ANALYSIS

FALL 2011

① 405 846 515

Suppose $\exists$ measurable $f_n, f : [0,1] \to \mathbb{R}$ s.t. $f_n \to f$ a.e.
We aim to show that $\forall \varepsilon > 0$ $\exists$ measurable $E \subset [0,1]$ w/ $|E| < \varepsilon$ s.t. $f_n \rightrightarrows f$ on $[0,1] \setminus E$.

Define $E_n(k) = \{x \in [0,1] : |f_m(x) - f(x)| < 1/k \ \forall m \geq n\}$. ← Note this is measurable b/c f_m, f are measurable.
Because $f_n \to f$ a.e, w/ k fixed, for a.e. x $\exists$ n s.t. $|f_m(x) - f(x)| < 1/k$ $\forall m \geq n$ $\Rightarrow x \in E_n(k)$. Therefore $\left|[0,1] \setminus \bigcup_{n=1}^{\infty} E_n(k)\right| = 0$.
By monotone convergence, this implies $\lim_{N\to\infty} \left|\bigcup_{n=1}^{N} E_n(k)\right| = 1$.
By construction, we have $E_n(k) \subset E_{n+1}(k)$ and so $\bigcup_{n=1}^{N} E_n(k) = E_N(k)$.
Therefore $\lim_{n\to\infty} |E_n(k)| = 1$ $\forall k$.

Fix some $\varepsilon > 0$. For all k, choose n_k sufficiently large so that $|E_{n_k}(k)| > 1 - 2^{-k}\varepsilon$. Define $E = \bigcup_{k=1}^{\infty} E_{n_k}(k)^c$. Then E is measurable and $|E| \leq \sum_{k=1}^{\infty} |E_{n_k}(k)^c| \leq \varepsilon \sum_{k=1}^{\infty} 2^{-k} = \varepsilon$. We claim $f_n \rightrightarrows f$ on E^c.
Suppose $\exists$ $\delta > 0$. Then $\exists$ k s.t. $1/k < \delta$. Since $E^c \subset E_{n_k}(k)$, $\forall x \in E^c$, $\forall m \geq n_k$, $|f_m(x) - f(x)| < 1/k < \delta$. As such an n_k can be found $\forall \delta > 0$, this implies $f_n \rightrightarrows f$ on E^c.
As this can be done $\forall \varepsilon > 0$, this completes the claim. $\square$

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(a). B/c $d\sigma$ is invariant under rotations and $\forall$ rotations R

$$Rx\cdot\xi = x\cdot R^T\xi$$

it follows that it suffices to consider $\xi = (0,0,|\xi|)$.

Then

$$\int_{S^2} e^{ix\cdot\xi}\,d\sigma(x) = \int_{S^2} e^{i|\xi|x_3}\,d\sigma = \int_{S^2}\left(\cos(|\xi|x_3) + i\sin(|\xi|x_3)\right)d\sigma(x)$$

B/c S^2, $d\sigma$ have even parity and $\sin(|\xi|x_3)$ has odd,

$$\int_{S^2} e^{ix\cdot\xi}\,d\sigma(x) = \int_{S^2}\cos(|\xi|x_3)\,d\sigma(x)$$

(spherical coordinates) $$= \int_0^{\pi}\int_0^{2\pi}\cos(|\xi|\cos\varphi)\sin\varphi\,d\theta\,d\varphi$$

$$= 2\pi\int_0^{\pi}\cos(|\xi|\cos\varphi)\sin\varphi\,d\varphi$$

$$= \frac{-2\pi}{|\xi|}\sin(|\xi|\cos\varphi)\Big|_{\varphi=0}^{\pi}$$

$$= \frac{-2\pi}{|\xi|}\left(-\sin(|\xi|) - \sin(|\xi|)\right)$$

$$= \frac{4\pi}{|\xi|}\sin|\xi|$$

as desired.

(b) Define $L: f \mapsto \int_{S^2}\int_{S^2} f(x+y)\,d\sigma(x)\,d\sigma(y)$. We aim to show that L extends uniquely ~~to a bound~~ from $C_c^\infty(\mathbb{R}^3)$ to a bounded linear functional on $L^2(\mathbb{R}^3)$. B/c C_c^∞ is dense in L^2, it suffices to show that $\|Lf\| \lesssim \|f\|_{L^2}$ uniformly in f $\forall f \in C_c^\infty$.

Fix some $f \in C_c^\infty$ and note that f is Schwartz. Then Fourier inversion and Fubini's can be applied to see

$$|Lf| = \int_{S^2}\int_{S^2}\int_{\mathbb{R}^3} e^{i(x+y)\cdot\xi}\hat{f}(\xi)\,d\xi\,d\sigma(x)\,d\sigma(y)$$

(Fubini's) $$= \int_{\mathbb{R}^3}\hat{f}(\xi)\left(\int_{S^2} e^{ix\cdot\xi}\,d\sigma(x)\right)\left(\int_{S^2} e^{iy\cdot\xi}\,d\sigma(y)\right)d\xi$$

(part a) $$\sim \int_{\mathbb{R}^3}\hat{f}(\xi)\,\frac{\sin^2|\xi|}{|\xi|^2}\,d\xi$$

$\longrightarrow$

Hölder and Plancherel thus imply

$$|Lf| \lesssim \|\hat{f}\|_{L^2} \left\| \frac{\sin^2|\xi|}{|\xi|^2} \right\|_{L^2} = \|f\|_{L^2} \left\| \frac{\sin^2|\xi|}{|\xi|^2} \right\|_{L^2}$$

It remains to show $\left\| \frac{\sin^2|\xi|}{|\xi|^2} \right\|_{L^2} < \infty$.

We recall that $\lim_{|\xi| \to 0} \frac{\sin^4|\xi|}{|\xi|^4} = 1$ so $\left|\frac{\sin^4|\xi|}{|\xi|^4}\right|$ is bounded near 0. Moreover, $\left|\frac{\sin^4|\xi|}{|\xi|^4}\right| \leq 1/|\xi|^4$ so $\left|\frac{\sin^4|\xi|}{|\xi|^4}\right|$ is integrable away from 0 because $4 > 3 = \dim \mathbb{R}^3$. Therefore $\left|\frac{\sin^4|\xi|}{|\xi|^4}\right|$ is integrable over $\mathbb{R}^3$ and so $|Lf| \lesssim \|f\|_{L^2}$ uniformly in $f \in C_c^\infty$.

B/c C_c^∞ is dense in L^2, and L is linear by construction, this implies L extends uniquely to a linear functional on L^2 by continuity. $\square$

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(a) Note that since $f \in L^p$ and $g \in L^q$, Hölder's inequality implies that

$$|f*g(x)| \le \int |f(x-y)g(y)|\,dy \le \|f(x-y)\|_{L^p(dy)} \|g\|_{L^q(dy)} = \|f\|_{L^p}\|g\|_{L^q} < \infty.$$

Therefore we may manipulate the integral form of $f*g$ as needed.

Suppose $\exists\ x, y \in \mathbb{R}^3$. By direct calculation,

$$|f*g(x) - f*g(y)| = \left| \int f(x-z)g(z)\,dz - \int f(y-z)g(z)\,dz \right|$$
$$\le \int |f(x-z) - f(y-z)|\,|g(z)|\,dz$$
$$\le \|f(x-z) - f(y-z)\|_{L^p(dz)} \|g\|_{L^q} \quad \text{(Hölder's)}$$
$$\le \|f(z) - f(z+(y-x))\|_{L^p(dz)} \quad (z \mapsto x-z).$$

Let $\tau_w : L^p \to L^p : h(z) \mapsto h(z+w)$. We recall that τ_w is continuous in w in the sense that $\tau_w h \to h$ in L^p as $|w| \to 0$. If time permits, this is shown in a lemma following this proof. Then

$$|f*g(x) - f*g(y)| \le \|f - \tau_{y-x} f\|_{L^p} \to 0 \quad \text{as } |y-x| \to 0.$$

Therefore $f*g$ is continuous.

(b) Fix $N > 0$ and consider

| After time |

$$\|f(x-y)g(y) - f(x-y)g(y)\chi_{\{|x|<N\}}(x)\|_{L^1(dy)}$$
$$\le \|f\|_{L^p}\|g - g\chi$$

In particular, for fixed N

$$\limsup_{|x|\to\infty} |f*g(x)| \le \limsup_{|x|\to\infty} \Big(\|f(x-y)g(y) - f(x-y)g(y)\chi_{\{|x|<N\}}(x)\|_{L^1(dy)}$$
$$+ \limsup_{|x|\to\infty} \|f(x-y)g(y)\chi_{|x|<R}(x)\|_{L^1(dy)}\Big) < \infty$$

$\le$

(b) Define $f_n = f\chi_{\{|x|<n\}}$ and $g_n = g\chi_{\{|x|<n\}}$. Then:
$f_n \to f$ in L^p and $g_n \to g$ in L^q b/c of MCT. We claim that $f_n * g_n \rightrightarrows f*g$.
By Hölder's, $\forall x$

$$
\begin{aligned}
|f_n * g_n(x) - f*g(x)| &\leq \int |f_n(x-y)g_n(y) - f(x-y)g(y)|\,dy \\
&\leq \|f_n(x-y)g_n(y) - f(x-y)g_n(y)\|_{L^1} \\
&\quad + \|f(x-y)g_n(y) - f(x-y)g(y)\| \\
&\leq \|f_n - f\|_{L^p}\|g_n\|_{L^q} + \|f\|_{L^p}\|g_n - g\|_{L^q} \\
&\leq \|f_n - f\|_{L^p}\|g\|_{L^q} + \|f\|_{L^p}\|g_n - g\|_{L^q} \to 0
\end{aligned}
$$

where the convergence is uniform in x. Therefore $f_n * g_n \rightrightarrows f*g$.
We claim $f_n * g_n$ is compactly supported on $\operatorname{supp} f_n + \operatorname{supp} g_n$.
Consider $x \notin \operatorname{supp} f_n + \operatorname{supp} g_n$. Then $\forall y \in \operatorname{supp} g_n$, $x-y \notin \operatorname{supp} f_n$.
Therefore

$$f_n * g_n(x) = \int_{\operatorname{supp} g_n} f_n(x-y)\,g_n(y)\,dy = 0$$

and so $f_n * g_n$ is compactly supported.
In ~~partialu~~ particular, $\lim_{|x|\to\infty} f_n * g_n(x) = 0 \quad \forall n$.
By uniform convergence, this implies $\lim_{|x|\to\infty} f*g(x) = 0$ as desired. $\square$

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B/c $L^2[0,1]$ is complete, it suffices to show $\{f(t,\cdot)\}_t$ is Cauchy w.r.t. t.

Fix $t_1 \leq t_2$. Then

$$\|f(t_2,\cdot)-f(t_1,\cdot)\|_{L^2}^2 = \int_0^1 |f(t_2,x)-f(t_1,x)|^2\,dx$$

$$= \int_0^1 \left| \int_{t_1}^{t_2} \partial_t f(t,x)\,dt \right|^2 dx$$

([illegible]) $$= \int_0^1 \left| \int_{t_1}^{t_2} \partial_t f(t,x) \sqrt{\frac{1+t^2}{1+t^2}}\,dt \right|^2 dx$$

(Cauchy-Schwarz) $$\leq \int_0^1 \left(\int_0^\infty |\partial_t f(t,x)|^2 (1+t^2)\,dt \right) \left(\int_{t_1}^{t_2} \frac{1}{1+t^2}\,dt \right) dx$$

(FTC + Fubini) $$= (\arctan(t_2)-\arctan(t_1)) \underbrace{\int_0^\infty \int_0^1 |\partial_t f(t,x)|^2 (1+t^2)\,dt\,dx}_{<\infty}$$

$$\lesssim \arctan(t_2)-\arctan(t_1)$$

As $t \to \infty$, $\arctan(t) \to \pi/2$. Therefore

$$\lim_{t_1,t_2\to\infty} \|f(t_2,\cdot)-f(t_1,\cdot)\|_{L^2} = 0$$

and so $\{f(t,\cdot)\}_t$ is Cauchy in L^2. This completes the proof, b/c L^2 is complete.

□

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Suppose for now that we have proven the Vitali covering lemma.

Fix $\lambda > 0$. Then $\forall x \in \{x \in \mathbb{R} : Mf(x) > \lambda\} = E_\lambda$, the definition of Mf implies that $\exists\, h(x) > 0$ s.t. $\frac{1}{2h(x)}\int_{x-h(x)}^{x+h(x)} |f| > \lambda$.

Consider the collection $\mathcal{B} = \{B(x, h(x)) : x \in E_\lambda\}$.

Then $\mathcal{B}$ is an open cover of E_λ. Additionally, $\forall x \in E_\lambda$,

$$\lambda < \frac{1}{2h(x)}\int_{x-h(x)}^{x+h(x)} |f| \leq \frac{1}{2h(x)}\int_{\mathbb{R}} |f| = \frac{1}{2h(x)} \|f\|_{L^1}$$

$$\Rightarrow h(x) < \frac{1}{2\lambda}\|f\|_{L^1}$$

Therefore $\sup_{x \in E_\lambda} h(x) < \infty$, and so the Vitali covering lemma can be applied.

Then $\exists$ disjoint $\{B(x_1, h_1), \dots\} \subset \mathcal{B}$ w/ $h_i = h(x_i)$ s.t.

$$E_\lambda \subset \bigcup_{B \in \mathcal{B}} B \subset \bigcup_{i \geq 1} B(x_i, 5h_i)$$

Then

$$|E_\lambda| \leq \sum_{i \geq 1} |B(x_i, 5h_i)|$$
$$= \sum_{i \geq 1} 5|B(x_i, h_i)|$$

By construction $\lambda < \frac{1}{|B(x_i,h_i)|}\int_{B(x_i,h_i)} |f| \Rightarrow |B(x_i,h_i)| < \frac{1}{\lambda}\int_{B(x_i,h_i)} |f|$.

Therefore

$$|E_\lambda| \leq \frac{5}{\lambda}\sum_i \int_{B(x_i,h_i)} |f|$$

Since $\{B(x_i,h_i)\}$ are disjoint, this implies

$$|E_\lambda| \leq \frac{5}{\lambda}\int_{\bigcup_i B(x_i,h_i)} |f| \leq \frac{5}{\lambda}\|f\|_{L^1}$$

as desired.

It remains to show the Vitali covering lemma.

Lemma: Suppose $\mathcal{B}$ is a collection of balls s.t. $\sup_{B\in\mathcal{B}} \mathrm{rad}(B) < \infty$.

Then $\exists$ a countable pairwise disjoint subcollection $\{B_1, \dots\}$ s.t.

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{i\geq 1} 5B_i$$

where $5B_i$ is the ball w/ the same center but $5\times$ the radius.

proof: Let $R = \sup_{B\in\mathcal{B}} \mathrm{rad}(B) < \infty$. Partition $\mathcal{B}$ into $\{\mathcal{B}_i\}$ where

$$\mathcal{B}_i = \{B\in\mathcal{B} : R2^{-i} < \mathrm{rad}(B) \leq R2^{-i+1}\}$$

By Zorn's lemma, $\exists$ a maximal pairwise disjoint subcollection $G_1 \subset \mathcal{B}_1$. Note that since $\mathbb{R}$ is separable, G_1 is countable.

Now consider $\widehat{\mathcal{B}}_2 = \{B\in\mathcal{B}_1 : B\cap B' = \emptyset \ \forall B'\in G_1\}$.

By Zorn's lemma, $\exists$ a maximal pairwise disjoint subcollection $G_2 \subset \widehat{\mathcal{B}}_2$. Iterating this process, we construct $G_n \subset \widehat{\mathcal{B}}_n$ where $\widehat{\mathcal{B}}_n = \{B\in\mathcal{B}_n : B\cap B' = \emptyset \ \forall B'\in G_1\cup\dots\cup G_{n-1}\}$.

Define $G = \bigcup_{n\geq 1} G_n$. By construction, each G_n is pairwise disjoint and pairwise disjoint from G_k for $k\leq n$. Therefore G is pairwise disjoint. Because $\mathbb{R}$ is separable this implies G is countable.

Consider $B\in\mathcal{B}_n$. By construction, since G_n is maximal, $\exists B'\in G_k$ for $k\leq n$ s.t. $B\cap B' \neq \emptyset$.

Since $k\leq n$, $\mathrm{rad}(B') \geq \frac{1}{2}\mathrm{rad}(B)$. Geometry then implies

$$B\subset 5B'$$

B/c $\{\mathcal{B}_n\}$ partitions $\mathcal{B}$, this implies that $\forall B\in\mathcal{B} \ \exists B'\in G$ s.t. $B\subset 5B'$. In particular,

$$\bigcup_{B\in\mathcal{B}} B \subset \bigcup_{B\in G} 5B$$

as desired. $\square$

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~~We show the full Portmanteau theorem, for practice.~~

Let $K \subset X$ be compact, ~~and let K° be its interior~~.
Then $\exists$ $g_K, f_K \in C(X)$ s.t. $\cancel{f_K \uparrow \chi_{K^\circ}}$ ~~and~~ $g_K \downarrow \chi_K$.

For completeness, these can be ~~cons~~ constructed explicitly as

$$g_K = \max(1 - k\,d(x, K), 0)$$

$$\cancel{f_K = \min(1, k\,d(x, X \setminus K^\circ))}$$

Then by monotonicity, $\forall k, n$

$$\int g_K \, d\mu_n \geq \int \chi_K \, d\mu_n = \mu_n(K)$$

By weak* convergence, this implies

$$\limsup_{n\to\infty} \mu_n(K) \leq \int g_K \, d\mu$$

Finally, since $g_K \downarrow \chi_K$ and $|g_K| \leq 1 \Rightarrow g_K \in L^1(d\mu)$ $\forall k$, downwards monotone convergence implies

$$\limsup_{n\to\infty} \mu_n(K) \leq \lim_{k\to\infty} \int g_K \, d\mu = \int \chi_K \, d\mu = \mu(K)$$

as desired.

$\square$

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We first claim that all zeros of $f(z) = z-2-e^{-z}$ in $\mathbb{H}$ are also in $D(0,)$.

Suppose $\exists z \in H$ s.t. $f(z)=0$. Then

$$\begin{aligned} 0 &= |z-2-e^{-z}| \\ &\geq |z|-2-e^{-Re(z)} \\ &> |z|-3 \quad (\text{b/c } Re(z)>0 \Rightarrow e^{-Re(z)}<1) \end{aligned}$$

$$\Rightarrow |z|<3$$

and so $z \in D(0,3)$. Therefore it suffices to count all zeros in $H \cap D(0,3)$.

To do so, we aim to apply Rouche's theorem to show that f and $z-2$ have the same # of zeros inside $H \cap D(0,3)$.

To show this, Rouche's theorem implies that it suffices to show

$$|e^{-z}| < |z-2| \quad \text{on } \partial(H \cap D(0,3))$$

For $z \in i\mathbb{R} \cap \partial(H \cap D(0,3))$, simple geometry implies $|z-2|>2$ and so

$$Re(z)=0 \Rightarrow |e^{-z}| = 1 < 2 < |z-2|. \quad \text{as desired.}$$

For $z \in \{|z|=3\} \cap \partial(H \cap D(0,3)) \setminus i\mathbb{R}$,

$$Re(z)>0 \Rightarrow |e^{-z}| = e^{-Re(z)} < 1 = 3-2 = |z|-2 \leq |z-2|$$

as desired.

Therefore $|e^{-z}| < |z-2|$ on $\partial(H \cap D(0,3))$ and so Rouche's theorem implies that $z-2$ and $z-2-e^{-z} = f(z)$ have the same # of zeros in $H \cap D(0,3)$. B/c $z-2$ has one zero of multiplicity 1 at $2 \in H \cap D(0,3)$, this implies f has 1 zero inside $H \cap D(0,3)$ and hence f has 1 zero inside H. $\square$

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We aim to show that f has a removable singularity at 0. To do so, the Riemann removable singularity theorem implies that it suffices to show f is bounded on a neighborhood of 0.

Consider $\frac{1}{2}\mathbb{D} = D(0, 1/2)$. For all $z \in \frac{1}{2}\mathbb{D}$, basic geometry implies that $D(z, 1/2) \subset \frac{1}{2}\mathbb{D}$. The mean

$|f(z)| > |a_n||z|^n$

$- \underbrace{|\sum_{k \neq n} a_n|}_{\leq}$

z^{α} is integrable near 0 if $\alpha > -d$

$\alpha > -2$

$z^{-n} f(z)$

$= a_n + a_{n+1} z$

$+ a_{n-1} z^{-1} \ldots$

$|z^{\alpha}|^2$ is integrable near 0 if $2\alpha > -2$

$\alpha > -1$

This follows immediately from the fact that $|z|^{\alpha}$ is integrable near 0 iff $\alpha > -2$. Therefore $|f(z)|^2$ is integrable near 0 iff $|f(z)|$ grows strictly slower ~~that~~ than $|z|^{-1}$ near 0 and hence has a removable singularity. We now make this reasoning rigorous.

Let $f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$ be the Laurent expansion of f on $\mathbb{D}^*$. Suppose for the sake of contradiction that $\exists\, n < 0$ s.t. $a_n \neq 0$. By definition of the Laurent series,

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Consider $zf(z)$. We claim that $zf(z)$ has a removable singularity at 0. By the Riemann removable singularity theorem, it suffices to show that $zf(z)$ is bounded near 0.

For all $z \in \frac{1}{2}\mathbb{D}^* = D(0,1/2)\setminus\{0\}$, it follows that $D(z,|z|) \subseteq \mathbb{D}^*$. Then by the mean value property, $\forall z \in \frac{1}{2}\mathbb{D}^*$

$$|zf(z)| \leq \frac{1}{|D(z,|z|/2)|}\int_{D(z,\ )} |w||f(w)|\,d\lambda(w)$$

(Hölder) $$\lesssim \frac{1}{|z|^2}\|f\|_{L^2(\mathbb{D}^*)}\,\||w|\chi_{D(z,|z|/2)}\|_{L^2}$$

$$\lesssim \frac{1}{|z|^2}\,\||w|\chi_{D(0,\frac{3|z|}{2})}\|_{L^2}$$

$$\lesssim \frac{1}{|z|^2}\sqrt{|z|^4} = 1 \quad \text{uniformly in } z$$

Therefore $zf(z)$ is bounded on $\frac{1}{2}\mathbb{D}^*$ and hence near 0.

By the Riemann removable singularity theorem, $zf(z)$ is holomorphic on $\mathbb{D}^*$ and hence f has at most a simple pole at 0.

Suppose f has a simple pole at 0. Then $|f| \gtrsim |z|^{-1}$ on a neighborhood of 0. However, $(|z|^{-1})^2$ is not integrable near 0, which contradicts $f \in L^2$. Therefore ~~f has a removable~~ our supposition was incorrect and f has a removable singularity at 0.

□

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By the Riemann mapping theorem, $\exists$ a conformal map φ s.t. φ takes Ω to $\mathbb{D}$ and $z_1 \longmapsto 0$.

Then consider

$$\varphi \circ f \circ \varphi : \mathbb{D} \to \mathbb{D}$$

By construction, w/ $\tilde{z}_2 = \varphi(z_2)$,

$$\varphi^{-1} \circ f \circ \varphi^{-1}(0) = \varphi \circ f(z_1) = \varphi(z_1) = 0$$

$$\varphi \circ f \circ \varphi^{-1}(\tilde{z}_2) = \varphi \circ f(z_2) = \varphi(z_2) = \tilde{z}_2$$

By the Schwarz lemma, since $\varphi \circ f \circ \varphi^{-1} : \mathbb{D} \to \mathbb{D}$ and $\varphi \circ f \circ \varphi^{-1}(0) = 0$, $|\varphi \circ f \circ \varphi^{-1}(z)| \leq 1$ $\forall z \in \mathbb{D}$. Moreover, since

$$|\varphi \circ f \circ \varphi^{-1}(\tilde{z}_2)| = |\tilde{z}_2|,$$

the Schwarz lemma implies $\varphi \circ f \circ \varphi^{-1} = az$ for some $|a| = 1$.

Because $\tilde{z}_2 = \varphi \circ f \circ \varphi^{-1}(\tilde{z}_2) = a\tilde{z}_2$, it follows that

$$\varphi \circ f \circ \varphi^{-1} = id_{\mathbb{D}}$$

$$\Rightarrow \quad f = id_{\Omega} \quad \text{b/c } \varphi : \Omega \to \mathbb{D} \text{ is conformal.}$$

$\square$

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Let f be entire w/ f non-vanishing. Define $U = \{z : |f(z)| < 1\}$. Suppose for the sake of contradiction that U has a bounded connected component V. Then V is open and connected.

Consider $z \in \partial V$ and take $\{z_n\} \subset V$ s.t. $z_n \to z$. Then by continuity, $|f(z)| = \lim_{n \to \infty} |f(z_n)| \leq 1$. If $|f(z)| < 1$ then $z \in V$ which would contradict the fact that V is open. Therefore $|f(z)| = 1$ on ∂V.

Since f is non-vanishing, $1/f$ is entire. Then $|1/f(z)| = 1/|f(z)| = 1$ on ∂V. Since V is bounded, the maximum modulus principle implies that $1/|f(z)| \leq 1$ on V. However, then $|f(z)| \geq 1$ on V, contradicting the fact that $|f(z)| < 1$ on V. Therefore ~~we have~~ a our supposition was incorrect and all connected components of U are unbounded.

$\square$

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~~Suppose that~~

($\Rightarrow$) Suppose that f is of exponential type w/ $|f(z)| \le c_1 e^{c_2|z|}$.

By the generalized Cauchy integral formula, $\forall z \in \mathbb{C}$,

$$f'(z) = \frac{1}{2\pi i}\int_{\partial D(z,1)} \frac{f(w)}{(w-z)^2}\,dw$$

$$\Rightarrow \quad |f'(z)| \le \frac{2\pi}{2\pi} \max_{w\in\partial D(z,1)} \left|\frac{f(w)}{(w-z)^2}\right|$$

$$= \max_{w\in\partial D(z,1)} |f(w)| \qquad (|w-z|=1)$$

$$\le \max c_1 e^{c_2|w|}$$

$$= \max_{0\le\theta\le 2\pi} c_1 e^{c_2|z+e^{i\theta}|}$$

$$\le \max c_1 e^{c_2|e^{i\theta}|} e^{c_2|z|}$$

$$= c_1 e^{c_2} e^{c_2|z|}$$

Therefore f' is of exponential type w/ $c_1' = c_1 e^{c_2}$ and $c_2' = c_2$.

($\Leftarrow$) Suppose f' is of ~~exponte~~ exponential type w/ $|f'(z)| \le c_1 e^{c_2|z|}$

By the previous direction, this implies that $\forall\, n \ge 1$,

$$|f^{(n)}(z)| \le c_1 e^{nc_2} e^{c_2|z|}$$

Taylor expanding f about 0, we then find that $\forall z$,

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, z^n$$

$$\Rightarrow \quad |f(z)| \le |f(0)| + \sum_{n\ge 1} \frac{1}{n!} |f^{(n)}(0)|\,|z|^n$$

$$\le |f(0)| + \sum_{n\ge 1} \frac{1}{n!} c_1 e^{nc_2} |z|^n$$

$$= |f(0)| + c_1 \sum_{n\ge 1} \frac{1}{n!} |e^{c_2} z|^n$$

$$= |f(0)| + c_1\left(e^{e^{c_2}|z|} - 1\right) \qquad (\text{Taylor expansion } e^z)$$

$$\le (|f(0)| + c_1)\, e^{e^{c_2}|z|}$$

Therefore f is of exponential type. $\square$

ANALYSIS

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a) ~~Pass to a subsequence $\{f_{n_k}\}$ s.t.~~

Let f be the limit of f_n in L^p.

Then $\exists$ a subsequence f_{n_k} s.t. $\|f_{n_k}-f\|_{L^p} < 2^{-k/p}$.

Define

$$F_N = \sum_{k=1}^{N} |f_{n_k}-f|^p \quad \text{and} \quad F = \lim_{N\to\infty} F_N = \sum_{k=1}^{\infty} |f_{n_k}-f|^p$$

By construction, $F_N \in L^1$ $\forall N$ b/c $f_{n_k}, f \in L^p$ $\forall k$. Then by MCT,

$$\|F\|_{L^1} = \lim_{N\to\infty} \|F_N\|_{L^1} \le \sum_{k=1}^{\infty} \|f_{n_k}-f\|_{L^p}^p$$
$$\le \sum_{k=1}^{\infty} 2^{-k}$$
$$\le 1$$

so $F \in L^1$. In particular, $F < \infty$ a.e $\Rightarrow \sum_{k=1}^{\infty} |f_{n_k}(x) - f(x)|^p < \infty$ for a.e x. By standard series properties, this implies $|f_{n_k}(x) - f(x)| \to 0$ for a.e x and so $f_{n_k} \to f$ a.e. as desired. $\square$

b) Typewriter $\square$.

① 405 846 515

b) We define a standard typewriter sequence. Define

$$f_n^k = \chi_{\left[\frac{k-1}{n}, \frac{k}{n}\right]} \quad \text{for } k = 1, \dots, n.$$

Then $\forall n, \forall k,$

$$\|f_n^k\|_{L^2} = \sqrt{\frac{1}{n}} \quad \dots$$

Then consider the sequence $f_1^1, f_2^1, f_2^2, f_3^1, f_3^2, f_3^3, \dots$

As shown, since $n^{-1/2} \to 0$ as $n \to \infty$, the sequence converges to 0 in L^2. However, $\forall\ x \in [0,1]$, $f_n^k(x) = 1$ for infinitely many k, n. Therefore f_n^k does not converge to 0 on $[0,1]$, and therefore does not converge to 0 almost everywhere.

③ 405846515

Vitali Covering Lemma (5R covering lemma) on $\mathbb{R}$

Let $C = \{B(x_\kappa, r_\kappa) : \kappa \in I\}$ be a collection of open balls in $\mathbb{R}$ s.t. $\{r_\kappa\}$ is bounded. Then $\exists$ a countable disjoint subcollection $\mathcal{F} = \{B_1, B_2, \dots\} \in C$ s.t. $\bigcup_{B \in C} B \subset \bigcup_{n=1}^{\infty} 5B_i$ w/ $5B(x,r) = B(x,5r)$.

Proof: Let $R = \sup_\kappa r_\kappa < \infty$. Partition C into $\{C_n\}_{n=0}^{\infty}$ w/ $C_n = \{B(x,r) \in C : r \in (\frac{R}{2^{n+1}}, \frac{R}{2^n}]\}$. By Zorn's lemma, $\exists$ a maximal disjoint subcollection $\mathcal{F}_1 \subset C_1$. Iteratively construct $\mathcal{F}_n \subset C_n$ via Zorn's lemma to be a maximal pairwise disjoint collection of balls that are disjoint from all balls in $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_{n-1}$. Let $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$. By construction, $\mathcal{F}$ is a pairwise disjoint countable subcollection of C.

Consider some $B \in C$. Since $\{C_n\}$ partitions C, $\exists$ n s.t. $B \in C_n$. By maximality, $\exists$ $\tilde{B} \in \mathcal{F}_1 \cup \dots \cup \mathcal{F}_n$ s.t. $B \cap \tilde{B} \neq \emptyset$. Since $\mathrm{rad}(B) \in [\frac{R}{2^{n+1}}, \frac{R}{2^n}]$ and $\mathrm{rad}(\tilde{B}) \in (\frac{R}{2^{n+1}}, R]$, it follows that $\mathrm{rad}(B) \leq 2\,\mathrm{rad}(B)$. Then since $B \cap \tilde{B} \neq \emptyset$, it follows that $B \subset 5\tilde{B}$. Then $\bigcup_{B \in C} B \subset \bigcup_{\tilde{B} \in \mathcal{F}} \tilde{B}$ as desired. $\square$

We now apply Vitali's covering lemma to prove the Hardy-Littlewood maximal inequality.

Fix some $f \in L^1$ and $\lambda > 0$. Consider the set

$$E = \{x : Mf(x) > \lambda\}.$$

By definition of Mf, $\forall x \in E$ $\exists$ some $r_x > 0$ s.t.

$$\frac{1}{2r_x} \int_{x-r_x}^{x+r_x} |f(y)| dy > \lambda.$$

Note that the above inequality implies $r_x < \frac{1}{2\lambda} \int_{x-r_x}^{x+r_x} |f| \leq \frac{1}{2\lambda} \|f\|_{L^1} < \infty$.

Consider the collection of balls $\mathcal{B} = \{B(x, r_x) : x \in E\}$.

Note that $\mathcal{B}$ covers E. By Vitali's covering lemma, $\exists$ a subcollection $\{B(x_n, r_{x_n})\}_{n \in \mathbb{N}}$ s.t. $E \subset \bigcup_{n=1}^{\infty} B(x_n, 5r_{x_n})$.

Then by countable subadditivity,

$$|E| \leq \sum_{n=1}^{\infty} |B(x_n, 5r_{x_n})| = 10 \sum_{n=1}^{\infty} r_{x_n} \leq \frac{5}{\lambda} \sum_{n=1}^{\infty} \int_{x_n - r_{x_n}}^{x_n + r_{x_n}} |f|$$

$$\leq \frac{5}{\lambda} \|f\|_{L^1}$$

where the final inequality follows from the pairwise disjointness of $\{B(x_n, r_{x_n})\}$.

$\square$

(4) 405 340 515

a) Since f is holomorphic on $\mathbb{D}$, $\log|f(z)|$ is subharmonic on $\mathbb{D}$. Then by the sub-mean value property of subharmonic functions, $\forall\ r \in (0,1)$

$$\log|f(0)| \le \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta$$

b/c $\log|f(z)|$ is defined both at 0 and on $\partial D(0,r)$.

b) Since f is holomorphic in $\mathbb{D}$ and not identically 0, f has at most countable zeros in $\mathbb{D}$. Then $\exists$ a sequence $r_n \in (0,1)$ s.t. $r_n \to 1$ and $\inf_{|z|=r_n} |f(z)| > 0$. Then $\forall n$, by part a,

$$\log|f(0)| \le \frac{1}{2\pi}\int_0^{2\pi} \log|f(r_n e^{i\theta})|\,d\theta.$$

Define $E = \{\theta \in [0,2\pi] : f(e^{i\theta}) = 0\}$. ~~Then~~ Since f is cont on $\overline{\mathbb{D}}$, which is compact, f is unif. cont on $\overline{\mathbb{D}}$. Then since $r_n \to 1$, $\forall\ \varepsilon > 0\ \exists\ N$ s.t. $n \ge N \Rightarrow |f(r_n e^{i\theta})| < \varepsilon\ \forall\ \theta \in E$.
Then for $n \ge N$,

$$\log|f(0)| \le \frac{1}{2\pi}\int_{[0,2\pi]\setminus E} \log|f(r_n e^{i\theta})|\,d\theta + \frac{1}{2\pi}\int_E \log|f(r_n e^{i\theta})|\,d\theta$$
$$\le \frac{\log M}{2\pi} + \frac{|E|}{2\pi}\log|\varepsilon|$$

w/ $M = \max_{z \in \overline{\mathbb{D}}} |f(z)|$. As this holds $\forall \varepsilon > 0$, taking $\varepsilon \to 0$ sends $\log|\varepsilon| \to -\infty$. Therefore since $f(0) \ne 0 \Rightarrow \log|f(0)| > -\infty$, it follows that $|E| = 0$ as desired.

$\square$

⑤ 405 846 515

a) Consider $f \in C_c(\mathbb{R})$. Then

$$\|f-f_n\|_{L^2}^2 = \int |f(x)-f(x+x_n)|^2 dx$$

~~Since $f \in C_c(\mathbb{R})$, f is uniformly continuous. Therefore~~

Let K be the compact support of f. Then $f-f_n$ is supported on $(K+x_n) \cup K$. Since K is bounded and $x_n \to 0$, $\tilde{K} = \bigcup_n (K+x_n) \cup K$ is bounded $\Rightarrow |\tilde{K}| < \infty$. Then

$$\|f-f_n\|_{L^2}^2 \leq |\tilde{K}| \quad \|f(x)-f(x+x_n)\|_{L^\infty}^2$$

Since $f \in C_c$, f is uniformly continuous. Therefore $\|f(x)-f(x+x_n)\|_{L^\infty} \to 0$ as $n \to \infty$. Then $\|f-f_n\|_{L^2}^2 \to 0$ as $n \to \infty$ and so $f_n \to f$ in L^2.

We now extend this to all of L^2. Fix some $f \in L^2$. By density of $C_c \subset L^2$, $\forall \varepsilon > 0$ $\exists \varphi \in C_c$ s.t. $\|\varphi - f\|_{L^2} < \varepsilon$. By the translational invariance of L^2 this implies $\|\varphi_n - f_n\|_{L^2} < \varepsilon$. Then

$$\|f-f_n\|_{L^2} \leq \|f-\varphi\|_{L^2} + \|\varphi-\varphi_n\|_{L^2} + \|\varphi_n - f_n\|_{L^2} \leq 2\varepsilon + \|\varphi-\varphi_n\|_{L^2}$$

Since $\|\varphi-\varphi_n\|_{L^2} \to 0$ and $\varepsilon > 0$ is arbitrary, this implies $\|f-f_n\|_{L^2} \to 0$ and so $f_n \to f$ in L^2.

b) Consider χ_W and define $\chi_W^\varepsilon = \chi_W(x+\varepsilon)$ ~~as above~~. By part a, $\chi_W^\varepsilon \to \chi_W$ in L^2. By construction,

$$\chi_W^\varepsilon(x) = \chi_W(x+\varepsilon) = \chi_{W-\varepsilon}(x). \text{ So}$$

$$|\chi_W(x) - \chi_W^\varepsilon(x)| = \begin{cases} 1 & x \in W \Delta W-\varepsilon \\ 0 & \text{else} \end{cases}$$

Then

$$\|\chi_W - \chi_W^\varepsilon\|_{L^2}^2 = \int |\chi_W - \chi_W^\varepsilon|^2 dx = |W \Delta W-\varepsilon| \to 0$$

Disjoint, ~~additivity~~ this implies $|W \cap W-\varepsilon| \to |W|$. In particular, since $|W|>0$, $\exists\ \varepsilon > 0$ s.t. $\forall h \in (0,\varepsilon)$, $|W \cap W-h| > 0$. $\Rightarrow$ $\exists\ x_n \in W \cap W-h$. Then ~~$\forall$~~ $x_n \in W$ and $x_n \in W-h$ $\Rightarrow$ $x_n + h \in W$. Then $h = x_n + h - x_n \in W-W$. Then $(0,\varepsilon) \subset W-W$. A symmetric argument shows $\exists\ \varepsilon' > 0$ s.t. $(-\varepsilon', 0) \subset W-W$ and $W \neq \emptyset \Rightarrow 0 \in W-W$. Then take $\tilde{\varepsilon} = \min(\varepsilon, \varepsilon') \Rightarrow (-\tilde{\varepsilon}, \tilde{\varepsilon}) \subset W-W$.

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Let $\alpha = \inf_{y \in E} \|y\|$ ~~and let~~ $\cancel{f : H \to [0,\infty) : x \mapsto \|x\|}$. ~~Then~~ $\cancel{E \subset f^{-1}[\alpha,\infty)}$ First we show uniqueness.

Suppose $\exists$ $x, y \in E$ s.t. $\|x\| = \|y\| = \alpha$. Since E is convex, $\forall t \in [0,1]$, $tx + (1-t)y \in E$. Then by the triangle inequality,

$$\|tx+(1-t)y\| \le t\|x\| + (1-t)\|y\| = \alpha$$

w/ equality iff x, y are linearly dependent. Since $tx + (1-t)y \in E$, $\|tx+(1-t)y\| \geq \alpha$ and so $\|tx+(1-t)y\| = \alpha$. Therefore equality holds and $x = \lambda y$ for some λ. Then since $\frac{x+y}{2} \in E$,
$\delta \le \|\frac{x+y}{2}\| = |\frac{1+\lambda}{2}| \|x\| = |\frac{1+\lambda}{2}| \delta \Leftrightarrow \lambda = 1 \Rightarrow x = y$. Therefore uniqueness holds.

We now show existence. Let x_n be a sequence in E s.t. $\|x_n\| \to \inf_{y \in E} \|y\| = \delta$. We claim that $\{x_n\}$ is Cauchy. By the parallelogram identity,

$$\|x_n - x_m\|^2 = 2\|x_n\|^2 + 2\|x_m\|^2 - \|x_n + x_m\|^2$$

By the triangle inequality and the convexity of E,

$$\delta \le \left\|\frac{x_n+x_m}{2}\right\| \le \frac{\|x_n\| + \|x_m\|}{2} \to \delta \text{ as } n,m \to \infty. \Rightarrow \|x_n + x_m\| \to 2\delta$$

Therefore, $\lim_{n,m\to\infty} \|x_n - x_m\|^2 = 2\delta^2 + 2\delta^2 - 4\delta^2 = 0$ and so $\{x_n\}$ is Cauchy. Since H is complete and E is closed, this implies $x_n \to x \in E$. Since $\|x_n\| \to \delta$, this implies $\|x\| = \delta$, as desired. $\square$

(8) 405 846 515

We assume Q is the square w/ vertices $z_0, z_0+1, z_0+i, z_0+1+i$, for some fixed z_0 to avoid confusion. ~~WLOG, suppose $z_0 = 0$ so that F has no zeros on ∂Q where Q has vertices $0, 1, i, 1+i$. The periodicity of F then implies that~~

~~Suppose on the contrary that F has a different # of zeros and poles. Since $1/F$ will still be meromorphic and periodic and will swap the # of zeros and poles, we may assume WLOG that #poles < # zeros in Q.~~

Let $p_1, \ldots, p_n$ denote the poles of F in Q, repeated according to mult.

Define $g(z) = \prod_{k \in \mathbb{Z} + i\mathbb{Z}} \prod_{i=1}^{n} (z - p_i - k)$.

Let Z = # of ~~poles~~ zeros of F in Q, counting multiplicities and P = # of poles of F in Q, counting multiplicities.

The argument principal implies

$$\frac{1}{2\pi i} \int_{\partial Q} \frac{f'(z)}{f(z)}\, dz = Z - P$$

$$\Rightarrow \frac{1}{2\pi i}\left(\int_{z_0}^{z_0+1} + \int_{z_0+1}^{z_0+1+i} + \int_{z_0+1+i}^{z_0+i} + \int_{z_0+i}^{z_0} \right) \frac{f'(z)}{f(z)} = Z - P$$

By the periodicity of f, $\frac{f'(z)}{f(z)} = \frac{f'(z+i)}{f(z+i)}$. Therefore

$$\int_{z_0}^{z_0+1} f'/f\, dz = \int_{z_0+i}^{z_0+1+i} f'/f$$

Similarly, $\frac{f'(z)}{f(z)} = \frac{f'(z+1)}{f(z+1)}$ so $\int_{z_0}^{z_0+i} f'/f\, dz = \int_{z_0+1}^{z_0+1+i} f'/f\, dz$.

Then

$$Z - P = \frac{1}{2\pi i}\left(\left(\int_{z_0}^{z_0+1} - \int_{z_0+i}^{z_0+1+i} + \int_{z_0+1}^{z_0+1+i} - \int_{z_0}^{z_0+i} \right) f'/f\, dz \right) = 0$$

so $Z = P$ as desired. $\square$

(9) 405 846 515

a) We aim to show A is compact by showing it is complete and totally bounded.

We first show completeness. Since l^2 is complete, it suffices to show A is closed. Suppose $\exists \{x^n\} \subset A$ s.t. $x^n \to x \in l^2$. By definition, $\|x^n - x\|_{l^2} \to 0$ implies that $|x^n_m - x_m| \to 0$ $\forall$ m. Then, adapting Fatou's lemma to apply to series,

$$\sum_{m \geq 1} m|x_m|^2 \leq \liminf_{n \to \infty} \sum_{m \geq 1} m|x^n_m|^2 \leq 1$$

and so $x \in A$. Therefore A is closed.

