}{\partial x} = \frac{\partial V}{\partial x}$ and $\frac{\partial\alpha}{\partial y} = \frac{\partial V}{\partial y}$. Therefore, $\operatorname{Re}\varphi$ and V differ by a constant. As we defined φ such that $\varphi(z_0) = V(z_0)$, this implies that $\operatorname{Re}\varphi = V$. Therefore, φ is a holomorphic function with real part V, and so is a holomorphic extension of V.

We now aim to show that φ is unique up to a constant. Suppose there exists a holomorphic φ^* such that $\operatorname{Re}\varphi^* = V$. As the complex derivative can be completely expressed by the partial derivatives of the real part, this implies that φ and φ^* have the same derivative and therefore differ by a constant. As their real parts are equal, this constant must be purely imaginary. □

From this extension, we define the complex electric potential as follows

Definition 3.2 (Complex Electric Potential)**.** [1] Let there exist an open $D \subset \mathbb{C}$ and let $\overline{D}$ be the closure of D in $\mathbb{C}_\infty$. Suppose that there exists an electric potential $V : \overline{D} \to R$. We define the *complex electric potential* on D associated with V to be the holomorphic extension $\Phi : D \to \mathbb{C}$ of V.

By theorem 3.1, we know that the complex electric potential is well-defined and unique up to an imaginary constant for any electric potential. Moreover, as the real and imaginary part of any holomorphic function are harmonic, any holormophic function uniquely generates two electric potentials. Therefore, the problem of finding an electric potential on some region with given boundary conditions is equivalent to finding a holomorphic function whose real or complex part satisfies the same boundary conditions.

3.1 Consequences of Holomorphic Extension

By working with the complex electric potential, we now have access to the tools of complex analysis when solving electrostatics. From this, basic properties of holomorphic functions can imply powerful properties of electric fields. For our purposes, the most important of these is the uniqueness of the electric potential, which will follow from the uniqueness of the complex electric potential. In order to prove the uniqueness of the complex potential, we first must remind the reader of the open mapping property and then show the maximum modulus principle of holomorphic functions.

Lemma 3.3 (Open Mapping Property)**.** *Let f be holomorphic and non-constant on some open, connected $U \subset \mathbb{C}$. Then f is an open map.*

Proof. As this result is only used to show the maximum modulus principle, which is then used to show the uniqueness of the complex electric potential, we omit this proof for the sake of compactness. We direct the reader instead to Stein-Shakarchi's proof of the same result in Theorem 4.4 [4]. □

With the open mapping property, we may prove the maximum modulus principle of holomorphic functions as follows.

Lemma 3.4 (Maximum Modulus Principle)**.** *Let f be a holomorphic function on an open connected $U \subset \mathbb{C}$. Then either f is constant or $|f|$ does not attain a local maximum on U.*

Proof. Suppose that f is non-constant and suppose for the sake of contradiction that $|f|$ attains a local maximum at $z \in U$. Then there exists some open $B \subset U$ such that $z \in B$ and $|f(w)| \leq |f(z)|$ for all $w \in B$.

Consider $f(B)$. Because $|f(w)| \leq |f(z)|$ for all $w \in B$, $f(B) \subset \overline{D(0, |f(z)|)}$. As $f(z) \in f(B)$, this implies that there does not exist an open neighborhood of $f(z)$ contained in $f(B)$, as any neighborhood of $f(z)$ will extend outside $\overline{D(0, |f(z)|)}$. However, this contradicts the fact that $f(B)$ is open by the open mapping property. Therefore our supposition was incorrect and $|f|$ does not attain a local maximum on U. [4] □

With the maximum modulus principle, we may now establish the uniqueness of complex electric potentials as follows.