We now show A is totally bounded. Let $\exists\ \varepsilon > 0$. Pick $N \geq 1$ s.t. $1/N < \varepsilon/2$. Then $\forall x \in A$,

$$\sum_{n \geq N} N|x_n|^2 \leq \sum_{n \geq N} n|x_n|^2 \leq 1 \implies \sum_{n \geq N} |x_n|^2 \leq 1/N < \varepsilon/2.$$

Note that $\forall x \in A$, $\sum n|x_n|^2 \leq 1 \implies n|x_n|^2 \leq 1\ \forall n \implies x_n \in [-1,1]$.
Pick $z_1, \ldots, z_m \in [-1,1]$ s.t. $\{B(z_i, \sqrt{\varepsilon/2N})\}$ cover $[-1,1]$.
Let $\mathcal{F} = \{(x_1, \ldots, x_N, 0, \ldots) : x_i \in \{z_1, \ldots, z_m\}\}$. Then $\forall x \in A$, $\exists\ y \in \mathcal{F}$ w/ $|x_i - y_i| < \sqrt{\varepsilon/N2}$. Then $\|x-y\|_{l^2} = \sum_{n=1}^{N} |x_n - y_n|^2 + \sum_{n=N+1}^{\infty} |x_n|^2$
$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon$

so $\{B(z, \varepsilon) : z \in \mathcal{F}\}$ ~~covers~~ is a finite cover of A. As this can be constructed $\forall \varepsilon > 0$, this implies A is totally bounded

b) Since A is compact, it suffices to show the mapping

$$f : x \longmapsto \int_0^{2\pi} \left| \sum_{n\geq 1} x_n e^{in\theta} \right| \frac{d\theta}{2\pi}$$

is continuous wrt l^2 norm. For all $x, y \in l^2$,

$$|f(x) - f(y)| = \left| \int_0^{2\pi} \left(\left| \sum_{n\geq 1} x_n e^{in\theta} \right| - \left| \sum_{n\geq 1} y_n e^{in\theta} \right| \right) \frac{d\theta}{2\pi} \right|$$

reverse triangle inequality $\leq \int_0^{2\pi} \left| \sum_{n\geq 1} (x_n - y_n) e^{in\theta} \right| \frac{d\theta}{2\pi}$

Hölder's $\leq \left\| \sum_{n\geq 1} (x_n - y_n) e^{in\theta} \right\|_{L^2((0,2\pi), \frac{d\theta}{2\pi})} \cancel{\| 1 \|_{L^2(\text{'' ''})}}$

$$\leq \left\| \sum_{n\geq 1} (x_n - y_n) e^{in\theta} \right\|_{L^2 \ldots}$$

Direct computation implies

$$\int_0^{2\pi} e^{in\theta} e^{im\theta} \frac{d\theta}{2\pi} = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

Therefore $\{e^{in\theta}\}$ is orthonormal in L^2. Then

$$\left\| \sum_{n\geq 1} (x_n - y_n) e^{in\theta} \right\|_{L^2}^2 = \int \left| \sum_{n\geq 1} (x_n - y_n) e^{in\theta} \right|^2 \frac{d\theta}{2\pi}$$

$$= \sum_{n\geq 1} (x_n - y_n)^2$$

$$\Rightarrow \left\| \sum_{n\geq 1} (x_n - y_n) e^{in\theta} \right\|_{L^2} = \| x - y \|_{l^2}$$

so

$$|f(x) - f(y)| \leq \|x - y\|_{l^2}$$

Then f is continuous, which concludes.

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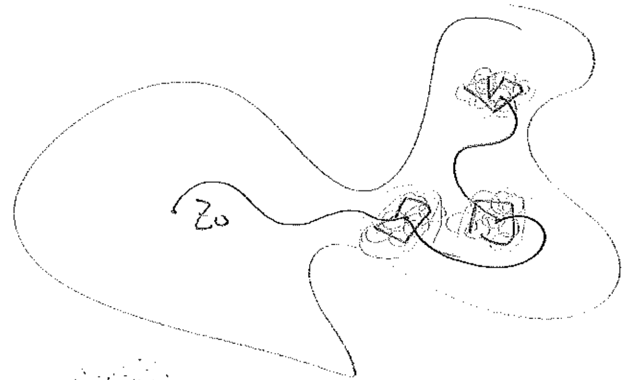

Let K be a compact subset of Ω. We recall Harnack's inequality for the disk, which states

Let u be a non-negative harmonic function on $D(z_0,R)$ that continuously extends to $\overline{D(z_0,R)}$ then $\forall\, z \in D(z_0,R)$ w/ $|z-z_0|=r$,

$$\frac{R-r}{R+r}\,u(z_0) \le u(z) \le \frac{R+r}{R-r}\,u(z_0)$$

Since K is compact, $\mathbb{C}\setminus\Omega$ is closed, ~~open~~, ~~df~~ and $K \subset \Omega$, it follows that $d(K, \mathbb{C}\setminus\Omega) = d > 0$. Consider the cover $\{D(z, d/2) : z \in K\}$ ~~since~~ of K. Since K is compact, $\exists$ a finite subcover $D_1, \dots, D_n$ of K. Since $d(K, \mathbb{C}\setminus\Omega) = d$ and D_i has radius $d/2$, it follows that $\hat{K} = \bigcup_{i=1}^{n} \overline{D_i} \subset \Omega$, ~~so~~ Moreover, $\hat{K}$ is compact.

By construction, $\hat{K}$ consists of at most finite connected components. Since Ω is connected, it is path connected. Therefore we can connect each connected component of $\hat{K}$ via a path.* Since the image of each path is compact we can let $K' = \hat{K} \cup \{\text{connecting paths}\}$ ~~so then~~ and thus K' is ~~connected. By constr~~ compact and connected. By construction, $K \subset K' \subset \Omega$ and K' is compact w/ $z_0 \in K'$. We now bound $u(z)$ for all $z \in K'$.

$\longrightarrow$

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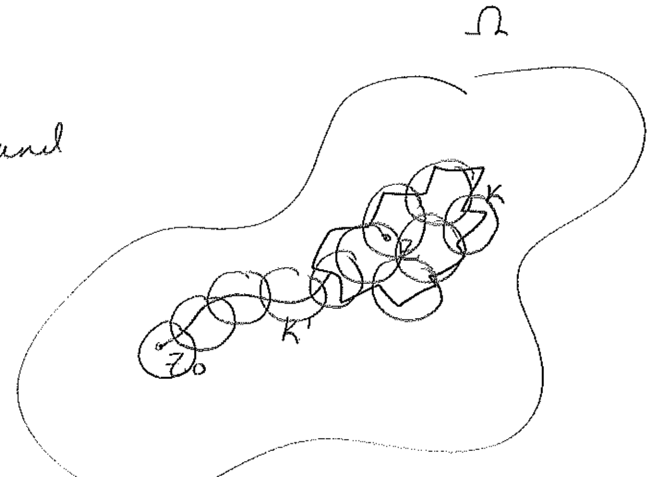
* Moreover, we can connect z_0 to $\hat{K}$ via a path.

Since K' is compact and $K' \subset \Omega$, it follows that $d(K', \mathbb{C} \setminus \Omega) = R > 0$. Consider the open cover $\{D(z, R/2) : z \in K'\}$ of K'. Since K' is compact, $\exists$ a finite subcover $D_1, \dots, D_m$ of K' of disks of radius $R/2$. ~~Up to reordering, we may assume WLOG that~~ ~~$z_0 \in D_1$. then~~



Consider some $z \in K$. Since $K \subset K'$ and K' is connected, $\exists$ a finite chain of disks $D_{K_1}, D_{K_2}, \dots, D_{K_j} \in \{D_i\}$ s.t. $D_{K_i} \cap D_{K_{i+1}} \neq \emptyset$ and

$z \in D_{K_1}$, $z_0 \in D_{K_j}$.

Let c_{K_i} = center of D_{K_i} and let $z_{K_i} \in D_{K_i} \cap D_{K_{i+1}}$.

By construction, since D_{K_i} has radius $R/2$ and $D(c_{K_i}, R) \subset \Omega$, it follows that $|c_{K_i} - z_{K_i}| \leq R/2$ and $|z_{K_i} - c_{K_{i+1}}| \leq R/2$ and we may apply Harnack's inequality on $D(c_{K_i}, R)$ to any $u \in \mathcal{U}$ to see that $\forall i$

$$\frac{R - |c_{K_i} - z_{K_i}|}{R + |c_{K_i} - z_{K_i}|} u(c_{K_i}) \leq u(z_{K_i})$$

$$\Rightarrow \frac{R - R/2}{R + R} u(c_{K_i}) \leq u(z_{K_i}) \Rightarrow u(c_{K_i}) \leq 4 u(z_{K_i}) \quad \forall i$$

and similarly

$$u(z_{K_i}) \leq 4 u(c_{K_{i+1}}) \quad \forall i.$$

Chaining these inequalities together, we find

$$u(c_{K_1}) \leq 4^{2j} u(c_{K_j})$$

Two final applications of Harnack's inequality implies

$$u(z) \leq 4^{2j+2} u(z_0) = 4^{2j+2} \leq 4^{2m+2}$$

As this holds $\forall\, z \in K$ ~~and m depends~~, $\forall\, u \in \mathcal{U}$, and m depends only on K, this implies that

$$\sup_{u \in \mathcal{U}} \sup_{z \in K} u(z) \leq 4^{2m+2}$$

as desired. $\square$

(12) To show F is analytic, it suffices to show that

$$\lim_{z \to w} \frac{F(z)-F(w)}{z-w}$$

exists for all $w \in \mathbb{D}$.

Claim 1: First, we show that $\frac{F(z)-F(w)}{z-w}$ is bounded $\forall\, z \neq w \in K$ for fixed compact $K \subset \mathbb{D}$. Let $K \subset \mathbb{D}$ be compact. Then $\exists\, R<1$ s.t. $K \subset D(0,R)$. Consider $z \neq w \in K$. Then

$$|F(z)-F(w)| = |f_{z,w,*}(z) - f_{z,w,*}(w)|$$
$$\leq \int_w^z |f'_{z,w,*}(\zeta)|\, d\zeta$$

Since $f_{z,w,*} : \mathbb{D} \to \mathbb{D}$, the ~~cauchy estimates~~ imply that on $K \subset D(0,R)$, $|f'_{z,w,*}| \leq \frac{1}{R} \sup_{\partial D(0,R)} |f_{z,w,*}| \leq \frac{1}{R}$. Then

$$|F(z)-F(w)| \leq |z-w| \frac{1}{R}$$

$$\Rightarrow \quad \left| \frac{F(z)-F(w)}{z-w} \right| \leq 1/R$$

Taking $R \to 1$, this implies that $\forall\, z \neq w \in \mathbb{D}$,

$$\left| \frac{F(z)-F(w)}{z-w} \right| \leq 1.$$

Claim 2: Fix some $w \in \mathbb{D}$. ~~We now aim to show that $\lim_{z \to w} \frac{F(z)-F(w)}{z-w}$ exists.~~ ~~To do so, it suffices to show that $\forall\, z_n \in \mathbb{D}$ w/ $z_n \neq w$ and $z_n \to w$~~ ~~$\lim_{n \to \infty} \frac{F(z_n)-F(w)}{z_n - w}$ exists and is independent of $\{z_n\}$.~~ Consider $z_n, w_n \in \mathbb{D}$ w/ $z_n, w_n \neq w$ s.t. $z_n \to w$, $w_n \to w$. Suppose

$$\left\{ \frac{F(z_n)-F(w)}{z_n - w} \right\}_n, \quad \left\{ \frac{F(w_n)-F(w)}{w_n - w} \right\}_n$$

converge. We aim to show these limits are equal. Define $g_n = f_{z_n, w_n, w}$. Since $g_n : \mathbb{D} \to \mathbb{D}$ $\forall n$, $\{g_n\}$ is a bounded family of holomorphic functions. Therefore $\exists$ a ~~locally~~ ~~locally~~ uniformly convergent on compact subsets subsequence $g_{n_k} \to g$. Since g_{n_k} is holomorphic and $\mathbb{D} \to \mathbb{D}$, g is holomorphic $\mathbb{D} \to \mathbb{D}$.

$\longrightarrow$

*

In particular, b/c $\dfrac{F(z_{n_k})-F(w)}{z_{n_k}-w} = \dfrac{g_{n_k}(z_{n_k})-g_{n_k}(w)}{z_{n_k}-w}$

and $\{F(z_{n_k})-F(w)$

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$\Leftarrow$: Suppose for every bounded sequence $\{x_n\} \in X$ $\exists$ a subsequence x_{n_j} and $\phi \in X$ s.t. $x_{n_j} = \phi + r_{n_j}$ and $Ar_{n_j} \to 0$ in Y.

We aim to show A is compact.

Suppose that $\{x_n\} \subset X$ is bounded. Then by supposition, $\exists$ a subsequence $\{x_{n_j}\}$ and $\phi \in X$ s.t. $x_{n_j} = \phi + r_{n_j}$ and $Ar_{n_j} \to 0$ in Y.

Then by ~~bounded~~ linearity, $Ax_{n_j} = A\phi + Ar_{n_j}$.

Since $Ar_{n_j} \to 0$ in Y, $Ax_{n_j} \to A\phi$ in Y. Therefore $\{Ax_n\}$ has a convergent subsequence in Y. As this holds $\forall$ bounded $\{x_n\}$, A is compact.

$\Rightarrow$: Suppose A is compact.

Let $\{x_n\} \subset X$ be a bounded sequence. By Banach-Alaoglu, since $X^{**} = X$, ~~the~~ any closed ball ~~contain~~ centered at the origin is ~~weakly~~ weak-* compact. B/c X is reflexive, weak-* is equivalent to weak. Therefore, since $\{x_n\}$ is bounded, it is contained in a weakly-compact ball.

~~Since X^* Since~~

Since we are working in Banach spaces, weakly-compact implies weakly-sequentially compact and so $\exists$ a weakly convergent subsequence of $\{x_n\}$, that converges to some $\phi \in X$.

By definition of compact, we may pass to another subsequence $\{x_{n_j}\}$ so that Ax_{n_j} converges in Y to some y.

Take $r_{n_j} = x_{n_j} - \phi$. It now suffices to show that $A\phi = y$.

Consider some $f \in Y^*$. Then

$$
\begin{aligned}
f(y) &= \lim_{j\to\infty} f(Ax_{n_j}) && (Ax_{n_j} \to y \text{ in } Y) \\
&= \lim_{j\to\infty} A^T f(x_{n_j}) \\
&= A^T f(\phi) && (~~wk~~\ x_{n_j} \rightharpoonup \phi) \\
&= f(A\phi) && \text{As this holds } \forall\, y \in Y^* \Rightarrow y = A\phi. \quad \square
\end{aligned}
$$

FALL 2010

~~1~~ ✓ ✓

~~2~~ ✓ ✓

3 ≈ ✓

~~? (very last detail)~~ ✓

4 ✓

5 ≈ ✓

~~6~~ ✓ ✓

~~7~~ ✓

8 ✓

~~9~~ ✓

10 ✓

~~11~~ ✓ ✓

12 ✓ ✓

① 405 846 515

a) Fatou's lemma states:

Let $f_1, f_2, \ldots$ be measurable $\overset{a.e.}{\checkmark}$ non-negative functions on $[0,1]$. Then

$$\int_{[0,1]} \liminf_{n\to\infty} f_n(x)\,dx \leq \liminf_{n\to\infty} \int_{[0,1]} f_n(x)\,dx$$

b) The DCT states:

Let $\exists$ measurable functions $f_1, f_2, \ldots : [0,1] \to \mathbb{R}$ s.t. $|f_n| \leq g$ a.e for some absolutely integrable g and $f_n \to f$ a.e Then $f_n \to f$ in $L^1([0,1])$ which is to say

$$\lim_{n\to\infty} \int_{[0,1]} |f_n - f|\,dx = 0$$

proof: Let our setup be as above. Since $|f_n| \leq g$ a.e $\forall n$, a.e pointwise convergence implies $|f| \leq g$. Then $|f_n - f| \leq 2g \Rightarrow 2g - |f_n - f| \geq 0$

Fatou's lemma then implies

$$\int \liminf_{n\to\infty} (2g - |f_n - f|)\,dx \leq \liminf_{n\to\infty} \int 2g - |f_n - f|\,dx$$

$$= 2\|g\|_{L^1} - \limsup_{n\to\infty} \int |f_n - f|\,dx \qquad \text{since } g \in L^1$$

Since $f_n \to f$, $|f_n - f| \to 0$. Then the LHS is equal to $2\|g\|_{L^1} < \infty$.

Therefore

$$2\|g\|_{L^1} \leq 2\|g\|_{L^1} - \limsup_{n\to\infty} \int |f_n - f|\,dx$$

$$\Rightarrow \limsup_{n\to\infty} \int |f_n - f|\,dx \leq 0$$

Since $\int |f_n - f|\,dx \geq 0$ this implies $\lim_{n\to\infty} \int |f_n - f|\,dx = 0$ as desired.

c) Define

$$f_n = n\chi_{[0,1/n]}$$

Then $f_n \to 0$ everywhere except $x = 0$ and $\int f_n\,dx = n(1/n) = 1$ $\forall n$.

So $\lim_{n\to\infty} \int f_n\,dx = 1$.

② 405 846 515

Recall by Riemann-Vitali that f is Riemann integrable over $[0,1]$ if it is bounded and continuous a.e. Therefore since $f:[0,1]\to\mathbb{R}$ is continuous, it is bounded and hence Riemann integrable. Moreover, the Riemann and Lebesgue integrals of f agree so it suffices to show the result for Riemann integrals. The same reasoning applies for $e^{f(x)}$ to show it is Riemann integrable.

We recall that since f is Riemann integrable

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(i/n)$$

Then by the continuity of exp,

$$\exp\left(\int_0^1 f(x)dx\right) = \lim_{n\to\infty} \exp\left(\frac{1}{n}\sum_{i=1}^{n} f(i/n)\right)$$

By the strict convexity of exp, this implies

$$\exp\left(\int_0^1 f(x)dx\right) \le \liminf_{n\to\infty} \sum_{i=1}^{n}\frac{1}{n}\exp(f(i/n)) \qquad (1)$$

~~with equality iff $f(i/n) = f(j/n)$ $\forall\, i,j = 1,\dots,n$ for all sufficiently large n.~~

Since $\exp\circ f$ is Riemann integrable, the RHS $\to \int_0^1 e^{f(x)}dx$ and so

$$\exp\left(\int_0^1 f(x)dx\right) \le \int_0^1 e^{f(x)}dx$$

as desired. ~~As noted earlier, we have equality iff $f(i/n)=f(j/n)$ $\forall\, i,j=1,\dots,n$ for all sufficiently large n~~ It remains to show the equality condition. Since exp is strictly convex,

Equality condition doesn't work w/ this solution.

③ 405 846 515

b) ~~Suppose $f_n \to f$ in L^1. Then, since $1 \in (L^1([0,1]))^* \cong L^\infty([0,1])$,~~

$$\int_0^1 f\,dx = \lim_{n\to\infty} \int_0^1 f_n\,dx = \lim_{n\to\infty} \int_0^1 \exp(\sin(2\pi n x))\,dx$$

$$= \lim_{n\to\infty} n \int_0^{1/n} \exp(\sin(2\pi n x))\,dx$$

$$(u = nx) \quad = \lim_{n\to\infty} \int_0^1 \exp(\sin(2\pi u))\,du$$

$$= \int_0^1 f_1(u)\,du$$

Let $c = \int_0^1 f_1(u)\,du$. We claim that $f_n \overset{*}{\rightharpoonup} c$ in $(L^1([0,1]))^* \cong L^\infty([0,1])$.

Consider the indicator function of an interval $\chi_{[a,b]}$.

Let $a_n = \frac{\lceil na \rceil}{n}$ and $b_n = \frac{\lfloor bn \rfloor}{n}$. Note that $|a_n - a|, |b_n - b| \leq 1/n$.

Then

$$\left| \int_0^1 f_n \chi_{[a,b]}\,dx - c(b-a) \right| \leq \left(\int_a^{a_n} + \int_{b_n}^{b} \right) |f_n|\,dx + \left| \int_{a_n}^{b_n} f_n\,dx - c(b_n - a_n) \right|$$
$$+ c|b_n - b| + c|a_n - a|$$

(since $|f_n| \leq e$)
$$\leq \frac{2e}{n} + \frac{2c}{n} + \left| \int_{a_n}^{b_n} f_n\,dx - c(b_n - a_n) \right|$$

($\frac{1}{n}$-periodicity of f_n)
$$= \frac{2(e+c)}{n} + \left| (\lfloor nb \rfloor - \lceil na \rceil) \int_0^{1/n} \exp(\sin(2\pi n x))\,dx - c(b_n - a_n) \right|$$

$$= \frac{2(e+c)}{n} + \left| \frac{\lfloor nb \rfloor - \lceil na \rceil}{n} \int_0^1 \exp(\sin(2\pi u))\,du - c(b_n - a_n) \right|$$

$$= \frac{2(e+c)}{n} + \left| (b_n - a_n)c - (b_n - a_n)c \right|$$

$$= \frac{2(e+c)}{n} \underset{n\to\infty}{\longrightarrow} 0$$

Therefore $\left| \int_0^1 f_n \chi_{[a,b]}\,dx - c(b-a) \right| \to 0$ as $n \to \infty$. The linearity of the integral then implies that $\int_0^1 f_n h\,dx \to c \int_0^1 h\,dx$ $\forall$ step functions h. B/c step functions are dense in $L^1([0,1])$, it now suffices to show that if $\int_0^1 f_n g_k\,dx \overset{n\to\infty}{\longrightarrow} c \int_0^1 g_k\,dx$ $\forall k$ and $g_k \to g$ in $L^1([0,1])$, then $\int_0^1 f_n g\,dx \to c \int_0^1 g\,dx$.

$\longrightarrow$

(4) 405 846 515 $\qquad |\{\ldots |>t\}| \le \frac{\|f\|_{L^1}}{t}$

~~By the layer cake representation,~~

$$\cancel{\int |Tf|^2 dx = \int_0^\infty 2t |\{x : |Tf| > t\}| dt}$$

~~Since $f \in C_c^0(\mathbb{R})$, f is bounded $\Rightarrow \|f\|_{L^\infty} < \infty$.~~ ~~Since $\|Tf\|_{L^\infty} \le \|f\|_{L^\infty} < \infty$,~~

~~it follows that $|\{x : |Tf| > t\}| = 0 \ \forall t \geq \|f\|_{L^\infty}$. Then~~

$$\cancel{\int |Tf|^2 dx = \int_0^{\|f\|_{L^\infty}} 2t |\{x : |Tf(x)| > t\}| dt}$$

$$\cancel{\le \int_0^{\|f\|_{L^\infty}} 2\|f\|_{L^1} dt}$$

$$\cancel{= 2\int_0^{\|f\|_{L^\infty}} \int_{\mathbb{R}} |f| dx \, dt}$$

Suppose $f \in C_c^0(\mathbb{R})$ w/ $f \geq 0$.

Define $h_t = \min(|f|, t/2)$ and $g_t = f - h_t = \begin{cases} 0 & f \le t/2 \\ f - t/2 & f > t/2 \end{cases}$.

Then $f = h_t + g_t \Rightarrow |Tf| \le |Th_t| + |Tg_t|$. Therefore if $|Tf| > t$, then $|Th_t| > t/2$ or $|Tg_t| > t/2$. Then

$$\{x : |Tf| > t\} \subset \{x : |Tg_t| > t/2\} \cup \{x : |Th_t| > t/2\}$$

As given, $\|Th_t\|_{L^\infty} \le \|h_t\|_{L^\infty}$ and $\|h_t\|_{L^\infty} \le t/2$ by construction so $\{x : |Th_t| > t/2\} = \emptyset \Rightarrow \{x : |Tf| > t\} \subset \{x : |Tg_t| > t/2\}$. Then by the layer cake representation

$$\int |Tf|^2 dx = \int_0^\infty 2t |\{x : |Tf| > t\}| dt \le \int_0^\infty 2t |\{x : |Tg_t| > t/2\}| dt$$

$$\le 4\int_0^\infty \|g_t\|_{L^1} dt = 4\int_0^\infty \int_{\mathbb{R}} (f - t/2) \chi_{\{x : f(x) > t/2\}}(x) \, dx \, dt$$

$$\text{(Fubini-Tonelli)} = 4\int_{\mathbb{R}} \int_0^{2f(x)} (f - t/2) \, dt \, dx$$

$$= 4\int_{\mathbb{R}} (2f^2(x) - f^2(x)) dx = 4\int_{\mathbb{R}} |f(x)|^2 dx$$

For general $f \in C_c^0(\mathbb{R})$, take f^+, f^- as the positive and negative parts of f. Then

? $$\int |Tf|^2 dx = \int |Tf^+ - Tf^-|^2 dx \le \int (2\max(|Tf^+|, |Tf^-|))^2 dx = \int 4\max(|Tf^+|^2, |Tf^-|^2) dx$$

$$\le 4\int |Tf^+|^2 + |Tf^-|^2 dx \le 16\int |f^+|^2 + |f^-|^2 dx = 16\int |f|^2 dx \qquad \square$$

(5) 405 846 515

a) Consider $f(x) = e^{2\pi i k x}$ for $k \in \mathbb{Z}$. Then

$$\frac{1}{N}\sum_{n=0}^{N-1} f(n\alpha+\mathbb{Z}) = \frac{1}{N}\sum_{n=0}^{N-1} e^{2\pi i k n \alpha} = \frac{1}{N}\,\frac{1-e^{2\pi i k \alpha N}}{1-e^{2\pi i k \alpha}} \to 0 \text{ as } N\to\infty$$

and $\int_0^1 f(x+\mathbb{Z})dx = \int_0^1 e^{2\pi i k x}dx = 0$

so $\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(n\alpha+\mathbb{Z}) = \int_0^1 f(x+\mathbb{Z})dx \quad \forall f = e^{2\pi i k x}$. Linearity then implies the same result for all trig polynomials.

Let $\mathcal{F} = \{\sum_{k=-n}^{n} a_k e^{2\pi i k x} : \{a_k\}\subset\mathbb{R} \text{ and } n\geq 0\}$. As shown, the property holds $\forall f\in\mathcal{F}$. B/c $\mathcal{F}$ is closed under linear combination and $e^{2\pi i k x}\cdot e^{2\pi i l x} = e^{2\pi i (k+l)x} \Rightarrow \mathcal{F}$ is closed under multiplication, it follows that $\mathcal{F}$ is an algebra. Since $\mathcal{F}$ is non-vanishing (contains 1) and separates points, Stone-Weierstrass implies $\mathcal{F}$ is uniformly dense in $C(\mathbb{R}/\mathbb{Z})$.

Let $\exists f\in C(\mathbb{R}/\mathbb{Z})$ and let $f_k \rightrightarrows f$ w/ $f_k\in\mathcal{F}$. Then

$$\left|\frac{1}{N}\sum_{n=0}^{N-1} f(n\alpha+\mathbb{Z}) - \int_0^1 f(x)dx\right| \leq \frac{1}{N}\sum_{n=0}^{N-1}|f(n\alpha+\mathbb{Z}) - f_k(n\alpha+\mathbb{Z})|$$
$$+ \left|\frac{1}{N}\sum_{n=0}^{N-1} f_k(n\alpha+\mathbb{Z}) - \int_0^1 f_k(x)dx\right|$$
$$+ \int_0^1 |f_k(x)-f(x)|dx$$

Since $f_k\in\mathcal{F}$ and $f_k\rightrightarrows f$ on $[0,1]$, taking $N\to\infty$ and then $k\to\infty$ implies $\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} f(n\alpha+\mathbb{Z}) = \int_0^1 f(x)dx \quad \forall f\in C(\mathbb{R}/\mathbb{Z})$.

b) Let $[a,b]$ be a closed interval in $\mathbb{R}/\mathbb{Z}$. There exists $u_k, l_k \in C(\mathbb{R}/\mathbb{Z})$ s.t. $u_k \downarrow \chi_{[a,b]}$ and $l_k \uparrow \chi_{[a,b]}$. Then

$$\int_0^1 l_k\,dx = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N-1} l_k(n\alpha+\mathbb{Z})$$
$$\leq \liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N-1}\chi_{[a,b]}(n\alpha+\mathbb{Z}) \leq \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N-1}\chi_{[a,b]}(n\alpha+\mathbb{Z})$$
$$\leq \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N-1} u_k(n\alpha+\mathbb{Z}) = \int_0^1 u_k\,dx$$

Taking $k\to\infty$ and applying MCT for l_k and DCT for u_k then implies the desired result. □

⑥ 405 846 515

a) By definition and Cauchy-Schwarz,

$$|Lf| = |f(1)| \le \sum_{k=0}^{\infty} |\hat{f}(k)|$$
$$= \sum_{k=0}^{\infty} \left(\sqrt{1+|k|^2}|\hat{f}(k)|\right)\left(1+|k|^2\right)^{-1/2}$$
$$\le \left(\sum_{k=0}^{\infty} (1+|k|^2)|\hat{f}(k)|\right)^{1/2}\left(\sum_{k=0}^{\infty}(1+|k|^2)^{-1}\right)^{1/2}$$
$$= \left(\sum_{k=0}^{\infty} \frac{1}{1+|k|^2}\right)^{1/2} \|f\|$$

since $\left(\sum_{k=0}^{\infty} \frac{1}{1+|k|^2}\right) < \infty$, this implies L is bounded.

b) By definition, we know that the inner product on $\mathcal{H}$ is given by

$$\langle f, g\rangle = \sum_{k=0}^{\infty} (1+|k|^2)\, \hat{f}(k)\, \overline{\hat{g}(k)}$$

Define $g(z) = \sum_{k=0}^{\infty} \frac{1}{1+|k|^2} \cancel{e^{2\pi i k z}}\, z^k$ * Then $\hat{g}(k) = \frac{1}{1+|k|^2}$ by ~~the~~ definition ~~orthonormality of $e^{2\pi i k x}$ in L^2~~. Therefore

(* since $|g| \le \sum_{k=0}^{\infty} \frac{1}{1+|k|^2} < \infty$, $g: \overline{\mathbb{D}} \to \mathbb{C}$ is well-defined.)

$$\|g\|^2 = \sum_{k=0}^{\infty} \frac{1}{1+|k|^2} < \infty$$

and so $g \in \mathcal{H}$. Additionally, $\forall f \in \mathcal{H}$,

$$\langle f, g\rangle = \sum_{k=0}^{\infty} (1+|k|^2)\hat{f}(k)\frac{1}{1+|k|^2} = \sum_{k=0}^{\infty} \hat{f}(k) = f(1)$$

and so $g \in \mathcal{H}$ represents L.

c) By linearity, the maximum on B must occur w/ $\|f\|=1$ as otherwise a scaling would yield a higher value.

The condition $f(0)=0 \iff \hat{f}(0)=0$. So, we define

$$h(z) = \sum_{k=1}^{\infty} \frac{1}{1+|k|^2} z^k$$

and then $L(f) = \langle f, h\rangle$ by part b. By construction $h(0)=0$ so $h/\|h\| \in B$. By Cauchy-Schwarz

$$|\mathrm{Re}(L(f))| \le |L(f)| = |\langle f, h\rangle| \le \|f\|\|h\| \le \|h\|$$

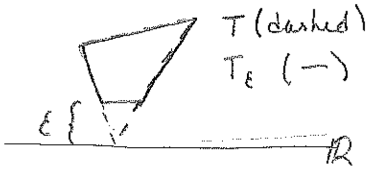
w/ equality occurring first iff $L(f) \in \mathbb{R}$, second iff $f = \lambda h$ and third iff $\|f\| = 1$. The only element of B satisfying these conditions is $h/\|h\|$, which achieves the maximum of $\|h\|$. $\square$

7 405 840 515

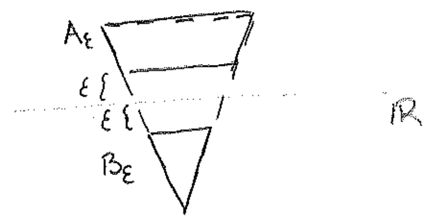
By Morera's theorem, it suffices to show that $\int_T f(z)dz = 0$ $\forall$ triangles T. Since f is holomorphic on $\mathbb{C}\setminus\mathbb{R}$ $\int_T f(z)dz = 0$ $\forall$ triangles $T \subset \mathbb{C}\setminus\mathbb{R}$. Therefore we only consider triangles T that intersect $\mathbb{R}$.

Consider a triangle T that intersects $\mathbb{R}$. Then either T intersects $\mathbb{R}$ at one point, two points or an entire side. In these cases we define T_ε, A_ε or B_ε as follows

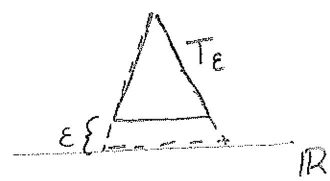
<u>one point</u>



<u>two points</u>



<u>side</u>



w/ the obvious generalization. Since T is a triangle, we can contain T in a compact set K. Then f is uniformly continuous on K and so, in the 3 cases

<u>one point/side</u>: $\displaystyle \int_T f(z)dz = \lim_{\varepsilon\to 0} \int_{T_\varepsilon} f(z)dz = 0$

<u>two points</u>: $\displaystyle \int_T f(z)dz = \lim_{\varepsilon\to 0} \int_{A_\varepsilon + B_\varepsilon} f(z)dz = 0$

where the 0 equality follows from the holomorphicity of f in $\mathbb{C}\setminus\mathbb{R}$. Therefore f is entire by Morera's.

$\square$

(8) 405 846 515

Define $z_0 = L(z)$. We claim $L(f) = f(z_0) \quad \forall f \in A(\mathbb{D})$.

Since $L \neq 0$, $\exists f \in A(\mathbb{D})$ s.t. $L(f) \neq 0$. Then by multiplicativity, $L(f) = L(1)L(f) \Rightarrow L(1) = 1$ since $L(f) \neq 0$.

Then by linearity, $L(c) = c \quad \forall c \in \mathbb{C}$.

Since $L(z) = z_0$, and $L(c)=c$ the multiplicativity and linearity of L implies that

$L(P) = P(z_0)$ for any polynomial P.

Consider some $f \in A(\mathbb{D})$. Then $f(z) - f(z_0) \in A(\mathbb{D})$ w/ a zero at z_0. Then $\exists$ $g \in A(\mathbb{D})$ s.t. $f(z) - f(z_0) = (z - z_0) g(z)$.

This implies

$$L(f) - f(z_0) = L(f - f(z_0)) = L(z - z_0) L(g) = 0$$

since $L(z) = z_0$. Therefore $L(f) = f(z_0) \quad \forall f \in A(\mathbb{D})$.

It now remains to show $z_0 \in \mathbb{D}$. Suppose on the contrary. Then $1/z - z_0 \in A(\mathbb{D})$ since $z_0 \notin \mathbb{D}$. However,

$$L\left(\frac{1}{z - z_0}\right) = \frac{1}{z_0 - z_0} \notin \mathbb{C}$$

which contradicts the definition of L. Therefore $z_0 \in \mathbb{D}$. $\square$

(9) 405 846 515

Suppose $\sum_{n=2}^{\infty} n|a_n| \le |a_1|$. Let $\exists\ z_0 \neq z_1 \in \mathbb{D}$. Since f is holomorphic on $\mathbb{D}$, its power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges absolutely and uniformly on $\mathbb{D}$. Then

$$|f(z_0) - f(z_1)| = \left|\sum_{n=1}^{\infty} a_n (z_0^n - z_1^n)\right|$$

The reverse triangle inequality then implies

$$\begin{aligned}|f(z_0) - f(z_1)| &\ge |a_1||z_0 - z_1| - \left|\sum_{n=2}^{\infty} a_n (z_0^n - z_1^n)\right| \\ &\ge |a_1||z_0 - z_1| - \sum_{n=2}^{\infty} |a_n| \underbrace{|z_0^n - z_1^n|}_{\substack{\le 2 \\ < n < 2n}}\end{aligned}$$

Recall that since $z_0 \neq z_1$,

$$z_0^n - z_1^n = (z_0 - z_1)(z_0^{n-1} + z_0^{n-2} z_1 + \dots + z_1^{n-1})$$

Then

$$|z_0^n - z_1^n| \le |z_0 - z_1|(|z_0|^{n-1} + |z_0|^{n-2}|z_1| + \dots + |z_1|^{n-1})$$

Since $z_0, z_1 \in \mathbb{D}$, this implies that

$$|z_0^n - z_1^n| < |z_0 - z_1| n.$$

Then

$$\begin{aligned}|f(z_0) - f(z_1)| &> |a_1||z_0 - z_1| - \sum_{n=2}^{\infty} |a_n| n |z_0 - z_1| \\ &= \left(|a_1| - \sum_{n=2}^{\infty} n|a_n|\right)|z_0 - z_1| \\ &\ge 0\end{aligned}$$

where the final inequality follows b/c $z_0 \neq z_1$ and $\sum_{n=2}^{\infty} n|a_n| \le |a_1|$.

Therefore $|f(z_0) - f(z_1)| > 0$ $\forall z_0 \neq z_1$ and so $f(z_0) \neq f(z_1)$ for $z_0 \neq z_1 \Rightarrow f$ is injective. $\square$

---

inverse function theorem and open mapping theorem imply that it suffices to show $f' \neq 0$ on $\mathbb{D}$, which follows from inequality.

(10) 405 846 515

Suppose on the contrary that $\exists$ conformal $\varphi: \mathbb{D}\setminus\{0\} \to A_1^2$. Since A_1^2 is bounded, φ is bounded and so the Riemann mapping theorem implies that the singularity of φ at 0 is removable. We may then extend φ to a holomorphic map ~~$\tilde{\varphi}: \mathbb{D}\setminus\{0\} \to \overline{A_1^2}$ where the image of $\tilde{\varphi}$ is $\subset \overline{A_1^2}$ by continuity.~~ $\tilde{\varphi}$ on $\mathbb{D}$. By the maximum/minimum modulus principle, $1<|\tilde{\varphi}(0)|<2$ and so $\tilde{\varphi}: \mathbb{D} \to A_1^2$. ~~Since φ is surjective onto A_1^2, $\tilde{\varphi}$ is surjective. Therefore to show $\tilde{\varphi}$ is still conformal, it suffices to show $\tilde{\varphi}$ is injective.~~

We claim $\tilde{\varphi}$ is injective. ~~By the open mapping theorem, $\forall r>0$ $\tilde{\varphi}(D_r(0))$ is open~~ Since φ is conformal, it suffices to show $\tilde{\varphi}(0) \neq \tilde{\varphi}(z)$ $\forall z \in \mathbb{D}\setminus\{0\}$. $*$

There $\exists$ disjoint open neighborhoods U, V of $0, z$ respectively. Then $\tilde{\varphi}(U) \cap \tilde{\varphi}(V) \neq \emptyset$ and by the open mapping theorem, $\tilde{\varphi}(U) \cap \tilde{\varphi}(V)$ is open since $\tilde{\varphi}(U), \tilde{\varphi}(V)$ are open.

Then $\exists$ $w \in U\setminus\{0\}$ and $w' \in V$ s.t. $\tilde{\varphi}(w) = \varphi(w) = \varphi(w') = \tilde{\varphi}(w')$.

However, this contradicts the ~~injectivity~~ of φ and so no such φ exists. $\square$

$*$ Since $\tilde{\varphi}(0) \in A_1^2$ and $\varphi: \mathbb{D}\setminus\{0\} \to A_1^2$ is surjective, $\exists$ some $z \in A_1^2$ s.t. $\tilde{\varphi}(0) = \tilde{\varphi}(z)$.

(11) 405 846 515

Direct computation implies

$$0 = \Delta f = \Delta u + i\Delta v \Rightarrow \Delta u = \Delta v = 0$$

and
$$0 = \Delta(f^2) = \Delta(u^2 - v^2 + 2iuv) = \Delta(u^2) - \Delta(v^2) + 2i\,\Delta(uv)$$

$$\Rightarrow 0 = \Delta(u^2 - v^2) = \Delta(uv)$$

Then

$$0 = \Delta(u^2 - v^2) = u_x^2 + u_y^2 + u\underbrace{(u_{xx} + u_{yy})}_{0} - v_x^2 - v_y^2 - v\underbrace{(v_{xx} + v_{yy})}_{0}$$

$$= u_x^2 + u_y^2 - v_x^2 - v_y^2 \Rightarrow u_x^2 - v_x^2 = -u_y^2 + v_y^2$$

and
$$0 = \Delta(uv) = uv_{xx} + u_{xx}v + 2u_x v_x + uv_{yy} + u_{yy}v + 2u_y v_y$$

$$= u\underbrace{(v_{xx} + v_{yy})}_{0} + v\underbrace{(u_{xx} + u_{yy})}_{0} + 2(u_x v_x + u_y v_y)$$

$$\Rightarrow u_x v_x = -u_y v_y.$$

Consider $u_x + iv_x$. With the above inequalities,

$$(u_x + iv_x)^2 = u_x^2 - v_x^2 + 2iu_x v_x = -u_y^2 + v_y^2 - 2iu_y v_y = (v_y - iu_y)^2$$

$$\Rightarrow u_x + iv_x = \pm(v_y - iu_y)$$

$$\Rightarrow u_x = \pm v_y \quad \text{and} \quad u_y = \mp v_x$$

Since f is holomorphic $\Leftrightarrow$ $u_x = v_y$ and $u_y = -v_x$ and $\bar{f}$ is holomorphic $\Leftrightarrow$ $u_x = -v_y$ and $u_y = v_x$, this completes the claim. $\square$

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1) Montel's, suffices to show
compact uniform convergence

2) computation

3) contradiction

$\mathcal{F} = \{f : \|f\|_{L^2(\mathbb{D})} < 1\}$

Let K compact suppose not then $\exists \{f_n\}$ w/ $|f_n| \geq n$ for some $z_n \in K$ ~~on K~~

$z_n \to z \in K$

Consider a sequence f_n then $\|f_n\|_{L^2} < 1 \ \forall n$

Let $\exists$ some compact $K \subset \mathbb{D}$. Let $r = d(K, \mathbb{C} \setminus \mathbb{D})/2$. Then $\forall z \in K$, $\forall f \in \mathcal{F}$ the mean value property of holomorphic functions implies, since $D_r(z) \subset \mathbb{D} \ \forall z \in K$,

$$
\begin{aligned}
|f(z)| &\leq \frac{1}{\pi r^2} \int_{D_r(z)} |f(w)| \, dw \\
&= \frac{1}{\pi r^2} \int_{\mathbb{D}} \chi_{D_r(z)} |f(w)| \\
&\leq \frac{1}{\pi r^2} \|f\|_{L^2(\mathbb{D})} \|\chi_{D_r(z)}\|_{L^2} \\
&= \frac{\sqrt{\pi r^2}}{\pi r^2} \|f\|_{L^2(\mathbb{D})} \\
&< \frac{1}{\sqrt{\pi r^2}}
\end{aligned}
$$

As this holds $\forall f \in \mathcal{F}$, and all compact K w/ r only depending on K, this completes the claim. D

Area integral => think mean value!

Analyses

~~SH~~ 5.09

(1) 405 846 515

Suppose first that X is σ-finite w.r.t. μ. Then Tonelli's theorem implies that

$$\int_{-\infty}^{\infty} \mu\big((F_t \setminus G_t) \cup (G_t \setminus F_t)\big)\,dt = \int_{-\infty}^{\infty} \mu\{x : f(x) > t \geq g(x) \text{ or } g(x) > t \geq f(x)\}\,dt$$

$$(l = \min(f,g),\ u = \max(f,g)) \quad = \int_{-\infty}^{\infty} \mu\{x : l(x) \leq t < u(x)\}\,dt$$

$$= \int_{-\infty}^{\infty}\int_X \chi_{\{l \leq t < u\}}(x)\,d\mu(x)\,dt$$

(Tonelli's) $*$

$$= \int_X \int_{-\infty}^{\infty} \chi_{\{l \leq t < u\}}(x)\,dt\,d\mu(x)$$

$$= \int_X \int_{l(x)}^{u(x)} dt\,d\mu$$

$$= \int_X (\max(f,g) - \min(f,g))\,d\mu$$

$$= \int_X |f-g|\,d\mu$$

as desired.

We now generalize. Let $A = \{f \neq g\}$. Then $A = \bigcup_{n=1}^{\infty} \{|f-g| > 1/n\}$. Since $f, g \in L^1(\mu)$ it follows that $|f-g| \in L^1(\mu)$ and so $\mu\{|f-g| > 1/n\} < \infty$ $\forall n$. Therefore A is σ-finite w.r.t. μ. We then repeat the above w/ the addition

$$\int_{-\infty}^{\infty}\int_X \chi_{\{l \leq t < u\}}\,d\mu\,dt = \int_{-\infty}^{\infty}\int_A \chi_{\{l \leq t < u\}}\,d\mu\,dt$$

$$= \int_A \int_{-\infty}^{\infty} \chi_{\{l \leq t < u\}}\,d\mu\,dt$$

$$= \int_X \int_{-\infty}^{\infty} \chi_{\{l \leq t < u\}}\,d\mu\,dt$$

at $(*)$, which follows since $l = u$ on A^c.

$\square$

② 405 846 515

~~H ∞-dim Hilbert space~~

~~(a) ~~If $x =$~~ Fix $x \in B \setminus \{0\}$~~

~~$\{ \frac{x}{\|x\|}, e_1, \dots \}$ orthonormal~~

~~anything if $x = 0$~~

~~$x_n = x + \sqrt{1 - \|x\|}\, e_n$~~

~~$\|x_n\| = \|x\| + 1 - \|x\| = 1$ ✓~~

~~$x_n = \frac{x}{\|x\|}$~~

~~for any $y \in H$~~

~~$\langle x_n,$~~ ~~$\langle e_n, y \rangle$~~

~~$\|y\|^2 \geq \sum |\langle e_n, y\rangle|^2$~~

~~$\langle x, y \rangle - \langle$~~

~~$\Rightarrow \sum |\langle e_n, y \rangle|^2$~~

(a) Fix some $x \in B$. Let $\{x, e_1, e_2, \dots\} \subset B$ be an infinite orthogonal set w/ $e_n \in S$ $\forall n$. We claim $e_n \rightharpoonup 0$.

Fix some $y \in H$. Then by Bessel's inequality,

$$\sum_{n=1}^{\infty} |\langle e_n, y \rangle|^2 \leq \|y\|^2 < \infty$$

$$\Rightarrow \quad |\langle e_n, y\rangle|^2 \to 0 \qquad\qquad (*)$$

$$\Rightarrow \quad \langle e_n, y \rangle \to 0$$

As this holds $\forall y \in H$, $e_n \rightharpoonup 0$.

Define $x_n = x + \sqrt{1 - \|x\|}\, e_n$. Then since $x \perp e_n$, $\|x_n\| = \|x\| + 1 - \|x\| = 1$.

Since $e_n \rightharpoonup 0$, it follows that $x_n \rightharpoonup 0$ as desired. Therefore $S \subset B$ is weakly dense.

(b) Let $\{e_1, \dots\}$ be ~~ano~~ an orthonormal infinite set.

Define $T_n : H \to H : x \mapsto \langle x, e_n \rangle e_n$. Then T is linear by linearity of $\langle \cdot, \cdot \rangle$, ~~and~~ $\|T_n(x)\| = |\langle x, e_n \rangle| \leq \|x\| \Rightarrow \|T_n\| \leq 1$ and $\|T_n(e_n)\| = \|e_n\| = 1$ so $\|T_n\| = 1$.

However, by the reasoning in $(*)$, $T_n(x) \to 0$ $\forall x$ since

$$\|T_n(x)\| = |\langle x, e_n \rangle| \to 0 \text{ as } n \to \infty.$$

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Let $\{\varphi_n\} \subset X^*$ be a countable dense subset of X^*.
Choose $\{x_n\} \subset X$ s.t. $\|\varphi_n(x_n)\| \gtrsim \|\varphi_n\|$ uniformly in n.
We claim that ~~$\mathbb{Q}\langle x_n \rangle$ span$\{x_n\}$; $\mathbb{Q}\langle\{x_n$~~

$$M = \{\text{linear combinations of } \{x_n\} \text{ over } \mathbb{Q}\}$$

is dense in X.

Suppose otherwise. Then $\overline{M} = \text{span}\{x_n\} \neq X$ so $\exists\ y \in X \setminus \overline{M}$.
Since $\overline{M}$ is a closed linear subspace of X, Hahn-Banach implies $\exists\ \psi \in X^*$ w/ $\psi(x_n) = 0\ \forall n$ and $\psi(y) = 1$.

Since $\{\varphi_n\}$ is dense in X^*, $\exists$ a sequence $\varphi_1, \varphi_2, \ldots$ s.t. $\varphi_k \to \psi$. Then

$$\|\varphi_k\| \lesssim \|\varphi_k(e_k)\| = \|\varphi_k(e_k) - \psi(e_k)\| \leq \|\varphi_k - \psi\| \to 0$$

and so $\varphi_k \to 0$. However, this would imply $\psi = 0$, contradicting the fact that $\psi(y) = 1$. Therefore our supposition was incorrect and $\overline{M} = X$. Since M is countable, this concludes.

□

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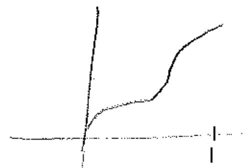
f non-decreasing on $[0,1]$



(a) ~~By definition~~ and Fatou's lemma, $\forall\, \delta \in (0,1)$

$$\cancel{\int_0^{1-\delta} f' = \int_0^{1-\delta} \lim_{\substack{h\to 0^+\\ h<\delta}} \frac{f(x+h)-f(x)}{h} \le \liminf_{\substack{h\to 0\\ |h|<\delta}} \int_0^{1-\delta} \frac{f(x+h)-f(x)}{h}} \le \int_0^{1-\delta} \frac{f(1)-f(0)}{h}$$

(a) We set $f(x)=f(1)\ \forall\, x>1$. Then by definition + Fatou's lemma,

$$\begin{aligned}
\int_0^1 f' &= \int_0^1 \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \\
&\le \liminf_{h\to 0^+} \frac{1}{h}\int_0^1 \big(f(x+h)-f(x)\big) \\
&= \liminf_{h\to 0^+} \frac{1}{h}\left(\int_h^{1+h} f(x) - \int_0^1 f(x)\right) \\
&= \liminf_{h\to 0^+} \frac{1}{h}\left(\int_1^{1+h} f(x) - \int_0^h f(x)\right) \\
&\le \liminf_{h\to 0^+} \frac{1}{h}\big(h f(1) - h f(0)\big) \\
&= f(1) - f(0)
\end{aligned}$$

As desired.

(b) ~~Let $\{f_n\}$ be a sequence of non-decreasing functions~~ on $[0,1]$ s.t.
$F(x) = \sum f_n(x)$ converges $\forall x$. We aim to show $F' = \sum f_n'$
Let $r_n(x) = \sum_{k\ge n} f_k(x)$. It then suffices to show $r_n' \to 0$ a.e.

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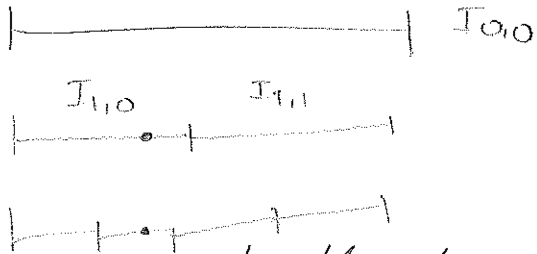

$E_n f$ = average of f over $1/2^n$ length intervals

~~Since E_n is linear, it suffices to consider non-negative $f \in L^1$.~~

Since a.e x is a Lebesgue point of f, it suffices to show this holds for all Lebesgue points. Moreover, since $|\mathbb{Q}| = 0$, it suffices to show this holds for all irrational Lebesgue points $x \in I$.

Fix some Lebesgue point $x \in I \setminus \mathbb{Q}$. Since $x \notin \mathbb{Q}$, $\forall n \; \exists \; j_n$ s.t.

$$x \in (I_{n,j_n})^{\circ}$$

w/ E° being the interior of E. Then

$$E_n f(x) = 2^n \int_{I_{nj}} f \, dt$$

For all n, geometry implies $I_{nj} \subset B(x, 2^{-n})$, w/ $2|I_{nj}| = |B(x, 2^{-n})|$

Then

$$\begin{aligned} |E_n f(x) - f(x)| &= \left| 2^n \int_{I_{nj}} f(t)\,dt - f(x) \right| \\ &\leq 2^n \int_{I_{nj}} |f(t) - f(x)|\,dt \\ &\leq 2^n \int_{B(x,2^{-n})} |f(t) - f(x)|\,dt \\ &\sim \frac{1}{|B(x,2^{-n})|} \int_{B(x,2^{-n})} |f(t) - f(x)|\,dt \to 0 \end{aligned}$$

where the ~~limiting behavior~~ limit is given by the Lebesgue differentiation theorem. Therefore $E_n f(x) \to f(x)$ for a.e $x \in I$.

$\square$

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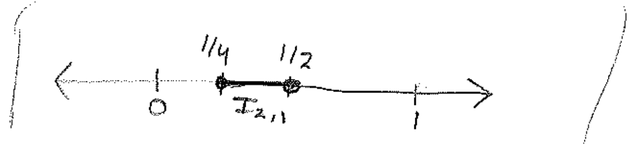
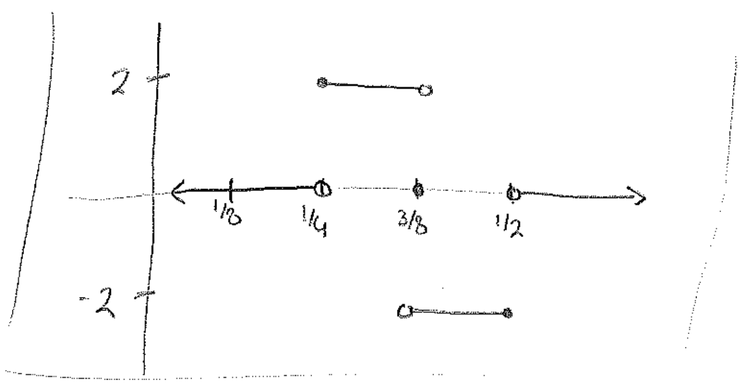
(a) $I_{2,1}$:



$$h_{21} = 2^{2/2}\left(\chi_{I_{3,2}} - \chi_{I_{3,3}}\right)$$
$$= 2\left(\chi_{[1/4,\,3/8]} - \chi_{[3/8,\,1/2]}\right)$$
$$= \begin{cases} 2 & \text{for } x \in [1/4, 3/8] \\ -2 & \text{for } x \in [3/8, 1/2] \\ 0 & \text{else} \end{cases}$$

=



(b) We claim that $\{h_{n,j}\}$ is an orthonormal basis of $M = \{f \in L^2 : \int f = 0\} \subset L^2$. First we note that M is a closed linear subspace of L^2 since the integral is linear and L^2 convergence implies L^1 convergence b/c $|I| = 1 < \infty$.

Consider $h_{n,j} \neq h_{m,k}$. WLOG, $m \geq n$. We note that by ~~construction~~, $\operatorname{supp} h_{n,j} = I_{n,j}$. Hence, we are working w/ dyadic intervals and $m \geq n \Rightarrow |I_{m,k}| \leq |I_{n,j}|$, we know that either $I_{n,j} \cap I_{m,k} = \emptyset$ or $I_{m,k} \subset I_{n,j}$. If $I_{n,j} \cap I_{m,k} = \emptyset$ then $\operatorname{supp} h_{n,j} \cap \operatorname{supp} h_{m,k} = \emptyset$ $\Rightarrow \langle h_{n,j}, h_{m,k} \rangle = 0$. If $I_{m,k} \subset I_{n,j}$ then either $I_{m,k} = I_{n,j}$, which contradicts $h_{n,j} \neq h_{m,k}$, or $I_{mk} \subset I_{n+1,2j}$ or $I_{mk} \subset I_{n+1,2j+1}$. Hence $h_{n,j} \neq h_{m,k}$, either $I_{mk} \subset I_{n+1,2j}$ or $I_{mk} \subset I_{n+1,2j+1}$.
In either case, $h_{n,j}$ is constant $\overset{\text{a.e}}{}$ on $\operatorname{supp} h_{m,k}$, so
$$\langle h_{n,j}, h_{m,k} \rangle = \int h_{n,j} h_{m,k} \sim \int h_{m,k} \sim |I_{n+1,2k}| - |I_{n+1,2k+1}| = 0$$

so $\{h_{n,j}\}$ are orthogonal.

By direct computation,

$$\|h_{n,j}\|_{L^2} = \int |h_{n,j}|^2 = 2^n\left(|I_{n+1,2j}| + |I_{n+1,2j+1}|\right)$$
$$= 2^n\left(2^{-n-1} + 2^{-n-1}\right)$$
$$= 1$$

Therefore $\{h_{n,j}\}$ is orthonormal.

To show $\{h_{n,j}\}$ is a basis of M, it suffices to show that for $f \in M$ if $\langle f, h_{n,j}\rangle = 0$ $\forall n,j$ then $f = 0$.

Fix some $f \in M$. We claim $\int_{I_{n,j}} f = 0$ $\forall n,j$. To show this, we induct on n.

Base case: $n=0$, $j=0$

here $I_{00} = I$ and $f \in M$, $\int_I f = 0$.

Inductive step:

Suppose that $\int_{I_{mk}} f = 0$ for all $m \leq n$ and valid k.

Then by our assumptions, $\forall$ valid j

$$0 = 2^n \int_{I_{nj}} f + \langle f, h_{nj}\rangle = 2^n\left(\int_{I_{nj}} + \int_{I_{n+1,2j}} - \int_{I_{n+1,2j+1}}\right) f = 2^{n+1}\int_{I_{n+1,2j}} f$$

$$0 = 2^n \int_{I_{nj}} f - \langle f, h_{nj}\rangle = 2^{n+1}\int_{I_{n+1,2j+1}} f$$

Therefore the claim holds for $n+1$ and all valid j.

By induction this implies $\int_{I_{nj}} f = 0$ $\forall n,j$. As any closed interval can be written as the countable disjoint union of dyadic intervals, this implies $\int_a^b f = 0$ $\forall a,b$. Lebesgue diff. then implies that $f = 0$ a.e. As this holds $\forall f \in M$ s.t. $\langle f, h_{nj}\rangle = 0$ $\forall n,j$, this implies that $\{h_{n,j}\}$ is an orthonormal basis of M.

Finally, Parseval's theorem yields

$$\|f\|_{L^2} = \sum_{n,j} |\langle f, h_{n,j}\rangle|^2 = \sum_{n,j} \left|\int f h_{nj}\right|^2 \quad \forall f \in M$$

as desired.

(c) Suppose $f \in L^1$ and $\int f = 0$. Consider the n^{th} partial sum

$$S_n(x) = \sum_{k=1}^{n} \sum_{j=0}^{2^n-1} \left(\int f h_{kj}\right) h_{kj}(x)$$

Consider $x \in I \setminus \mathbb{Q}$. Then $x \notin \partial I_{n,j}$ $\forall$ n,j.

Let $j(n)$ be s.t. $x \in I_{n,j(n)}$ and define $\delta(n)$ s.t. $I_{n,j(n)} \cup I_{n,\delta(n)} = I_{n-1,j(n-1)}$. Then

$$S_n(x) = \sum_{k=1}^{n} \left(\int f h_{kj(k)}\right) h_{kj(k)}(x)$$

$$= \sum_{k=1}^{n} 2^k \left(\int_{I_{k+1,2j(k)}} f - \int_{I_{k+1,2j(k)+1}} f\right) 2^{k/2} (-1)^{j(k+1)}$$

$$= \sum 2^{k+k/2} (-1)^{j(k+1)} \left(\int_{I_{k+1,2j(k)}} f - \left(\int_{I_{k,j(k)}} - \int_{I_{k+1,2j(k)}}\right) f\right)$$

$$= \sum 2^{k+k/2} (-1)^{j(k+1)} \left(2\int_{I_{k+1,2j(k)}} f - \int_{I_{k,j(k)}} f\right) h_{k,j(k)}(x)$$

I THINK THEY'RE WRONG

⑦

(a) Since $\frac{1}{z-w}$ is Borel measurable, to show ~~$|F(z)|<\infty$~~ F exists a.e, it only needs to be shown that $|F|<\infty$ a.e.

It then suffices to show $\int_K |F(z)|\,dxdy<\infty$ $\forall$ compact K.

Fix some compact K and let $R>0$ be sufficiently large so that $K\subset \overline{D(z,R)}$ $\forall z\in K$. Direct computation then yields

$$\int_K |F(z)|\,dxdy \le \int_K\int_{\mathbb{C}} \frac{1}{|z-w|}\,d\mu(w)\,dxdy$$

$$\text{(Fubini's)} \quad = \int_{\mathbb{C}}\int_K \frac{1}{|z-w|}\,dxdy\,d\mu(w)$$

Let $K_R=\{z: d(z,K)\le R\}$ be an R-extension of K. Note that $\forall z\in K_R$, $K_R\subset \overline{D(z,2R)}$. We can then decompose our integral as

$$\int_K |F(z)|\,dxdy \le \left(\int_{K_R}\int_K + \int_{\mathbb{C}\setminus K_R}\int_K\right)\frac{1}{|z-w|}\,dxdy\,d\mu(w)$$

$$= I_1 + I_2$$

For I_1, direct computation and polar coordinates yields

$$I_1 = \int_{K_R}\int_K \frac{1}{|z-w|}\,dxdy\,d\mu(w) \le \int_{K_R}\int_{\overline{D(w,2R)}}\frac{1}{|z-w|}\,dxdy\,d\mu(w)$$

$$\le \int_{K_R}\int_0^{2R}\int_0^{2\pi}\frac{1}{r}\,r\,dr\,d\theta\,d\mu(w) = 4\pi R\,\mu(K_R)<\infty$$

since $\mu(\mathbb{C})<\infty$.

For I_2, $\forall z\in K$ and $w\in\mathbb{C}\setminus K_R$, $|w-z|\ge R$. Therefore

$$I_2 = \int_{\mathbb{C}\setminus K_R}\int_K \frac{1}{|z-w|}\,dxdy\,d\mu(w) \le \frac{1}{R}|K|\,\mu(\mathbb{C}\setminus K_R)<\infty$$

since K is compact and $\mu(\mathbb{C})<\infty$.

Therefore $F(z)$ exists for a.e z.

(b) It suffices to show that $\int_{-n}^{n} \int_{\mathbb{C}} \frac{1}{|z-w|} d\mu(w)\, dx < \infty$ for a.e y and all $n \in \mathbb{N}$.

By part a, we have that $\forall\, m \in \mathbb{N}$,

$$\int_{[-n,n]\times[-m,m]} \int_{\mathbb{C}} \frac{1}{|z-w|} d\mu(w)\, dx dy < \infty$$

$$\Rightarrow \int_{-m}^{m} \left(\int_{-n}^{n} \int_{\mathbb{C}} \frac{1}{|z-w|} d\mu(w) dx \right) dy < \infty$$

$$\Rightarrow \int_{-n}^{n} \int_{\mathbb{C}} \frac{1}{|z-w|} d\mu(w)\, dx < \infty \quad \text{for a.e } y \in [-m,m] \quad (1)$$

Let Y_n^m denote the set of all y ~~s.t.~~ $\in [-m,m]$ s.t. (1) ~~is finite~~ holds.

Then $|[-m,m] \setminus Y_n^m| = 0$.

Let $Y^m = \bigcap_{n \in \mathbb{N}} Y_n^m$. Then continuity of measure from below implies $|[-m,m] \setminus Y^m| = 0$

Finally, let $Y = \bigcup_{m \in \mathbb{N}} Y^m$. Then

$$|\mathbb{R} \setminus Y| \leq \sum_m |[-m,m] \setminus Y^m| = 0$$

and by construction, $\forall y \in Y$, $\int_{-n}^{n} \int_{\mathbb{C}} \frac{1}{|z-w|} d\mu(w) dx < \infty \quad \forall n \in \mathbb{N}$.

Therefore for a.e $y \in \mathbb{R}$, $\int_{-n}^{n} \int_{\mathbb{C}} \frac{1}{|z-w|} d\mu(w) dx < \infty\ \forall n$. This implies that for a.e y, $\int_K |F(z)| dx < \infty$, $\forall$ compact $K \in \mathbb{R} \times \{y\}$.

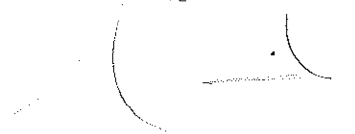
$\square$

(b)

$$\int_K |F(x+iy)|\,dx \le \int_K \int_{\mathbb{C}} \frac{1}{|z-w|}\,d\mu(w)\,dx$$

$$= \int_{\mathbb{C}} \int_K \frac{1}{|x+iy-w|}\,dx\,d\mu(w)$$

WLOG $K = [-R, R]$ $$= \int_{\mathbb{C}} \int_{-R}^{R} \frac{1}{|x+iy-w|}\,dx\,d\mu(w)$$



(c) Suppose that part b has been shown. By the same reasoning, the result for vertical lines can be shown.

~~Let~~ Then for a.e square $S \subset \mathbb{C}$, we know that F is absolutely integrable over ∂S. Then for all such squares, Fubini's theorem implies that

$$\int_{\partial S} F(z)\,dz = \int_{\partial S} \int_{\mathbb{C}} \frac{1}{z-w}\,d\mu(w)\,dz$$

$$= \int_{\mathbb{C}} \int_{\partial S} \frac{1}{z-w}\,dz\,d\mu(w)$$

The CIF then implies that $\int_{\partial S} \frac{1}{z-w}\,dz = 2\pi i$ if $w \in S$ and 0 otherwise. Therefore

$$\int_{\partial S} F(z)\,dz = \int_{\mathbb{C}} 2\pi i\, \chi_S(w)\,d\mu(w) = 2\pi i\,\mu(S)$$

which is what was to be shown.

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f entire, non-constant, $f(1-z) = 1-f(z)$

suppose for the sake of contradiction that $f(\mathbb{C}) \neq \mathbb{C}$.

Then $\exists$ $w \in \mathbb{C}$ s.t. $f(z) \neq w$ $\forall$ z.

Then $f(1-z) \neq w$ $\forall z$ since $z \mapsto 1-z$ is an automorphism of $\mathbb{C}$.

$\Rightarrow$ $1-f(z) \neq w$ $\forall z$

$\Rightarrow$ $f(z) \neq 1-w$ $\forall z$

since $1-w \neq w$ (if $w \neq 1/2$), this implies that f avoids two distinct complex numbers and so the little Picard theorem implies f is constant $\divideontimes$, given $w \neq 1/2$. If $w = 1/2$, then $f(1/2) = 1 - f(1/2) \Rightarrow f(1/2) = 1/2$ which is a $\divideontimes$.

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Suppose for the sake of contradiction that f is not injective on $\mathbb{D}$. Then $\exists$ w_1, w_2 s.t. $f(w_1) = f(w_2)$.

Define $g(z) = f(z) - f(w_1)$. The argument principle then implies
$$\frac{1}{2\pi i}\int_C \frac{g'(z)}{g(z)}\,dz = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)-f(w_1)}\,dz = \#\{f^{-1}(f(w_1)) \cap \operatorname{int} C$$
where C is a circle of radius $\rho \in (r,1) \cap (|w_1|,1) \cap (|w_2|,1)$.

* Since f is injective on $A\cancel{(r,1)} = \{r < |z| < 1\}$, g is injective on A.

Therefore $g(C)$ is a simple Jordan curve in $\mathbb{C}$. In particular, since a simple closed curve ~~only~~ can only wind around a point at most once, $\# f^{-1}(f(w_1)) \cap \operatorname{int} C \leq 1 \Rightarrow \#\{w_1, w_2\} \leq 1 \Rightarrow w_1 = w_2$.

Therefore f is injective on $\mathbb{D}$.

$\square$

---

* By definition of the winding #, this implies

$\operatorname{wind}(g(C), 0) = \#\{f^{-1}(f(w_1)) \cap \operatorname{int} C$

$\operatorname{wind}(f(C), f(w_1)) = \#$

(9) 405 846 515

a) Fix some $R>0$ s.t. $f \neq 0$ on $\{|z|=R\}$.

If $\#\{|a_n| \leq R\} = \infty$, then $\{a_n\}$ has a limit point which would imply $f \equiv 0$. Since $f(0) \neq 0$, $\exists$ finitely many zeros of f in $D(0,R)$. Let $a_1, \ldots, a_n$ enumerate said zeros, repeated according to multiplicity.

$a \in \mathbb{D}$ $\quad \varphi_a(z) = \frac{z-a}{1-\bar{a}z}$

We recall the Blaschke factor φ_a: which is a meromorphic function s.t. $\varphi_a : \mathbb{D} \to \mathbb{D}$ is holomorphic, ~~$\varphi_a(a) = 0$~~ $\varphi : \partial\mathbb{D} \to \partial\mathbb{D}$ the only zero of φ_a is at a, w/ order 1 and the only pole of φ_a is at $\frac{1}{\bar{a}}$ also of order 1. Utilizing these factors, we consider

$$\varphi_{a_i/R}(z/R)$$

Then $\varphi_{a_i/R}(z/R) : R\mathbb{D} \to R\mathbb{D}$ is holomorphic w/ its only zero being simple at a_i, and $|\varphi_{a_i/R}(z/R)| = 1 \ \forall \ z \in \partial R\mathbb{D}$.

Consider the product $g(z) = f(z) / \prod_{i=1}^{n} \varphi_{a_i/R}(z/R)$. Then g has no zeros in $R\mathbb{D}$, is holomorphic, and satisfies $|g| = |f|$ on $\partial R\mathbb{D}$. Then $\log|g(z)|$ is harmonic in $R\mathbb{D}$ and cont in $\overline{R\mathbb{D}}$. The mean value theorem then implies that

$$\frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\theta})|\,d\theta = \log|g(0)|$$

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta = \log|f(0)| - \sum_{i=1}^{n} \log|\varphi_{a_i/R}(0)|$$

$$= \log|f(0)| - \sum_{|a_i|<R} \log|a_i/R|$$

$$= \log|f(0)| + \sum_{|a_i|<R} \log|R/a_i|$$

Which is what was to be found. $\square$

(b) Suppose $|f(z)| \le Ce^{|z|^\lambda}$ for $C, \lambda > 0$. ~~Then $|f(0)| \le C$~~

Part a then implies that ~~$\forall R>0$~~, for a.e $R>0$,

$$\log|f(0)| + \sum_{|a_i|<R} \log(R/|a_i|) \le \frac{1}{2\pi}\int_0^{2\pi} (\log C + R^\lambda)\, d\theta$$
$$= \log C + R^\lambda$$

$$\Rightarrow \quad \sum_{|a_i|<R} \log(R/|a_i|) \lesssim R^\lambda$$

In particular, taking the above equation w/ $2R$ we find

$$\sum_{|a_i|<R} \log(2R/|a_i|) \le \sum_{|a_i|<2R} \log(2R/|a_i|) \lesssim (2R)^\lambda \lesssim R^\lambda$$

For all $|a_i|<R$, $\log(2R/|a_i|) \ge \log 2$. Therefore

$$\sum_{|a_i|<R} \log(2R/|a_i|) \ge \log 2 \, \#\{i : |a_i|<R\}$$

$$\Rightarrow \quad \#\{i : |a_i|<R\} \lesssim R^\lambda$$

* Then $\forall \varepsilon>0$,

$$\sum (1/|a_i|)^{\lambda+\varepsilon} = \sum_{k\ge m} \sum_{2^k \le |a_i| < 2^{k+1}} (1/|a_i|)^{\lambda+\varepsilon}$$
$$\sim \sum_{k\ge m} (2^{-k})^{\lambda+\varepsilon} \sum_{2^k\le|a_i|<2^{k+1}} 1$$
$$\le \sum_{k\ge m} (2^{-k})^{\lambda+\varepsilon} \sum_{|a_i|<2^{k+1}} 1$$
$$\lesssim \sum_{k\ge m} (2^{-k})^{\lambda+\varepsilon} (2^{k+1})^\lambda$$
$$\sim \sum_{k\ge m} 2^{-k\varepsilon} < \infty$$

as desired. $\square$

---

* Since $f(0)\neq 0$, $\exists\, m \in \mathbb{Z}$ s.t. $|a_i| \ge 2^m$ $\forall i$ as otherwise 0 would be a limit point of $\{a_i\}$ ~~and then~~ contradiction $f(0)\neq 0$.

10) 405 846 515

Let $\{f_n\}\subset H$ be a Cauchy sequence w.r.t. L^2.
We first show that $\{f_n\}$ converges pointwise on $\mathbb{D}$.

Fix some $z\in\mathbb{D}$ and let $U\subset\mathbb{D}$ be a ~~neighborhood~~ disk centered at z.
The mean value property implies that $\forall n$,

$$f_n(z)=\frac{1}{\mu U}\int_U f(w)\,d\mu(w) \qquad (1)$$

Since $\{f_n\}\subset L^2(\mathbb{D})$ is Cauchy, it follows that $\{f_n\}\subset L^2(U)$ is Cauchy. Since $\mu U<\infty$, this implies that $\{f_n\}\subset L^1(U)$ is Cauchy. By equation 1, this implies $\{f_n(z)\}$ is Cauchy. Since $\mathbb{C}$ is complete, $\exists$ some $f(z)$ s.t. $f_n(z)\to f(z)$.

Let $\{f_n\}\subset H$ be Cauchy.
~~We aim~~ It suffices to show $\{f_n\}$ converges ~~locally~~ uniformly ~~con~~ on compact subsets of $\mathbb{D}$. To do so, it suffices to show $\{f_n\}$ converges uniformly on $\overline{\mathbb{D}(0,r)}\subset\mathbb{D}$ $\forall\, r\in(0,1)$.
Fix some $r\in(0,1)$.
For all $z\in\overline{D(0,r)}=K$, ~~standard~~, it follows that $D(z,1-r)\subset\mathbb{D}$.
The mean value theorem then implies that $\forall n$

$$f_n(z)=\frac{1}{\mu D(z,1-r)}\int_{D(z,1-r)} f_n(w)\,d\mu(w)=\frac{1}{\pi(1-r)^2}\int_{D(z,1-r)} f_n(w)\,d\mu(w).$$

Then $\forall\, n,m$, monotonicity implies

$$|f_n(z)-f_m(z)|\le\frac{1}{\pi(1-r)^2}\int_{\mathbb{D}}|f_n-f_m|\,d\mu$$

$$\text{(Cauchy-Schwarz)}\quad \le \frac{\sqrt{\mu(\mathbb{D})}}{\pi(1-r)^2}\,\|f_n-f_m\|_{L^2}\to 0$$

~~Therefore~~ As this bound is independent of $z\in K$, this implies that $\{f_n\}$ is uniformly Cauchy on K and so converges uniformly ~~to some f~~ on K.

~~As uniform convergence preserves holomorphicity, fn converges f is holo~~

Therefore $\{f_n\}$ converges uniformly on compact subsets of $\mathbb{D}$, to some holomorphic $f: \mathbb{D} \to \mathbb{C}$.

~~As $\mu\mathbb{D} < \infty$, this implies~~ We claim ~~$f_n \to f$ in L^2.~~

Since $L^2(\mathbb{D}, \mu)$ is already complete, $f_n \to \tilde{f}$ in L^2.

Since ~~$\exists$ a subsequence~~ L^2 convergence implies $\exists$ a subsequence pointwise convergence, $\underbrace{\tilde{f} = f \text{ a.e}}_{\text{cont} \Rightarrow \tilde{f} = f.}$ This concludes. $\square$

Messy, but I am really

(12) 405 040 515

If we assume knowledge of conformal modulus, then this is immediate since a conformal map is 1-quasiconformal and mod(Q) = 1 while mod(R) = 2, 1/2 depending on ordering.

Suppose for the sake of contradiction that such a map exists. Then $f: \partial Q \to \partial R$. By pre-composing w/ a rotation/reflection we may assume wlog that $f(0)=0$ and $f(i)=i$.

By iteratively applying the Schwarz reflection principle, we may extend f to a cont holomorphic map $f: \{0 \le \mathrm{Im}(z) \le 1\} \to \{0 \le \mathrm{Im}(z) \le 1\}$, that is holomorphic on the interior.

Again applying Schwarz reflection, we may extend f to a holomorphic map $f: \{-1 < \mathrm{Im}(z) < 2\} \to \{-1 < \mathrm{Im}(z) < 2\}$.

Let φ be a conformal map $\{-1 < \mathrm{Im}(z) < 2\} \to \mathbb{D}$, that takes $0 \mapsto 0$ and $i \mapsto i/2$

Then $\varphi \circ f \circ \varphi^{-1}: \mathbb{D} \to \mathbb{D}$ w/

$$\varphi \circ f \circ \varphi^{-1}(0) = \varphi \circ f(0) = \varphi(0) = 0$$
$$\varphi \circ f \circ \varphi^{-1}(i/2) = \varphi \circ f(i) = \varphi(i) = i/2$$

Schwarz lemma then implies $\varphi \circ f \circ \varphi^{-1} = id \Rightarrow f = id$.

However $id(Q) = Q \neq R$ ※.

<u>Note</u>: Schwarz lemma can only be applied iteratively here since the image of each boundary segment of Q is a straight line $\subseteq \mathbb{C}$.

FALL 2009

ANALYSIS

① 405 846 615

Define $f_n = n\chi_{[0, 1/n^2 + 1/n^3]}$. Then

$$\|f_n\|^2_{L^2[0,1]} = n^2\left(\frac{1}{n^2} + \frac{1}{n^3}\right) = 1 + \frac{1}{n}$$

Let $\mathcal{F} = \{f_n : n \in \mathbb{N}\} \subset L^2$. Then $\mathcal{F}$ does not contain an element of smallest norm. ~~Since $f_n \in \mathcal{F}$ is isolated~~

To show $\mathcal{F}$ is closed, we aim to show that there are no convergent subsequences in $\mathcal{F}$. Suppose on the contrary that $\{f_{n_k}\} \subset \mathcal{F}$ is a convergent sequence by passing to a subsequence we may assume n_k is increasing w.r.t. k s.t. f_{n_k} is a subsequence of $\{f_n\}$. Passing again to a subsequence we may assume $\{f_{n_k}\}$ is a.e. pointwise convergent. By ~~construction~~ $f_{n_k} \to 0$ a.e. however, $\|f_{n_k} - 0\|_{L^2} = 1 + \frac{1}{n_k} \not\to 0$. Therefore f_{n_k} is not convergent in L^2, which is a contradiction. Then our supposition was incorrect and $\mathcal{F}$ has no convergent ~~sub~~ sequences, and so $\mathcal{F}$ is a closed subset of $L^2[0,1]$. □

Thought process:

① Want sequence $f_n \in L^2$ w/ $\|f_n\|_{L^2}$ decreasing

(2) Can't decrease to 0 since that would imply $f_n \to 0$.

③ Let's decrease to 1 instead

(4) Want $\|f_n\|_{L^2} = 1 + 1/n$

(5) $f_n = n\chi_{[0, 1/n^2 + 1/n^3]}$

(6) $f_n \to 0$ pointwise a.e but $\|f_n - 0\|_{L^2} \not\to 0$ so no convergent sequences.

(2) 405 846 515

Note that $a_{n,m} = 0$ except for finitely many n, m. Therefore no justification is needed for any manipulations/derivatives.

Direct computation implies

$$u = v - \Delta v = \sum_{n,m} a_{n,m} (1 - \Delta) e^{2\pi i (nx + my)}$$

$$= \sum_{n,m} a_{n,m} \left(1 + 4\pi^2 n^2 + 4\pi^2 m^2 \right) e^{2\pi i (nx+my)}$$

Since $\{ e^{2\pi i (nx+my)} : n, m \in \mathbb{Z} \}$ are orthonormal in $L^2([0,1]^2)$, it follows that

$$\| u \|^2_{L^2([0,1]^2)} = \sum_{n,m} |a_{n,m}|^2 \left(1 + 4\pi^2 n^2 + 4\pi^2 m^2 \right)^2$$

Since $|e^{2\pi i (nx+my)}| \leq 1$, Cauchy-Schwartz implies

$$\| v \|_{L^\infty} \leq \sum_{n,m} |a_{n,m}|$$

$$= \sum_{n,m} |a_{n,m}| \left(\frac{1 + 4\pi^2 n^2 + 4\pi^2 m^2}{1 + 4\pi^2 n^2 + 4\pi^2 m^2} \right)^2$$

$$\leq \left(\sum_{n,m} |a_{n,m}|^2 \left(1 + 4\pi^2 n^2 + 4\pi^2 m^2 \right)^2 \right)^{1/2} \underbrace{\left(\sum_{n,m} \frac{1}{(1 + 4\pi^2 n^2 + 4\pi^2 m^2)^2} \right)^{1/2}}_{< \infty}$$

$$\lesssim \| u \|_{L^2([0,1]^2)}$$

As desired. $\square$

(4) 405 846 515

Let $\mathcal{L}$ denote the Lebesgue measure. We aim to show that

$$\lim_{\varepsilon\to 0} \frac{\mu([x-\varepsilon, x+\varepsilon])}{\mathcal{L}([x-\varepsilon, x+\varepsilon])} = 0 \qquad (1)$$

for μ-a.e $x \in \mathbb{R}$.

As given, $\mu \perp \mathcal{L}$. Therefore $\exists\, A \subset \mathbb{R}$ s.t. $\mathbb{R} = A \cup A^c$ and $\mu(A^c) = 0$ and $\mathcal{L}(A) = 0$. It then suffices to show (1) holds for μ-a.e $x \in A$. Following the idea of the Lebesgue differentiation theorem, define

$$E_k = \left\{ x \in A : \limsup_{\varepsilon\to 0} \frac{\mu[x-\varepsilon, x+\varepsilon]}{\mathcal{L}[x-\varepsilon, x+\varepsilon]} > 1/k \right\}$$

We claim $\mu(E_k) = 0$. Fix $\varepsilon > 0$. Since $E_k \subset A$, $\mathcal{L}(E_k) = 0$. Outer regularity of $\mathcal{L}$ then implies $\exists$ open $U \supset E_k$ w/ $\mathcal{L}(U) < \varepsilon$.

By definition, $\forall x \in E_k\ \exists\, r(x) > 0$ s.t. $\frac{\mu[x-r(x), x+r(x)]}{\mathcal{L}[x-r(x), x+r(x)]} > 1/k$

where $r(x)$ is taken sufficiently small so that $(x-r(x), x+r(x)) \subset U$.

Then $\{(x - \frac{r(x)}{5}, x + \frac{r(x)}{5}),\ x \in E_k\}$ is an open cover of E_k. By the Vitali covering lemma, $\exists\, x_1, x_2, \ldots \in E_k$ s.t. $\{(x_n - r(x_n), x_n + r(x_n)),\ n \in \mathbb{N}\}$ is an open cover of E_k and $\{(x_n - \frac{r(x_n)}{5}, x_n + \frac{r(x_n)}{5}),\ n \in \mathbb{N}\}$ are pairwise disjoint. Then

$$\mu(E_k) \le \sum_{n=1}^{\infty} \mu(x_n - r(x_n), x_n + r(x_n)) < k \sum_{n=1}^{\infty} \mathcal{L}(x_n - r(x_n), x_n + r(x_n))$$

$$= 5k \sum_{n=1}^{\infty} \mathcal{L}\left(x_n - \frac{r(x_n)}{5}, x_n + \frac{r(x_n)}{5}\right)$$

$$= 5k\, \mathcal{L}\left(\bigcup_{n=1}^{\infty} \left(x_n - \frac{r(x_n)}{5}, x_n + \frac{r(x_n)}{5}\right)\right) \quad \text{(pairwise disjoint)}$$

$$\le 5k\, \mathcal{L}(U)$$

$$\le 5k\varepsilon$$

Taking $\varepsilon \to 0$ concludes. $\square$

(slight difference here, problem asks for $\mathcal{L}$-a.c, not μ-a.e)

(6)

Define μ_R by $d\mu_R = \frac{h(Re^{i\theta})}{2\pi} d\theta$ for $0<R<1$.

Then $\|\mu_R\| = \mu_R[0,2\pi] = \frac{1}{2\pi}\int_0^{2\pi} h(Re^{i\theta})\,d\theta = h(0)$.

Therefore $\{\mu_R : 0<R<1\}$ is a bounded subset of the space of Borel measures on $[0,2\pi]$. Banach-Alaoglu implies that the ball of radius $h(0)$ in $M[0,2\pi]$ is ~~compact~~ weak-* compact when viewed as the dual $M[0,2\pi] \cong (C[0,2\pi])^*$. Let $\{R_n\}\subset(0,1)$ be an increasing sequence converging to 1. By weak-* compactness, we may pass to a subsequence and assume $\mu_{R_n} \xrightarrow{*} \mu$ for some μ.

By Portmanteau's theorem, since $[0,2\pi]$ is open and closed in $[0,2\pi]$, $\mu[0,2\pi] = \lim_{n\to\infty} \mu_{R_n}[0,2\pi] = h(0)$.

We claim that μ is the desired measure.

Fix some $re^{i\eta}\in\mathbb{D}$. Then for sufficiently large n s.t. $R_n>r$,

$$h(R_n re^{i\eta}) = \frac{1}{2\pi}\int_0^{2\pi} P_{r/R_n}(\eta-\theta)\, h(R_n e^{i\theta})\,d\theta$$

$$= \int_0^{2\pi} P_{r/R_n}(\eta-\theta)\,d\mu_{R_n}(\theta)$$

Since $R_n\to 1$ and $\mu_{R_n} \xrightarrow{*} \mu$, the continuity of h implies

$$h(re^{i\eta}) = \int_0^{2\pi} P_r(\eta-\theta)\,d\mu(\theta)$$

as desired. $\square$

(7) 405 846 515

a) A unitary operator is a bounded linear ~~unitary~~ operator $U : H \to H$, w/ H a Hilbert space, that satisfies $UU^* = U^*U = id$ w/ U^* the adjoint ~~(or complex transp~~ of U.

b) By definition, S is invertible, then it suffices to show that $(I - \lambda S^*)$ is invertible as that will imply $S - \lambda I = S(I - \lambda S^*)$ is invertible. We claim that

$$(I - \lambda S^*)^{-1} = \sum_{n=0}^{\infty} (\lambda S^*)^n.$$

Consider the RHS. Since $\|S\| = \|S^*\| = 1$,

$$\Big\| \sum_{n=N}^{M} (\lambda S^*)^n \Big\| \le \sum_{n=N}^{M} |\lambda|^n \|S^*\|^n = \sum_{n=N}^{M} |\lambda|^n \le \frac{|\lambda|^N}{1 - |\lambda|^M}$$

as $N, M \to \infty$, this goes to 0. Therefore the partial sums $\sum_{n=1}^{N} (\lambda S^*)^n$ are Cauchy. Since H is complete, these converge to an operator $\sum_{n=1}^{\infty} (\lambda S^*)^n$ in H. Moreover,

$$(I - \lambda S^*) \sum_{n=0}^{\infty} (\lambda S^*)^n = \lim_{N\to\infty} (I - \lambda S^*) \sum_{n=0}^{N} (\lambda S^*)^n$$

$$= \lim_{N \to \infty} I - (\lambda S^*)^{N+1}$$

Since $\|\lambda S^*\| = |\lambda| < 1$, $\|(\lambda S^*)^{N+1}\| \to 0$ as $N \to \infty$ and so $\lim_{N\to\infty} (\lambda S^*)^{N+1} = 0$.

Then $(I - \lambda S^*) \sum_{n=0}^{\infty} (\lambda S^*)^n = I$. A symmetric argument shows that $\left(\sum_{n=0}^{\infty} (\lambda S^*)^n\right)(I - \lambda S^*) = I$ and so $(I - \lambda S^*)^{-1} = \sum_{n=0}^{\infty} (\lambda S^*)^n$.