Theorem 3.5 (Uniqueness of Complex Electric Potential Given Boundary Conditions)**.** *Let D be an open, simply connected subset of $\mathbb{C}$ and let ∂D be the boundary of D in $\mathbb{C}_\infty$. Let Φ be a complex electric potential on D satisfying the boundary conditions Φ_0 on ∂D. Then Φ is the unique complex potential satisfying the boundary conditions Φ_0.*

Proof. Suppose there exists complex electric potential Φ, Φ^* on D satisfying the boundary conditions Φ_0. Then $\Phi - \Phi^* : \overline{D} \to \mathbb{C}$ is holomorphic on D and 0 on ∂D. As $|\Phi - \Phi^*| = 0$ on ∂D, either $|\Phi - \Phi^*|$ is constant on D or $|\Phi - \Phi^*|$ attains a local maximum on D. By the maximum modulus principle of holomorphic functions, $|\Phi - \Phi^*|$ does not attain a local maximum on D. Therefore, $|\Phi - \Phi^*|$ is constant on D. By continuity, this implies that

$|\Phi - \Phi^*| = 0$ and so $\Phi = \Phi^*$. This implies that Φ is the unique complex potential satisfying Φ_0. □

By relaxing the boundary condition to solely a condition on the real part of Φ, we find that Φ is unique up to an imaginary constant. Therefore, by chasing uniqueness, this has the immediate corollary of

Corollary 3.6 (Uniqueness of the Electric Potential)**.** *Let $D \subset \mathbb{C}$ be open and simply connected and let ∂D be the boundary of D in $\mathbb{C}_\infty$. Let there exist $V_0 : \partial D \to \mathbb{R}$ and suppose that there exists an electric potential $V : \overline{D} \to \mathbb{R}$ satisfying V_0 on ∂D. Then V is unique.*

Continuation of Electric Potential

Though not strictly required for the purposes of this paper, we can use the holormorphic extension to show the interesting result that any electric potentials agreeing on an open subset of a connected set must agree everywhere. To do so, we can use the similar property of analytic (holomorphic) functions.

Lemma 3.7 (Analytic Continuation)**.** *Let O be an open, connected subset of $\mathbb{C}$. Let $f, g : O \to \mathbb{C}$ be holomorphic functions. If f and g agree on some non-empty, open subset of O, then $f = g$ on O.*

Proof. The reader is directed to Stein-Shakarchi's proof of this in Corollary 4.9 [4]. □

By utilizing the complex electric potential, we can extend this result to electric potentials.

Theorem 3.8 (Harmonic Continuation (Uniqueness of Electric Potentials))**.** *Let O be an open, connected subset of $\mathbb{C}$. Let V_1, V_2 be electric potentials (harmonic) on O. If V_1 and V_2 agree on some non-empty, open subset of O, then $V_1 = V_2$ on O.*

Proof. Suppose that there exists some nonempty, open subset $D \subset O$ such that $V_1 = V_2$ on D. Let Φ_1, Φ_2 be complex electric potentials associated with V_1, V_2 respectively on O. Let there exist some $z_0 \in D$. Because $\operatorname{Re}\Phi_1 = \operatorname{Re}\Phi_2$ on O and a complex potential can be adjusted by an imaginary constant while remaining associated to an electric potential, we may assert that $\Phi_1(z_0) = \Phi_2(z_0)$. Because $\operatorname{Re}\Phi_1 = \operatorname{Re}\Phi_2$ on O and the complex derivative can be written entirely as a function of the real part, $\Phi_1' = \Phi_2'$ on O. Therefore Φ_1 and Φ_2 differ by a constant on O. As $\Phi_1(z_0) = \Phi_2(z_0)$ for some $z_0 \in D$, this implies that $\Phi_1 = \Phi_2$ on D. Therefore, by analytic continuation, $\Phi_1 = \Phi_2$ on O. This implies that $V_1 = \operatorname{Re}\Phi_1 = \operatorname{Re}\Phi_2 = V_2$ on O. □

This has the interesting implication that any electric potential is globally defined by its local behavior.