Then $(S - \lambda I)^{-1} = \left(\sum_{n=0}^{\infty} (\lambda S^*)^n\right) S^*$ and is invertible.

$\longrightarrow$

(8) 405 846 515

a) Suppose on the contrary. Then $\exists$ distinct $z_0, z_1 \in \Omega$ w/ $f(z_0) = f(z_1)$.
Let $\gamma : [0,1] \to \Omega : t \mapsto (1-t)z_0 + t z_1$ this is well-defined since Ω is convex. Then

$$f(z_1) - f(z_0) = \int_\gamma f'(z)\,dz$$

$$= \int_0^1 f'((1-t)z_0 + t z_1)(z_1 - z_0)\,dt$$

Since $$\mathrm{Re}\left(\frac{f(z_1)-f(z_0)}{z_1 - z_0}\right) = \int_0^1 \mathrm{Re}(f'((1-t)z_0 + t z_1))\,dt$$

By assumption, $\mathrm{Re}(f') > 0$ and so

$$\mathrm{Re}\left(\frac{f(z_1)-f(z_0)}{z_1 - z_0}\right) > 0$$

B/c $z_1 \neq z_0$ this implies $f(z_1) - f(z_0) \neq 0$ $\divideontimes$.

Therefore our supposition was incorrect and f is 1-to-1.

b) Let $\Omega = \mathbb{C} \setminus (-\infty, 0]$. Then Ω is open, connected and simply connected. Define $f : \Omega \to \mathbb{C}$ by $f(z) = z^{3/2} = (z^{1/2})^3$ where we've taken the principal branch of the square root.

Then $f'(re^{i\theta}) = \frac{3}{2}(re^{i\theta})^{1/2} = \frac{3}{2}\sqrt{r}\,e^{i\theta/2} = \frac{3}{2}\sqrt{r}\cos(\theta/2) + \frac{3}{2}\sqrt{r}\,i\sin(\theta/2)$ for all $\theta \in (-\pi, \pi)$. Since $\theta \in (-\pi, \pi)$, $\cos(\theta/2) > 0$.

Then $\mathrm{Re}(f') > 0$. However,

$$f(1) = 1$$

$$f(e^{\frac{4\pi i}{3}}) = e^{2\pi i} = 1$$

so f is not 1-to-1. $\square$

(9) 405 846 515

a) By assumption, f has at most one pole in $\overline{\mathbb{D}}$, so it suffices to show that f has a pole in $\overline{\mathbb{D}}$.
Since f is $\sqrt{2}$-periodic, the behavior of f on the square $R = [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]^2 \subset \overline{\mathbb{D}}$ determines the behavior of f on $\mathbb{C}$ and vice versa.
Suppose for the sake of contradiction that f does not have a pole in R. Then b/c R is compact, f is bounded on R. Periodicity thus implies that f has no poles and is bounded, so it must be constant. But that contradicts the fact that f is non-constant.
Therefore f has a pole, and consequently exactly one pole, in $\overline{\mathbb{D}}$.

b) Consider R as above. If ∂R has a pole, then ∂R has >1 poles by periodicity, which contradicts part a. Therefore ∂R contains no poles.
Suppose for the sake of contradiction that f has a simple pole in $\overline{\mathbb{D}}$. By part a, this simple pole lies in R°. ~~Direct computation, implies and periodicity imply~~
Then $\int_{\partial R} f\,dz = \left(\int_{\partial R_1} + \int_{\partial R_2} + \int_{\partial R_3} + \int_{\partial R_4}\right) f\,dz$
w/ ∂R_1 the bottom side, ∂R_2 the right, etc.
By periodicity, $\int_{\partial R_1} f\,dz = -\int_{\partial R_3} f\,dz$ and $\int_{\partial R_2} f\,dz = -\int_{\partial R_4} f\,dz$ so $\int_{\partial R} f\,dz = 0$. However, since f has a simple pole and no other poles in R, Cauchy integral formula implies $\int_{\partial R} f\,dz = 2\pi i \operatorname{Res}(f,p) \neq 0$ where p is the pole. $\divideontimes$ $\square$

c) Define

$$h(\lambda) = \langle (S+\lambda I)(S-\lambda I)^{-1} v, v \rangle$$

for some fixed $v \in H$ and $\forall \lambda$ w/ $|\lambda|<1$.

By the continuity of inner products,

$$h(\lambda) = \langle (S+\lambda I)\left(\sum_{n=0}^{\infty} (\lambda S^*)^n\right) S^* v, v \rangle$$

$$= \sum_{n=0}^{\infty} \langle (\lambda^n (S^*)^n + \lambda^{n+1}(S^*)^{n+1}) v, v \rangle$$

$$= \sum_{n=0}^{\infty} \lambda^n \langle (S^*)^n v, v \rangle + \sum_{n=1}^{\infty} \lambda^n \langle (S^*)^n v, v \rangle$$

$$= \|v\|^2 + \sum_{n=1}^{\infty} 2 \lambda^n \langle (S^*)^n v, v \rangle$$

which is a power series in λ. Since $\langle (S^*)^n v, v \rangle \leq \|v\|^2$, the coefficients $2\langle (S^*)^n v, v \rangle$ are bounded and so the power series has radius of convergence at least 1. In particular, h is analytic on $D(0,1)$. Therefore $Re(h)$ is harmonic on $D(0,1)$. It remains to show $Re(h) > 0$.

If $v=0$ then the problem fails as $h=0$. So we assume $v \neq 0$. Then

$$2 Re(h) = h(\lambda) + \overline{h(\lambda)}$$

$$= \sum_{n=1}^{\infty} 2\lambda^n \langle (S^*)^n v, v \rangle + \overline{\sum_{n=1} 2 \lambda^n \langle (S^*)^n v, v \rangle}$$

$$= 2 \sum_{n=1}^{\infty} Re(\lambda^n \langle (S^*)^n v, v \rangle)$$

(12)

We claim that f has at most simple poles.

Suppose on the contrary that f has a pole of order $k > 1$ at $n\pi$. Then f grows at a rate of $|z|^{-k}$ near $n\pi$.

However, this contradicts the bound of $|f(z)| \leq (1 + |Im(z)|^{-1}) e^{-|Im(z)|}$, which grows at most at a rate of $|z|^{-1}$. Therefore f has at most simple poles.

Since f has at most simple poles at $\pi\mathbb{Z}$, it follows that $f(z)\sin(z)$ extends to an entire function $\varphi(z)$.

If f were holomorphic, then ~~continuity~~ would imply bounded and hence constant. Therefore since f is non-constant, it has a pole. WLOG, suppose its pole is at 0.

Then $\varphi(0) \neq 0$.

We claim that φ has no zeros. Suppose for the sake of contradiction that φ has a zero at a. Then Jensen's formula implies that for sufficiently large R,

$$\frac{1}{2\pi} \int_0^{2\pi} \log|\varphi(Re^{i\theta})|\, d\theta \geq \log|\varphi(0)| + \log\left(\frac{R}{|a|}\right)$$

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi} \log|\varphi(Re^{i\theta})|\, d\theta \gtrsim \log R$$

By the bound on f, $\forall z \in \mathbb{C} \setminus \mathbb{R}$,

$$|\varphi(z)| \leq (1 + 1/|Im(z)|)\, e^{-|Im(z)|} \left|\frac{1}{2i}(e^{iz} - e^{-iz})\right|$$

$$\leq \frac{1}{2}(1 + 1/|Im(z)|)\, e^{-|y|}(2e^{|y|})$$

$$= \left(1 + \frac{1}{|Im(z)|}\right)|$$

Then

$$\log R \le \frac{1}{2\pi}\int_0^{2\pi} \log\left(1 + {}^{1}/_{R|\sin\theta|}\right) d\theta$$

$$\sim \frac{1}{2\pi}\int_0^{\pi/2} \log\left(1 + {}^{1}/_{R\sin\theta}\right) d\theta$$

For $\theta \in [0, \pi/2]$, $\sin(\theta) \geq \frac{2\theta}{\pi}$. Then

$$\log R \le \int_0^{\pi/2} \log\left(1 + {}^{\pi}/_{2R\theta}\right) d\theta$$

Taking $R \to \infty$, DCT implies that

$$\int_0^{\pi/2} \log(1 + \pi/2R\theta)\, d\theta \to 0$$

which is a contradiction. Therefore φ has no zeros.

This implies that f has a simple pole at all $\pi\mathbb{Z}$

since sine has a simple zero at all $\pi\mathbb{Z}$. $\square$

4/13/2023

To: Matthew Kowalski

From: Inwon Kim, Graduate Vice Chair

Re: 23S Qualifying Exam Results

You may review your examnination(s) in the Student Services Office. Examinations are kept on file for three years and then destroyed.

ExamName	ExamDate	ExamResult
Analysis	23S	PASS
Geo/Top	23S	PASS

Quarter entered: 21F

Exam	Date	Result
Basic	21F	PASS
Analysis	22S	FAIL
Geo/Top	22S	FAIL
Analysis	22F	FAIL
Geo/Top	22F	FAIL
Analysis	23S	PASS
Geo/Top	23S	PASS

MA Students:

_____ You have passed your qualifying exam requirement

_____ You have not passed your qualifying exam requirement within 7 quarters

PhD Students:

__√__ You have passed all the required written qualifying exams

_____ You have not passed the Basic examination at the PhD level within 4 quarters

_____ You have not passed at least one area examination within 5 quarters

_____ You have not passed all the required qualifying examinations within 7 quarters

All Students:

_____ You have not met previous conditions of the Graduate Studies Committee

_____ Please talk to GVC ASAP

Questions about exam grading and results must be addressed in writing to the Graduate Studies Committee. **You have one week to review your exam and submit any concerns to the committee**. For specific questions about the procedure, see Martha or Helen.

Students falling behind satisfactory progress are subject to loss of support and dismissal from the program. Exceptions may be granted by the Graduate Studies Committee. Exceptions to loss of support will be given only in extreme circumstances. If a student falls one year behind any of the

Analysis

Spring 2017

① 405 846 515

Fix $x, h \in \mathbb{R}$. Then by definition,

$$|F(x+h) - F(x)| \doteq \frac{1}{|K|} \left| \int_K f(x+h+t)\,dt - \int_K f(x+t)\,dt \right|$$

Making the change of variables $x \to x+h$

$$|F(x+h) - F(x)| = \frac{1}{|K|} \left| \int_{K+h} f(x+t)\,dt - \int_K f(x+t)\,dt \right|$$

$$= \frac{1}{|K|} \left| \int_{(K+h)\Delta K} f(x+t)\,dt \right|$$

$$\leq \frac{1}{|K|} \int_{(K+h)\Delta K} |f(x+t)|\,dt$$

Since $f \in L^\infty$, this implies $|F(x+h) - F(x)| \leq \frac{\|f\|_{L^\infty}}{|K|} |(K+h)\Delta K| \leq |(K+h)\Delta K|$ unif in x.

Let $\tau_h : L^1 \to L^1$ denote the translation operator $\tau_h g(x) = g(x-h)$.

We recall that τ_h is continuous w.r.t. h, in the sense that $\tau_h g \to g$ in L^1 for all $g \in L^1$.

By definition, $\forall x, h$

$$|F(x+h) - F(x)| \leq |(K+h)\Delta K|$$

$$= \int_{\mathbb{R}} |\chi_{K+h} - \chi_K|\,dt$$

$$= \int_{\mathbb{R}} |\tau_h \chi_K - \chi_K|\,dt$$

$$= \|\tau_h \chi_K - \chi_K\|_{L^1}$$

Since K is compact, $\chi_K \in L^1$. Therefore $\|\tau_h \chi_K - \chi_K\|_{L^1} \to 0$ as $h \to 0$.

Since this bound is uniform in x, $F(x+h) \to F(x)$ uniformly in x as $h \to 0$. Therefore F is uniformly continuous.

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Throughout this problem, we use the convention of "passing to a subsequence" w/o a change in indices. This is to avoid ~~unwieldy~~ unwieldy notation down the line.

Let $\{f_n\}$ be as given. Since $\|f_n\|_{L^2}$ is uniformly bounded, by passing to a subsequence we may assume that

$$\|f_n\|_{L^2} \to M$$

monotonically as $n \to \infty$.

We claim that by passing to a subsequence, we may assume that f_n converges pointwise a.e. to some f.

For the moment, assume that the claim holds. Then by Fatou's lemma,

$$\|f\|_{L^2} \le \liminf_{n\to\infty} \|f_n\|_{L^2} = M$$

By the triangle inequality, $\forall n$,

$$|f_n - f| \le |f_n| + |f|$$

Therefore $|f_n| + |f| - |f_n - f| \geq 0$ $\forall n$. Fatou's lemma then implies

$$\liminf_{n\to\infty} \int |f_n| + |f| - |f_n - f| \geq \int \liminf_{n\to\infty} |f_n| + |f| - |f_n - f|$$

Since $f_n \to f$ a.e., this implies

$$\liminf_{n\to\infty} \int |f_n| + |f| - |f_n - f| \geq 2\int |f| \qquad (1)$$

B/c $|[0,1]| = 1 < \infty$, the L^1 norm is bounded by the L^2 norm. Therefore $\|f\|_{L^1} \le \|f\|_{L^2} \le M$ and $\limsup_{n\to\infty} \|f_n\|_{L^1} \le M$.

Rearranging (1) then yields

$$\liminf_{n\to\infty} \int |f_n| - \int |f| \geq \limsup_{n\to\infty} \int |f_n - f|$$

hence f_n is non-decreasing $\forall n$.

$$f_n(x) \leq f_n(1) \quad \forall x$$

In particular, this implies that f_n, f are bounded above.

DCT then implies $\int |f_n| \to \int |f|$ and so

$$\limsup_{n\to\infty} \int |f_n - f| \leq 0$$

Therefore $f_n \to f$ in L^1. By the convention of passing to a subsequence,

~~It remains~~ this concludes.

It remains to show the claim. This follows an argument similar to Dini's theorem, but I am out of time.

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~~Since $\mathcal{F}$ is a σ-algebra, it suffices to~~

We recall that $C[0,1]$ is separable w.r.t the supremum norm. This can be shown easily by considering polynomials w/ rational coefficients and then applying Stone-Weierstrass.

Therefore, the topology on $C[0,1]$ is countably generated by open balls around countably many functions.

~~Since $\mathcal{F}$ is a σ-algebra~~ In particular, any open set in $C[0,1]$ can be written as the countable union of open balls. B/c $\mathcal{F}$ is a σ-algebra and hence closed under countable union, it thus suffices to show that $\mathcal{F}$ contains all open balls. $B_r(f) = \{g : \|g-f\| < r\}$.

Since any open ball can be written as the countable intersection of closed balls via $B_r(f) = \bigcap_{n\geq 1} \{g : \|g-f\| \leq r+1/n\} = \bigcap_{n\geq 1} \overline{B_{r+1/n}(f)}$, it suffices to show that $\mathcal{F}$ contains all closed balls.

Fix some $f \in C[0,1]$ and $r>0$. Since $\mathbb{Q}\cap[0,1]$ is dense in $[0,1]$, continuity implies that $\|f-g\| = \sup_{x\in[0,1]} |f(x)-g(x)| = \sup_{x\in\mathbb{Q}\cap[0,1]} |f(x)-g(x)|$.

Therefore

$$\begin{aligned}
\overline{B_r(f)} &= \{g : \|g-f\| \leq r\} \\
&= \{g : \sup_{x\in\mathbb{Q}\cap[0,1]} |g(x)-f(x)| \leq r\} \\
&= \{g : |g(x)-f(x)| \leq r \ \forall x \in \mathbb{Q}\cap[0,1]\} \\
&= \bigcap_{x\in\mathbb{Q}\cap[0,1]} \{g : |g(x)-f(x)| \leq r\}
\end{aligned}$$

By definition of L_x, this implies

$$\overline{B_r(f)} = \bigcap_{x\in\mathbb{Q}\cap[0,1]} L_x^{-1}([f(x)-r, f(x)+r])$$

Since L_x is $\mathcal{F}$-measurable and $x\in\mathbb{Q}\cap[0,1]$ is countable, this implies $\overline{B_r(f)} \in \mathcal{F}$ $\forall$ r,f. By the earlier reasoning, this concludes. $\square$

(4) 405 846 515

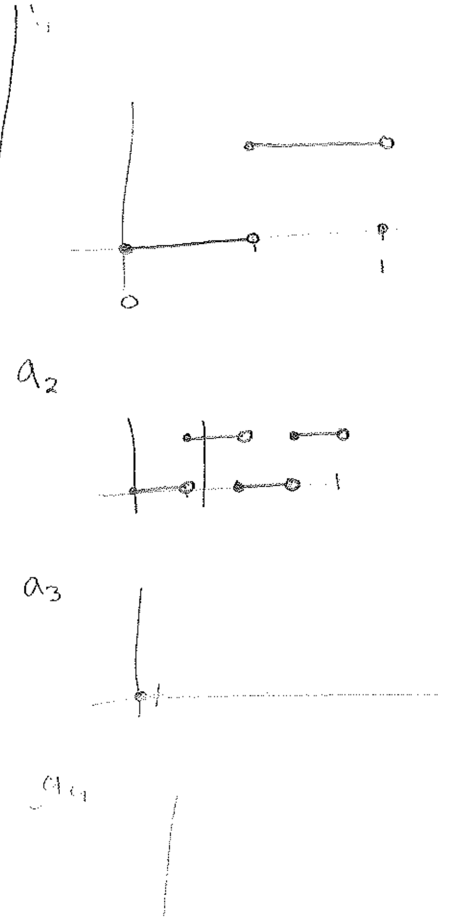
Let (L_{n_k}) be a subsequence of (L_n).
To show that L_{n_k} does not converge weak*
to some L, it suffices to find a
μ s.t. $L_{n_k}(\mu)$ does not converge.

Define

$$x_0 = \sum_{j \geq 1} 2^{-n_{2j}}.$$

Then $x_0 \in [0,1)$. B/c $n_{2j} \geq n_{2j-1} + 2$,
it follows that $\sum_{j\geq 1} 2^{-n_{2j}}$ is a valid
binary expansion of x_0. This is because
the expansion does not contain an infinite
string of 1's. Therefore

$$a_{n_k}(x_0) = \begin{cases} 0 & k \text{ is odd} \\ 1 & k \text{ is even} \end{cases}$$



Define $\mu = \delta_{x_0}$ to be the ~~delta~~ delta measure at x_0.
Then μ is finite and Borel, so $\mu \in M[0,1)$.
By construction,

$$L_{n_k}(\mu) = \int a_{n_k}(x)\,d\mu(x) = a_{n_k}(x_0) = \begin{cases} 0 & k \text{ is odd} \\ 1 & k \text{ is even} \end{cases}$$

and so $L_{n_k}(\mu)$ does not converge as $k \to \infty$.

Since such an x_0 can be constructed for any subsequence
L_{n_k} of L_n, this implies that no subsequence converges weak*. $\square$

(5) 405 346 515

Since $\nu << \mu$, $\exists\, f \in L^1(d\mu)$ s.t. $d\nu = f\,d\mu$ in the sense that

$$\int g\,d\nu = \int g f\,d\mu$$

for all g. This is guaranteed by the Radon-Nikodym theorem.

Therefore to show that $\hat{\nu}(n) \to 0$, ~~it suffi~~ we can equivalently show that $\int e^{2\pi i n x} f(x)\,d\mu(x) \to 0$ as $n \to \infty$ $\forall\, f \in L^1$.

Suppose first that $f = e^{2\pi i m x}$ for some $m \in \mathbb{Z}$.

Then for all n,

$$\int e^{2\pi i n x} f(x)\,d\mu(x) = \int e^{2\pi i (n+m)x}\,d\mu(x)$$

In particular, as $n \to \infty$,

$$\lim_{n\to\infty} \int e^{2\pi i n x} f(x)\,d\mu(x) = \lim_{n+m\to\infty} \int e^{2\pi i (n+m)x}\,d\mu(x) = 0$$

By linearity, this implies that for all finite linear combinations of $\{e^{2\pi i m x}\}_{m\in\mathbb{Z}}$, the claim holds. Let $\mathcal{F}$ denote the finite linear combination of $\{e^{2\pi i m x}\}_{m\in\mathbb{Z}}$. We claim that $\mathcal{F}$ is dense in L^1.

By definition, $\mathcal{F}$ is a sub-algebra in $C[0,1]$ since $e^{2\pi i m x} e^{2\pi i k x} = e^{2\pi i (m+k)x}$. Moreover, $\mathcal{F}$ contains the constant function 1 and the injective function $e^{2\pi i x}$. Therefore by Stone-Weierstrass, $\mathcal{F}$ is ~~dense~~ uniformly dense in $C[0,1]$.

B/c μ is finite, uniform convergence implies $L^1(d\mu)$ convergence. Therefore $\mathcal{F}$ is L^1-dense in $C[0,1]$. Since $C[0,1]$ is dense in $L^1[0,1]$, this implies that $\mathcal{F}$ is dense in $L^1[0,1]$ as desired.

Fix some $f \in L^1(d\mu)$. Then $\exists$ $g_k \in \mathcal{F}$ s.t. $g_k \to f$ in L^1.

Then $\forall$ $n \geq 0$,

$$\left| \int_0^1 e^{2\pi i n x} f(x) d\mu \right| \leq \left| \int_0^1 e^{2\pi i n x} (f - g_k) d\mu \right| + \left| \int_0^1 e^{2\pi i n x} g_k d\mu \right|$$
$$\leq \|f - g_k\|_{L^1(d\mu)} + \left| \int_0^1 e^{2\pi i n x} g_k d\mu \right|$$

Since $g_k \in \mathcal{F}$, $\left| \int_0^1 e^{2\pi i n x} g_k d\mu \right| \to 0$ as $n \to \infty$.

Therefore $\limsup_{n \to \infty} \left| \int_0^1 e^{2\pi i n x} f(x) d\mu \right| \leq \|f - g_k\|_{L^1(d\mu)}$ $\forall$ k.

Since $g_k \to f$ in $L^1(d\mu)$, this implies $\int_0^1 e^{2\pi i n x} f(x) d\mu \to 0$ as $n \to \infty$.

By the earlier reasoning, this concludes.

$\square$

(7) 405 846 515

This is immediate if we assume knowledge of quasi-conformal maps. The function f is conformal and ~~extends continuous~~ so is 1-quasiconformal. Therefore $A_R = \{1<|z|<R\}$ and $A_S = \{1<|z|<S\}$ have the same modulus and hence $R=S$.

We now proceed w/ an elementary proof.
Let A_R, A_S be as above. Suppose $\exists$ f, one-to-one and continuous ~~on~~ from $\overline{A_R}$ to $\overline{A_S}$ s.t. f is analytic on A_R.
Since f is one-to-one and analytic, f is conformal $A_R \to A_S$.
Therefore f maps ∂A_R to ∂A_S. By ~~composin~~ pre-composing w/ a reflection, we may assume that $f: \partial\mathbb{D} \to \partial\mathbb{D}$.

B/c f fixes $\partial\mathbb{D}$, we may extend f holomorphically to from A_R to $\{1/R < |z| < R\}$ via $f(1/z) = 1/f(z)$ for $z \in A_R$.
The fact that this extension is holomorphic follows from 2 methods. ~~Fist~~ First, mapping $\mathbb{D}$ to $\mathbb{H}$ conformally, ~~and no~~ noting that f maps to a ~~function~~ analytic function $\check{f}$ from a subset of $-\mathbb{H}$ to $\mathbb{H}$, which fixes $\mathbb{R}$.
The Schwarz reflection principle then extends $\check{f}$ to a subset of $\mathbb{H}$ via $\check{f}(\bar{z}) = \overline{\check{f}(z)}$, which pulls back to an extension of f across $\partial\mathbb{D}$ via $f(1/z) = 1/f(z)$. Alternatively, we may follow the proof of Schwarz reflection for a reflection across $\partial\mathbb{D}$ instead, in which case the proof is identical. ~~We do not~~

Let f also denote the extension to $\{1/R < |z| < R\}$ via $f(1/z) = 1/f(z)$.
Since $f: A_R \to A_S$, $f: \{1/R < |z| < R\} \to \{1/S < |z| < S\}$.
Repeating this extension, f extends to a holomorphic map

$$f: \{0 < |z| < R\} \to \{0 < |z| < S\}.$$

Since f is bounded, f has a removable singularity at 0. By the maximum modulus principle, since $f: \partial\mathbb{D} \to \partial\mathbb{D}$, $f: \mathbb{D} \to \mathbb{D}$. Moreover, since $f: \{0<|z|<R\} \to \{0<|z|<S\}$ preserves boundaries, $f(0) = 0$.
Therefore by the Schwarz lemma, $|f(z)| \leq |z|$ on $\mathbb{D}$.
For $1 < |z| < R$, we recall that $f(z) = 1/f(1/z)$.
Therefore $|f(z)| \geq 1/|1/z| \geq |z|$ for $1 < |z| < R$.
In particular, by continuity, $S \geq R$.
Repeating this argument w/ f^{-1} instead of f yields $R \geq S$.
Therefore $R = S$.

$\square$

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X

We proceed by induction on n.

Suppose $n=1$. Then by direct computation,

$$B(z) = \frac{z-a_1}{1-\bar{a}_1 z}$$

$$\Rightarrow B'(z) = \frac{(1-\bar{a}_1 z) + (z-a_1)\bar{a}_1}{(1-\bar{a}_1 z)^2} = \frac{1-|a_1|^2}{(1-\bar{a}_1 z)^2}$$

Then $B'(z)$ has $n-1=0$ zeros as desired.

Suppose that the claim holds for k points $a_1, \dots, a_k \in \mathbb{D}$.

Suppose that this extends to $k+1$ points $a_1, \dots, a_k, a \in \mathbb{D}$.

We claim that

For ease of notation, let

$$B(z) = \frac{z-a}{1-\bar{a}z} \prod_{j=1}^{k} \frac{z-a_j}{1-\bar{a}_j z} = \frac{z-a}{1-\bar{a}z} B_k(z)$$

Then

$$B'(z) = B_k(z) \frac{1-|a|^2}{(1-\bar{a}z)^2} + B_k' \frac{z-a}{1-\bar{a}z}$$

By construction, $B(z)$ is holomorphic on a neighborhood of $\bar{\mathbb{D}}$.

Therefore by the argument principle,

$$Z = \int_{\partial\mathbb{D}} \frac{B'(z)}{B(z)} dz = \int_{\partial\mathbb{D}} \frac{\left(\frac{1-|a|^2}{(1-\bar{a}z)^2}\right)}{\frac{z-a}{1-\bar{a}z}} dz + \int_{\partial\mathbb{D}} \frac{B_k'(z)}{B(z)} dz$$

where Z = # zeros of B in $\mathbb{D}$. By the inductive assumption and since $\frac{z-a}{1-\bar{a}z}$ has 1 zero in $\mathbb{D}$, this implies

$$Z = 1 + k - 1 = k$$

which is the claim.

By induction, this concludes.

This would be zeros of B of B which is incorrect.

(9) 405 846 515

(a) We recall the Blaschke factor φ_a for $a \in \mathbb{D}$ defined by

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$$

Since $a \in \mathbb{D}$, φ_a has a simple zero at $a \in \mathbb{D}$ and a simple pole at $1/\bar{a} \notin \mathbb{D}$. Therefore φ_a is holomorphic on a neighborhood of $\mathbb{D}$ and meromorphic on $\mathbb{C}$. Moreover, for $|z|=1$, $1/z = \bar{z}$ and so

$$|\varphi_a(z)| = \frac{|z-a|}{|1-\bar{a}z|} = \frac{|z-a|}{|\frac{1}{z}-\bar{a}|} = \frac{|z-a|}{|\bar{z}-\bar{a}|} = 1.$$

~~Since f is non not identically~~

Fix R as given. Since f is not identically zero ($f(0)\neq 0$), f has at most finitely many zeros in $\{|z|<R\}$.

Define

$$g(z) = \frac{f(z)}{\prod_{|a_n|<R} \varphi_{a_n/R}(z/R)}$$

Since φ_a has a simple zero at a, $\varphi_{a_n/R}(z/R)$ has a simple zero at a_n. Therefore, by the properties above, g is holomorphic on a neighborhood of $\{|z|<R\}$, ~~and~~ nonzero on a neighborhood of $\{|z|<R\}$, and $|g(z)| = |f(z)|$ for $|z|=R$.

Since g is nonvanishing on a neighborhood of $\{|z|<R\}$, $\log|g|$ is harmonic on a neighborhood of $\{|z|<R\}$.

~~Then~~ The mean value theorem then implies

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\theta})|\,d\theta$$

Since $|g|=|f|$ for $|z|=R$, this implies

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta$$

By construction,

$$g(0) = \frac{f(0)}{\prod_{|a_n|<R} \varphi_{a_n/R}(0)}$$

$$= \frac{f(0)}{\prod_{|a_n|<R} a_n/R}$$

So, since $|a_n/R| \neq 1 \Rightarrow \log|a_n/R| \neq 0$,

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta = \log|g(0)| = \log|f(0)| + \sum_{|a_n|<R} \log(R/|a_n|)$$

as desired.

(b) Suppose that $|f(z)| \leq e^{|z|^\lambda}$ for some λ.

Then by part (a), for all R s.t. $|f(z)| \neq 0$ for $|z| = 2R$,

$$\sum_{|a_n|<2R} \log\frac{2R}{|a_n|} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(2Re^{i\theta})|\,d\theta - \log|f(0)|$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} |2Re^{i\theta}|^\lambda\,d\theta - \log|f(0)|$$

$$\leq R^\lambda - \log|f(0)|$$

If $|a_n| < R$ then $\log\frac{2R}{|a_n|} \geq \log 2 \geq 1$. Therefore

$$\#\{|a_n|<R\} \leq \sum_{|a_n|<2R} \log\frac{2R}{|a_n|} \leq R^\lambda - \log|f(0)| \leq R^\lambda + |\log|f(0)||$$

Since f is not identically zero, f has at most countably many zeros.

Therefore $|f(z)| \neq 0$ for $|z| = R$ for a.e. $R > 0$.

In particular, $\exists$ some constant $b > 1$ s.t. $|f(z)| \neq 0$ for all $|z| = b^k$ w/ $k \geq 0$.

Then $$\sum (1/|a_n|)^{\lambda+\varepsilon} = \sum_{|a_n|<1} (1/|a_n|)^{\lambda+\varepsilon} + \sum_{k\in\mathbb{N}} \sum_{b^k \leq |a_n| < b^{k+1}} (1/|a_n|)^{\lambda+\varepsilon}$$

Since f has finitely many zeros in $\{|z|<1\}$, the first term is finite.

For the second term,

$$\sum_{k\in\mathbb{N}} \sum_{b^k \le |a_n| < b^{k+1}} \left(1/|a_n|\right)^{\lambda+\varepsilon} \sim \sum_{k\in\mathbb{N}} \sum_{|a_n|\sim b^k} \left(b^{-k}\right)^{\lambda+\varepsilon}$$

$$\le \sum_{k\in\mathbb{N}} \#\{|a_n| < b^{k+1}\} \left(b^{-k}\right)^{\lambda+\varepsilon}$$

$$\lesssim \sum_{k\in\mathbb{N}} \left(b^{\lambda k+\lambda} + |\log|f(0)||\right)\left(b^{-k}\right)^{\lambda+\varepsilon}$$

$$\lesssim \sum_{k\in\mathbb{N}} \left(b^{-\varepsilon} + |\log|f(0)||\, b^{-k(\lambda+\varepsilon)}\right)$$

$$< \infty$$

w/ the final inequality coming from $\varepsilon > 0$ and $(\lambda+\varepsilon) > 0$.

Therefore $\sum \left(1/|a_n|\right)^{\lambda+\varepsilon} < \infty$ as desired. $\square$

(11) 405 846 515

As given, $u \in L^p$. Let $\int_{\mathbb{R}\times\mathbb{R}} |u|^p dxdy = M < \infty$. If ~~$M=0$ then $u=0$ and we're done. So assume $M>0$.~~

By the mean value theorem for harmonic functions, $\forall z_0 \in \mathbb{C}$, $\forall r > 0$

$$u(z_0) = \frac{1}{\pi r^2} \int_{D(z_0,r)} u(x+iy)\, dxdy$$

$$\Rightarrow \quad |u(z_0)| \le \frac{1}{\pi r^2} \int_{D(z_0,r)} |u|\, dxdy$$

If $p=1$, then this implies that $\forall r > 0$

$$|u(z_0)| \le \frac{1}{\pi r^2} \int_{\mathbb{R}\times\mathbb{R}} |u| dxdy = \frac{1}{\pi r^2} M$$

Taking $r \to \infty$ then implies $u(z_0) = 0$.

If $p > 1$, then $\frac{p}{p-1}$ is the Hölder conjugate of p. By Hölder's inequality $\forall r > 0$

$$|u(z_0)| \le \frac{1}{\pi r^2} \int_{\mathbb{R}\times\mathbb{R}} \chi_{D(z_0,r)} |u|\, dxdy$$

$$\le \frac{1}{\pi r^2} \|u\|_{L^p} \|\chi_{D(z_0,r)}\|_{L^{\frac{p}{p-1}}}$$

$$= \frac{M}{\pi r^2} (\pi r^2)^{\frac{p-1}{p}}$$

Since $p > 1$, $0 < \frac{p-1}{p} < 1$. Then $r^{2\frac{p-1}{p}} / r^2 \to 0$ as $r \to \infty$.

Taking $r \to \infty$ this implies $|u(z_0)| \le 0$.

As this holds $\forall z_0 \in \mathbb{C}$ ~~and all $1 \le p < \infty$,~~ in either case this implies $u \equiv 0$. $\square$

(12) 405 846 515

~~Since adding a constant does not affect the derivative,~~
~~we may assume WLOG that $f(0)=0$.~~
~~Then $\forall z \in \mathbb{D}$,~~

$$\cancel{|f(z)| = |f(z)-f(0)| \leq C|z-0|^K = C|z|^K}$$

For all $z \in \mathbb{D}$, we know that $\forall \varepsilon > 0$, $\{w : |w-z| = 1-|z|-\varepsilon\} \subset \mathbb{D}$.
Therefore by the Cauchy estimates,

$$|f'(z)| \lesssim \frac{1}{(1-|z|-\varepsilon)} \max_{|w-z|=1-|z|-\varepsilon} |f(w)| \qquad \text{uniformly in } z, \varepsilon$$

$$\lesssim_C \frac{1}{(1-|z|-\varepsilon)} (1-|z|-\varepsilon)^K$$

$$= (1-|z|-\varepsilon)^{K-1}$$

Taking $\varepsilon \to 0$ then implies that $|f'(z)| \lesssim_C (1-|z|)^{K-1}$ uniformly in z.
In particular, $\exists$ a constant $A(C)$ s.t.

$$|f'(z)| \leq A(1-|z|)^{K-1}$$

as desired.

$\square$

Analysis
Fall 2016

① 405 846 515

By the triangle inequality,

$$|f_n - f| \leq |f_n| + |f|$$

Therefore $\forall n$, $|f_n| + |f| - |f_n - f| \geq 0$. Fatou's lemma then implies that

$$\int \liminf_{n\to\infty} (|f_n| + |f| - |f_n - f|)\, d\mu \leq \liminf_{n\to\infty} \int |f_n| + |f| - |f_n - f|\, d\mu$$

Since $f_n \to f$ a.e and $\|f_n\|_{L^1} \to \|f\|$, this implies that

$$2\int |f|\, d\mu \leq 2\int |f|\, d\mu - \limsup_{n\to\infty} \int |f_n - f|\, d\mu$$

$$\Rightarrow \limsup_{n\to\infty} \int |f_n - f|\, d\mu \leq 0$$

Therefore $\|f_n - f\|_{L^1} \to 0$ as desired.

$\square$

② 405 846 515

By definition, $\exists$ ~~E~~ a Borel set $E \subset \mathbb{R}$ s.t. $|E|=0$ and $\mu(\mathbb{R}\setminus E)=0$.
Therefore it suffices to show the limit for a.e $x \in E$.

~~Equivalently, we show that~~

Define $\Omega = \{x \in E : \limsup_{r \to 0^+} \frac{\mu[x-r, x+r]}{2r} < \infty\}$.

Then it suffices to show that $\mu(\Omega) = 0$.

Define $\Omega_k = \{x \in E : \limsup_{r \to 0} \frac{\mu[x-r,x+r]}{2r} < k-1\}$. Then $\Omega = \bigcup_{k \geq 1} \Omega_k$.

Fix some $k \geq 1$. (*) By definition, $\forall x \in \Omega_k$, $\exists r(x) \leq 1$ s.t. $B(x, r(x)) \subset U_k$ and

$$\frac{\mu[x-r(x), x+r(x)]}{2r(x)} < k$$

$$\Rightarrow \mu[x-r(x), x+r(x)] < k \, |[x-r(x), x+r(x)]|$$

The collection $\{B(x, r(x)/5), x \in \Omega_k\}$ covers Ω_k and has radii bounded by 1. By the Vitali covering lemma, this implies $\exists$ a countable disjoint subcollection $B(x_1, r(x_1)/5), B(x_2, r(x_2)/5), \ldots$ s.t. $\Omega_k \subset \bigcup_{j \geq 1} B(x_j, r(x_j))$.

Then

$$\mu(\Omega_k) \leq \sum_{j \geq 1} \mu(B(x_j, r(x_j))) \leq \sum_j 2k\, r(x_j) = 5k \sum_{j \geq 1} 2 \cdot \frac{r(x_j)}{5}$$

$$= 5 \sum_{j \geq 1} k \, |B(x_j, r(x_j)/5)|$$

Since $\{B(x_j, r(x_j)/5)\}$ are disjoint and $B(x_j, r(x_j)/5) \subset U_k$ $\forall j$,

$$\mu(\Omega_k) \leq 5k \left|\bigcup_{j \geq 1} B(x_j, r(x_j)/5)\right| \leq 5k |U_k| \leq \frac{5\, 2^{-k} \varepsilon k}{5k} \leq 2^{-k} \varepsilon$$

Then $\mu(\Omega) \leq \sum_{k \geq 1} \mu(\Omega_k) \leq 2\varepsilon$. Taking $\varepsilon \to 0$ then concludes. $\square$

(*) Fix some $\varepsilon > 0$. Since E is Borel and $|E| = 0$, $\exists$ an open set $U_k \supset E$ s.t. $|U_k| < \frac{2^{-k}\varepsilon}{5k}$.

③ 405 846 515

(a) We note that weak* convergence of measure implies that $\forall\ f \in C(X)$,

$$\int f\, d\mu_n \to \int f\, d\mu$$

Suppose that $\varphi : X \to [0,\infty]$ is lower semi-continuous. Then $\exists\ f_k \in C(X)$ s.t. $f_k \uparrow \varphi$. Then $\forall k$, since $\mu_n \xrightarrow{*} \mu$,

$$\lim_{n\to\infty} \int f_k\, d\mu_n = \int f_k\, d\mu$$

Since $f_k \le \varphi\ \forall\ k$, monotonicity implies that $\forall k$,

$$\int f_k\, d\mu \le \liminf_{n\to\infty} \int \varphi\, d\mu_n$$

Finally, taking $k \to \infty$, MCT implies

$$\int \varphi\, d\mu \le \liminf_{n\to\infty} \int \varphi\, d\mu_n$$

As desired. $\square$

(b) Let $\mu_n \in P(K)$ be a minimizing sequence for E.
~~Since $P(K)$ is contained in the closed unit ball of~~
Let $M(K)$ denote the space of ~~com measured~~ finite Borel measures on K.
Then $M(K) \cong (C(K))^*$ via Riesz representation.
By Banach-Alaoglu, the closed unit ball in $M(K)$ is weak* compact. Since $M(K)$ is a metric space, ~~comp~~ the closed unit ball is sequentially weak* compact.
In particular, since $\{\mu_n\} \subset P(K)$ and $P(K)$ is contained in the closed unit ball, by passing to a subsequence we may assume that μ_n converges weak* to some $\mu \in M(K)$.
~~Since $f \in C(K)$, we have~~

~~$\mu(K) =$~~

$\longrightarrow$

We claim that $\mu \in P(K)$. To show this, we must show that μ is non-negative and $\mu(K)=1$.

Let $U \subset K$ be open. ~~Define~~ Then $\exists$ $f_k \in C(K)$ s.t. $f_k \uparrow \chi_U$.
These can be constructed explicitly by

$$f_k(x) = \max\{k\, d(x, K\setminus U), 1\}$$

Since $K\setminus U$ is closed in K, $d(x, K\setminus U) > 0$ iff $x \in U$. Thus $f_k \uparrow \chi_U$.
By weak* convergence and MCT

$$\mu(U) = \lim_{k\to\infty} \int f_k \, d\mu = \lim_{k\to\infty} \lim_{n\to\infty} \int f_k \, d\mu_n \geq 0$$

Since open sets generate the Borel σ-algebra on K, this implies that μ is non-negative.
Finally, since $1 \in C(K)$,

$$\mu(K) = \int 1 \, d\mu = \lim_{n\to\infty} \int 1 \, d\mu_n = 1$$

and so $\mu \in P(K)$.

We now claim that $\mu_n \otimes \mu_n \xrightarrow{*} \mu \otimes \mu$ on $K \times K$.
For all $f, g \in C(K)$, since $\mu_n \xrightarrow{*} \mu$

$$\begin{aligned}\iint f(x) g(y) \, d\mu_n\otimes\mu_n(x,y) &= \left(\int f(x)\, d\mu_n(x)\right)\left(\int g(y)\, d\mu_n(y)\right) \\ &\to \left(\int f \, d\mu\right)\left(\int g \, d\mu\right) \qquad (1) \\ &= \iint f(x) g(y) \, d\mu\otimes\mu(x,y).\end{aligned}$$

By linearity, this result holds $\forall$ finite linear combinations of separable continuous functions on $C(K\times K)$, which we denote by $\mathcal{F}$.
We claim that $\mathcal{F}$ is uniformly dense in $C(K\times K)$.
Since $\mathcal{F}$ is a sub-algebra and $1 \in \mathcal{F}$, Stone-Weierstrass implies that it suffices to show $\mathcal{F}$ separates points. $\longrightarrow$

Because $f(x,y)=x$ and $g(x,y)=y$ are in $\mathcal{F}$, it follows immediately that $\mathcal{F}$ separates points. Therefore $\mathcal{F}$ is uniformly dense in $C(K\times K)$.

We now extend (1) to all of $C(K\times K)$ to show that $\mu_n\otimes\mu_n \rightharpoonup^* \mu\otimes\mu$.

Suppose $\exists$ $f\in C(K\times K)$. Fix $\varepsilon>0$.

Then $\exists$ $g\in\mathcal{F}$ s.t. $\|f-g\|_{L^\infty(K)}<\varepsilon$ and N s.t. $\forall n\geq N$, $\left|\int g\,d\mu_n\otimes d\mu_n - \int g\,d\mu\otimes\mu\right|<\varepsilon$. For all $n\geq N$, this implies

$$\left|\int f\,d\mu_n\otimes\mu_n - \int f\,d\mu\otimes\mu\right| \leq \left|\int (f-g)\,d\mu_n\otimes\mu_n\right| + \left|\int g\,d\mu_n\otimes\mu_n - \int g\,d\mu\otimes\mu\right| + \left|\int (g-f)\,d\mu\otimes\mu\right|$$

$$\leq 3\varepsilon$$

As such an N can be found $\forall\varepsilon>0$, this concludes $\mu_n\otimes\mu_n \rightharpoonup^* \mu\otimes\mu$.

Finally, we show μ minimizes E.

We can write

$$\frac{1}{|x-y|} = \sup_{n\geq 1} \max\left(\frac{1}{|x-y|}, n\right)$$

Since $\max\left\{\frac{1}{|x-y|}, n\right\}$ is continuous $\forall n$, this implies that $\frac{1}{|x-y|}$ is lower semi-continuous. Thus by part (a), since $\mu_n\otimes\mu_n \rightharpoonup^* \mu\otimes\mu$

$$E(\mu) \leq \liminf_{n\to\infty} \iint \frac{1}{|x-y|}\,d\mu_n(x)\,d\mu_n(y)$$

Since μ_n is a minimizing sequence for n, this implies that μ minimizes E.

$\square$

(5) 405 846 515 Let Σ be the σ-algebra generated by $\{S(t,B)\}$.

We first claim that Σ contains all open balls in $C[0,1]$.
We note that since $\mathbb{Q}\cap[0,1]$ is dense,

$$\|f\| = \max_{x\in[0,1]} |f(x)| = \max_{\substack{x\in[0,1]\\ x\in\mathbb{Q}}} |f(x)| \quad \forall f\in C([0,1]).$$

Fix some $f\in C[0,1]$, and some $r>0$. Then

$$\begin{aligned} B(f,r) &= \{g\in C[0,1] : \|f-g\|<r\} \\ &= \{g\in C[0,1] : \max_{x\in[0,1]\cap\mathbb{Q}} |f(x)-g(x)|<r\} \\ &= \bigcap_{x\in\mathbb{Q}\cap[0,1]} \{g\in C[0,1] : |f(x)-g(x)|<r\} \\ &= \bigcap_{x\in\mathbb{Q}\cap[0,1]} S(x,(f(x)-r, f(x)+r)) \end{aligned}$$

Since $(f(x)-r, f(x)+r)$ is Borel and $\mathbb{Q}\cap[0,1]$ is countable, this implies that $B(f,r)\in S$. Therefore Σ contains all open balls in $C[0,1]$. Since Σ is a σ-algebra, this implies that Σ contains $\mathcal{A}$, so $\mathcal{A}\subset\Sigma$.

We now claim that $\Sigma\subset\mathcal{A}$. Define $L_t : C[0,1]\to\mathbb{R}$ by $f\mapsto f(t)$. Then L_t is continuous w.r.t the uniform topology on $C[0,1]$ since $\forall f_n\in C[0,1]$ w/ $f_n\to f$ uniformly,

$$\lim_{n\to\infty} L_t(f_n) = \lim_{n\to\infty} f_n(t) = f(t) = L_t(f)$$

Therefore L_t is Borel measurable. Then $\forall$ Borel $B\subset\mathbb{R}$,

$$L_t^{-1}(B) = \{f\in C[0,1] : f(t)\in B\} = S(t,B)$$

is Borel. Therefore $\Sigma\subset\mathcal{A}$.

Since inclusion has been shown in both directions, $\Sigma=\mathcal{A}$, which concludes. $\square$

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(a) By the mean value theorem and Hölder $\therefore$ $\forall z_0 \in \mathbb{D}, \forall f \in \mathcal{H}$,

$$|L_{z_0}(f)| = |f(z_0)|$$
$$= \left| \frac{1}{|D(z_0, 1-|z_0|)|} \int_{D(z_0,1-|z_0|)} f(z)\, dA(z) \right|$$
$$\le \frac{1}{\pi(1-|z_0|)^2} \int_{\mathbb{D}} |f(z)|\, dA(z)$$

Hölder $\lesssim_{z_0} \|f\|_{L^2(\mathbb{D})} \|1\|_{L^2(\mathbb{D})}$

$$\lesssim_{z_0} \|f\|_{L^2(\mathbb{D})}$$

Therefore L_{z_0} is a bounded linear functional $\forall z_0$.

(b) We claim that $\{\sqrt{\frac{n+1}{\pi}} z^n\}_{n\ge 0}$ forms an orthonormal basis for $\mathcal{H}$.

By direct computation,

$$\|z^n\|_{L^2} = \int_{\mathbb{D}} z^n \bar{z}^n\, dA(z)$$
$$= \int_{\mathbb{D}} |z|^{2n}\, dA(z)$$
$$= \frac{2\pi}{2n+2} |z|^{2n+2} \Big|_{|z|=0}^{|z|=1}$$
$$= \frac{\pi}{n+1}$$

Therefore $\|\sqrt{\frac{n+1}{\pi}} z^n\|_{L^2} = 1 \quad \forall n$.

For $n > m$, by the rotational symmetry of $dA(z)$

$$|\langle z^n, z^m\rangle| = \int_{\mathbb{D}} |z|^{2m} z^{n-m}\, dA(z)$$
$$(w = e^{\frac{-\pi i}{n-m}} z) \quad = -\int_{\mathbb{D}} |w|^{2m} w^{n-m}\, dA(w)$$
$$(z = w) \quad = -\langle z^n, z^m\rangle$$

Therefore $\langle z^n, z^m\rangle = 0$ for $n > m$ and so $\{\sqrt{\frac{n+1}{\pi}} z^n\}$ is orthonormal.

By the Riesz representation theorem, $\forall z_0 \in \mathbb{D}$ $\exists g_{z_0} \in \mathcal{H}$ s.t. $L_{z_0}(f) = \langle f, g_{z_0} \rangle$. Then

$$\begin{aligned}
g_{z_0}(z) &= \sum_{n=0}^{\infty} \left\langle g_{z_0}, \sqrt{\tfrac{n+1}{\pi}}\, z^n \right\rangle \sqrt{\tfrac{n+1}{\pi}}\, z^n \\
&= \sum_{n=0}^{\infty} \frac{n+1}{\pi}\, \overline{z_0}^n z^n \\
&= \frac{1}{\pi} \sum_{n=0}^{\infty} (n+1) (\overline{z_0} z)^n \\
&= \frac{1}{\pi} \frac{d}{dz} \left(\sum_{n=0}^{\infty} \frac{1}{\overline{z_0}} (\overline{z_0} z)^{n+1} \right) && (1) \\
&= \frac{1}{\pi} \frac{d}{dz} \left(\frac{1/\overline{z_0}}{1 - \overline{z_0} z} \right) && (2) \\
&= \frac{1}{\pi (1 - \overline{z_0} z)^2}
\end{aligned}$$

where line (1) follows since the sum converges locally uniformly and (2) follows from a geometric series.

Therefore $g_{z_0} = \dfrac{1}{\pi(1 - \overline{z_0} z)^2}$. $\square$

Ⓑ 405 846 515

(a) We note that this ~~is precisely~~ follows immediately from Jensen's ~~inequality~~ formula. So that is what we prove.

Fix $0<r<1$ s.t. $f(z)\neq 0$ for $|z|=r$.

Method 1:

Since f is holomorphic on $\mathbb{D}$, $\log|f(z)|$ is sub-harmonic on $\mathbb{D}$. Then by the sub-mean value property,

$$\log|f(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta$$

as desired.

Method 2:

Since $f(0)\neq 0$, f is not identically 0. Therefore, f has at most countably many 0s in $\mathbb{D}$, which we denote $\{a_1,\dots\}$. If $\{a_n\}$ has an accumulation point, then since $f\not\equiv 0$, it is on $\partial\mathbb{D}$. Therefore $\exists$ finitely many zeros of f inside radius r. Denote these zeros as $a_1,\dots,a_n$ repeated according to multiplicity.

Let φ_a denote the Blaschke ~~product~~ factor

$$\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$$

for $a\in\mathbb{D}$. We recall that $|\varphi_a(z)|=1$ $\forall$ $z\in\partial\mathbb{D}$, φ_a is holomorphic except a simple pole at $1/\bar{a}\notin\mathbb{D}$. Define

$$g(z) = f(z)\Big/\prod_{i=1}^{n}\varphi_{a_i/r}(z/r)$$

Then g is nonvanishing ~~on~~ a neighborhood of $\{|z|<r\}$ and $|g(z)|=|f(z)|$ $\forall$ $|z|=r$.

Then $\log|g(z)|$ is harmonic on a neighborhood of $\{|z|<r\}$.

~~Then~~ The mean value property thus implies

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi}\log|g(re^{i\theta})|\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta$$

$\longrightarrow$

Direct computation then yields

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|d\theta = \log|f(0)| - \sum_{i=1}^{n}\log|a_i/r| = \log|f(0)| + \sum_{i=1}^{n}\log(r/a_i)$$ Jensen's formula

Since $a_i < r \ \forall i$, this implies

$$\log|f(0)| \leq \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|d\theta$$

as desired.

(b) Define $E = \{\theta \in [0,2\pi] : f(e^{i\theta}) = 0\}$.

Since f is continuous on $\overline{\mathbb{D}}$, f is bounded above by some $M > 0$.

Then $\forall$ r as in part (a),

$$2\pi\log|f(0)| \leq \int_E \log|f(re^{i\theta})|d\theta + \int_{[0,2\pi]\setminus E}\log|M|d\theta$$

$$\leq 2\pi\log M + \int_E \log|f(re^{i\theta})|d\theta \leq 2\pi\log M + |E|\max_{\theta\in E}\log|f(re^{i\theta})|$$

Since f is continuous on $\overline{\mathbb{D}}$, f is uniformly continuous.

Therefore as $r \to 1$, $|f(re^{i\theta})| \to 0$ uniformly for all $\theta \in E$.

Since f has at most countably many zeroes, part (a) can be applied for all but countable $r \in (0,1)$.

Therefore $\exists$ a sequence $r_n \in (0,1)$ s.t. $r_n \to 1$ and r_n satisfies part (a).

Then $\forall n$,

$$2\pi\log|f(0)| \leq 2\pi\log M + \max_{\theta\in E}|E|\log|f(r_ne^{i\theta})|$$

Since $f(r_ne^{i\theta}) \to 0$ uniformly for $\theta \in E$,

this implies that either $\log|f(0)| = -\infty \Rightarrow f(0) = 0$

or $|E| = 0$. Therefore $|E| = 0$. $\square$

(9) 405 846 515

(a) This is a standard Monotone / DCT / Fubini's theorem on bounded subset argument.

(b) Suppose $\exists$ $n \in \mathbb{N}$ s.t.

$$\limsup_{n\to\infty} |f(z)|/|z|^n < \infty$$

Then for sufficiently large $|z|$, $|f(z)| \lesssim |z|^n$ uniformly in z.
By Cauchy ~~estimates~~, this implies that for sufficiently large R, for any $w \in \mathbb{C}$,

$$\left|f^{(n+1)}(w)\right| \lesssim \frac{1}{R^{n+1}} \max_{|z-w|=R} |f(z)|$$

$$\lesssim \frac{1}{R^{n+1}} \max_{|z-w|=R} |z|^n$$

$$\lesssim \frac{1}{R^{n+1}} \max_{|z-w|=R} \left(|z-w|+|w|\right)^n$$

$$= \frac{1}{R^{n+1}}\left(R+|w|\right)^n \to 0 \quad \text{as } R\to\infty$$

Therefore $f^{(n+1)} \equiv 0$ and so $f = a_0 + a_1 z + \dots + a_n z^n$.
From the formula provided,

$$|f(z)| \le \int_{[0,1]} |e^{izt}|\, d\mu(t)$$

$$= \int_0^1 e^{-t\,\mathrm{Im}(z)}\, d\mu(t)$$

$$\le \underbrace{\mu[0,1]}_{\to 1} \max\left(e^{-\mathrm{Im}(z)}, 1\right)$$

$$\le 1 + e^{-\mathrm{Im}(z)}$$

~~Therefore,~~ so B/c f is a polynomial, this implies that f is constant.

Fix some $k \in \mathbb{N}$. Then

$$f(z) = \int_{[0,1/k]} e^{izt}\,d\mu(t) + \int_{(1/k,1]} e^{izt}\,d\mu(t)$$

and so

$$\begin{aligned} \operatorname{Im}(f(1)) &= \int_{[0,1/k]} \sin(t)\,d\mu(t) + \int_{(1/k,1]} \sin(t)\,d\mu(t) \\ &\geq \int_{(1/k,1]} \sin(t)\,d\mu(t) \\ &\geq \sin(1/k)\,\mu(1/k,1] \end{aligned}$$

Similarly,

$$\begin{aligned} \operatorname{Im}(f(-1)) &= \int_{[0,1/k]} \sin(-t)\,d\mu(t) + \int_{(1/k,1]} \sin(-t)\,d\mu(t) \\ &\leq -\sin(1/k)\,\mu(1/k,1] \end{aligned}$$

Since f is constant, this implies that $\mu(1/k,1] = 0 \ \forall k$.

Therefore $\mu\{0\} = 1$ and so $\mu = \delta_0$ as desired. $\square$

(10) 405 846 515

(a) We claim that $M=2$ satisfies the constraints.

~~Suppose~~ Fix $z \in \mathbb{C}$ and suppose that $\exists\, n$ s.t. $|f^n(z)| \geq 2$. Then by the inverse triangle inequality,

$$
\begin{aligned}
|f^{n+1}(z)| &= |(f^n(z))^2 - 1| \\
&\geq |f^n(z)|^2 - 1 \\
&\geq 2^2 - 1 = 3
\end{aligned}
$$

Moreover, $\forall\, k \in \mathbb{N}$, if $|f^m(z)| \geq 2+k$, then

$$
\begin{aligned}
|f^{m+1}(z)| &= |(f^m(z))^2 - 1| \\
&\geq |f^m(z)|^2 - 1 \\
&\geq |2+k|^2 - 1 \\
&\geq k^2 + 2^2 - 1 \\
&\geq (k+2) + 1
\end{aligned}
$$

simpler: if $|z| > 2$ then
$$
\begin{aligned}
|f(z)| &= |z^2 - 1| \\
&= |z - 1/z||z| \\
&\geq (|z| - 1/|z|)|z| \\
&\geq \tfrac{3}{2}|z|
\end{aligned}
$$

Therefore by induction, $\forall\, j \geq 0$,

$$|f^{n+j}(z)| \geq 2 + j$$

and so $\lim_{n\to\infty} |f(z)| = \infty$.

This concludes that either $|f^n(z)| \leq 2 \;\forall n$ or $\lim_{n\to\infty} |f^n(z)| = \infty$.

(b) Since $\mathbb{C} = U \cup K$, it suffices to show that K is compact w/o holes.

By ~~definition~~ part (a), and definitions,

$$K = \bigcap_{n \geq 0} \{|f^n(z)| \leq 2\}$$

since $|f^n(z)|$ is continuous $\forall\, n$, $\{|f^n(z)| \leq 2\}$ is closed $\forall\, n$ and so K is closed.

$\longrightarrow$

As shown in part a, if $|z| \geq 2$ then $|f^n(z)| \to \infty$.
Therefore $K \subset D(0,2)$ and so is bounded.
Therefore K is compact.

Finally, suppose for the sake of contradiction that
$\mathbb{C} \setminus K$ has a bounded connected component V.
Then by the maximum modulus principle, $\forall z \in K$ $\forall n$,

$$|f^n(z)| \leq \max_{z \in \partial V} |f^n(z)|$$

since $\partial V \subset K$ this implies

$$|f^n(z)| \leq 2$$

and so $z \in K$. Therefore no such connected bounded component of K exists. $\square$

(11) 405 846 515

(a) Suppose $f(0) \neq 0$. Let $M > 0$ be an upper bound of g. Since $f(0) \neq 0$, continuity implies that for sufficiently small $|z|$, $f(z) \neq 0$. Therefore, for sufficiently small $|z|$, $|f(1/z)| = \left|\frac{g(z)}{f(z)}\right| \leq \frac{M}{|f(z)|}$.

In particular,

$$\limsup_{|z|\to\infty} |f(z)| = \limsup_{|z|\to 0} |f(1/z)| \leq \limsup_{|z|\to 0} \frac{M}{|f(z)|} = M/|f(0)|$$

Therefore f is bounded as $|z|\to\infty$. Continuity then implies f is bounded everywhere and hence is constant.

(b) Suppose instead that $f(0)=0$. If $f \equiv 0$ then the claim holds trivially. Therefore assume $f \not\equiv 0$. Let n be the order of the zero of f at 0. Then on a neighborhood of 0, $|f(z)| \sim |z|^n$. Since the zero of f at 0 is discrete, this implies

$$\limsup_{|z|\to\infty} \frac{|f(z)|}{|z|^n} = \limsup_{|z|\to 0} |z|^n |f(1/z)| \leq \limsup_{|z|\to 0} \frac{M|z|^n}{|f(z)|} \sim M$$

By Cauchy estimates, this implies that $f(z) = az^n + a_{n-1}z^{n-1} + \dots + a_0$. Since f has a zero of order n at 0, this implies $f(z) = az^n$ as desired.

Alternate: apply part (a) w/ $h(z) = \frac{f(z)}{z^n}$ $\square$

(12) 405 846 515

(a) For each $z \in K$, $\exists$ an ~~ot~~ ~~oo~~ open rectangle R_z s.t.

$z \in R_z \subset U$ and $\bar{R}_z \subset U$.

Then $\{R_z : z \in K\}$ is an open cover for K and so ~~admits~~ admits a finite subcover, which we denote $R_1, \ldots, R_n$. Let $V = R_1 \cup \ldots \cup R_n$.

Then $K \subset V \subset \bar{V} \subset U$ and ∂V ~~is~~ consists of finitely many line segments ~~sinc~~ since V is the finite union of rectangles.

(b)

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(a) We recall Young's inequality which states

$$\|g * h\|_{L^r} \le \|g\|_{L^p} \|h\|_{L^q}$$

provided $1 + 1/r = 1/p + 1/q$. Therefore

$$\|t^{1/2} K_t * f\|_{L^\infty} \le \|f\|_{L^3} \|t^{1/2} K_t\|_{L^{3/2}} = \|f\|_{L^3} t^{1/2} \|K_t\|_{L^{3/2}}$$

L^∞ case is just triangle inequality + Hölder

By direct computation,

$$\|K_t\|_{L^{3/2}}^{3/2} = \int |(4\pi t)^{-3/2} e^{-|x|^2/4t}|^{3/2} dx$$

$$= \frac{1}{(4\pi t)^{9/4}} \int e^{-3|x|^2/8t} dx$$

$$(u = x/t^{1/2}) \quad = \frac{1}{(4\pi t)^{9/4}} \int e^{-3|u|^2/8} t^{3/2} du$$

$$\sim t^{3/2}/t^{9/4} = t^{-1/3}$$

Therefore $\|K_t\|_{L^{3/2}} \sim t^{-1/2}$ and so

$$\|t^{1/2} K_t * f\|_{L^\infty} \lesssim t^{1/2} t^{-1/2} \|f\|_{L^3} = \|f\|_{L^3}$$

uniformly in t, as desired.

(b) Fix $t \le 1$. Then $t \le \sqrt{t}$ and so $e^{-\frac{|x|^2}{4t}} \le e^{-\frac{|x|^2}{4\sqrt{t}}}$.

Direct computation then yields

$$\|K_t\|_{L^{3/2}}^{3/2} = \int |(4\pi t)^{-3/2} e^{-|x|^2/4t}|^{3/2} dx$$

$$\le \frac{1}{(4\pi t)^{9/4}} \int e^{-\frac{3|x|^2}{8\sqrt{t}}} dx$$

$$(u = x/t^{1/4}) \quad \simeq \frac{t^{3/4}}{t^{9/4}} \int e^{-\frac{3|u|^2}{8}} du$$

$$\sim t^{-3/2}$$

and so $\|K_t\|_{L^{3/2}} \lesssim t^{-1}$

This should follow from the fact that $K_t \to 0$ a.e as $t \to 0$.

(1b) Fix $\varepsilon > 0$. ~~By density,~~ By the monotone convergence theorem, we know that bounded functions are dense in L^3. Therefore $\exists\, g$ s.t. $|g| \lesssim 1$ and $\|g - f\|_{L^3} < \varepsilon$. Then

$$\begin{aligned}
\|t^{1/2} K_t * f\|_{L^\infty} &\le \|t^{1/2} K_t * (f-g)\|_{L^\infty} + \|t^{1/2} K_t * g\|_{L^\infty} \\
&\lesssim \|f-g\|_{L^3} + \left\|\int t^{1/2} K(x-y) g(y)\, dy\right\|_{L^\infty} \\
&\lesssim \varepsilon + \left\|\int t^{1/2} K(x-y)\, dy\right\|_{L^\infty} \\
&= \varepsilon + \left\|\int t^{1/2} K(y)\, dy\right\|_{L^\infty} \\
&\sim \varepsilon + t^{-1} \int e^{-|y|^2/4t}\, dy \\
(u = y/\sqrt{t}) \quad &\sim \varepsilon + t^{1/2} \int e^{-|u|^2/4}\, du \\
&\lesssim \varepsilon + t^{1/2}
\end{aligned}$$

Therefore $\limsup_{t \to 0} \|t^{1/2} K_t * f\|_{L^\infty} \lesssim \varepsilon$. Taking $\varepsilon \to 0$ then yields $\lim_{t \to 0} \|t^{1/2} K_t * f\|_{L^\infty} = 0$ as desired.

$\square$

2) 405 846 515

It suffices to show that $\sum_{n\geq 1} \frac{1}{\sqrt{n}}|f(x-\sqrt{n})|$ is integrable over $[k,k+1]$.

By Tonelli's and rearranging,

$$\int_k^{k+1} \sum_{n\geq 1} \frac{1}{\sqrt{n}}|f(x-\sqrt{n})|\,dx = \sum_{n\geq 1} \frac{1}{\sqrt{n}} \int_k^{k+1} |f(x-\sqrt{n})|\,dx$$

$$(u = x-\sqrt{n}) \qquad = \sum_{n\geq 1} \frac{1}{\sqrt{n}} \int |f(u)|\,\chi_{\{k+\sqrt{n} \leq u \leq k+1+\sqrt{n}\}}\,du$$

$$= \sum_{n\geq 1} \frac{1}{\sqrt{n}} \int |f(u)|\,\chi_{\{(u-k-1)^2 \leq n \leq (u-k)^2\}}\,du$$

$$= \int |f(u)| \Big(\sum_{n\geq 1} \frac{1}{\sqrt{n}} \chi_{\{(u-k-1)^2 \leq n \leq (u-k)^2\}} \Big)\,du$$

We recall that we can bound

$$\sum_{n\geq 1} \frac{1}{\sqrt{n}} \chi_{\{(u-k-1)^2 \leq n \leq (u-k)^2\}} \leq \int_{\min((u-k-1)^2,1)}^{(u-k)^2} 1/\sqrt{y}\,dy \leq 2(u-k-u+k+1) = 2$$

Therefore,

$$\int_k^{k+1} \sum_{n\geq 1} \frac{1}{\sqrt{n}} |f(x-\sqrt{n})|\,dx \leq 2\int |f(u)|\,du = 2\|f\|_{L^1} < \infty$$

As this holds $\forall$ k, this implies that $\sum_{n\geq 1} \frac{1}{\sqrt{n}}|f(x-\sqrt{n})| < \infty$ for a.e x. $\square$

(3) 405 846 515

Let $\Omega = \{x : f(x) \leq f(x+1/n) \ \forall n \geq 1\}$.

Consider some $k/m \in \mathbb{Q}_{\geq 0}$. If $x, x+1/m, x+2/m, \ldots, x+\frac{k-1}{m} \in \Omega$,

then $f(x) \leq f(x+1/m) \leq f(x+2/m) \leq \ldots \leq f(x+k/m)$.

Equivalently, if $x \in \Omega \cap (\Omega - 1/m) \cap \ldots \cap (\Omega - \frac{k+1}{m})$, then $f(x) \leq f(x+k/m)$.

Iterating this for all k, m, we find that

$$\Omega^* = \{x : f(x) \leq f(x+q) \ \forall q \in \mathbb{Q}_{\geq 0}\} \supset \bigcap_{q \in \mathbb{Q}_{\geq 0}} (\Omega - q)$$

since $|\mathbb{R} \setminus \Omega| = 0$, ~~the~~ this implies that for a.e x, $f(x) \leq f(x+q) \ \forall q \in \mathbb{Q}_{\geq 0}$.

~~Consider some $x \in \Omega^*$ s.t. x is a Lebesgue point of f.~~

~~Fix some $a \geq 0$. and let $q_n \in \mathbb{Q}_{\geq 0}$ be s.t. $q_n \to a$.~~

Fix some $a \geq 0$. Consider some x ~~s.t.~~ $x, x+a$ are Lebesgue points of f. since $f, f(x+a) \in L^1_{loc}$, we note ~~that~~ a.e x satisfies these requirements.

Let $\exists q_n \in \mathbb{Q}_{\geq 0}$ s.t. $q_n \to a$. Then since translation is continuous in L^1, $\forall [c,d] \subset \mathbb{R}$,

$$\int_c^d (f(y+q_n) - f(y))\,dy \to \int_c^d (f(y+a) - f(y))\,dy.$$

since $|\mathbb{R} \setminus \Omega^*| = 0$, $\int_c^d (f(y+q_n) - f(y))\,dy \geq 0 \ \forall n$. Therefore

$$\int_c^d (f(y+a) - f(y))\,dy \geq 0$$

$\forall [c,d] \subset \mathbb{R}$.

In particular, since $x, x+a$ are Lebesgue points of f,

$$f(x-a) - f(x) = \frac{1}{2r} \lim_{r \to 0} \int_{x-r}^{x+r} (f(y+a) - f(y))\,dy$$
$$\geq \frac{1}{2r} \lim_{r \to 0} 0$$
$$\geq 0$$

~~which is what~~. As this holds for a.e x, this concludes. $\square$

(4) 405 846 515

~~If $V_1 = V$, then the identity map $V \to V$ satisfies the desired properties. Therefore we assume otherwise.~~

Suppose $V_1 \neq V$. Then $\exists$ some $u \in V \setminus V_1$. By normalizing, we assume ~~$\|u\|=1$.~~

Let $\{e_1, \dots, e_n\}$ be a basis for V_1 s.t. $\|e_i\| = 1$ $\forall i$.
~~Define $U = \text{span}\{e_1, \dots, e_n, u\}$.~~ Then $\forall v \in V_1$ $\exists!$ constants $\alpha_1, \dots, \alpha_n$ s.t. $v = \alpha_1 e_1 + \dots + \alpha_n e_n$.
For each i, define $\tilde{P}_i : V_1 \to \mathbb{R}$ (can be made $\mathbb{C}$ if needed) by

$$\tilde{P}_i(\alpha_1 e_1 + \dots + \alpha_n e_n) = \alpha_i$$

Then $\tilde{P}_i$ is linear and continuous by the continuity of basis expansions.* since V_1 is a linear subspace. Hahn Banach implies that $\tilde{P}_i$ extends to a bounded linear functional P_i on all of V. ~~we denote the extension again by P_i~~

Define $P : V \to V$ by

$$P(v) = \sum_{i=1}^{n} P_i(v) e_i$$

Then P is a continuous linear map since P_i is cont. + linear $\forall i$.
Additionally, since $V_1 = \text{span}\{e_1, \dots, e_n\}$, $\text{im} P \subset V_1$.
Finally, $\forall \alpha_1 e_1 + \dots + \alpha_n e_n \in V_1$,

$$P(\alpha_1 e_1 + \dots + \alpha_n e_n) = \sum_{i=1}^{n} \alpha_i e_i = \alpha_1 e_1 + \dots + \alpha_n e_n$$

Therefore $P = \text{id}$ on V_1, so $P^2 = P$ and $\text{im} P = V_1$. $\square$

* alternatively b/c linear on finite dimension $\to$ continuous

(5) 405 846 515

We note that since $f, \frac{\sin(t|\xi|)}{|\xi|} \in C_0$ and so their product is as well. Therefore we may write

$$u(x,t) = \left(\frac{\sin(t|\xi|)}{|\xi|} f(\xi)\right)^{\vee}(x)$$

In particular, this implies that by Plancherel,

$$\|u(\cdot,t)\|_{L^2} = \|\hat{u}(\xi,t)\|_{L^2(d\xi)}$$

$$= \left\| \frac{\sin(t|\xi|)}{|\xi|} f(\xi)\right\|_{L^2(d\xi)}$$

Stopped here.

We recall that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. and that $\frac{\sin x}{x}$ is decreasing on $[0,\pi/2]$ w/ $\frac{\sin x}{x} \geq \frac{2}{\pi}$.

Therefore, for $t|\xi| \leq \frac{\pi}{2}$, $\frac{\sin(t|\xi|)}{t|\xi|} \gtrsim 1$. Equivalently $\frac{\sin(t|\xi|)}{|\xi|} \gtrsim t \;\; \forall\, |\xi| \leq \frac{\pi}{2t}$.

Then

$$\|u(\cdot,t)\|_{L^2}^2 \geq \int_{|\xi|\leq\frac{\pi}{2t}} t^2 |f(\xi)|^2\, d\xi \gtrsim \operatorname*{ess\,inf}_{|\xi|\leq\frac{\pi}{2t}} |f(\xi)|^2$$

In particular,

$$\liminf_{t\to\infty} \|u(\cdot,t)\|_{L^2} \gtrsim \liminf_{|\xi|\to 0} |f(\xi)|$$

Therefore $\lim_{t\to\infty} \|u(\cdot,t)\|_{L^2} = \infty \;\; \forall f$ s.t. $\liminf_{|\xi|\to 0} |f(\xi)| = \infty$.

We claim that $\mathcal{F} = \{f : \liminf_{|\xi|\to 0} |f(\xi)| = \infty\}$ is dense in L^2.

Let $g \in L^2$. Consider $\varphi = |\xi|^{-1/2}\chi_{|\xi|\leq 1}$. Then

$$\|\varphi\|_{L^2}^2 = \int_{|\xi|\leq 1} |\xi|^{-1}\, d\xi = 2\pi < \infty$$

$$g + \varepsilon\varphi \in \mathcal{F}$$

and $\|g - (g+\varepsilon\varphi)\|_{L^2} = \|\varepsilon\varphi\|_{L^2} \sim \varepsilon$.

Therefore $\forall \varepsilon > 0$ $\forall g \in L^2$ $\exists h \in \mathcal{F}$ s.t. $\|g-h\|_{L^2} < \varepsilon$, so $\mathcal{F}$ is dense in L^2. $\square$

(8) 405 346 515

Let φ denote the Cayley transform $\varphi(z) = \frac{z-i}{z+i}$ which maps $\mathbb{C}_+$ conformally ~~H~~ to the unit disk $\mathbb{D}$. We note that $\varphi(\infty) = 1$.

Define $g_n = \varphi \circ f_n \circ \varphi^{-1} : \mathbb{D} \to \mathbb{D}$. Since $|g_n| < 1$, Montel's theorem implies that $\exists$ a subsequence g_{n_k} of g_n which converges uniformly on compact subsets to some holomorphic function g, $|g| \leq 1$.

~~We aim to bring this convergence back to~~

We claim that f_{n_k} also converges uniformly on compact subsets (potentially to ∞).

Fix a compact subset $K \subset \mathbb{C}_+$. Since φ is conformal, $\varphi(K) \subset \mathbb{D}$ is compact and so $\|g_{n_k} - g\|_{L^\infty} = \|\varphi \circ f_{n_k} \circ \varphi^{-1} - g\|_{L^\infty} \to 0$ on $\varphi(K)$.

~~Since $K, \varphi(K)$ are compact, it follows that φ, φ^{-1} respectively are uniformly bounded on $K, \varphi(K)$.~~

Then $\|\varphi \circ f_{n_k} - g \circ \varphi\|_{L^\infty} \to 0$ on K.

As noted, $\varphi(\infty) = 1$. Therefore if $g(\varphi(z)) \neq 1 \ \forall z \in K$, then $\varphi^{-1} \circ g \circ \varphi$ is holomorphic on K.

Suppose $\exists\ z \in K$ s.t. $g(\varphi(z)) = 1$.

(a) 405 846515

If f has infinite zeros in $\overline{\mathbb{D}}$, then they must have an accumulation point which would imply that $f \equiv 0$, contradicting ~~that~~ the fact that $|f|=1$ on $\partial\mathbb{D}$. Therefore f has finitely many zeros in $\overline{\mathbb{D}}$, which we enumerated $a_1, \dots, a_n$, repeated according to multiplicity. We note that since $|f|=1$ on $\partial\mathbb{D}$, $a_1, \dots, a_n \in \mathbb{D}$.

We recall the Blaschke factor

$$\varphi_a = \frac{z-a}{1-\bar{a}z}$$

for $a \in \mathbb{D}$. By ~~construction~~: φ is meromorphic on $\mathbb{C}$ w/ a simple zero at $a \in \mathbb{D}$ and a simple pole at $1/\bar{a} \in \mathbb{C} \setminus \overline{\mathbb{D}}$ if $a \neq 0$. Additionally, $\forall z \in \partial\mathbb{D}$,

$$|\varphi_a(z)| = \left|\frac{z-a}{1-\bar{a}z}\right| = \frac{1}{|z|}\left|\frac{z-a}{\frac{1}{z}-\bar{a}}\right|$$

$$= \left|\frac{z-a}{\bar{z}-\bar{a}}\right| = \left|\frac{z-a}{\overline{z-a}}\right| = 1$$

since $|z|=1$ implies $\bar{z} = \frac{1}{z}$. Therefore $|\varphi_a| = 1$ on $\partial\mathbb{D}$.

Consider the ~~quotient~~ $g = f / \prod_{i=1}^{n} \varphi_{a_i}$. Then g is holomorphic on a neighborhood of $\overline{\mathbb{D}}$ w/ no zeros in $\mathbb{D}$ and $|g|=1$ on $\partial\mathbb{D}$. By maximum modulus principle, $|g| \leq 1$ on $\mathbb{D}$. Since g is nonvanishing on a neighborhood of $\overline{\mathbb{D}}$, $1/g$ is holomorphic on $\overline{\mathbb{D}}$ w/ $|1/g| = 1/|g| = 1$ on $\partial\mathbb{D}$. Maximum modulus This implies that $|1/g| \leq 1$ on $\mathbb{D}$ and so $|g| \geq 1$ on $\mathbb{D}$.

Therefore $|g|=1$ on $\mathbb{D}$ $\Rightarrow$ $g \equiv c$ for some ~~constant~~ $|c|=1$.

Then $f = c\prod_{i=1}^{n} \varphi_{a_i}$ on $\mathbb{D}$. By meromorphic continuation, $f = c\prod_{i=1}^{n} \varphi_{a_i}$ on $\mathbb{C}$.

However, f is entire and φ_{a_i} has a pole at $1/\bar{a}_i$ if $a_i \neq 0$.

Therefore $a_i = 0$ $\forall i$ and so $f = cz^n$. $\square$

(10) 403 846 515

idea: Blaschke products remove all zeros but maintain $|f|\to\infty$ as $|z|\to 1$
Then f is nonvanishing $\Rightarrow$ $1/f$ is holomorphic and $|1/f|\to 0$ as $|z|\to 1$
By then $1/f = 0$ $\Rightarrow$ $f=\infty$.

Suppose that such an f exists.
Since $\lim_{|z|\to 1} |f(z)| = \infty$, $\exists\ R\in(0,1)$ s.t. $|f(z)|>0$ $\forall\ |z|>R$.
Therefore all the zeros of f are contained in $\overline{D(0,R)}$.
Since $\overline{D(0,R)}$ is closed, this implies that either f is identically zero or f has finitely many zeros. Since $|f|\to\infty$ as $|z|\to 1$, $f\not\equiv 0$ and so we may enumerate the zeros of f as $a_1,\dots,a_n$.
We recall the Blaschke products φ_a from problem 9.
Consider $g = f/\prod_{i=1}^{n}\varphi_{a_i}$. Since φ_{a_i} are holomorphic on ~~a nbhd~~ $\overline{\mathbb{D}}$ and $|\varphi_{a_i}|=1$ on $\partial\mathbb{D}$, $\lim_{|z|\to 1}|g| = \lim_{|z|\to 1}|f| = \infty$. Additionally, g is nonvanishing on $\mathbb{D}$. Therefore $1/g$ is holomorphic on $\mathbb{D}$ w/ $\lim_{|z|\to 1}|1/g| = 0$.
Then $\forall\ \varepsilon>0\ \exists\ R_1\in(0,1)$ s.t. $|1/g|<\varepsilon$ $\forall\ |z|\geq R$. By the maximum modulus principle, $|1/g|<\varepsilon$ $\forall\ |z|<R$. Therefore $|1/g|<\varepsilon$ on $\mathbb{D}$.
Taking $\varepsilon\to 0$ implies $|1/g|=0$ on $\mathbb{D}$.
Therefore $g\equiv\infty$ on $\mathbb{D}$. Since $\varphi_{a_i}\not\equiv 0$, this implies that $f\equiv\infty$ on $\mathbb{D}$. However this contradicts holomorphicity. Therefore no such f exists.
~~Therefore the only "holomorphic" functions f w~~

$\square$

(11) 405 846 5'5

Suppose on the contrary that

$$\max_{|z|=1} |f(z) - 1/z| < 1$$

Then

$$\max_{|z|=1} |z f(z) - 1| < 1$$

Rouché's theorem then implies that 1 and $1 + f(z)z - 1 = z f(z)$ have the same number of zeros in $\mathbb{D}$. Therefore $zf(z)$ has no zeros in $\mathbb{D}$ and in particular does not have a zero at 0. However, this implies that f has a simple pole at 0, which contradicts the fact that f is holomorphic on $\{|z|<2\}$. Therefore

$$\max_{|z|=1} |f(z) - 1/z| \geq 1$$

as desired. $\square$

(12) 405 846 515

(a) Define $u : \mathbb{H} \to \mathbb{R}_{>0}$ by $u(x+iy) = y$.

Then $\Delta u = u_{xx} + u_{yy} = 0$ and so u is harmonic on $\mathbb{H}$.

We recall the inverse Cayley transform

$$\varphi : z \longmapsto \frac{z+1}{iz-i}$$

which maps $\mathbb{D}$ conformally to $\mathbb{H}$.

Then $v : \mathbb{D} \to \mathbb{R}_{>0}$ defined by $v = u \circ \varphi$ is harmonic on $\mathbb{D}$. Moreover, by construction,

$$v(z) = \operatorname{Im}\left(\frac{z+1}{iz-i}\right)$$

$$\Rightarrow \lim_{z \to 1} v(z) = \operatorname{Im}\left(\lim_{z\to 1} \frac{z+1}{iz-i}\right) = \infty$$

as desired.

(b)

Nah

ANALYSIS

FALL 2015

① 405 846 515

Suppose that $K \subset \mathbb{R}^d$ is compact and $A \subset \mathbb{R}^d$ satisfies $|A| < \infty$.
Consider $f = \chi_A$. Then $\forall x \in K$,

$$|f * g_n(x)| \leq \int |g_n(x-y)||f(y)|dy$$
$$= \int_A |g_n(x-y)|dy$$
$$(z = x-y) \quad = \int_{x-A} |g_n|$$
$$(x \in K) \quad \leq \int_{K-A} |g_n|$$

As this bound is independent of $x \in K$ this implies

$$\|f * g_n\|_{sup(K)} \leq \int_{K-A} |g_n|$$

Since K is compact and $|A| < \infty$, it follows that $|K-A| < \infty$.
Therefore b/c $|g_n| \leq 1$, the DCT implies

$$\limsup_{n\to\infty} \|f * g_n\|_{sup(K)} \leq \int_{K-A} \lim_{n\to\infty} |g_n| = 0$$

and so $f * g_n \to 0$ uniformly on K.
By linearity of convolution and sublinearity of $\|\cdot\|_{sup}$, this extends to all simple functions.
Consider arbitrary $f \in L^1$. We recall that simple functions are dense in L^1, so $\exists$ simple function f_K s.t. $f_K \to f$ in L^1.
By Hölder's inequality, $\forall\ x \in K$

$$|f * g_n(x)| \leq |(f-f_K) * g_n(x)| + |f_K * g_n(x)|$$
$$\leq \|g_n\|_{L^\infty} \|f-f_K\|_{L^1} + \|f_K * g_n\|_{L^\infty(K)} \leq \|f-f_K\|_{L^1} + \|f_K * g_n\|_{L^\infty}$$

Therefore $\|f * g_n(x)\|_{sup(K)} \leq \|f-f_K\|_{L^1} + \|f_K * g_n\|_{L^\infty(K)}$. Since f_K is simple, taking $n \to \infty$ implies $\limsup_{n\to\infty} \|f * g_n\|_{sup(K)} \leq \|f-f_K\|_{L^1}$. Taking $k \to \infty$ then yields $\|f * g_n\|_{sup(K)} \to 0$. As this holds $\forall$ compact K, this concludes. ∎

(3) 405 846 515

$0<p<1$, $1/p>1$

if $g=\chi_{[a,b]}$

then $\left|\int_a^b f\,dx\right| \leq |b-a|^{1/p}$

as this holds $\forall$ intervals, $\left|\int_A f\,dx\right| \leq |A|^{1/p}$

Consider an open interval $(a,b)\subset\mathbb{R}$. We can approximate $\chi_{(a,b)}$ from below by $g_n\in C_0$ s.t. $g_n\nearrow\chi_{(a,b)}$. Explicitly, we construct

$$g_n(x) = \min(1, n\,d(x, \mathbb{R}\setminus(a,b)))$$

Since $\mathbb{R}\setminus(a,b)$ is closed, $0<g_n(x)\leq 1$ $\forall x\in(a,b)$ and $g_n(x)=0$ $\forall x\notin(a,b)$. By construction, we then see that $g_n\nearrow\chi_{(a,b)}$.

By the MCT, this implies

$$\left|\int_a^b f\right| = \lim_{n\to\infty}\left|\int f g\right| \leq \lim_{n\to\infty}\left(\int|g_n|^p\right)^{1/p} = \left(\int\chi_{(a,b)}\right)^{1/p} = (b-a)^{1/p}$$

for all intervals $(a,b)\subset\mathbb{R}$.

Now suppose that y is a Lebesgue point of f. Then

$$|f(y)| = \lim_{h\to 0}\frac{1}{2h}\left|\int_{y-h}^{y+h} f\right| \leq \liminf_{h\to 0}\frac{1}{2h}(2h)^{1/p} = \liminf_{h\to 0}(2h)^{1/p-1}$$

Since $0<p<1$, $1/p>1$ and so $\lim_{h\to 0}(2h)^{1/p-1}=0$.

Therefore $f(y)=0$ $\forall$ Lebesgue points y of f. Since $f\in L^1_{loc}$, a.e. y is a Lebesgue point, so $f(y)=0$ a.e.

$\square$

(4) 405 846 515

We note that (b) => (a), so it suffices to show (b).

after time

Suppose that $\sum_{n=1}^{\infty} \|f_n - e_n\|^2 = K < \infty$. If $K=0$ then $f_n = e_n$ and we're done. So assume $K>0$.

To show that $\{e_n\}$ is complete, it suffices to show that if $\exists\, g \in \mathcal{H}$ s.t. $\langle g, e_n\rangle = 0\ \forall n$, then $g=0$.