4 CONFORMAL MAPPINGS

For our purposes, we establish the following definition of a conformal map

Definition 4.1 (Conformal Map)**.** A map $f : U \to V$ where $U, V \subset \mathbb{C}_\infty$ is *conformal* if it is a holomorphic map with holomorphic inverse. This is equivalent to being a *complex diffeomorphism* and is sometimes called a *biholomorphism*.

Some sources generalize conformal maps to those which are locally complex diffeomorphisms. However, for our purposes, it is better to have the most restrictive definition as it makes our mappings clearer.

From this definition, we acquire the following theorem.

Theorem 4.2 (Conformal Maps Preserve Complex Potential)**.** *Suppose there exist simply connected, open subsets D_1, D_2 with closures $\overline{D}_1, \overline{D}_2 \subset \mathbb{C}_\infty$ such that there exists a continuous $f : \overline{D}_1 \to \overline{D}_2$ that is conformal on D_1 and takes $\partial D_1 \to \partial D_2$. Then given any boundary conditions Φ_1 on ∂D_1 that are satisfied by a complex potential Φ on D_1, $\Phi \circ f^{-1}$ is a complex potential on D_2 satisfying $\Phi_1 \circ f^{-1}$. Similarly, given any boundary conditions Φ_2 on ∂D_2 that are satisfied by a complex potential Φ on D_2, $\Phi \circ f$ is a complex potential on D_1 satisfying $\Phi_2 \circ f$.*

This is to say that conformal maps between spaces generate equivalences of complex potentials.

Proof. To show the first claim, suppose that there exists boundary conditions Φ_1 on ∂D_1 that are satisfied by a complex potential Φ on D_1. Because f is conformal, $\Phi \circ f^{-1} : \overline{D}_2 \to \mathbb{C}$ is the composition of holomorphic maps and is therefore holomorphic on D_2. Additionally, because f maps $\partial D_1 \to \partial D_2$, $\Phi \circ f^{-1}$ satisfies $\Phi_1 \circ f^{-1}$. This completes the proof of the first claim. The second claim follows from a symmetric argument. □

By removing one of the directions, we may relax the requirements on f such that it need only be holomorphic on the interior and continuously preserve the boundary. Doing so, we arrive at the following alteration of the previous theorem.

Theorem 4.3 (Holomorphic Maps Pullback Complex Potentials)**.** *Let there exist simply connected, open subsets $\tilde{D}, D$ with closures $\overline{\tilde{D}}, \overline{D} \subset \mathbb{C}_\infty$. Suppose that there exists a continuous map $f : \overline{\tilde{D}} \to \overline{D}$ that is holomorphic on $\tilde{D}$, surjective, and takes $\partial\tilde{D}$ to ∂D. Then given boundary conditions Φ_0 on ∂D that are satisfied by a complex potential Φ on D, $\Phi \circ f$ is a complex potential on $\tilde{D}$ satisfying $\Phi_0 \circ f$.*

This states that the pullback of holomorphic maps preserve complex potentials.

Proof. The proof of this follows the same logic as the proof of the previous theorem. □

4.1 Restriction to Electric Potential

While perhaps more elegant when in complex form, the benefits besides beauty are limited for the complex electric potential. Therefore, we restrict our conformal mapping theorem to the electric potential. Because the complex electric potential admits two unique electric potentials and an electric potential admits a unique complex potential, our conformal maps should also preserve electric potential. In fact, by noting that $\operatorname{Re} \Phi \circ f = V \circ f$ and $\operatorname{Re} \Phi \circ f^{-1} = V \circ f^{-1}$, we gain the preservation immediately. We then arrive at the following theorem.

Corollary 4.4 (Conformal Maps Preserve Electric Potential)**.** *Suppose there exist simply connected, open subsets D_1, D_2 with closures $\overline{D}_1, \overline{D}_2 \subset \mathbb{C}_\infty$ such that there exists a continuous $f : \overline{D}_1 \to \overline{D}_2$ that is conformal on D_1 and takes $\partial D_1 \to \partial D_2$. Then given any boundary conditions V_1 on ∂D_1 that are satisfied by an electric potential V on D_1, $V \circ f^{-1}$ is an electric potential on D_2 satisfying $V_1 \circ f^{-1}$. Similarly, given any boundary conditions V_2 on ∂D_2 that are satisfied by an electric potential V on D_2, $V \circ f$ is an electric potential on D_1 satisfying $V_2 \circ f$.*

5 Generalization to Open Sets

As stated, our conformal mapping theorem relies on our region being simply connected. We now aim to extend this to open sets in general.

Theorem 5.1 (Conformal Maps Preserve Electric Potential)**.** *Suppose there exist open subsets D_1, D_2 with closures $\overline{D}_1, \overline{D}_2 \subset \mathbb{C}_\infty$ such that there exists a continuous $f : \overline{D}_1 \to \overline{D}_2$ that is conformal on D_1 and takes $\partial D_1 \to \partial D_2$. Then given any boundary conditions V_1 on ∂D_1 that are satisfied by an electric potential V on D_1, $V \circ f^{-1}$ is an electric potential on D_2 satisfying $V_1 \circ f^{-1}$. Similarly, given any boundary conditions V_2 on ∂D_2 that are satisfied by an electric potential V on D_2, $V \circ f$ is an electric potential on D_1 satisfying $V_2 \circ f$.*

Proof. We will only show the first claim, as the second claim will follow the same logic. To this end, suppose that V_1 are boundary conditions on ∂D_1 that are satisfied by an electric potential V on D_1. Consider the function $V \circ f^{-1} : D_2 \to \mathbb{R}$. Because the fact that $V \circ f^{-1}$ satisfies the boundary conditions $V_1 \circ f^{-1}$ follows immediately from the fact that f preserves boundaries, it suffices to show that $V \circ f^{-1}$ is harmonic on D_2.

Let there exist some $z \in D_2$. Because D_2 is open, there exists an open ball $B' \subset D_2$ containing z. Because f is continuous, $f^{-1}(B')$ is open. Then there exists some open $B \subset f^{-1}(B')$ such that $f^{-1}(z) \in B$. Because f is conformal, it is a homeomorphism and so preserves simply connectedness. Because the ball B is simply connected, this implies that $f(B)$ is simply connected. Then, as defined, $f : B \to f(B)$ is a conformal map between simply connected, open subset of $\mathbb{C}$. Corollary 4.4 then implies that $V \circ f^{-1}$ is harmonic on $f(B)$. In particular, this implies that V is harmonic at z. As this holds for all $z \in D_2$, $V \circ f^{-1}$ is harmonic on D_2. Therefore, $V \circ f^{-1}$ is an electric potential on D_1 satisfying the boundary conditions $V_2 \circ f^{-1}$. □

6 Expansion to Higher Dimensions

Though interesting in its own right, two dimensional electrostatics has limited applications as it generally has different potentials for seemingly equivalent situations. Therefore, we wish to extend our conformal mapping method to some class of three dimensional problems. To do so, we exploit the translational symmetry of many electrostatic systems.

Let F be an open region of $\mathbb{R}^3$ such that F is invariant under translation along a fixed direction. Without loss of generality, we may choose our basis of $\mathbb{R}^3$ such that this translational symmetry is along the z axis. We may then associate $F \cong D \times \mathbb{R}$ where D is an open subset of $\mathbb{C}$.

Suppose that there exists $V_0 : \partial D \times \mathbb{R} \to \mathbb{R}$ satisfying the same symmetry as F such that there exists an electric potential V on $D \times \mathbb{R}$ satisfying V_0. Further, suppose that there exists an open $\tilde{D} \subset \mathbb{C}$ such that there exists a conformal map $f : D \to \tilde{D}$ taking boundary to boundary. Then $V \circ (f \times \mathrm{id})$ is an electric potential on $\tilde{D} \times \mathbb{R}$ satisfying the boundary conditions $V_0 \circ (f \times \mathrm{id})$. The proof of this claim follows the same logic as the two-dimensional case, with the added annoyance of mapping the additional dimension identically in each step.