Suppose that such a g exists. Then

$$\|g\|^2 = \sum_{n=1}^{\infty} |\langle g, f_n\rangle|^2$$
$$= \sum_{n=1}^{\infty} |\langle g, e_n\rangle + \langle g, f_n - e_n\rangle|^2$$
$$= \sum_{n=1}^{\infty} |\langle g, f_n - e_n\rangle|^2$$
$$\le \|g\|^2 \sum_{n=1}^{\infty} \|f_n - e_n\|^2$$
$$\le K\|g\|^2$$

$\hat{g} = \sum \langle g, e_n\rangle e_n = 0$

$\|g - \hat{g}\|^2 = \sum_{n=1}^{\infty} |\langle g - \hat{g}, f_n\rangle|^2$

$= \sum$

~~Suppose $g \neq 0$. Then by scaling g by κ, the above calculation yields~~

$$\kappa\|g\| \le K\kappa^2\|g\|^2$$

$$\Rightarrow \frac{1}{K\|g\|} \le \kappa$$

~~Since κ is arbitrary, this bound is impossible. Therefore we have a contradiction and $g=0$.~~

~~Then $\{e_n\}$ is complete.~~

Taking $K=1$ yields part a. $\square$

⑥ 405 846 515

$$\underbrace{\iint \frac{|u(x+y)-u(x)|^2}{|y|^{d+1}}\,dx\,dy}_{\text{Plancherel}}$$

$$\iint \frac{|e^{2\pi i y\cdot\xi}\hat{u}(\xi)-\hat{u}(\xi)|^2}{|y|^{d+1}}\,d\xi\,dy$$

Let

We recall that $[u(x+y)]^\wedge(\xi) = e^{2\pi i y\cdot\xi}\hat{u}(\xi)$ where the Fourier transform is taken in x. Therefore by Plancherel,

$$\iint \frac{|u(x+y)-u(x)|^2}{|y|^{d+1}}\,dx\,dy = \iint \frac{|e^{2\pi i y\cdot\xi}-1|^2|\hat{u}(\xi)|^2}{|y|^{d+1}}\,d\xi\,dy$$

Fubini then implies that

$$\iint \frac{|u(x+y)-u(x)|^2}{|y|^{d+1}}\,dx\,dy = \int |\hat{u}(\xi)|^2 \int \frac{|e^{2\pi i y\cdot\xi}-1|^2}{|y|^{d+1}}\,dy\,d\xi$$

$$= \int |\hat{u}(\xi)|^2 \int \frac{|e^{\pi i y\cdot\xi}-e^{-\pi i y\cdot\xi}|^2}{|y|^{d+1}}\,dy\,d\xi$$

$$\sim \int |\hat{u}(\xi)|^2 \int \frac{\sin^2(\pi y\cdot\xi)}{|y|^{d+1}}\,dy\,d\xi$$

Consider only the inner integral. Up to a rotation, we may assume that $\xi = (0,\dots,|\xi|)$ so that $y\cdot\xi = |\xi| y_d$. Then

$$\int \frac{\sin^2(\pi y\cdot\xi)}{|y|^{d+1}}\,dy = \int \frac{\sin^2(\pi|\xi|y_d)}{|y|^{d+1}}\,dy$$

$(z = |\xi|y)$

$$= \int \frac{\sin^2(\pi z_d)}{|z|^{d+1}}\frac{|\xi|^{d+1}}{|\xi|^d}\,dz$$

$$= |\xi| \int \frac{\sin^2(\pi z_d)}{|z|^{d+1}}\,dz$$

We note that $\dfrac{\sin^2(\pi z_d)}{|z|^{d+1}} \lesssim \dfrac{1}{|z|^{d+1}}$ and so it is integrable near ∞.

Near 0, $\sin^2(\pi z_d) \lesssim |z|^2$, so $\frac{\sin^2(\pi z_d)}{|z|^{d+1}} \lesssim \frac{1}{|z|^{d-1}}$

and so it is integrable near 0.

Therefore $\int \frac{\sin^2(\pi y\cdot\xi)}{|y|^{d+1}}\,dy \sim |\xi|$ which implies

$$\iint \frac{|u(x+y)-u(x)|^2}{|y|^{d+1}}\,dx\,dy \sim \| |\xi|^{1/2}\hat{u}(\xi)\|_{L^2}^2$$

This implies that $\iint \frac{|u(x+y)-u(x)|^2}{|y|^{d+1}}\,dx\,dy<\infty$ iff $|\xi|^{1/2}\hat{u}(\xi)\in L^2$.

Therefore for $u\in L^2$, $(1+|\xi|^{1/2})\hat{u}\in L^2$ iff $\iint \frac{|u(x+y)-u(x)|^2}{|y|^{d+1}}\,dx\,dy<\infty$,

~~which concludes~~, since $u\in L^2 \iff \hat{u}\in L^2$ by Plancherel.

$\square$

⑦ 405 846 515

Define $g : \mathbb{C} \to \mathbb{C}$ by

$$g(z) = \begin{cases} f(z) & z \in \overline{\mathbb{D}} \\ f(1/z) & z \notin \overline{\mathbb{D}}. \end{cases}$$

By the given symmetry of f, g is continuous on $\mathbb{C}$.
Since $1/z$ is holomorphic except at 0 and f is holomorphic, g is holomorphic on $\mathbb{C} \setminus \partial\mathbb{D}$.

Following the same proof as the Schwarz reflection principle, g is entire by Morera's theorem.
By construction, $\sup_{z \in \mathbb{C}} |g(z)| = \sup_{z \in \overline{\mathbb{D}}} |f(z)| < \infty$ since f is continuous on $\overline{\mathbb{D}}$. Therefore g is entire and bounded so it is constant. Then f is constant. $\square$

(8) 405 846 515

~~$g(z) = f\left(\frac{\log z}{i}\right)$ well-defined ✓~~

~~$|g(x+iy)| = \left|f\left(\frac{1}{i}\ln\sqrt{x^2+y^2} + i\,\mathrm{Arg}(x+iy)\right)\right|$~~

~~$= |f(\arg(x+iy) - i\ln\sqrt{x^2+y^2})|$~~

~~$\leq |e^{-\kappa\ln\sqrt{x^2+y^2}}|$~~

~~$= r^{\kappa}$~~

~~$\leq |z|^{\kappa} \leq |z| \Rightarrow g = az$~~

~~$f\left(\frac{\log(z)}{i}\right) = az$~~

B/c f is 2π-periodic, $g(z) = f\left(\frac{\log z}{i}\right)$ is well-defined and entire.
Then by direct computation, $\forall z$,

$$\frac{\log(z)}{i} = \frac{1}{i}(\ln|z| + i\arg z) = \arg z - i\ln|z|$$

$$\Rightarrow |g(z)| \leq e^{\kappa\ln|z|} = |z|^{\kappa}$$

Since $\kappa<1$, this implies $|g(z)| \leq |z|$.
In particular, $|g(z)|$ has a zero at 0, so $\frac{g(z)}{z}$ is entire.
Then $\left|\frac{g(z)}{z}\right| \leq 1$ and ~~so~~ Liouville's implies $g(z) = az$.
B/c $|g(z)| \leq |z|^{\kappa}$ for ~~$0<\kappa$~~ $\kappa<1$, this implies that $a=0$ and so $g \equiv 0$.
By construction, this implies that $f=0$ and hence is constant. □

Need to account for pole at origin, but that is immediate.

(9) 405 846 515

We first show that (f_j) is uniformly bounded on compact subsets of $\mathbb{C}$. To show this, it suffices to show (f_j) is uniformly bounded on $\overline{D(0,R)}$ $\forall$ $R>0$.

Consider $w \in \overline{D(0,R)}$ for fixed R. By the mean value theorem, $\forall j$

$$|f_j(w)| = \left| \iint_{\overline{D(w,R)}} f_j(z)\,dx\,dy \right|$$

$$\leq \iint_{\overline{D(w,R)}} |f_j(z)|\,dx\,dy$$

(Hölder's) $$\leq \|f_j\|_{L^2(\overline{D(w,R)})} \|1\|_{L^2(\overline{D(w,R)})}$$

$$\lesssim_R \|f_j\|_{L^2(\overline{D(w,R)})}$$

For all $w \in \overline{D(0,R)}$, $\overline{D(w,R)} \subset \overline{D(0,2R)}$. Therefore

$$|f_j(w)| \lesssim_R \|f_j\|_{L^2\{|z|\leq 2R\}}$$

On $\{|z| \leq 2R\}$, $e^{-|z|^2} \gtrsim_R 1$. Therefore

$$|f_j(w)| \lesssim_R \|e^{-|z|^2/2} f_j\|_{L^2\{|z|\leq 2R\}} \leq C$$

As this holds $\forall w, j$, (f_j) is uniformly bounded on $\overline{D(0,R)}$ $\forall R>0$.
Therefore (f_j) is uniformly bounded on compact subsets.
Montel's theorem then implies that f_{jk} converges uniformly on compact subsets to an entire function f, for some subsequence (f_{jk}) of (f_j).

We claim that

$$\iint |f_{jk}(z) - f(z)| e^{-2|z|^2}\,dx\,dy \to 0 \quad \text{as } k \to \infty.$$

By Fatou's lemma, since $f_{jk} \to f$ pointwise,

$$\iint |f|^2 e^{-|z|^2} dx dy \le \liminf_{k \to \infty} \iint |f_{jk}|^2 e^{-|z|^2} dx dy \le C$$

Therefore $\|f_{jk} e^{-|z|^2/2}\|_{L^2}, \|f e^{-|z|^2/2}\|_{L^2} \le \sqrt{C}$ $\forall k$. *

~~Since bounded~~ Fix $\varepsilon > 0$. Since bounded in L^2 implies uniform integrability, $\exists R > 0$ s.t. $\|f_{jk} e^{-|z|^2/2}\|_{L^2\{|z|>R\}} < \varepsilon$ and similar for j.
Therefore

$$\|(f_{jk} - f) e^{-|z|^2/2}\|_{L^2} \le \|(f_{jk} - f) e^{-|z|^2/2}\|_{L^2\{|z| \le R\}} + \|f_{jk} e^{-|z|^2/2}\|_{L^2\{|z|>R\}} + \|f e^{-|z|^2/2}\|_{L^2\{|z|>R\}}$$

$$\le \|(f_{jk} - f) e^{-|z|^2/2}\|_{L^2\{|z| \le R\}} + 2\varepsilon$$

B/c $\{|z| \le R\}$ is compact, $f_{jk} \rightrightarrows f$ on $\{|z| \le R\}$. Therefore $f_{jk} e^{-|z|^2/2} \to f e^{-|z|^2/2}$ in $L^2\{|z| \le R\}$. Therefore

$$\limsup_{k \to \infty} \|(f_{jk} - f) e^{-|z|^2/2}\|_{L^2} \le 2\varepsilon$$

Taking $\varepsilon \to 0$ then implies that

$$\iint |f_{jk} - f|^2 e^{-|z|^2} dx dy \to 0 \quad \text{as } k \to \infty$$

Because $e^{-|z|^2} \ge e^{-2|z|^2}$, this concludes. □

* This part fails. Instead, note that $\forall R > 0$,

$$\iint_{\{|z|>R\}} |f_{jk} - f|^2 e^{-2|z|^2} dx dy \le e^{-R} \iint_{\{|z|>R\}} |f_{jk} - f|^2 e^{-|z|^2} dx dy$$

$$\le 2C e^{-R} \to 0 \quad \text{as } R \to \infty$$

This concludes by the same reasoning.

Bounded in L^p doesn't mean shit, smh.
Unless your measure is finite. Then uniform integrability.

DON'T IGNORE WEIRDNESS

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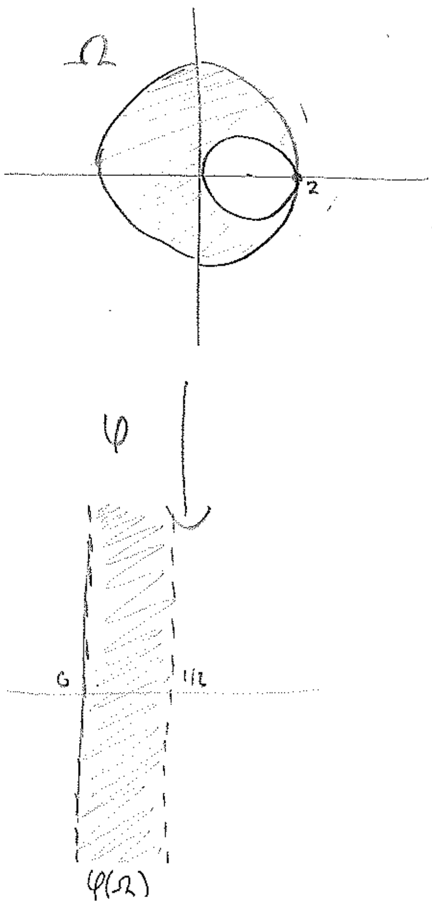

Let $\Omega = \{ |z|<2,\ |z-1|>1 \}$.

Consider the Möbius transformation
$$\varphi(z) = \frac{z}{z-2}.$$
Since $\varphi(2)=\infty$ and Möbius transformations preserve generalized circles, $\{|z|=2\}$ and $\{|z-1|=1\}$ are mapped to ~~straight~~ parallel straight lines.

Since $\varphi: 0 \mapsto 0$, $-2 \mapsto 1/2$ and $2 \mapsto \infty$, φ takes $\mathbb{R}$ to $\mathbb{R}$. Conformality then implies that $\{|z|=2\}$ and $\{|z-1|=1\}$ are mapped to vertical lines at $1/2$ and 0 respectively.

Since Ω is connected, this implies that $\varphi(\Omega)$ is the strip $\{0 < \mathrm{Im}(z) < 1/2\}$

Define u on $\varphi(\Omega)$ by
$$u(x+iy) = 2y$$
Then u is harmonic, ~~and~~ 0 when $y=0$ and 1 when $y=1/2$.

Define v on Ω by $u \circ \varphi^{-1}$. Then v is harmonic since u is harmonic and φ is conformal and v ~~sat~~ satisfies the desired boundary condition.

For explicitness, we compute
$$\varphi^{-1}(z) \cong \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \cong \left(\frac{2z}{-z+1}\right)\frac{1}{2}$$
Therefore
$$v(z) = \mathrm{Im}\left(\frac{2z}{1-z}\right)$$
is the desired map.

$\square$

Analysis

Spring 2015

① 405 846 515

We first claim that it suffices to show the result for a dense subset of L^1.

By the triangle inequality, $\forall n$,

$$\sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{\frac{k+1}{n}} f \right| \leq \sum_{k=-n^2}^{n^2} \int_{k/n}^{\frac{k+1}{n}} |f| = \int_{-n}^{n} |f| \leq \int |f| \qquad (0)$$

Therefore, $\forall g \in L^1$

$$\left| \int |f| - \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{\frac{k+1}{n}} f \right| \right| = \int |f| - \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{\frac{k+1}{n}} f \right|$$

$$\leq \int |f-g| + \int |g| - \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{\frac{k+1}{n}} g \right| + \sum_{k=-n^2}^{n^2} \left(\left| \int_{k/n}^{\frac{k+1}{n}} g \right| - \left| \int_{k/n}^{\frac{k+1}{n}} f \right| \right)$$

(reverse triangle inequality) $\leq \|f-g\|_{L^1} + \int |g| - \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{\frac{k+1}{n}} g \right| + \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{\frac{k+1}{n}} (g-f) \right|$

(triangle inequality) $\leq \|f-g\|_{L^1} + \int |g| - \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{\frac{k+1}{n}} g \right| + \int_{-n}^{n} |g-f|$

$$\leq 2\|f-g\|_{L^1} + \int |g| - \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{\frac{k+1}{n}} g \right| \qquad (1)$$

Suppose that the result holds on a dense subset of L^1. Then $\forall f \in L^1$, $\exists g_k \in L^1$ s.t. the result holds for g_k and $\|g_k - f\|_{L^1} \to 0$. Taking $n \to \infty$ and then $k \to \infty$ in ~~the~~ equation (1) above then extends the result to f. Therefore it suffices to show the result on a dense subset of L^1.

$\longrightarrow$

We claim that the result holds $\forall\ g \in C_c(\mathbb{R})$.

Fix some $g \in C_c(\mathbb{R})$. ~~and~~ Then $\exists\ m \geq 1$ s.t. $\operatorname{supp} g \subset [-m, m]$.

Then $\forall\ n \geq m$,

$$\sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{\frac{k+1}{n}} g \right| = \sum_{k=-mn}^{mn} \left| \int_{k/n}^{\frac{k+1}{n}} g \right|$$

~~Since $g, |g| \in C_c$, we recall that their Riemann sums converge to their integral. That is to say that~~

$$\cancel{\int g = \int_{-m}^{m} g = \lim_{n\to\infty} \sum}$$

Since $g, |g| \in C_c$, they are uniformly continuous.

Therefore $\forall\ \varepsilon > 0\ \exists\ N$ s.t. $\forall\ n \geq N$, $|g(x) - g(y)| < \varepsilon$ if $|x-y| < 1/n$.

By the reverse triangle inequality, this implies $||g(x)| - |g(y)|| < \varepsilon$ if $|x-y| < 1/n$

Then $\forall\ n \geq N$,

$$\begin{aligned}
\sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{\frac{k+1}{n}} g \right| &= \sum_{k=-mn}^{mn} \left| \int_{k/n}^{\frac{k+1}{n}} g \right| \\
&\geq \sum_{k=-mn}^{mn} \left(\left| \int_{k/n}^{\frac{k+1}{n}} g(k/n) \right| - \left| \int_{k/n}^{\frac{k+1}{n}} g(x) - g(k/n) \right| \right) \\
&\geq \sum_{k=-mn}^{mn} \left(\frac{1}{n} |g(k/n)| - \frac{\varepsilon}{n} \right) \\
&= \sum_{k=-mn}^{mn} \frac{1}{n} |g(k/n)| - 2m\varepsilon
\end{aligned}$$

Since $|g| \in C_c$, its Riemann sums converge to its integral.

Therefore $\sum_{k=-mn}^{mn} \frac{1}{n} |g(k/n)| \to \int_{-m}^{m} |g|$ as $n \to \infty$.

Then $\forall\ \varepsilon > 0$,

$$\liminf_{n\to\infty} \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{\frac{k+1}{n}} g \right| \geq \int |g| - 2m\varepsilon$$

Taking $\varepsilon \to 0$ yields $\liminf_{n\to\infty} \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{\frac{k+1}{n}} g \right| \geq \int |g|$.

By equation (0), this implies $\lim_{n\to\infty} \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{\frac{k+1}{n}} g \right| = \int |g|$. Therefore the

result holds $\forall$ $g \in C_c(\mathbb{R})$.
Since $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, this concludes. $\square$

② 405 846 515

Decomposing $\|fg\|_{L^1}$ into dyadic annuli, we find that

$$\int_{\mathbb{R}^n} |fg| = \int_{\{|x|<1\}} |fg| + \sum_{k\geq 0} \int_{2^k \leq |x| < 2^{k+1}} |fg|$$

By Hölder's inequality, since $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$, this implies

$$\int |fg| \leq \left(\int_{|x|<1} |f|^2\right)^{1/2} \left(\int_{|x|<1} |g|^3\right)^{1/3} \left(\int_{|x|<1} 1^6\right)^{1/6}$$
$$+ \sum_{k\geq 0} \left(\int_{2^k \leq |x| < 2^{k+1}} |f|^2\right)^{1/2} \left(\int_{2^k \leq |x| < 2^{k+1}} |g|^3\right)^{1/3} \left(\int_{2^k \leq |x| < 2^{k+1}} 1^6\right)^{1/6} \qquad (1)$$

Since $f \in L^2_{loc}$ and $g \in L^3_{loc}$, $\left(\int_{|x|<1} |f|^2\right)^{1/2} \leq \left(\int_{|x|\leq 1} |f|^2\right)^{1/2} < \infty$
and $\left(\int_{|x|<1} |g|^3\right)^{1/3} \leq \left(\int_{|x|\leq 1} |g|^3\right)^{1/3} < \infty$. We note $\left(\int_{|x|<1} 1\right)^{1/6} = |B(0,1)|^{1/6} < \infty$.

~~Let~~ I denote the second term in equation (1).
Then by the given assumption, and since $|\{2^k \leq |x| < 2^{k+1}\}| \sim 2^{nk}$

$$I \leq \sum_{k\geq 0} (2^{ka})^{1/2} (2^{kb})^{1/3} \left(|\{2^k \leq |x| < 2^{k+1}\}|\right)^{1/6}$$
$$\lesssim \sum_{k\geq 0} 2^{\frac{ka}{2}} 2^{\frac{kb}{3}} 2^{\frac{nk}{6}}$$
$$= \sum_{k\geq 0} (2^{1/6})^{(3a+2b+n)k}$$

B/c $2^{1/6} > 1$ and $3a+2b+n < 0$, this implies $I < \infty$.
Therefore $\int |fg| < \infty$ and so $fg \in L^1$

□

(3) 405 846 515

(a) We recall the Hardy-Littlewood maximal theorem, which states that

$$m(\{Mg > s\}) \leq \frac{a_n}{s} \|g\|_{L^1}. \qquad (1)$$

Fix some $s > 0$ and $f \in L^1_{loc}(\mathbb{R})$. Define

$$l = |f| \chi_{\{|f| \leq s/2\}}$$

$$u = |f| \chi_{\{|f| > s/2\}}$$

Then $|f| = l + u$. By the triangle inequality, M is a subadditive operator. Therefore $Mf \leq Ml + Mu$. Then

$$\{Mf > s\} \subset \{Ml + Ms > s\}$$

If $Ml + Mu > s$, then $Ml > s/2$ or $Mu > s/2$. Therefore

$$\{Mf > s\} \subset \{Ml > s/2\} \cup \{Mu > s/2\}$$

By construction, $l \leq s/2$. Therefore $Ml \leq s/2$ and so $\{Ml > s/2\} = \emptyset$.
Monotonicity and (1) then imply

$$|\{Mf > s\}| \leq |\{Mu > s/2\}| \leq \frac{2a_n}{s} \|u\|_{L^1}$$

By definition of u, this implies

$$|\{Mf > s\}| \leq \frac{2a_n}{s} \int_{|f| > s/2} |f|$$

Taking $C_n = 2a_n$ then concludes.

$\longrightarrow$

(b) Since $\varphi(0)=0$ and $\varphi'>0$,

$$\varphi : [0,\infty) \to (0,\infty)$$

is a non-decreasing bijection. Then by the Layer Cake decomposition, we since $Mf \geq 0$,

$$\int_{\mathbb{R}} \varphi(Mf)\,dx = \int_0^\infty |\{\varphi(Mf) > s\}|\,ds$$

$$= \int_0^\infty |\{Mf > \varphi^{-1}(s)\}|\,ds$$

By the previous part,

$$\int \varphi(Mf)\,dx \leq \int_0^\infty \frac{C_n}{\varphi^{-1}(s)} \int_{\{|f| > \varphi^{-1}(s)/2\}} |f(x)|\,dx\,ds$$

By Fubini,

$$= C_n \int_{\mathbb{R}} |f(x)| \int_0^\infty \frac{1}{\varphi^{-1}(s)} \chi_{\{|f| > \varphi^{-1}(s)/2\}}\,ds\,dx$$

Making the change of variables $t = \varphi^{-1}(s)$ or $s = \varphi(t)$,

$$\int \varphi(Mf)\,dx \leq C_n \int_{\mathbb{R}} |f(x)| \int_0^\infty \frac{\varphi'(t)}{t} \chi_{\{2|f| > t\}}\,dt\,dx$$

$$= C_n \int_{\mathbb{R}} |f(x)| \int_{\{0 < t < 2|f|\}} \frac{\varphi'(t)}{t}\,dt$$

As claimed.

$\square$

(4) 405 846 515

Let $\mathcal{F} \subset L^1(0,2\pi)$ denote the linear combinations of translations $f(x-a)$.

($\Rightarrow$) Suppose $\mathcal{F}$ is dense in $L^1(0,2\pi)$.

Then $\forall k \in \mathbb{Z}$, $\exists$ a sequence $f(x-a_n) \in \mathcal{F}$ s.t. $f(x-a_n) \to e^{ikx}$ in L^1. Taking the Fourier transform, we find that $\forall m \in \mathbb{Z}$

$$\widehat{f(\cdot - a_n)}(m) = e^{-ima_n}\hat{f}(m)$$

$$\widehat{e^{ik\cdot}}(m) = \begin{cases} 0 & m \neq k \\ 1 & m = k \end{cases}$$

Since $f(x-a_n) \to e^{ikx}$ in L^1, the Fourier transforms converge in l^∞.

In particular, $e^{-ika_n}\hat{f}(k) \to 1$ as $n \to \infty$. Therefore $\hat{f}(k) \neq 0$ $\forall k \in \mathbb{Z}$.

($\Leftarrow$) Suppose that $\hat{f}(k) \neq 0$ $\forall k$. To show that $\mathcal{F}$ is dense in $L^1(0,2\pi)$, Hahn-Banach implies that it suffices to show that if $\varphi \in (L^1(0,2\pi))^*$ satisfies $\varphi = 0$ on $\mathcal{F}$ then $\varphi \equiv 0$.

By Riesz representation it suffices to show that if $\varphi \in L^\infty(0,2\pi)$ satisfies $\int \varphi g\,dx = 0$ $\forall g \in \mathcal{F}$ then $\varphi \equiv 0$.

Fix $\varphi \in L^\infty(0,2\pi)$ s.t. $\int \varphi g\,dx = 0$ $\forall g \in \mathcal{F}$. Then $\forall a$,

$$\int \varphi(x) f(x-a)\,dx = 0$$

$$\Rightarrow \quad \varphi * f(a) = 0 \quad \forall a$$

Taking a Fourier transform, this implies

$$0 = \widehat{\varphi * f}(k) = \hat{\varphi}(k)\hat{f}(k)$$

Since $\hat{f}(k) \neq 0$ $\forall k$, this implies $\hat{\varphi} \equiv 0$. Therefore $\varphi \equiv 0$ as desired. $\square$

By earlier reasoning, this concludes.

⑤ 405 846 515 - MY INITIAL ATTEMPT

To show that U is well-defined on $\mathbb{R}$, it must be shown that $|U| < \infty \ \forall x, \xi \in \mathbb{R}$. ~~To show this, it suffices to show that $U \in L^1_{loc}(\mathbb{R}^2)$.~~

~~To show $U \in L^1_{loc}(\mathbb{R}^2)$, it suffices to show that~~

$$\cancel{\|U(x,\xi)\|_{L^1(\mathbb{R}\times[-n,n])} < \infty}$$

~~$\forall n \geq 0$.~~ By the triangle inequality and Hölder's, $\forall x, \xi$,

$$|U(x,\xi)| \leq \int |e^{-(x+i\xi-y)^2/2}||u(y)|dy$$

$$= \int e^{-Re(x+i\xi-y)^2/2}|u(y)|dy$$

(Hölder's) $$\leq \|u\|_{L^2}\left(\int e^{-(x-y)^2}dy\right)^{1/2}$$

(translational invariance) $$= \|u\|_{L^2}\left(\int e^{-y^2}dy\right)^{1/2}$$

$$< \infty$$

since the Gaussian $e^{-y^2} \in L^1$. Therefore U is well-defined and uniformly bounded by some $M > 0$.

We now compute $\iint |U|^2 e^{-\xi^2} dx\, d\xi$ explicitly.

~~Since $|U|$ is uniformly bounded, $|U|^2 e^{-\xi^2}$ is absolutely~~

~~For now, we work w/ the assumption that Fubini's theorem may be applied.~~

⟶

By definition,

$$I = \iint |U(x,\xi)|^2 e^{-\xi^2} dx\, d\xi = \iint U(x,\xi)\,\overline{U(x,\xi)}\, e^{-\xi^2} dx\, d\xi$$

$$= \iint \left(\int e^{-(x+i\xi-y)^2/2} u(y)\,dy\right)\overline{\left(\int e^{-(x+i\xi-z)^2/2} u(z)\,dz\right)} e^{-\xi^2} dx\, d\xi$$

$$= \iiiint e^{-(x+i\xi-y)^2/2} e^{-(x-i\xi-z)^2/2} u(y)u(z)\,dy\,dz\, e^{-\xi^2} dx\, d\xi$$

$$= \iiiint \exp\left(\frac{-1}{2}\left((x-y)^2+(x-z)^2+2i(z-y)\xi-2\xi^2\right)\right) u(y)u(z)\,dy\,dz\, e^{-\xi^2} dx\, d\xi$$

We claim that Fubini's theorem can be applied. Assuming this claim,

$$I = \iiint\left(\int e^{-\frac{((x-y)^2+(x-z)^2)}{2}} dx\right) e^{-i(z-y)\xi} u(y)u(z)\,dy\,dz\,d\xi$$

Consider only the ~~innermost integral~~ innermost integral.

By completing the square,

$$e^{-\frac{(x-y)^2+(x-z)^2}{2}} = e^{\frac{-1}{2}(x^2-2xy+y^2+x^2-2xz+z^2)}$$

$$= e^{\frac{-1}{2}(2x^2-2x(y+z)+y^2+z^2)}$$

$$= e^{-\frac{y^2}{2}} e^{-\frac{z^2}{2}} e^{-\frac{(y+z)^2}{4}} e^{-(x-\frac{y+z}{2})^2}$$

Then

$$\int e^{\frac{-(x-y)^2+(x-z)^2}{2}} dx = \pi\, e^{\frac{-1}{2}(2y^2+2z^2+(y+z)^2)}$$

and so

$$I = \pi \iiint e^{\frac{-1}{2}(2y^2+2z^2+(y+z)^2)} e^{-i(z-y)\xi} u(y)u(z)\,dy\,dz\,d\xi$$

I do not have time ~~f~~ to finish this problem.
What should happen is that the $e^{-i(z-y)\xi}$ term will be used ~~to~~ along w/ the integrals over dy, dz to change $u(y)$ and $u(z)$ into $|\hat{u}(\xi)|^2$. Plancherel along w/ the integral over $d\xi$ will then conclude.

~~I have not figured ou~~

(5) 405 846 515 - FINAL

By definition, for all $u \in L^2$,

$$U(x,\xi) = \int e^{-(x+i\xi-y)^2/2} u(y)\,dy$$

$$= e^{-\frac{1}{2}(x+i\xi)^2+\frac{x^2}{2}} \int e^{-\frac{(y-x)^2}{2}} e^{+i\xi y} u(y)\,dy$$

Since $u \in L^2$ and $e^{-\frac{(y-x)^2}{2}} \in L^2$ $\forall x$, the product $e^{-\frac{(y-x)^2}{2}} u(y) \in L^1$.

Therefore by definition of the Fourier transform,

$$U(x,\xi) = e^{\frac{\xi^2}{2} - ix\xi} \sqrt{2\pi}\, \left[e^{-\frac{(\cdot-x)^2}{2}} u(\cdot)\right]^\wedge(-\xi)$$

In particular, this implies that $U(x,\xi)$ is well-defined.

Moreover,

$$\iint |U(x,\xi)|^2 e^{-\xi^2} dx\,d\xi = 2\pi \iint \left|\left[e^{-\frac{(\cdot-x)^2}{2}} u(\cdot)\right]^\wedge(-\xi)\right|^2 dx\,d\xi$$

$$(\text{Tonelli's}) \quad = 2\pi \iint \left|\left[e^{-\frac{(\cdot-x)^2}{2}} u(\cdot)\right]^\wedge(-\xi)\right|^2 d\xi\,dx$$

$$= 2\pi \int \left\|\left[e^{-\frac{(\cdot-x)^2}{2}} u(\cdot)\right]^\wedge\right\|^2_{L^2(\xi)} dx$$

$$(\text{Plancherel}) \quad = 2\pi \int \left\| e^{-\frac{(y-x)^2}{2}} u(y)\right\|^2_{L^2(dy)} dx$$

$$= 2\pi \iint e^{-(y-x)^2} |u(y)|^2\,dy\,dx$$

$$(\text{Tonelli's}) \quad = 2\pi \iint |u(y)|^2 e^{-(y-x)^2}\,dx\,dy$$

$$= 2\pi^{3/2} \int |u(y)|^2\,dy$$

As desired. $\square$

⑥ 405 846 515

For all $n \geq 1$, since T is compact,

$$K_n = T(B(0,n)) \subset B_2$$

is sequentially pre-compact. Therefore $\overline{K_n}$ is sequentially compact. ~~b/c B_2 is~~ since B_2 is a metric space, this implies that $\overline{K_n}$ is compact.

Consider the open cover $\{B_{1/n}(T(x))\}_{x \in \overline{K_n}}$ of $\overline{K_n}$. Since $\overline{K_n}$ is compact, $\exists$ a finite subcover $B_{1/n}(T(x_1^n)), \ldots, B_{1/n}(T(x_{k_n}^n))$.

We claim that $A = \{T(x_i^n) : n \geq 1, i = 1, \ldots, k_n\}$ is a countable dense subset of $T(B_1)$.

To show this, it suffices to show that $\forall x \in B_1$, $\forall \varepsilon > 0$ $\exists T(y) \in A$ s.t. $d(T(x), T(y)) < \varepsilon$

Fix some $x \in B_1$ and $\varepsilon > 0$. Then ~~for suff~~ $\exists$ $n \geq 1$ sufficiently large so that $x \in B(0,n)$ and $1/n < \varepsilon$.

Since $\overline{K_n}$ is covered by $B_{1/n}(T(x_1^n)), \ldots, B_{1/n}(T(x_{k_n}^n))$,

$\exists$ some $j \in \{1, \ldots, k_n\}$ s.t. $T(x) \in B_{1/n}(T(x_j^n))$. ~~and~~ Then

$T(x_j^n) \in A$ and $d(T(x), T(x_j^n)) < 1/n < \varepsilon$.

Since A is the countable union of finite points, A is countable.

Therefore A is a countable dense subset of $T(B_1)$. $\square$

⑦ 405 846 515

We aim to apply Montel's theorem.

We recall the Cayley transform $\varphi(z) = \frac{z-i}{z+i}$ which conformally maps the upper half plane $\mathbb{C}^+$ to the unit ~~disk~~ disk $\mathbb{D}$.

Consider the sequence of functions $(\varphi \circ f_n)$. By definition,

$$\varphi \circ f_n : \mathbb{D} \to \mathbb{D} \qquad \forall n$$

and since $\varphi(0) = -1$, $\varphi \circ f_n(0) \to -1$ as $n \to \infty$.

Since $\mathbb{D}$ is bounded, $(\varphi \circ f_n)$ is uniformly bounded.

Therefore $(\varphi \circ f_n)$ is a normal family of holomorphic functions.

This implies that all subsequences of $(\varphi \circ f_n)$ admit a further subsequence which converges uniformly on compact subsets of $\mathbb{D}$.

We recall the Urysohn subsequence lemma which states that: if every subsequence admits a subsequence which converges to some fixed limit, then the sequence itself converges to said limit.

Therefore, to show that $\varphi \circ f_n$ converges uniformly on compact sets, it suffices to show that all convergent subsequences converge to the same limit.

Suppose that $\varphi \circ f_{n_k}$ is a subsequence which converges uniformly on compact ~~sets~~ sets to some g. We claim that $g \equiv -1$.

Since $\varphi \circ f_{n_k} : \mathbb{D} \to \mathbb{D}$ $\forall k$, continuity implies that $g : \mathbb{D} \to \overline{\mathbb{D}}$.

B/c $\varphi \circ f_{n_k}(0) \to -1$ as $k \to \infty$, we know that $g(0) = -1 \Rightarrow |g(0)| = 1$.

Therefore by the maximum modulus principle, since $|g(z)| \le 1$ on $\mathbb{D}$, $g \equiv -1$, as desired.

By the earlier reasoning, this implies that $\varphi \circ f_n$ converges uniformly on compact subsets to -1.

Since $\varphi(-i) = \infty$, and φ has no other poles, φ is uniformly continuous on ~~on~~ a neighborhood of 0. Therefore f_n converges uniformly on compact subsets to $\varphi^{-1}(-1) = 0$ as desired. $\square$

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This appears to be a straightforward application of Phragmen-Lindelof on the sector. Unfortunately, I do not recall the statement of that theorem.

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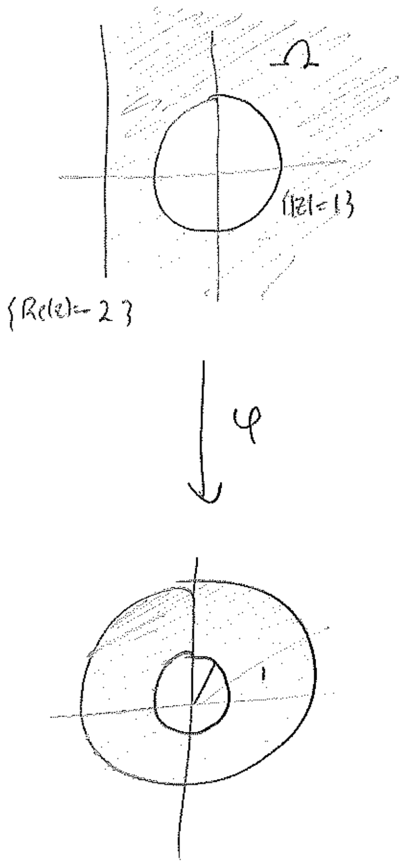

We aim to map Ω conformally to a nicer region. Namely, an annulus, via a möbius transformation.

Consider the point $z_0 = -2-\sqrt{3}$. Under reflections across $\{Re(z)=-2\}$ and across $\{|z|=1\}$, we see that

$$z_0 \longmapsto z_0 - 2(z_0-(-2)) = -2-\sqrt{3}+4+2\sqrt{3}-4 = -2+\sqrt{3}$$

$$z_0 \longmapsto \frac{1}{\bar{z}_0} = \frac{-2+\sqrt{3}}{4-3} = -2+\sqrt{3}$$

Therefore the points $-2-\sqrt{3}$ and $-2+\sqrt{3}$ are symmetric under reflection across both the circle $\{|z|=1\}$ and the line $\{Re(z)=-2\}$.

Therefore if we apply a Möbius transform which sends $-2-\sqrt{3}$ to ∞ and $-2+\sqrt{3}$ to 0, the circle $\{|z|=1\}$ and line $\{Re(z)=-2\}$ will be mapped to concentric circles, centred at 0. *

Consider $\varphi(z) = \dfrac{z+2-\sqrt{3}}{z+2+\sqrt{3}}$. This satisfies the above restrictions. Then φ maps $\{|z|=1\}$ to the circle of radius $|\varphi(1)| = \left|\frac{3-\sqrt{3}}{3+\sqrt{3}}\right| = \left|\frac{6-2\sqrt{3}}{6}\right| = \left|1-\frac{1}{3}\sqrt{3}\right|$

and $\{Re(z)=-2\}$ to the circle of radius $|\varphi(-2)| = \left|\frac{-\sqrt{3}}{\sqrt{3}}\right| = |-1| = 1$.

Since φ preserves connectedness $\varphi(\Omega)$ is then the annulus $\{1-\frac{\sqrt{3}}{3} < |z| < 1\} =: A$.

* This is because Möbius transformations preserve reflection and circles reflect 0 to ∞.

Consider $u:\bar{\Omega}\to\mathbb{R}$ s.t. $u=1$ on $\{|z|=1\}$ and $u=0$ on $\{\mathrm{Re}(z)=-2\}$. Then $u\circ\varphi^{-1}:\bar{A}\to\mathbb{R}$ is harmonic b/c φ is conformal and

$$u\circ\varphi^{-1}=1 \text{ on } \{|z|=1-\tfrac{\sqrt{3}}{3}\}$$

$$u\circ\varphi^{-1}=0 \text{ on } \{|z|=1\}$$

Since the solution to the Dirichlet problem is unique, this implies that

$$u\circ\varphi^{-1}(z)=\frac{\log|z|}{\log(1-\frac{\sqrt{3}}{3})}$$

since $\log|z|$ is harmonic on A and $\dfrac{\log(1)}{\log(1-\frac{\sqrt{3}}{3})}=0$, $\dfrac{\log(1-\frac{\sqrt{3}}{3})}{\log(1-\frac{\sqrt{3}}{3})}=1$.

Then

$$u(z)=u\circ\varphi^{-1}\circ\varphi(z)=\frac{\log\left|\frac{z+2-\sqrt{3}}{z+2+\sqrt{3}}\right|}{\log(1-\frac{\sqrt{3}}{3})}.$$

In particular,

$$u(2)=\frac{\log\left|\frac{4-\sqrt{3}}{4+\sqrt{3}}\right|}{\log(1-\frac{\sqrt{3}}{3})}$$

which is what was to be found. $\square$

(12) 405 346 515

Define

$$\mathcal{F} = \{\pm\cos(z-\alpha), \pm\sin(z-\alpha) : \alpha \in \mathbb{C}\}.$$

Since $(\pm\sin(z-\alpha))' = \pm\cos(z-\alpha)$ and $(\pm\cos(z-\alpha))' = \mp\sin(z-\alpha)$ $\forall \alpha$,
it follows that all $f \in \mathcal{F}$ satisfy $(f')^2 + f^2 = 1$.

We claim that $\mathcal{F}$ contains all entire functions which satisfy $(f')^2 + f^2 = 1$.

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It sufficient to show that u is the real part of a holomorphic function. $V(z)$ $\int$ [illegible]

Define $g(z) = u_x - iu_y$. Then the real and imaginary parts of g are continuously differentiable and satisfy

$$u_{xx} + u_{yy} = 0 \Rightarrow (u_x)_x = (-u_y)_y$$

$$u_{xy} = u_{yx} \Rightarrow (u_x)_y = -(-u_y)_x$$

Therefore g satisfies the Cauchy Riemann equations and hence is holomorphic on Ω.

Let $c = \text{Res}(g, 0)$. Then $\forall$ closed curves $\gamma \subset \Omega$,

$$\int_\gamma g\,dz = 2\pi i c\,W(\gamma, 0)$$

where $W(\gamma, 0)$ is the winding # of γ about 0. Then

$$\int_\gamma g\,dz = c\int_\gamma \tfrac{1}{z}\,dz$$

$$\Rightarrow \int_\gamma \left(g - \tfrac{c}{z}\right)dz = 0$$

hence $\frac{c}{z}$ is continuous on Ω, this implies that $g - \frac{c}{z}$ is holomorphic on Ω and has a primitive h s.t $h' = g - \frac{c}{z}$.

I don't believe Raymond here

(11) 405 346 515

Since $\mathbb{D}$ is path connected, $\forall z \in \mathbb{D}$ $\exists$ a path $\gamma(z)$ from 0 to z. Define $v : \Omega \to \mathbb{C}$ via

$$v(z) = \int_{\gamma(z)} u_x \, dy - u_y \, dx$$

where $u_x = \frac{\partial u}{\partial x}$ and $u_y = \frac{\partial u}{\partial y}$.

We first show that v is independent of the choice of γ.

Suppose $\exists$ $\gamma_1, \gamma_2 : [0,1] \to \mathbb{D}$ s.t.

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Define $\mathcal{F} = \{\pm 1, \cos(z-\alpha) : \alpha \in \mathbb{C}\}$. We claim that $\mathcal{F}$ encompasses all solutions of $(f')^2 + f^2 = 1$.

Since $(\cos(z-\alpha))' = -\sin(z-\alpha)$ and $\sin^2 + \cos^2 = 1$, it follows that $\cos(z-\alpha)$ satisfies $(f')^2 + f^2 = 1$ $\forall \alpha \in \mathbb{C}$. Similarly, ± 1 satisfies the equation.

It then remains to show that all solutions are in $\mathcal{F}$.

Suppose that f is entire satisfying $(f')^2 + f^2 = 1$.

Taking derivatives, we find that

$$2f'(f''+f) = 0$$

Since $\mathbb{C}$ cannot be written as the disjoint union of 2 discrete sets and f', $f''+f$ are entire, this implies that $f' \equiv 0$ or $f''+f \equiv 0$.

If $f' \equiv 0$ then f is a constant. Then $(f')^2 + f^2 = f^2 = 1 \Rightarrow f = \pm 1 \in \mathcal{F}$.

Suppose instead that $f''+f = 0$. Then, recalling basic differential equations, $f = \beta \cos(z-\alpha)$ for some $\alpha, \beta \in \mathbb{C}$.

By direct calculation,

$$\begin{aligned} 1 &= f^2 + (f')^2 \\ &= \beta^2(\cos^2(z-\alpha) + \sin^2(z-\alpha)) \\ &= \beta^2 \end{aligned}$$

and so $\beta = \pm 1$. By adding π to α, we may assume $\beta = 1$. Then $f \in \mathcal{F}$. This concludes. $\square$

Analysis

Fall 2013

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Since f is proper, $f^{-1}\{w\}$ is compact $\forall$ $w \in V$.

Therefore f is non-constant since otherwise there would exist $w \in V$ s.t. $f^{-1}\{w\} = U$ which is not compact.

By the open mapping theorem, this implies that f is open and so $f(U) \subset V$ is open.

We claim that $f(U) \subset V$ is closed. If this can be shown then $f(U) \subset V$ is both open and closed and so $f(U) = V$ since V is connected.

To show $f(U) \subset V$ is closed, it suffices to show $V \setminus f(U)$ is open. ~~Suppose~~ ~~$\exists$ $\tilde{z} \in V \setminus f(U)$.~~

Suppose ~~for the sake of contradiction~~ that $\exists\, \tilde{z} \in \overline{f(U)}$. Then $\exists$ $z_n \in U$ s.t. $f(z_n) \to \tilde{z}$. Then $\{f(z_n), \tilde{z}\}_n$ is compact and so $f^{-1}\{f(z_n), \tilde{z}\}$ is compact. Since $(z_n) \subset f^{-1}\{f(z_n), \tilde{z}\}$, $\exists$ a ~~sub~~ subsequence z_{n_k} s.t. $z_{n_k} \to z \in f^{-1}\{f(z_n), \tilde{z}\}$.

Then by continuity, $f(z) = \tilde{z}$ and so $\tilde{z} \in f(U)$. Therefore $\overline{f(U)} \subset f(U)$ and hence $f(U) = \overline{f(U)}$. Therefore $f(U)$ is closed.

Since V is connected and $f(U)$ is non-empty, open, and closed, this implies $f(U) = V$. ~~so~~ $\square$

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Suppose on the contrary that such an f exists.

Define g by

$$g(z) = f(z^2).$$

Then g is holomorphic near 0 and satisfies

$$g(1/n) = f(1/n^2) = \frac{n^2-1}{n^5}$$

for all sufficiently large n.

Define $p(z) = z^3 - z^5$. Then $\forall\, n > 0$,

$$p(1/n) = n^{-3} - n^{-5} = \frac{n^2-1}{n^5}$$

Therefore p and g agree on a set w/ an accumulation point. Holomorphic continuation then implies $p = g$.

In particular, near 0,

$$f(z^2) = z^3 - z^5$$

However, this would imply

$$f(z^2) = f((-z)^2) = (-z)^3 - (-z)^5 = -z^3 + z^5 = -f(z^2)$$

and so $f \equiv 0$. This contradicts the fact that $f(1/n^2) = \frac{n^2-1}{n^5} \neq 0$ for sufficiently large n. Therefore no such f exists. $\square$

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Suppose $\exists$ a holomorphic function $f: \mathbb{D} \to \mathbb{C}$ s.t.

$$\lim_{n\to\infty} |f(z_n)| = +\infty \qquad (1)$$

for all $z_n \in \mathbb{D}$ s.t. $|z_n| \to 1$.

We claim that f has at most finitely many zeros.

By (1), f is not identically 0. Therefore f has at most countably many zeros in $\mathbb{D}$ s.t. the zeros of f do not accumulate in $\mathbb{D}$. Suppose $\exists\, z_n \in \mathbb{D}$ w/ $f(z_n) = 0$ s.t. $z_n \to z \in \partial\mathbb{D}$ Then $\lim_{n\to\infty} |f(z_n)| = 0$ which contradicts (1). Therefore the zeros of f do not accumulate in $\overline{\mathbb{D}}$ and hence must be finite. Let $a_1, \dots, a_n$ enumerate the zeros of f, repeated by multiplicity.

We recall the Blaschke factor φ_a for $a \in \mathbb{D}$ defined by

$$\varphi_a = \frac{a - z}{1 - \bar{a} z}$$

Then φ_a is holomorphic on a neighborhood of $\overline{\mathbb{D}}$, $|\varphi_a| = 1$ on $\partial\mathbb{D}$, and φ_a has a simple zero at a.

Define $g = f / \prod_{i=1}^{n} \varphi_{a_i}$. Then g is non-vanishing on $\mathbb{D}$ and $\forall\, |z_n| \to 1$,

$$|g(z_n)| = |f(z_n)| / \prod_{i=1}^{n} |\varphi_{a_i}(z_n)| \to \infty$$

b/c $|\varphi_{a_i}(z_n)| \to 1$ $\forall a_i$.

Since g is nonvanishing and holomorphic on $\mathbb{D}$, $\log|g|$ is harmonic on $\mathbb{D}$. Then $\forall\, 0<r<1$, the mean value property implies

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|g(re^{i\theta})|\, d\theta$$

Then by Fatou's lemma,

$$\log|g(0)| \geq \frac{1}{2\pi}\int_0^{2\pi} \liminf_{r\to\infty} \log|g(re^{i\theta})|\, d\theta = \infty$$

Which contradicts the holomorphicity of g and f. Therefore no such f exists. $\square$

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By Montel's theorem, it suffices to show that $\{f_n\}$ is uniformly bounded on compact subsets of $\mathbb{D}$.

Let $K \subset \mathbb{D}$ be compact. Then since $\mathbb{C} \setminus \mathbb{D}$ is closed, define

$$r = d(K, \mathbb{C} \setminus \mathbb{D}) > 0$$

Then $\forall$ $z \in K$, $D(z,r) \subset \mathbb{D}$. The mean value theorem then implies that $\forall n$, $\forall z \in K$,

$$\begin{aligned} |f_n(z)| &= \left| \frac{1}{\pi r^2} \int_{D(z,r)} f_n(w)\, dA(w) \right| \\ &\leq \frac{1}{\pi r^2} \int_{D(z,r)} |f_n(w)|\, dA(w) \\ &\leq \frac{1}{\pi r^2} \int_{\mathbb{D}} |f_n| \\ &\leq \frac{1}{\pi r^2} \end{aligned}$$

Since this is uniform $\forall n$, $\forall z \in K$, this implies that $\{f_n\}$ is uniformly bounded on compact subsets of $\mathbb{D}$. Montel's theorem then concludes. $\square$

(7) 405 846 515

Let $\mathcal{J}$ denote the L^2 closure of finite linear combinations of $\{x^{1/2}e^{2\pi ikx}\}_{k\in\mathbb{Z}}$. If we want to avoid complex numbers then we may consider finite linear combination of $\{\sin(2\pi kx), \cos(2\pi kx)\}$, but the proof will not work out as nicely, though it will work.

For all $k\in\mathbb{Z}$, $x^{-1/2}x^{1/2}e^{2\pi ikx} = e^{2\pi ikx} \in L^1$ and (0)

$$\int x^{-1/2}x^{1/2}e^{2\pi ikx}dx = 0$$

Then by linearity, the same holds $\forall$ finite linear combination of $\{x^{1/2}e^{2\pi ikx}\}_{k\in\mathbb{Z}}$.

We claim $\mathcal{J} = L^2[0,1]$. Since Schwartz functions are dense in $L^2[0,1]$, it suffices to show that $S\subset\mathcal{J}$. Since S is a linear subspace of $L^2[0,1]$ it suffices to show that $S\cap\mathcal{J}^\perp = \{0\}$. Suppose $\exists$ $h\in S\cap\mathcal{J}^\perp$. Then $\forall k$

$$0 = \langle h, x^{1/2}e^{2\pi ikx}\rangle = \int_0^1 e^{-2\pi ikx}x^{1/2}h(x)\,dx \qquad (1)$$

Since $h\in S$, $x^{1/2}h\in L^1[0,1]$. Therefore $x^{1/2}h(x)$ has a well-defined Fourier transform. By equation (1), $[\cdot^{1/2}h(\cdot)]^\wedge(k) = 0$ $\forall k$. Therefore, since $\sum_{k\in\mathbb{Z}}\hat{g}(k)e^{2\pi ikx} \to g(x)$ in L^1, $x^{1/2}h(x) = 0$ a.e. Then $h=0$ almost everywhere and so $S\cap\mathcal{J}^\perp = \{0\}$. Therefore $S\cap\mathcal{J}^\perp = \{0\}$ and so $S\subset\mathcal{J}^\perp$. Since S is dense in $L^2[0,1]$, this implies $\mathcal{J} = L^2[0,1]$.

Therefore the finite linear combinations of $\{x^{1/2}e^{2\pi ikx}\}_{k\in\mathbb{Z}}$ are dense in $L^2[0,1]$ and satisfy (0) as desired. $\square$

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Define $\mathcal{F} = \{x \mapsto B(x,y) : \|y\| \leq 1\}$. Then $\forall$ fixed $x \in X$,

$$\sup_{\|y\| \leq 1} |B(x,y)| \leq \sup_{\|y\| \leq 1} C_x \|y\| \leq C_x < \infty$$

Then $\mathcal{F}$ is pointwise bounded. By Banach-Steinhaus, the uniform boundedness principle, this implies that $\mathcal{F}$ is uniformly bounded. So $\exists$ $C > 0$ s.t.

$$\sup_{\substack{\|x\| \leq 1 \\ \|y\| \leq 1}} |B(x,y)| \leq C$$

Then $\forall$ $x \in X$, $y \in Y$, $x, y \neq 0$

$$|B(x,y)| = \|x\| \, \|y\| \, |B(\tfrac{x}{\|x\|}, \tfrac{y}{\|y\|})| \leq C \|x\| \, \|y\| \qquad (1)$$

For $x, y = 0$, $|B(0,0)| = 0 \leq C \|0\| \, \|0\|$. Therefore (1) holds $\forall$ $x \in X$, $y \in Y$. $\square$

$*$ need to note $x \mapsto B(x,y)$ is bounded $\forall y$ since

$$|B(x,y)| \leq C_y \|x\|$$

so $\mathcal{F} \subset X^*$.

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(a) Fix $f \in L^2(\mathbb{R})$. Let $g(x) = \frac{\check{f}(x)\check{f}(x)}{\sqrt{2\pi}}$. Then by Hölder and Plancherel,

$$\|g\|_{L^1} \le \|\check{f}\|_{L^2}^2 = \|f\|_{L^2}^2 < \infty$$

so $g \in L^1$.

Moreover, since $\check{f}(x)\check{f}(x) = (f*f)^\vee(x) \in L^1$, Fourier inversion implies that

$$\begin{aligned} h(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} (f*f)^\vee(x)\,dx \\ &= \frac{1}{\sqrt{2\pi}} \left((f*f)^\vee\right)^\wedge(\xi) \\ &= f*f(\xi) \end{aligned}$$

as claimed.

(b) We claim this is false. Suppose for the sake of contradiction that this is true.

Take $g \in L^1 \setminus L^2$. Then $\exists\, f \in L^2$ s.t. $\hat{g}(\xi) = f*f(\xi)$.

Then by Hölder,

$$\|\hat{g}\|_{L^1} \le \|f\|_{L^2}^2 < \infty$$

and so $\hat{g} \in L^1$. By Riemann Lebesgue, we also have $\hat{g} \in L^\infty$. Therefore $\hat{g} \in L^1 \cap L^\infty$ and so $\hat{g} \in L^2$. However, since the Fourier transform is an L^2 isometry, up to constants this would imply that $g \in L^2$, which is a contradiction.

Therefore the claim must be false. $\square$

ANALYSIS

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we claim that (a) and (c) are false.

(b). Define $f_{nk} = \chi_{[k/n,\frac{k+1}{n}]}$ for $n \geq 1$ and $k = 0, 1, \ldots, n-1$.

Consider the sequence $(f_{00}, f_{10}, f_{11}, f_{20}, f_{21}, \ldots) =: (f_{nk})$.

Then $f_{nk} \to 0$ in L^3 since

$$\|f_{nk}\|_{L^3}^3 = \int \chi^3_{[k/n,\frac{k+1}{n}]} dx = \left|\left[\tfrac{k}{n}, \tfrac{k+1}{n}\right]\right| = \frac{1}{n} \to 0$$

However, f_{nk} does not converge pointwise a.e b/c $\forall x \in [0,1]$, $f_{nk}(x)$ takes both the values $0, 1$ infinitely often as $n, k \to \infty$.

(a), (c) Define $f_n = n\chi_{[0,1/n]}$.

Then $f_n \to 0$ a.e b/c $\forall x \neq 0$ $\frac{1}{n} < x$ for sufficiently large n and so $f_n(x) = 0$ for sufficiently large n.

Additionally, $f_n \to 0$ in measure b/c $\forall\ \varepsilon > 0$, $\{|f_n| > \varepsilon\} \subset [0,1/n]$ and so $|\{|f_n| > \varepsilon\}| \leq |[0,1/n]| = 1/n \to 0$ as $n \to \infty$.

However, $f_n \not\to 0$ in L^3 b/c $\|f_n\|_{L^1} = n\frac{1}{n} = 1$ $\forall n$ and so if $f_n \to 0$ in L^3 then $0 = \|0\|_{L^1} = \lim_{n\to\infty} \|f_n\|_{L^1} = 1$ which is impossible.

② 405 846 515

(a) Suppose that $f : X \to Y$ is continuous. Define

$$\mathcal{F} = \{E \subset Y : f^{-1}(E) \in \mathcal{B}(X)\}$$

We claim that $\mathcal{F}$ is a σ-algebra. To show this, it must be shown that $\mathcal{F}$ is closed under complements and countable unions and that $\emptyset \in \mathcal{F}$.

First, we note that $f^{-1}(\emptyset) = \emptyset \in \mathcal{B}(X)$ and so $\emptyset \in \mathcal{F}$.

Next, let $A_1, A_2, \ldots \in \mathcal{F}$. Then

$$\begin{aligned} f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) &= \{x : f(x) \in \bigcup_{n=1}^{\infty} A_n\} \\ &= \{x : f(x) \in A_n \text{ for some } n\} \\ &= \bigcup_{n=1}^{\infty} \{x : f(x) \in A_n\} = \bigcup_{n=1}^{\infty} f^{-1}(A_n) \end{aligned}$$

Since $\mathcal{B}(X)$ is closed under countable unions and $f^{-1}(A_n) \in \mathcal{B}(X)$, this implies $f^{-1}(\cup A_n) \in \mathcal{B}(X)$ and so $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$. Therefore $\mathcal{F}$ is closed under countable union.

Finally, suppose $\exists A \in \mathcal{F}$. Then

$$\begin{aligned} f^{-1}(Y \setminus A) &= \{x : f(x) \notin A\} \\ &= X \setminus \{x : f(x) \in A\} \\ &= X \setminus f^{-1}(A) \end{aligned}$$

Since $f^{-1}(A), X \in \mathcal{B}(X)$, this implies $Y \setminus A \in \mathcal{F}$ and so $\mathcal{F}$ is closed under complement.

Therefore $\mathcal{F}$ is a σ-algebra.

Moreover, since f is continuous, $\forall$ open $U \in Y$, $f^{-1}(U)$ is open. Since $\mathcal{B}(X)$ contains all open sets in X, this implies that $\mathcal{F}$ contains all open subsets of Y. B/c $\mathcal{B}(Y)$ is generated by the open sets in Y, and $\mathcal{F}$ is a σ-algebra, $\mathcal{B}(Y) \subset \mathcal{F}$.

Therefore $\forall E \in \mathcal{B}(Y)$, $f^{-1}(E) \in \mathcal{B}(X)$. $\square$

(b) Suppose $\exists A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$.

Define $f : X \times Y \to X$ by $(x,y) \mapsto x$. Then $\forall$ open $U \subset X$, $f^{-1}(U) = U \times Y$ which is open in $X \times Y$ b/c Y is open in Y.
Therefore f is continuous. By part (a), this implies that $f^{-1}(A) = A \times Y \in \mathcal{B}(X \times Y)$.

Similarly define $g : X \times Y \to Y$ by $(x,y) \mapsto y$. Then by the symmetric reasoning, g is continuous. By part (a), this implies $g^{-1}(E) = X \times E \in \mathcal{B}(X \times Y)$.

Since $\mathcal{B}(X \times Y)$ is closed under ~~intersections~~,

$$A \times Y \cap X \times E = A \times E \in \mathcal{B}(X, Y)$$

as desired. $\square$

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~~By definition,~~

$$\cancel{\|f_n - g_n\|_{L^1} = \sum_{k=0}^{n-1} \int_{k/n}^{\frac{k+1}{n}} \left| n \int_{k/n}^{\frac{k+1}{n}} f(y)\,dy - g(x) \right| dx}$$

$$\cancel{= \sum_{k=0} \int_{k/n}^{\frac{k+1}{n}} \left| n \underbrace{\int_{k/n}^{\frac{k+1}{n}} (f(y) - g(y))\,dy}_{\|f_n - g\|_{L^1}} \right| dx}$$

Fix some $\varepsilon > 0$. By the density of $C[0,1]$ in $L^1[0,1]$, $\exists$ a continuous function g s.t. $\|f-g\|_{L^1} < \varepsilon$. Define g_n as was done for f.
B/c $[0,1]$ is compact, g is equicontinuous. Therefore $\exists\ \delta > 0$ s.t. $|x-y| < \delta \Rightarrow |g(x) - g(y)| < \varepsilon$. Pick N sufficiently large so that $1/N < \delta$. Then $\forall\ n \geq N$, $1/n < \delta$. Direct computation then yields

$$\|g_n - g\|_{L^1} = \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \left| n \int_{\frac{k}{n}}^{\frac{k+1}{n}} (g(y) - g(x))\,dy \right| dx$$

$$\leq \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} n \int_{\frac{k}{n}}^{\frac{k+1}{n}} |g(y) - g(x)|\,dy\,dx$$

$$(|x-y| \leq 1/n < \delta) \quad < \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} n\varepsilon \int_{\frac{k}{n}}^{\frac{k+1}{n}} dy\,dx$$

$$= n^2 \varepsilon \frac{1}{n^2} = \varepsilon$$

Similarly,

$$\|f_n - g_n\|_{L^1} \leq \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} n \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(y) - g(y)|\,dy\,dx$$

$$\leq \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(y) - g(y)|\,dy$$

$$= \|f - g\|_{L^1} < \varepsilon.$$

Therefore $\forall\ n \geq N$,

$$\|f_n - f\|_{L^1} \leq \|f_n - g_n\|_{L^1} + \|g_n - g\|_{L^1} + \|g - f\|_{L^1} < 3\varepsilon.$$

As such an N can be found $\forall\ \varepsilon > 0$, this implies $f_n \to f$ in L^1. □

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(a) Suppose $\exists \{f_n\} \subset S$ s.t. $f_n \to f$ in L^1. Then $\forall n$

$$\left|\int f\,dx\right| = \left|\int f\,dx - \int f_n\,dx\right| \le \int |f-f_n|\,dx = \|f_n - f\|_{L^1} \to 0$$

and so $f \in S$. Therefore S is closed in the L^1 topology.

(b) By the linearity of the integral, $S \cap L^2(\mathbb{R}^3)$ is a linear subspace of $L^2(\mathbb{R}^3)$. Let $\mathcal{J}$ denote the closure of $S \cap L^2(\mathbb{R}^3)$ in L^2.
By the density of Schwartz functions in $L^2(\mathbb{R}^3)$ (or equivalently the density of L^1 functions in L^2), it suffices to show that $\mathcal{S} \subset \mathcal{J}$ where $\mathcal{S}$ is the space of Schwartz functions.
Suppose $\exists\, g \in \mathcal{J}^\perp \cap \mathcal{S}$. We claim that $g \equiv 0$. To show this, we aim to show that $\hat{g} \equiv 0$ a.e by approximating the Fourier transform w/ functions in S.

Fix $n > 0$ and $\xi \in \mathbb{R}^3 \setminus \{xyz = 0\}$. Let $A_n = \left[\frac{-n}{|\xi_1|}, \frac{n}{|\xi_1|}\right] \times \left[\frac{-n}{|\xi_2|}, \frac{n}{|\xi_2|}\right] \times \left[\frac{-n}{|\xi_3|}, \frac{n}{|\xi_3|}\right]$ and consider $\chi_{A_n} e^{-2\pi i x\cdot\xi}$. Since $\chi_{A_n} e^{-2\pi i x\cdot\xi}$ is compactly supported and bounded, it is in both L^1 and L^2. We now compute

$$\int \chi_{A_n} e^{-2\pi i x\cdot\xi}\,dx = \prod_{i=1}^{3} \int_{\frac{-n}{|\xi_i|}}^{\frac{n}{|\xi_i|}} e^{-2\pi i t \xi_i}\,dt$$

Making the change of variables $t \to \frac{t}{|\xi_i|}\operatorname{sign}(\xi_i)$ yields

$$\int \chi_{A_n} e^{-2\pi i x\cdot\xi}\,dx = \prod_{i=1}^{3} \frac{\operatorname{sign}(\xi_i)}{|\xi_i|} \int_{-n}^{n} e^{-2\pi i t}\,dt = 0$$

and so $\chi_{A_n} e^{-2\pi i x\cdot\xi} \in S$ $\forall n > 0$ and a.e $\xi \in \mathbb{R}^3$.
B/c A_n monotonically increases to $\mathbb{R}^3$ as $n \to \infty$ for fixed $\xi \in \mathbb{R}^3$, it follows that $\chi_{A_n} e^{-2\pi i x\cdot\xi} \to e^{-2\pi i x\cdot\xi}$ as $n \to \infty$.

$\longrightarrow$

Therefore $g\chi_{A_n}e^{-2\pi i x\cdot\xi} \to e^{-2\pi i x\cdot\xi}g$ as $n\to\infty$.
Moreover, $|g\chi_{A_n}e^{-2\pi i x\cdot\xi}| \le |e^{-2\pi i x\cdot\xi}g| = |g| \in L^1$ $\forall n$. Therefore DCT implies that for a.e $\xi \in \mathbb{R}^3$

$$\hat{g}(\xi) = \int e^{-2\pi i x\cdot\xi}g(x)\,dx = \lim_{n\to\infty}\int \chi_{A_n}e^{-2\pi i x\cdot\xi}g(x)\,dx = 0$$

since $g \in \mathcal{F}^{\perp} \cap \mathcal{S}$. Therefore $\hat{g}(\xi) = 0$ a.e and so $g = 0$.
Then $\mathcal{F}^{\perp} \cap \mathcal{S} = \{0\}$. Since $\mathcal{S}$ is a linear subspace of $L^2(\mathbb{R}^3)$, this implies that $\mathcal{S} \subset \mathcal{F}$. Because $\mathcal{S}$ is dense in $L^2(\mathbb{R}^3)$, this implies that $\mathcal{F} = L^2(\mathbb{R}^3)$ and so $S \cap L^2(\mathbb{R}^3)$ is dense in $L^2(\mathbb{R}^3)$.
$\square$

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Let $\mathcal{H}$ be a separable Hilbert space

Riesz Representation Theorem

Let, $\exists$ $L \in \mathcal{H}^*$ be a bounded linear functional on $\mathcal{H}$. Then $\exists!$ $h \in \mathcal{H}$ s.t. $L(f) = \langle f, h \rangle$ $\forall$ $f \in \mathcal{H}$.

Proof: We first show uniqueness. Suppose $\exists$ h, h' s.t. $L(\cdot) = \langle \cdot, h \rangle = \langle \cdot, h' \rangle$. Then

$$\langle h-h', h \rangle = \langle h-h', h' \rangle$$

$$\Rightarrow \quad 0 = \langle h-h', h-h' \rangle = \|h-h'\|^2$$

and so $h = h'$ since $\|\cdot\|$ is a norm. Therefore h is unique.

We now show existence. If $L(f) = 0$ $\forall$ f then $h = 0$ satisfies the desired properties. Otherwise, $\exists$ $g \in (\ker L)^\perp$ s.t. $\|g\| = 1$.

We claim that $L(f) = \langle f, \overline{L(g)}\, g \rangle$ $\forall$ f.

By direct calculation, $\forall$ $f \in \mathcal{H}$,

$$L\big(L(f) g - L(g) f\big) = L(f)L(g) - L(g)L(f) = 0$$

Then $L(f)g - L(g)f \in \ker L$ and so

$$0 = \langle L(f)g - L(g)f, g \rangle = L(f)\|g\|^2 - \langle f, \overline{L(g)}\, g \rangle$$

Therefore $L(f) = \langle f, \overline{L(g)}\, g \rangle$ $\forall$ $f \in \mathcal{H}$.

$\square$

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Let $f_y(x) = f(x-y)$ and let $\mathcal{F}$ be the closure of linear combination of f in $L^2(\mathbb{R})$.

~~To show B/c $L^2(\mathbb{R})$ is a Hilbert space, showing that $\mathcal{F} = L^2$ is~~

Suppose that $\exists\, h \in \mathcal{F}^\perp$. We claim that $h \equiv 0$.

Since $h \in \mathcal{F}^\perp$, it follows that $\langle h, f_y \rangle = \int f(x-y)h(x)\,dx = 0 \;\forall y$.

Let $\tilde{f}(x) = f(-x)$. Then by definition, $\forall\, y$,

$$0 = \int f(x-y)h(x)\,dx = \int \tilde{f}(y-x)h(x)\,dx = \tilde{f} * h(y)$$

Standard properties of the Fourier transform then imply

$$\widehat{\tilde{f}*h} = 0 \implies \hat{\tilde{f}}(\xi)\hat{h}(\xi) = 0 \;\forall \xi$$

By direct computation,

$$\hat{\tilde{f}}(\xi) = \int e^{-2\pi i x\xi} f(-x)\,dx = \hat{f}(-\xi)$$

Therefore $\hat{f}(-\xi)\hat{h}(\xi) = 0 \;\forall \xi$. As $\hat{f}(\xi) > 0$ a.e., this implies that $\hat{h}(\xi) = 0$ a.e and so $h = 0$ identically. Therefore $\mathcal{F}^\perp = \{0\}$ and so $\mathcal{F} = L^2(\mathbb{R})$ as desired.

□

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Define $u(z) = \inf_n u_n(z)$. We note that since $\{u_n\}$ is non-increasing and bounded below, $u(z) = \lim_{n\to\infty} u_n(z)$.
To show that u is harmonic, it suffices to show that u is continuous and satisfies the mean value property locally.

To show continuity, we show that $\{u_n\}$ is uniformly ~~cont~~ Cauchy on compact subsets of $\mathbb{D}$. Fix $R \in (0,1)$ and consider $\overline{D(0,R)} \subset \mathbb{D}$.
By Harnack's inequality, $\forall m > n$, $\forall z \in \overline{D(0,R)}$ since $u_m - u_n \geq 0$ is harmonic

$$|u_m(z) - u_n(z)| = u_m(z) - u_n(z)$$
$$\leq \frac{R+|z|}{R-|z|}\left(u_m(0) - u_n(0)\right) \to 0$$

since $u_k(0) \to u(0)$, $\{u_k(0)\}$ is Cauchy, implying this final limit.
As the bound is independent of $z \in \overline{D(0,R)}$, this implies $\{u_n\}$ is uniformly Cauchy on $\overline{D(0,R)}$ and hence converges uniformly to u on $\overline{D(0,R)}$.
Since $u_n \in C(\mathbb{D})$ $\forall n$, this implies u is continuous on $D(0,R)$ $\forall R$ and hence u is continuous on $\mathbb{D}$.
We now show u satisfies the mean value property.
Consider $z \in \mathbb{D}$ and fix $R > 0$ s.t. $\overline{D(z,R)} \subset \mathbb{D}$. Then by monotonicity,
$\forall w \in \overline{D(z,R)}$, $u(w), u_n(w) \leq u_1(w) \leq \max_{w \in \overline{D(0,R)}} u_1(w) = M < \infty$
Since M is integrable over $\overline{D(0,R)}$, the DCT implies that $\forall r \in (0,R)$,

$$\frac{1}{|D(z,r)|}\int_{D(z,r)} u(w)\,dx\,dy = \lim_{n\to\infty} \frac{1}{|D(z,r)|}\int_{D(z,r)} u_n(w)\,dx\,dy = \lim_{n\to\infty} u_n(z) = u(z)$$

Therefore u locally satisfies the mean value property and hence is harmonic on $\mathbb{D}$.

D

~~ALTERNATE: extend to holomorphic f_n consider $g_n = e^{f_n}$ and apply Montel~~

Alternate: Define $u = \inf_n u_n = \lim_{n\to\infty} u_n$

Since $\mathbb{D}$ is simply connected, $\forall n$ $\exists$ holomorphic f_n s.t. $u_n = \mathrm{Re}(f_n)$.

Define $g_n = \exp f_n$. We claim that $\{g_n\}$ is uniformly bounded on compact subsets of $\mathbb{D}$.

Fix a compact subset $K \subset \mathbb{D}$. Then $\forall n$, $\forall z \in K$

$$|g_n(z)| = e^{u_n(z)} \leq e^{u_1(z)} \leq e^{\max_{z\in K} u_1(z)} < \infty$$

Therefore $\{g_n\}$ is uniformly bounded on compact subsets of $\mathbb{D}$. Montel's theorem then implies that $\exists$ a subsequence which converges locally uniformly to some holomorphic g.

Since $u_n \geq 0$ $\forall n$, $|g_n| \geq 1$ $\forall n$ and so $|g| \geq 1$ $\forall n$.

Additionally, since $g_n \to g$, it follows that

$$e^{u_n} = |g_n| \to |g| \text{ pointwise}$$

and hence

$$u_n \to \log|g| \text{ pointwise.}$$

Since g is holomorphic w/ $|g| \geq 1$, $\log|g|$ is harmonic. By uniqueness of the ~~limits~~ limit, this implies that $\inf_n u_n(z) = \log|g(z)|$ is harmonic, as desired.

$\square$

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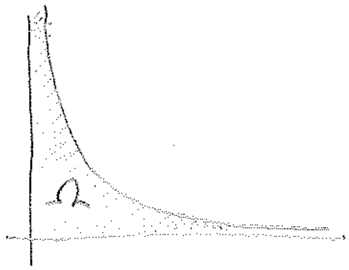
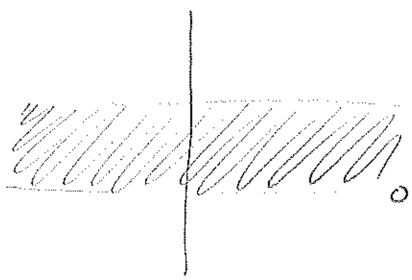
We intend to map Ω conformally to a strip, construct a harmonic function on the strip which satisfies the properties and then map it back to Ω.



$\varphi : z \mapsto z^2$



Consider the map $\varphi : \mathbb{C} \to \mathbb{C} : z \mapsto z^2$. We note that φ is a conformal map from the first quadrant to $\mathbb{H}$ and hence is conformal on Ω extending continuously to $\overline{\Omega}$. We claim φ takes Ω to the strip $S = \{z : 0 < \mathrm{Im}(z) < 2\}$. We note that φ takes $\{iy : y \geq 0\} \to \{-y : y \geq 0\}$ and $\{x : x \geq 0\} \to \{x \geq 0\}$. Moreover, $\forall\, z = x + \frac{i}{x}$,
$\varphi(z) = x^2 - \frac{1}{x^2} + 2i$. Therefore φ takes $\partial\Omega$ to ∂S.
To show φ takes $\Omega \to S$, it then suffices to check a point. Consider $\frac{1}{2} + \frac{i}{2} \in \Omega$, then
$$\varphi\left(\tfrac{1}{2} + \tfrac{i}{2}\right) = \frac{1}{4} - \frac{1}{4} + \frac{1}{2}i = \frac{1}{2}i \in S.$$
Therefore φ takes Ω to S.

Define $u : S \to \mathbb{R}$ by $u(x+iy) = e^{\pi x}\sin(\pi y)$. Then u is harmonic on $\mathbb{C}$ since $\Delta u(x+iy) = \partial_{xx}u + \partial_{yy}u = \pi^2 e^{\pi x}\sin(\pi y) - \pi^2 e^{\pi x}\sin(\pi y) = 0$.
In particular, u is harmonic on S, continuous on $\overline{S}$. ~~and~~ Additionally,

$$\left.\begin{aligned} y = 0 &\implies u(x) = 0 \\ y = 2 &\implies u(x+2i) = e^{\pi x}\sin(2\pi) = 0 \end{aligned}\right\} \implies u = 0 \text{ on } \partial S$$

and u is unbounded on S since $u(x+iy) \to \infty$ as $x \to \infty$ for $y \notin \mathbb{Z}$.

Define v on Ω by $v(z) = u(\varphi^{-1}(z)) = u(\sqrt{z}) = e^{\pi \mathrm{Re}(\sqrt{z})}\sin(\pi\,\mathrm{Im}(\sqrt{z}))$.
Since φ is conformal, v is harmonic on Ω, continuous on $\overline{\Omega}$.
~~and~~ By construction of u, φ satisfies the desired properties. $\square$

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By the triangle inequality,

$$\left|\int_{\Gamma} f(z)e^{ikz}dz\right| \leq \sup_{z\in\Gamma}|f(z)| \int_{\Gamma}|e^{ikz}||dz|$$
$$= \sup_{z\in\Gamma}|f(z)| \int_{\Gamma} e^{-k\operatorname{Im}(z)}|dz|$$

Parametrizing Γ as $\{Re^{i\theta} : 0\leq\theta\leq\pi/2\}$ implies $|dz| = R\,d\theta$.
Therefore

$$\left|\int_{\Gamma} f(z)e^{ikz}dz\right| \leq \sup_{z\in\Gamma}|f(z)|\; R\int_0^{\pi/2} e^{-k\operatorname{Im}(Re^{i\theta})}d\theta$$
$$= \sup_{z\in\Gamma}|f(z)|\; R\int_0^{\pi/2} e^{-kR\sin\theta}d\theta$$

We recall that $\sin\theta$ is concave on $[0,\pi/2]$ and hence

$$\sin\theta \geq \frac{2}{\pi}\theta \qquad \forall\, \theta\in[0,\pi/2].$$

then

$$\left|\int_{\Gamma} f(z)e^{ikz}dz\right| \leq R\sup_{z\in\Gamma}|f(z)| \int_0^{\pi/2} e^{-2kR\theta/\pi}d\theta$$
$$= R\sup_{z\in\Gamma}|f(z)|\left(\frac{-\pi}{2kR}e^{-2kR\theta/\pi}\Big|_{\theta=0}^{\pi/2}\right)$$
$$= R\sup_{z\in\Gamma}|f(z)|\;\frac{\pi}{2kR}\left(1-e^{-kR}\right)$$
$$\leq \frac{\pi}{2k}\sup_{z\in\Gamma}|f(z)|$$
$$\leq \frac{100}{k}\sup_{z\in\Gamma}|f(z)|$$

As desired. $\square$

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We first claim that Γ defines a holomorphic function on the right half-plane $H = \{Re(z) > 0\}$.

Consider a compact $K \subset H$, and let $R = \max\{Re(z) : z \in K\}$, $r = \min\{Re(z) : z \in K\}$. Then $\forall z \in K$, $\forall t \in (0, \infty)$,

$$
\begin{aligned}
|t^{z-1}e^{-t}| &= e^{-t}|e^{(z-1)\ln t}| \\
&= e^{-t}\, e^{Re(z-1)\ln(t)} \\
&= e^{-t}\, t^{Re(z)-1} \\
&\le e^{-t} t^{R-1} \chi_{\{Re(z) \ge 1\}} + e^{-t} t^{r-1} \chi_{\{Re(z) < 1\}} \\
&\le e^{-t}(t^{R-1} + t^{r-1})
\end{aligned}
$$

Since $r > 0$, t^{r-1} is integrable near 0, and so $e^{-t}t^{r-1}$ is integrable. Since $t^{R-1} \le e^{t/2}$, $e^{-t}t^{R-1}$ is integrable. Therefore $t^{z-1}e^{-t}$ is uniformly bounded by a function integrable over $(0, \infty)$ $\forall z \in K$. In particular, $\Gamma(z)$ is well-defined on K.

Because of the dominating function $e^{-t}(t^{R-1} + t^{r-1})$, DCT implies that $\forall z_0 \in K$,

$$\lim_{z \to z_0} \Gamma(z) = \int_0^\infty \lim_{z \to z_0} t^{z-1} e^{-t}\, dt = \int_0^\infty t^{z_0 - 1} e^{-t}\, dt = \Gamma(z_0)$$

Therefore Γ is continuous on K. Additionally, $\forall$ triangles $T \subset K$,

$$\int_T \int_0^\infty |t^{z-1}e^{-t}|\, dt\, |dz| \le \int_T \int_0^\infty e^{-t}(t^{R-1} + t^{r-1})\, dt\, |dz| \lesssim |T| < \infty.$$

Then Fubini's theorem implies

$$\int_T \Gamma(z)\, dz = \int_0^\infty \int_T t^{z-1} e^{-t}\, dz\, dt = \int_0^\infty 0\, dt = 0$$

b/c $t^{z-1}e^{-t}$ is holomorphic on K. As this holds $\forall$ triangles T, Morera's theorem concludes that Γ is holomorphic on K.

As this holds $\forall$ compact $K \subset H$, Γ is holomorphic on H.

$\longrightarrow$

We now aim to extend Γ meromorphically to $\mathbb{C}$.
To show this, we show that Γ can be extended meromorphically to $H-n-1$ $\forall n \geq 0$ by induction. Specifically, we claim that ~~on $H-n$~~, Γ can be extended meromorphically to $H-n-1$ via the formula $\dfrac{1}{z(z+1)\dots(z+n)}\int_0^\infty t^{z+n}e^{-t}\,dt$.

We note that the same reasoning as before implies that $\int_0^\infty t^{z+n}e^{-t}dt$ is holomorphic on $H-n-1$, so we do not repeat it here.

We begin w/ the base case. Suppose $n=0$. By integration by parts, on H,

$$\begin{aligned}\Gamma(z) &= \frac{1}{z}\int_0^\infty z t^{z-1}e^{-t}dt \qquad \begin{pmatrix} dv = zt^{z-1}dt & u = e^{-t} \\ v = t^z & du = -e^{-t}dt\end{pmatrix} \\ &= \frac{1}{z}\left(t^z e^{-t}\Big|_{t=0}^{\infty} \stackrel{\nearrow 0}{} + \int_0^\infty t^z e^{-t}dt\right) \\ &= \frac{1}{z}\int_0^\infty t^z e^{-t}dt\end{aligned}$$

As noted, $\int_0^\infty t^z e^{-t}dt$ is holomorphic on $H-1$ and so $\frac{1}{z}\int_0^\infty t^z e^{-t}dt$ is meromorphic on $H-1$. Since $\Gamma(z) = \frac{1}{z}\int_0^\infty t^z e^{-t}dt$ on H, this implies that Γ extends meromorphically to $H-1$.

Suppose for the sake of induction that Γ extends meromorphically to $H-n-1$ via the formula $\Gamma(z) = \frac{1}{z(z+1)\dots(z+n)}\int_0^\infty t^{z+n}e^{-t}dt$.
By integration by parts, $\forall z \in H-n-1$,

$$\begin{aligned}\Gamma(z) &= \frac{1}{z(z+1)\dots(z+n)(z+n+1)}\int_0^\infty (z+n+1)t^{z+n}e^{-t}dt \\ &= \frac{1}{z\cdots(z+n+1)}\left(\cancel{t^{z+n+1}e^{-t}\Big|_{t=0}^{\infty}} \stackrel{\nearrow 0}{} + \int_0^\infty t^{z+n+1}e^{-t}dt\right) \\ &= \frac{1}{z(z+1)\dots(z+n+1)}\int_0^\infty t^{z+n+1}e^{-t}dt\end{aligned}$$

As before, this defines a meromorphic continuation of Γ to $H-n-2$.
Therefore by induction, Γ extends meromorphically to $\mathbb{C}$. $\square$

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Suppose $P(z) = a_n z^n + \dots + a_0$.

Define $Q(z) = \overline{a_0} z^n + \overline{a_1} z^{n-1} + \dots + \overline{a_n}$.

Then whenever $|z| = 1$,

$$
\begin{aligned}
P(z)Q(z) &= P(z)\left(\overline{a_0} z^n + \overline{a_1} z^{n-1} + \dots + \overline{a_n}\right) \\
(|z|=1) \quad &= P(z)\left(\overline{a_0} z^n + \overline{a_1} |z|^2 z^{n-1} + \overline{a_2} |z|^4 z^{n-2} + \dots + \overline{a_n} |z|^{2n}\right) \\
&= P(z)\left(\overline{a_0} z^n + \overline{a_1}\, \overline{z} z^n + \overline{a_2}\, \overline{z}^2 z^n + \dots + \overline{a_n}\, \overline{z}^n z^n\right) \\
&= z^n P(z)\left(\overline{a_0} + \overline{a_1 z} + \dots + \overline{a_n z^n}\right) \\
&= z^n P(z) \overline{P(z)} \\
&= z^n |P(z)|^2
\end{aligned}
$$

As desired.

$\square$

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~~Suppose on the contrary that $\exists z_0 \in \mathbb{C}$ s.t. $f(z_0) \neq 0$.~~
~~Jensen's formula then implies that $\forall r > 0$~~

$$\cancel{\log|f(z_0)|}$$

First, we note that by the FTC,

$$|f(z)| = \left|\int_0^z f'(w)\,dw + \cancel{f(0)}^{0}\right|$$

$$\leq \int_0^z |f'(w)|\,|dw|$$

$$\leq \int_0^z e^{|w|}\,|dw|$$

$$= e^{|z|}$$

Therefore $|f(z)| \leq e^{|z|}$.

Suppose on the contrary that $\exists z_0 \in \mathbb{C}$ s.t. $f(z_0) \neq 0$. Then by Jensen's formula, w/ $\{a_n\}$ an enumeration of the zeros of f, $\forall r > 0$

$$\log|f(z_0)| + \sum_{|a_n - z_0| < r} \log\left(r/|a_n - z_0|\right) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(z_0 + re^{i\theta})|\,d\theta$$

$$\Rightarrow \log|f(z_0)| + \sum_{\left|\frac{n}{\sqrt{1+|n|}} - z_0\right| < r} \log\left(\frac{r}{\left|\frac{n}{\sqrt{1+|n|}} - z_0\right|}\right) \leq \frac{1}{2\pi}\int_0^{2\pi} |z_0 + re^{i\theta}|\,d\theta \leq |z_0| + r$$

Restricting our attention only to the zeros of f of the form $\frac{n}{\sqrt{1+|n|}}$ that are within r/e of z_0, we find that

$$\log|f(z_0)| + \sum_{\left|\frac{n}{\sqrt{1+|n|}} - z_0\right| < r/e} 1 \leq |z_0| + r$$

$$\Rightarrow \log|f(z_0)| + \#\left\{n : \left|\frac{n}{\sqrt{1+|n|}} - z_0\right| < r/e\right\} \leq |z_0| + r$$

Letting $R = r/e$, this implies that

$$\#\left\{n : \left|\frac{n}{\sqrt{1+|n|}} - z_0\right| < R\right\} \lesssim R$$

However, since $\frac{n}{\sqrt{1+|n|}} \sim \sqrt{|n|}$, there exists $\sim R^2$ zeros of that form in a disk of radius R about z_0 for sufficiently large R. Thus for sufficiently large R, $R^2 \lesssim R$ $\divideontimes$.

$\square$

Analysis

Fall 2022

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Suppose first that $f \in C_c(\mathbb{R}^d)$ w/ $\operatorname{supp} f \subset \{|x| \le R\}$ for some $R > 0$.

Then $\operatorname{supp} f(jx+ta) \subset \frac{\{|x|\le R\} - ta}{j}$. By direct computation,

$$\frac{\{|x|\le R\} - ta}{j} = \frac{\{|x+ta| \le R\}}{j}$$

$$= \{|x + \tfrac{ta}{j}| \le R/j\}$$

For fixed k, a, as $t \to \infty$, the balls $\{|x + \frac{ta}{j}| \le R/j\}$ for $j = 1, \dots, k$ become disjoint. Therefore for sufficiently large t, for fixed x,

$$\left|\sum_{j=1}^k f(jx+ta)\right| = \sum_{j=1}^k |f(jx+ta)|$$

Then, by making the change of variables $u = jx + ta$,

$$\Rightarrow \int \left|\sum_{j=1}^k f(jx+ta)\right| dx = \sum_{j=1}^k \int |f(jx+ta)|\, dx$$

$$(u = jx+ta) \quad = \sum_{j=1}^k j^{-d} \int |f|$$

$$= \boxed{\|f\|_{L^1} \sum_{j=1}^k j^{-d}}$$

Therefore $\forall f \in C_c(\mathbb{R}^d)$, $\lim_{t\to\infty} \int \left|\sum_{j=1}^k f(jx+ta)\right| dx = \|f\|_{L^1} \sum_{j=1}^k j^{-d}$.

We claim that this extends to all of L^1 by density. We recall that $C_c(\mathbb{R}^d)$ is dense in L^1. Therefore $\forall f \in L$ $\exists g_m \in C_c$ s.t. $g_m \to f$ in L^1.