7 Non-Concentric Circles

We conclude this paper with a brief example of the method of conformal mappings, namely non-concentric circles. To this end, let there exist $0 < r_0 < r_1$ and $z_0 \in D(0, r_1)$

such that $r_1 - |z_0| < r_0$. Define $D \subset \mathbb{C}$ as $D = \{z : |z| < r_1, |z - z_0| < r_1\}$. As defined, D is the area between two non-concentric circles. For $a_0, a_1 \in \mathbb{R}$, we establish the boundary conditions $V_0 : \partial D \to \mathbb{R}$ as

$$V_0 = \begin{cases} a_0, & |z - z_0| = r_0 \\ a_1, & |z| = r_1 \end{cases}$$

We wish to find an electric potential V on D satisfying V_0.

Though seemingly solvable in its current state, the lack of symmetry is concerning. Therefore, we will use a conformal map to reduce our problem to one with rotational symmetry. To this end, define $\tilde{D} = \{z : r_0 < |z| < r_1\}$. We aim to find a conformal map $f : D \to \tilde{D}$, which we will construct in steps. Define f_1, f_2, f_3, f_4, f_5 to be maps on the Riemann sphere such that

$$\begin{aligned} f_1 &: z \mapsto \frac{z}{r_1} & f_2 &: z \mapsto \frac{z - i}{z + i} \\ f_3 &: z \mapsto z - \operatorname{Re} f_2 \circ f_1(z_0) & f_4 &: z \mapsto \frac{z - f_3 \circ f_2 \circ f_1(z_0)}{z + f_3 \circ f_2 \circ f_1(z_0)} \\ f_5 &: z \mapsto r_1 z \end{aligned}$$

Direct computation shows that f_1 maps $\overline{D(0, r_1)}$ to $\overline{D(0, 1)}$ and then f_2 maps $\overline{D(0, 1)}$ to $\overline{\mathbb{H}}$, which takes the inner circle to a circle in the upper half plane and the outer circle to the real line. Then f_3 maps the center of the inner circle to the imaginary axis, f_4 maps the real line to the unit circle and the inner circle to a circle centered at 0, and finally f_5 scales the system back to the original scaling. As each of these functions are Mobius transformations, they are conformal. Therefore we define the conformal map $f = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$ that takes D to $\tilde{D}$ and preserves boundaries. We will now find an electric field $\tilde{V}$ satisfying $V \circ f^{-1}$

It follows from direct computation, or transformation arguments, that

$$V_0 \circ f^{-1} = \begin{cases} a_0, & |z| = r_0 \\ a_1, & |z| = r_1 \end{cases}$$

Which are rotationally symmetric boundary conditions. Therefore, from the polar form of Laplace's equations, we find that the electric potential on $\tilde{D}$ has the general form $\tilde{V}(z) = \alpha \log(|z|) + \beta$. Plugging in the boundary conditions then implies that

$$\tilde{V}(z) = \left(\frac{a_1 - a_0}{\log(r_1/r_0)} \right) \log |z| + a_0 - \frac{(a_1 - a_0) \log(r_0)}{\log(r_1/r_0)}$$

By our conformal map, this yields the electric potential $V = \tilde{V} \circ f$ on D that satisfies the boundary conditions $V_0 \circ f^{-1} \circ f = V_0$. The exact form of V is easily calculable, but is unwieldy and does not provide further insight.

This example illustrates the fact that electrostatics, and rather physics as a whole, is largely a study of symmetry. Students of physics are taught to find symmetries in any problem, both for the physical significance via Noether's theorem, but also for the computational shortcuts that they provide. As is shown in the example, the method of using conformal mappings allows one to take a problem and transform it into a similar problem with exploitable symmetries.

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Department of Mathematics, University of California, Los Angeles, CA 90095-1555,USA
Email address: mattkowalski@math.ucla.edu