We note that by the triangle inequality, $\forall t$,

$$\int \left|\sum_{j=1}^k f(jx+ta)\right| dx \le \sum_{j=1}^k \int |f(jx+ta)|\, dx = \|f\|_{L^1} \sum_{j=1}^k j^{-d}$$

Let $C = \sum_{j=1}^k j^{-d}$.

$\longrightarrow$

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(b) By direct computation, $\forall\ z = x+iy$, w/ $y>0$

$$u(z) - u(\bar{z}) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{t-z} - \frac{f(t)}{t-\bar{z}}\,dt$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{2iy\,f(t)}{(t-x)^2+y^2}\,dt$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y\,f(t)}{(t-x)^2+y^2}\,dt$$

Then by the triangle inequality,

$$\limsup_{y\to 0^+} |u(z)-u(\bar{z})| \le \limsup_{y\to 0^+} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y\,|f(t)|}{(t-x)^2+y^2}\,dt$$

For fixed x, t, $\frac{y\,|f(t)|}{(t-x)^2+y^2}$ is decreasing for sufficiently small $y>0$

Then by the MCT, $\int_{-\infty}^{\infty} \frac{y|f(t)|}{(t-x)^2+y^2}\,dt \to 0$ as $y\to 0^+$.

Therefore

$$\lim_{y\to 0^+} (u(z)-u(\bar{z})) = 0$$

$\forall\ x\in\mathbb{R}$ where $z = x+iy$. $\square$

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then by the triangle inequality, $\forall\, t$

$$\left| \int \left| \sum_{j=1}^{k} f(jx+ta) \right| dx - \|f\|_{L^1} \underbrace{\sum_{j=1}^{k} j^{-d}}_{C} \right| = \|f\|_{L^1} C - \int \left| \sum_{j=1}^{k} f(jx+ta) \right| dx$$

$$\leq C\left(\|f-g_m\|_{L^1} + \|g_m\|_{L^1}\right) - \int \left(\left| \sum_{j=1}^{k} g_m(jx+ta) \right| - \left| \sum_{j=1}^{k} f(jx+ta) - g_m(jx+ta) \right| \right) dx$$

$$= C\|f-g_m\|_{L^1} + \left(\|g_m\|_{L^1} - \int \left| \sum_{j=1}^{k} g_m(jx+ta) \right| \right) + \int \left| \sum_{j=1}^{k} f(jx+ta) - g_m(jx+ta) \right| dx$$

$$\leq C\|f-g_m\|_{L^1} + \left(\|g_m\|_{L^1} - \int \left| \sum_{j=1}^{k} g_m(jx+ta) \right| dx \right) + \sum_{j=1}^{k} \int \left| f(jx+ta) - g_m(jx+ta) \right| dx$$

$$= 2C\|f-g_m\|_{L^1} + \left(\|g_m\|_{L^1} - \int \left| \sum_{j=1}^{k} g_m(jx+ta) \right| dx \right)$$

B/c $g_m \in C_c$, taking $t \to \infty$ sends the second term to 0.
Then taking $m \to \infty$ sends the first term to 0.
Therefore

$$\int \left| \sum_{j=1}^{k} f(jx+ta) \right| dx \longrightarrow C\|f\|_{L^1}$$

for all $f \in L^1(\mathbb{R}^d)$.

D

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Viewing the sum as an integral w.r.t the counting measure, we may apply Hölder to find that

$$\left|\sum_{n=1}^{\infty}\left|\frac{f(x+n)}{\sqrt{n}}\right|\right| \le \left(\sum_{n\geq 1}|f(x+n)|^p\right)^{1/p}\left(\sum_{k\geq 1} k^{-\frac{p'}{2}}\right)^{1/p'} \qquad (1)$$

where $p' = \frac{p}{1-p}$ is the Hölder conjugate of p.

To show that $\sum_{n=1}^{\infty}\frac{f(x+n)}{\sqrt{n}}$, it suffices to show that $\sum\frac{f(x+n)}{\sqrt{n}} \in L^p_{loc}$.
In particular, it suffices to show $\int_m^{m+1}\left|\sum_{n\geq 1}\frac{f(x+n)}{\sqrt{n}}\right|^p dx < \infty \ \forall m\in\mathbb{Z}$.
By (1) above, $\forall m\in\mathbb{Z}$,

$$\int_m^{m+1}\left|\sum_{n\geq 1}\left|\frac{f(x+n)}{\sqrt{n}}\right|\right|^p dx \le \left(\sum_{k\geq 1} k^{-\frac{p'}{2}}\right)^{1/p'}\int_m^{m+1}\sum_{n\geq 1}|f(x+n)|^p$$

Since $1\le p<2$, $p'>2$. Therefore $p'/2>1$ and so $\sum_{k\geq 1} k^{-p'/2}<\infty$.
Then by Tonelli's, we swap the sum and integral to find

$$\int_m^{m+1}\left|\sum\left|\frac{f(x+n)}{\sqrt{n}}\right|\right|^p dx \lesssim \sum_{n\geq 1}\int_m^{m+1}|f(x+n)|^p dx$$

$$(u = x+n) \qquad = \sum_{n\geq 1}\int_{m-n}^{m+1-n}|f|^p$$

$$\le \|f\|_{L^p}^p < \infty$$

Therefore $\sum_{n\geq 1}\frac{f(x+n)}{\sqrt{n}}\in L^p_{loc}$ and so is finite a.e. as desired.

First consider $2<p\le\infty$. Take $f = \langle x\rangle^{-1/2}$ where $\langle x\rangle$ is the Japanese bracket $\langle x\rangle = \sqrt{1+|x|^2}$. Then f is bounded, so it is in L^∞. Also, $|f|^p = \langle x\rangle^{-p/2}$ for $2<p<\infty$ which is integrable since $p/2>1$. Therefore $f\in L^p \ \forall\ 2<p\le\infty$. However, $\forall x$, for sufficiently large n,

$$\frac{f(x+n)}{\sqrt{n}} \gtrsim_x \frac{n^{-1/2}}{\sqrt{n}} = n^{-1}$$

which is not summable. Therefore $\sum_{n\geq 1}\frac{f(x+n)}{\sqrt{n}}$ diverges for all x.

It remains to show the $p=2$ case. ⟶

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As given, $\forall k$,

$$\|f\|^p_{L^p(\mu_k)} \leq M^p \left(\sum_j \left(\int |f_j|^p \, d\nu_k \right)^{2/p} \right)^{p/2}$$

Since $p \geq 2$, $p/2 \geq 1$. Therefore by Minkowski's integral inequality,

$$\|f\|^p_{L^p(\mu_k)} \leq M^p \int \left(\sum_j |f_j|^{p \cdot 2/p} \right)^{p/2} d\nu_k$$

$$= M^p \int \left(\sum_j |f_j|^2 \right)^{p/2} d\nu_k$$

Then,

$$\|f\|^p_{L^p(\mu)} = \sum_k \|f\|^p_{L^p(\mu_k)} \leq M^p \sum_k \int \left(\sum_j |f_j|^2 \right)^{p/2} d\nu_k$$

$$\Rightarrow \|f\|_{L^p(\mu)} \leq M \left(\sum_k \int \left(\sum_j |f_j|^2 \right)^{p/2} d\nu_k \right)^{1/p} = M \left(\int \left(\sum_j |f_j|^2 \right)^{p/2} d\nu \right)^{1/p}$$

By Minkowski's integral inequality again, since $p/2 \geq 1$,

$$\|f\|^2_{L^p(\mu)} = M^2 \left(\int \left(\sum_j |f_j|^2 \right)^{p/2} d\nu \right)^{2/p}$$

$$\leq M^2 \sum_j \left(\int |f_j|^p d\nu \right)^{2/p}$$

Rearranging then yields

$$\|f\|_{L^p(\mu)} \leq M \left(\sum_j \|f_j\|^2_{L^p(\nu)} \right)^{1/2}$$

as desired. $\square$

(9a) 405 846 515

We recall Borel-Caratheodory which states that if f is holomorphic on $\overline{D(z_0,R)}$ then $\forall\ 0 \le r < R$,

$$\max_{|z-z_0|\le r} |f(z)| = \max_{|z-z_0|=r} |f(z)| = \frac{2r}{R-r} \max_{|z-z_0|\le R} |\operatorname{Re} f(z)| + \frac{R+r}{R-r} |f(z_0)|$$

An immediate consequence of this is that if f_j are holomorphic on $\overline{D(z_0,R)}$, $f_j(z_0)$ converges and $\operatorname{Re}(f_j(z))$ converges uniformly, then $\{f_j\}$ converges uniformly on $\overline{D(z_0,r)}$ $\forall\ 0 \le r < R$.

Since Ω is open, any compact subset of Ω can be covered by finitely many small closed disks contained in Ω. Therefore, to show that f_j converges uniformly on all compact subsets of Ω, it suffices to show that f_j converges uniformly on all ~~sufficiently small~~ closed disks in Ω. By choosing our cover of compact subsets so that the double of each disk is in Ω, ~~we must~~ it suffices to show that f_j converges uniformly on all disks $\overline{D(z_0,R)}$ s.t. $\overline{D(z_0,2R)} \subset \Omega$.

Combining our corollary to Borel Caratheodory w/ the fact that $\operatorname{Re}(f_j)$ converges uniformly on all compact subsets of Ω, it suffices to show that f_j converges pointwise on Ω.

Let $F = \{z \in \Omega : f_j(z) \text{ converges}\}$. Then $a \in F$ so F is non-empty. ~~By our reasoning above, $\forall z \in F$,~~ We claim that F is open. For all $z \in \Omega$ $\exists\ R > 0$ s.t. $\overline{D(z,2R)} \subset \Omega$. Then by our corollary above, f_j converges uniformly on $\overline{D(z,R)}$. Therefore $D(z,R) \subset F$. Then F is open.

We now claim that F is closed.

Suppose $\exists \{z_n\} \subset \mathcal{F}$ s.t. $z_n \to z$ in Ω.

Since Ω is open, $\exists R > 0$ s.t. $\overline{D(z, 4R)} \subset \Omega$.

Since $z_n \to z$, $\exists n$ s.t. $z_n \in D(z, R)$.

Then $z \in D(z_n, R)$ and $\overline{D(z_n, 2R)} \subset \overline{D(z, 4R)} \subset \Omega$.

As before, this implies that f converges uniformly on $\overline{D(z_n, R)}$ and so converges at z. Therefore $z \in \mathcal{F}$ and so $\mathcal{F}$ is closed.

Since $\mathcal{F} \subset \Omega$ is non-empty, open, and closed, and Ω is connected, $\mathcal{F} = \Omega$. Therefore f_1 converges pointwise on Ω, which concludes. ▫

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(9) alternate

Let $g_j = e^{f_j}$. Then $|g_j| = e^{\mathrm{Re} f_j}$. Since e^z is uniformly continuous on compact sets, this implies that $|g_j|$ converges uniformly on compact subsets of Ω. In particular, $|g_j|$ is ~~bounded~~ uniformly bounded on compact subsets of Ω.

Montel's theorem then implies that every subsequence (g_{j_k}) of (g_j) admits a further subsequence which converges uniformly on compact subsets.

To show that (g_j) converges uniformly on compact subsets, ~~it~~ Urysohn's subsequence principle then implies that it suffices to show that all convergent subsequences of (g_j) converge to the same limit.

Let g_{j_k} and g_{j_n} denote two convergent subsequences of (g_j), w/ limits g and $\tilde{g}$ respectively. Since $\mathrm{Re} f_j$ converges, ~~uniformly,~~ $|g_{j_k}| = e^{\mathrm{Re} f_{j_k}}$ and $|g_{j_n}| = e^{\mathrm{Re} f_{j_n}}$ converge to the same limit in particular, $|g| = |\tilde{g}|$ on Ω. Then $g = e^{i\theta}\tilde{g}$ for some $\theta \in \mathbb{R}$ since $g, \tilde{g}$ are holomorphic. Since $f_j(a)$ converges, it follows that $g(a) = e^{i\theta}\tilde{g}(a)$ and so $\theta = 0$. Therefore $g = \tilde{g}$.

Urysohn's subsequence lemma then concludes. $\square$

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Suppose that f has a zero of order k at 0. Then $f(z)/z^k$ is entire and does not have a zero at 0. If $f(z)/z^k = p(z)$ for some polynomial p, then $f(z) = z^k p(z)$ is a polynomial. Therefore it suffices to consider the case where $f(0) \neq 0$. *

Suppose $f(0) \neq 0$. Then by Jensen's formula, for all $r > 0$,

$$\log|f(0)| + \sum_{|a_n|<r} \log(r/|a_n|) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})| \leq \frac{1}{2\pi}\int_0^{2\pi} \log_+|f(re^{i\theta})|\,d\theta$$

$$\Rightarrow \log|f(0)| + \sum_{|a_n|<r} \log(r/|a_n|) \leq m(r)$$

where $\{a_n\}$ are the zeros of f, repeated according to multiplicity. Rearranging this, this implies that

$$\frac{\log|f(0)|}{\log r} \leq \frac{m(r)}{\log r} - \frac{\sum_{|a_n|<r} \log(r/|a_n|)}{\log r}$$

We note that $\sum_{|a_n|<r} \log(r/|a_n|) \gtrsim \#\{|a_n| < \sqrt{r}\} \log r$ for sufficiently large r. Therefore

$$\limsup_{r\to\infty} \frac{\log|f(0)|}{\log r} \leq \limsup_{r\to\infty} \frac{m(r)}{\log r} - \liminf_{r\to\infty} \#\{|a_n| < \sqrt{r}\}$$

Since $f(0) \neq 0$, $\limsup_{r\to\infty} \frac{\log|f(0)|}{\log r} = 0$. Therefore

$$\liminf_{r\to\infty} \#\{|a_n| < \sqrt{r}\} \leq \limsup_{r\to\infty} \frac{m(r)}{\log r} < \infty$$

This implies that f has finitely many zeros $a_1, \dots, a_n$, repeated according to multiplicity.

$\rightarrow$

* This is because $|g(re^{i\theta})| \leq |f(re^{i\theta})|$ for sufficiently large r and so

$$\limsup_{r\to\infty} \frac{\frac{1}{2\pi}\int_0^{2\pi} \log|g(re^{i\theta})|\,d\theta}{\log r} \leq \limsup_{r\to\infty} \frac{m(r)}{\log(r)} < \infty$$

Therefore we may work w/ g instead.

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We claim that f is of order 0. Suppose on the contrary. Then $\forall \varepsilon > 0$ ~~for inf~~ $\forall \kappa > 0$, for sufficiently large r,

$$|f(re^{i\theta})| \gtrsim e^{\kappa r^{\varepsilon}}$$

Then $m(r) \gtrsim r^{\varepsilon}$, for large r. However, then

$$\frac{m(r)}{\log r} \gtrsim \frac{r^{\varepsilon}}{\log r} \to \infty \quad \text{as} \quad r \to \infty$$

which is a contradiction. Therefore f is of order 0.

Since f is of order 0 and has finitely many zeros $a_1, \ldots, a_n$, Hadamard factorization implies that f can be written

$$f(z) = e^{a} \prod_{i=1}^{n} (z - a_i)$$

where $a \in \mathbb{C}$ is a constant. Therefore f is a polynomial. $\square$

(11) 405 846 515

(a) Let $z = x + iy$. Then $\forall\, t \in \mathbb{R}$,

$$\left|\frac{f(t)}{t-z}\right| = \frac{|f(t)|}{\sqrt{(t-x)^2+y^2}} \leq \frac{|f(t)|}{|y|} \qquad (1)$$

To show that U is holomorphic, it suffices to show that u is holomorphic on on compact subsets of $\mathbb{C} \setminus \mathbb{R}$.

Fix a compact subset $K \subset \mathbb{C}\setminus\mathbb{R}$. Let $I = \min_{z \in K} |\operatorname{Im}(z)| > 0$.

By Morera's theorem it suffices to show that u is continuous and conservative on K.

We first show continuous. By (1), $\forall\, t \in \mathbb{R}, z \in K$, $\left|\frac{f(t)}{t-z}\right| \leq \frac{|f(t)|}{I}$

Therefore by the DCT, $\forall\, z_0 \in K$, b/c $f \in L^1$,

$$\lim_{z \to z_0} U(z) = \int_{-\infty}^{\infty} \lim_{z \to z_0} \frac{f(t)}{t-z} \, dt = \int_{-\infty}^{\infty} \frac{f(t)}{t-z_0} \, dt = U(z_0)$$

and so u is continuous.

Now let $T \subset K$ be a triangle. Since $|T| < \infty$,

$$\int_T \int_{-\infty}^{\infty} \left|\frac{f(t)}{t-z}\right| dt \, |dz| \leq |T| \frac{\|f\|_{L^1}}{I} < \infty$$

Therefore Fubini's theorem can ~~be~~ applied to see that

$$\int_T u(z) \, dz = \frac{1}{2\pi i} \int_T \int_{-\infty}^{\infty} \frac{f(t)}{t-z} \, dt \, dz = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_T \frac{f(t)}{t-z} \, dz \, dt = 0$$

where the last inequality follows since $\frac{f(t)}{t-z}$ is holomorphic in $z \in K$.

Then by Morera's theorem, u is holomorphic on K.

By (1), we also see that

$$|u(z)| \leq \int_{-\infty}^{\infty} \frac{|f(t)|}{|\operatorname{Im}(z)|} dt \leq \frac{\|f\|_{L^1}}{|\operatorname{Im}(z)|} \to 0 \quad \text{as } |\operatorname{Im}(z)| \to \infty$$

as desired.

$\longrightarrow$

Analysis
F21

1) ~~405 846 515~~

We normalize the Fourier transform as

$$\hat{g}(k) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx} g(x)\,dx$$

with this, we note that $(\bar{g})^\wedge(k) = \overline{\hat{g}(-k)}$

and $\left(\frac{\partial g}{\partial x}\right)^\wedge(k) = -ik\,\hat{g}(k)$ whenever g is smooth.

Fix $f \in L^1$ s.t. $\int f(x)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4}\right)\varphi\,dx = 0$ $\forall$ smooth φ, 2π-periodic.

~~Then since $f \in L^1$, φ and its derivatives are bounded, and e^{ikx} is bounded~~. Since $f \in L^1$, $f(x) = \sum_{k\in\mathbb{Z}} \hat{f}(k)e^{ikx}$ where the sum converges in L^1. Since φ is smooth and 2π-periodic,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4}\right)\varphi(x) = \sum_{j\in\mathbb{Z}} \left((ij)^2 + (ij)^4\right)\hat{\varphi}(j)e^{ijx}$$

where the sum converges uniformly.

B/c $f \in L^1$ and φ is smooth, we may freely rearrange the sums and integrals to see that

$$0 = \int_0^{2\pi} f(x)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4}\right)\varphi(x)\,dx$$

$$= \sum_{j,k} \hat{f}(k)\hat{\varphi}(j)(j^4 - j^2)\int_0^{2\pi} e^{ikx}e^{ijx}\,dx$$

If $j = -k$ then $\int_0^{2\pi} e^{ikx}e^{ijx}dx = \int_0^{2\pi} 1\,dx = 2\pi$.

If $j \neq -k$ then $e^{ikx}e^{ijx} = e^{i(k+j)x}$, and so $\int_0^{2\pi} e^{ikx}e^{ijx}dx = \frac{1}{i(k+j)}e^{i(k+j)x}\Big|_0^{2\pi} = 0$.

Therefore

$$0 = 2\pi\sum_{k\in\mathbb{Z}} (k^4 - k^2)\hat{f}(k)\hat{\varphi}(-k)$$

$\longrightarrow$

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Consider $\varphi = \dfrac{e^{-ikx}}{2\pi}$. Then

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4}\right)\varphi(x) = \frac{1}{2\pi}\left(k^4 - k^2\right)e^{-ikx}$$

and so

$$0 = \int_0^{2\pi} f(x)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^4}\right)\varphi(x)\,dx$$

$$= \frac{1}{2\pi}(k^4 - k^2)\int_0^{2\pi} f(x)e^{-ikx}\,dx$$

$$= (k^4 - k^2)\hat{f}(k)$$

Therefore $\forall\, k \in \mathbb{Z}$, either $k^4 - k^2 = 0$ or $\hat{f}(k) = 0$. Since $k^4 - k^2 = 0$ iff $k = 0, \pm 1$, this implies that $\hat{f}(k) = 0 \ \forall\, |k| > 1$.

Since f is in L^1, we recall that

$$f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k)\, e^{ikx} \quad \text{a.e.}$$

Therefore

$$f(x) = \hat{f}(0) + \hat{f}(1)e^{ix} + \hat{f}(-1)e^{-ix}$$

Letting $a = \hat{f}(0)$, $b = \hat{f}(1)$, and $c = \hat{f}(-1)$ then concludes. $\square$

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By Lusin's theorem, $\forall\, n \geq 1$, $\exists$ a compact $K_n \subset [0,1]$ s.t.

$|[0,1] \setminus K_n| < 1/n$ and $\varphi|_{K_n}$ is continuous.

Then $\varphi(K_n) = \varphi|_{K_n}(K_n)$ is compact in $[0,1]$.

Define $B = \bigcup_{n=1}^{\infty} \varphi(K_n)$. Then $B \subset \varphi[0,1]$ and B is Borel.

By construction, $\forall\, n \geq 1$,

$$m(\varphi^{-1}(B)) \geq m(K_n) = 1 - m([0,1] \setminus K_n) \geq \frac{n-1}{n}.$$

Taking $n \to \infty$ then implies $m(\varphi^{-1}(B)) = 1$ as desired. $\square$

(4) 405 846 515

(a) Define a measure μ_n by

$$\mu_n = \frac{1}{2^n \pi r_n^2} \sum_{D \in C_n} \mathcal{L}|_D$$

where $\mathcal{L}$ is the Lebesgue measure on $[0,1]^2$. Since $\mathcal{L}$ is Borel and C_n is pairwise disjoint, μ_n is Borel and well-defined.
By construction, since $|C_n| = 2^n$

$$\mu_n[0,1]^2 = \frac{1}{2^n \pi r_n^2} \sum_{D \in C_n} \mathcal{L}|_D([0,1]^2)$$

$$= \frac{1}{2^n \pi r_n^2} \sum_{D \in C_n} \mathcal{L}(D)$$

$$= \frac{2^n \pi r_n^2}{2^n \pi r_n^2} = 1$$

Therefore μ_n is a Borel probability measure on $[0,1]^2$.
Moreover, since μ_n is supported on K_n, $\mu_n(K_n) = 1$.
Viewing $M([0,1]^2)$ as the dual of $C[0,1]^2$, Banach-Alaoglu implies that the closed unit ball in $M([0,1]^2)$ is weak* compact. Since $C[0,1]^2$ is separable, this implies that $\exists$ a subsequence μ_{n_k} s.t. $\mu_{n_k} \xrightarrow{*} \mu$ for some μ in $M([0,1]^2)$ w/ $\mu[0,1]^2 \leq 1$.

We first claim that μ is a probability measure. Since 1 is continuous on $[0,1]^2$, weak* convergence implies $\mu[0,1]^2 = \int 1 \, d\mu = \lim_{k\to\infty} \int 1 \, d\mu_{n_k} = 1$.
~~and~~ Thus μ is a Borel probability measure.

We now claim $\mu(D) = 2^{-n}$ $\forall$ $D \in C_n$.
~~Since D° is open, $\exists$ $f_j \in C[0,1]^2$ s.t. $f_j \uparrow \chi_D$. These can be constructed explicitly as~~

$$\cancel{f_j = \max(1 - jd(x,D), 0)}$$

Then $\mu(D) \leq \int f_j \, d\mu$ $\forall j$.

~~weak*~~ convergence then implies that

Since D is closed ~~and D° is open~~, $\exists\ f_j, \cancel{g_i} \in C[0,1]^2$ s.t. $f_j \le 1$

$f_j \searrow \chi_D$ ~~and $g_i \nearrow \chi_{D^\circ}$.~~ These can be constructed explicitly as

$$f_j = \max\left(1 - j\, d(x, D), 0\right)$$

$$\cancel{g_i = \min\left(i\, d(x, [0,1]^2 \setminus D^\circ), 1\right)}$$

Then by monotonicity, $\forall\ k, j$

$$\mu_{n_k}(D) \le \int f_j \, d\mu_{n_k}$$

By weak* convergence, this implies

$$\limsup_{k\to\infty} \mu_{n_k}(D) \le \int f_j \, d\mu$$

Taking $j \to \infty$, DCT implies

$$\limsup_{k\to\infty} \mu_{n_k}(D) \le \mu(D) \qquad (1)$$

For $n_k \ge n$, construction implies that

$$\mu_{n_k}(D) = \frac{1}{2^{n_k}\pi r_{n_k}^2} \sum_{\substack{D' \in C_{n_k} \\ D' \subset D}} \mathcal{L}(D \cap D')$$

By construction, $|\{D' \in C_{n_k} : D' \subset D\}| = 2^{n_k - n}$. Then

$$\mu_{n_k}(D) = \frac{1}{2^{n_k}\pi r_{n_k}^2} 2^{n_k - n} \pi r_{n_k}^2 = 2^{-n} \quad \forall D \in C_n$$

Therefore $\mu(D) \ge 2^{-n}$. Since $|C_n| = 2^n$ and C^n is pairwise disjoint,

$$1 \ge \mu\left(\bigcup_{D \in C_n} D\right) = 2^n \mu(D)$$

and so $\mu(D) \le 2^{-n}$. Therefore $\mu(D) = 2^n\ \forall\ D \in C_n$.

We note that the result (1) only relied on the fact that D is closed. Therefore it holds $\forall\ K_n$, and so

$$\limsup_{k\to\infty} \mu_{n_k}(K_n) \le \mu(K_n) \quad \forall n.$$

By construction, for $n_k \ge n$, since $D \subset K_n\ \forall\ D \in C_{n_k}$

$$\mu_{n_k}(K_n) = \frac{1}{2^{n_k}\pi r_{n_k}^2} \sum_{D \in C_{n_k}} \mathcal{L}(D \cap K_n)$$

$$= \frac{1}{2^{n_k}\pi r_{n_k}^2} \sum_{D \in C_{n_k}} \mathcal{L}(D) = 1 \qquad \longrightarrow$$

Therefore $1 \le \mu(K_n) \le \mu[0,1]^2 = 1$ and so $\mu(K_n) = 1$ $\forall n$.
Then by monotone convergence, $\mu(\bigcap_{n=1}^{\infty} K_n) = 1$, $\Rightarrow \mu(K) = 1$ as claimed.

(b) Suppose $\exists$ a closed set $V \subseteq [0,1]^2$ s.t. $\mu(V) = 1$.
We claim $K \subset V$. Suppose on the contrary.
Then $\exists\, x \in K$ s.t. $x \notin V$.
By definition of K, $\exists$ a sequence of closed disks $D_i \in C_i$ s.t. $x \in D_i$ $\forall i$. Since $D_i \in C_i$, D_i has radius r_i. By construction, $r_i \to 0$.
Since V is closed and $x \notin V$, $\exists$ some r_{i+1} s.t. $D(x, r_{i+1}) \subset [0,1]^2 \setminus V$.
Then since $r_i < r_{i+1}$, $D_i \subset D(x, r_{i+1}) \subset [0,1]^2 \setminus V$.
By part (a), $\mu(D_i) = 2^{-i}$ and so

$$\begin{aligned}\mu(V) &\le \mu([0,1]^2 \setminus D_i) \\ &= 1 - 2^{-i} \\ &< 1\end{aligned}$$

This contradicts $\mu(V) = 1$ and so $K \subset V$. Therefore K is the smallest closed set of full measure and so $\operatorname{supp} \mu = K$. $\square$

(7) 405 846 515

Since f, g are holomorphic, $u = \mathrm{Re}(f) - \mathrm{Re}(g)$ is harmonic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Since $\overline{\mathbb{D}}$ is compact, u is uniformly continuous on $\overline{\mathbb{D}}$.

As given, $u = 0$ on $\partial\mathbb{D}$. Therefore $\forall\, \varepsilon > 0\ \exists\, 0 < r < 1$ s.t. if $|z| \geq r$ then $|u| \leq \varepsilon$. By the maximum (resp minimum) principle, this implies that $|u| \leq \varepsilon\ \forall\, |z| < r$, and hence $|u| \leq \varepsilon$ on $\mathbb{D}$.
Taking $\varepsilon \to 0$ implies $u = 0$ on $\mathbb{D}$.
Therefore $f - g$ is a purely imaginary holomorphic function on $\mathbb{D}$. We claim that this implies that $f - g$ is constant.
Suppose not. Then ~~f-g is an~~ by the open mapping theorem, $f - g$ is an open map. Therefore $(f-g)(\mathbb{D})$ is open and contained in $i\mathbb{R}$. However, no ~~non-triv~~ non-empty subsets of $i\mathbb{R}$ are open in $\mathbb{C}$, so this is a contradiction. Therefore $f - g$ is constant and hence an imaginary constant.

$\square$

(9) 405 846 515

Let $f(z) = z^5 \cos z + 5iz^4 + 2$. We claim that f has 4 zeros, ~~including~~ counting multiplicity, inside $\mathbb{D}$.

By the triangle inequality, $\forall |z| = 1$,

$$\begin{aligned} |z^5 \cos z + 2| &\leq |z|^5 |\cos z| + 2 \\ &= 2 + \left| \frac{e^{iz} + e^{-iz}}{2} \right| \\ &\leq 2 + e^{|z|} \\ &= 2 + e < 5 = |5iz^4| \end{aligned}$$

Therefore by Rouché's theorem, $5iz^4$ and $5iz^4 + z^5 \cos z + 2 = f(z)$ have the same # of zeros in $\mathbb{D}$. Since $5iz^4$ has a single zero of multiplicity 4 at the origin, f has 4 zeros in $\mathbb{D}$. $\square$

(10) 405 846 515

If f is constant then the inequality holds. Therefore we work in the case where f is non-constant. In particular, where f, f' are not identically zero.

Suppose that f nonconstant satisfies the inequality.

Since $f \not\equiv 0$, f'/f is a meromorphic function on $\mathbb{C}$, w/ discrete singularities.

By the inequality, $|f'/f| \leq 2$ except potentially at the singularities.

Since the singularities are discrete, and f'/f is bounded the Riemann removeable singularity theorem implies that f'/f only has removeable singularities and hence is entire.

By Liouville's, this implies that $f'/f = c$ for some $|c| \leq 2$.

Then $f' = cf$. By the uniqueness of solutions to ODEs, this implies that $f = \kappa e^{cz}$ for some $\kappa \in \mathbb{C}$, $|c| \leq 2$.

We note that $c = 0$ handles the f constant f case.

If $f = \kappa e^{cz}$ for $|c| \leq 2$, then $f' = cf \Rightarrow |f'| \leq 2|f|$ as desired.

Therefore $|f'| \leq 2|f|$ iff $f = \alpha e^{cz}$ for some $\kappa \in \mathbb{C}$, $|c| \leq 2$. $\square$

(12) 405 846 515

(a) Recall the Blaschke factor φ_a for $a \in \mathbb{D}$ defined by

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a} z}$$

By ~~contruction~~, φ_a is ~~an~~ holomorphic on an open neighborhood of $\overline{\mathbb{D}}$ w/ a simple zero at a and $|\varphi_a| = 1$ on $\partial\mathbb{D}$.

Consider $\varphi_{a/2}(z/2)$ for $a \in B$. Then $\varphi_{a/2}(z/2)$ has only a simple zero at $a \in B$, is holomorphic on a neighborhood of $\overline{B}$. and satisfies $|\varphi_{a/2}(z/2)| = 1$ on ∂B. By maximum modulus, $|\varphi_{a/2}(z/2)| \leq 1$

Since f has zeros at $\pm i, \pm 1$,

$$g(z) = f(z) \Big/ \varphi_{1/2}(z/2)\, \varphi_{-1/2}(z/2)\, \varphi_{i/2}(z/2)\, \varphi_{-i/2}(z/2)$$

is holomorphic on B.

We claim that $|g| \leq 1$ on B. Since $|\varphi_{a/2}(z/2)| = 1$ on ∂B, $\lim_{|z| \to 1} |\varphi_{a/2}(z/2)| = 1$. Therefore

$$\limsup_{|z| \to 1} |g(z)| = \limsup_{|z| \to 1} |f(z)| \leq 1.$$

The maximum modulus principle then implies $|g| \leq 1$ on B. In particular, $|g(0)| \leq 1$ and so

$$|f(0)| \leq \prod_{a = \pm 1, \pm i} |\varphi_{a/2}(0)|$$

By definition,

$$|\varphi_{1/2}(0)| = |\varphi_{-1/2}(0)| = |\varphi_{i/2}(0)| = |\varphi_{-i/2}(0)| = 1/2.$$

~~and contradict~~ Therefore

$$|f(0)| \leq 1/16$$

as desired.

$\longrightarrow$

(b) Consider

$$f(z) = \varphi_{1/2}(z/2)\,\varphi_{-1/2}(z/2)\,\varphi_{i/2}(z/2)\,\varphi_{-i/2}(z/2)$$

Then f has zeros at $\pm 1, \pm i$ and, by the maximum modulus principle, satisfies $|f|<1$ on B since f is non-constant and $|f|_{\partial B}=1$. Therefore f satisfies the ~~constraint~~ of the problem and

$$|f(0)| = 1/16$$

as claimed.

$\square$

Analysis
Spring 2020

(2) 405 846 515

We recall the Riesz pre-compactness criterion for L^p which states

Let $\mathcal{F} \subset L^p$ for $1 \le p < \infty$. Then $\mathcal{F}$ is pre-compact in L^p iff $\mathcal{F}$ satisfies

(1) uniform boundedness: $\exists M \geq 0$ s.t. $\|f\|_{L^p} \le M \quad \forall f \in \mathcal{F}$.

(2) tightness: $\forall \varepsilon > 0 \ \exists R(\varepsilon) > 0$ s.t. $\sup_{f \in \mathcal{F}} \int_{|x| \ge R(\varepsilon)} |f| < \varepsilon$

(3) ~~equicontinuity~~: $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. if $|y| < \delta$ then $\forall f \in \mathcal{F}$

$$\|f(x+y) - f(x)\|_{L^p(x)} < \varepsilon$$

If time permits, we prove this. Though we note that it is a straightforward proof which mirrors Arzela-Ascoli, and we only need the backwards direction for this problem.

Therefore, to show that $\{f_n\}$ has a subsequence converging in L^1, it suffices to show (1), (2), (3) above. Since (1), (2) are given, ~~we ont~~ it only remains to show (3).

Fix some $y > 0$. Then $\forall n$,

$$\|f_n(x+y) - f_n(x)\|_{L^1(x)} = \int_{\mathbb{R}} |f_n(x+y) - f_n(x)| \, dx$$

Since f_n is differentiable $\forall n$, the FTC implies

$$\|f_n(x+y) - f_n(x)\|_{L^1(x)} = \int_{\mathbb{R}} \left| \int_x^{x+y} f_n'(t) \, dt \right| dx$$

$$\le \int_{\mathbb{R}} \int_{\mathbb{R}} |f_n'(t)| \, \chi_{[x, x+y]}(t) \, dt \, dx$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{[t-y, t]}(x) \, |f_n'(t)| \, dt \, dx$$

By Tonelli's,

$$\|f_n(x+y) - f_n(x)\|_{L^1} \le \int_{\mathbb{R}} \int_{\mathbb{R}} |f_n'(t)| \, \chi_{[t-y, t]}(x) \, dx \, dt = |y| \int |f_n'| \le |y|.$$

A symmetric argument shows the same result for $y < 0$.

Therefore $\|f_n(x+y) - f_n(x)\|_{L^1} \to 0$ uniformly as $y \to 0$ which concludes. ◻

(3) 405 846 515

We can write

$$L^\infty \cap L^3 = \bigcup_{m \geq 0} \underbrace{\{f \in L^3 : \|f\|_{L^\infty} \leq m\}}_{A_m}$$

We claim that A_m is closed in L^3.
Suppose $\exists$ $(f_n) \subset A_m$ s.t. $f_n \to f$ in L^3.
Then $f \in L^3$. By passing to a ~~subsequence~~ subsequence, we may assume that $f_n \to f$ pointwise a.e. Since $\|f_n\|_{L^\infty} \leq m$, definition implies that $|\{|f_n| > m+\varepsilon\}| = 0$ and so $|f_n| \leq m+\varepsilon$ a.e $\forall\, \varepsilon > 0$.
Therefore $|f| \leq m+\varepsilon$ a.e for all $\varepsilon > 0$ and so $\|f\|_{L^\infty} \leq m$.
Then $f \in A_m$ and so A_m is closed in L^3.

Therefore $L^\infty \cap L^3$ is the countable union of closed sets and hence is Borel. $\square$

④ 405 846 515

Suppose that the result has been shown on a dense subset $\mathcal{F} \subset L^1$. Then $\forall f \in L^1$ $\exists$ $g_k \in \mathcal{F}$ s.t. $g_k \to f$ in L^1.
By the triangle inequality, $\forall k$

$$\left|\int_0^2 f(x)\sin(x^n)dx\right| \le \left|\int_0^2 g_k(x)\sin(x^n)dx\right| + \|f-g_k\|_{L^1}$$

since $|\sin(x^n)| \le 1$. Taking $n\to\infty$ and then $k\to\infty$ implies that $\int_0^2 f(x)\sin(x^n)\,dx \to 0$ as $n\to\infty$ as desired.

Therefore, it suffices to show the result on a dense subset of L^1.

Consider the family of simple functions over intervals. This is dense in L^1, so it suffices to show the result for such functions.
By linearity, it then suffices to show the result for the indicator function of an interval $[a,b] \subset [0,2]$.

Then

$$\int_0^2 \chi_{[a,b]}\sin(x^n)\,dx = \int_a^b \sin(x^n)\,dx$$

$$= \left(\int_{[a,b]\cap[0,1)} + \int_{[a,b]\cap[1,2]}\right)\sin(x^n)\,dx$$

Consider first $[a,b]\cap[0,1]$. For $x\in[0,1)$, $x^n \to 0$ as $n\to\infty$.
Then $\sin(x^n)\to 0$ a.e. on $[0,1)$ as $n\to\infty$. Since $\sin(x^n)\le 1$,
DCT then implies that

$$\int_{[a,b]\cap[0,1)} \sin(x^n)\,dx \to 0 \quad \text{as } n\to\infty.$$

It then remains to consider $[a,b]\cap[1,2]$.

$\longrightarrow$

Let $[c,d] = [a,b] \cap [1,2]$. Then by integration by parts,

$$\int_c^d \sin(x^n)\,dx = \int_c^d \frac{nx^{n-1}}{nx^{n-1}} \sin(x^n)\,dx$$

$$\begin{pmatrix} du = -nx^{n-1}\sin(x^n)dx \\ v = 1/nx^{n-1} \end{pmatrix} = -\frac{\cos(x^n)}{nx^{n-1}}\Big|_c^d + \int_c^d \frac{1-n}{nx^n} \cos(x^n)\,dx$$

Then by the triangle inequality,

$$\left|\int_c^d \sin(x^n)\,dx\right| \leq \frac{1}{nd^{n-1}} + \frac{1}{nc^{n-1}} + \frac{1-n}{n}\int_c^d x^{-n}\,dx$$

$$= \frac{1}{nd^{n-1}} + \frac{1}{nc^{n-1}} + \frac{1-n}{n}\frac{1}{1-n} x^{-n+1}\Big|_c^d$$

$$= \frac{1}{nd^{n-1}} + \frac{1}{nc^{n-1}} + \frac{1}{n}\left(\frac{1}{d^{n-1}} - \frac{1}{c^{n-1}}\right)$$

Since $c,d \geq 1$, this implies $\left|\int_c^d \sin(x^n)dx\right| \to 0$ as $n \to \infty$.

Therefore $\int_0^2 \chi_{[a,b]} \sin(x^n)dx \to 0$ as $n \to \infty$ $\forall$ $[a,b] \subset (0,2)$ which concludes. $\square$

5) 405 846 515

(a) Suppose that $\{f_n\}$ is a bounded sequence in $L^2(0,1]$.

Then by Banach-Alaoglu, $\exists$ a subsequence $f_{n_K} \xrightarrow{*} f$ converges weak* on $L^2[0,1]$ when $L^2[0,1]$ is equipped w/ the standard inner product. Then $\forall$ x,

$$Tf_{n_K}(x) = \int_0^x f_{n_K}(t)\,dt = \langle f_{n_K}, \chi_{[0,x]}\rangle \to \langle f, \chi_{[0,x]}\rangle = Tf(x)$$

Therefore $Tf_n \to Tf$ pointwise.

We aim to upgrade pointwise convergence to L^2 via DCT.

Let M denote the bound on $\{f_n\}$. Then by Fatou, $\|f\|_{L^2} \le M$.

Then by Hölder, $\forall x$, $\forall K$

$$|Tf_{n_K}(x)| \le \int_0^1 |f_{n_K}(t)|\,dt \le \|f_{n_K}\|_{L^2} \le M$$

and similarly for f. Since $M \in L^2(0,1)$, DCT implies $Tf_{n_K} \to Tf$ in L^2.

(b) We note that $\forall$ $f \in L^2$, $Tf \in C[0,1]$. This is because $\forall$ $x \ge y$

$$|Tf(x) - Tf(y)| \le \int_y^x |f| \le |x-y|^{1/2}\|f\|_{L^2} \to 0 \text{ as } |x-y| \to 0$$

Now suppose for the sake of contradiction that $f \not\equiv 0$ is an eigenfunction of T w/ eigenvalue λ. Let x be a Lebesgue point of it. Then

$$f(x) = \lim_{h\to 0}\int_{x-h}^{x+h}\frac{f(t)}{2h}\,dt = \lim_{h\to 0}\frac{Tf(x+h)-Tf(x-h)}{2h} = \lambda\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h} = \lambda f'(x)$$

Therefore f is differentiable a.e. and satisfies $f = \lambda f'$.

Since $Tf \in C[0,1]$, $f \in C(0,1)$. If $\lambda = 0$ then $f \equiv 0$, which is a contradiction.

Assume $\lambda \ne 0$. Then $f = ae^{x/\lambda}$ by uniqueness of ODEs

for some $a \in \mathbb{R}$. However, then for all x,

$$a\lambda e^{x/\lambda} = \int_0^x ae^{t/\lambda}\,dt = a\lambda e^{t/\lambda}\Big|_{t=0}^{x} = a\lambda e^{x/\lambda} - a\lambda$$

which implies $a = 0 \Rightarrow f \equiv 0$, which is a contradiction.

Therefore T has no eigenvalues.

⑥ 405 846 515

(a) To show that no subsequence of L_n converges weak*, it suffices to define $f \in L^\infty$ s.t. $L_n(f)$ diverges to ∞. If such an f can be found then $L_{n_k}(f)$ diverges to ∞ $\forall$ subsequences L_{n_k} and so no subsequence converge.

We first aim to calculate the maximum of $\frac{x^n e^{-x}}{n!}$. Since $\frac{x^n e^{-x}}{n!} \to 0$ as $x \to \infty$ and $\frac{0^n e^{-0}}{n!} = 0$ while $\frac{x^n e^{-x}}{n!} \geq 0$, its maximum occurs at a nonzero critical point. By direct calculation,

$$0 = \left(\frac{e^{-x}x^n}{n!}\right)' = \frac{nx^{n-1} - x^n}{n!} e^{-x} \iff x = 0, n$$

and so $\frac{x^n e^{-x}}{n!}$ achieves its maximum of $\frac{n^n e^{-n}}{n!} = m_n$ at $x = n$.

Moreover, on $[n, n+1)$, $\frac{x^n e^{-x}}{n!} \geq \frac{n^n e^{-n}}{n!} e^{-1} = \frac{m_n}{e}$.

Define

$$f = \sum_{n \geq 0} \frac{n}{m_n} \chi_{[n,n+1)}$$

Then $\forall$ x, $f(x) \leq \frac{n}{m_n} = \frac{n e^n n!}{n^n} \to 0$ as $n \to \infty$,

and so $f(x)$ is bounded. Therefore $f \in L^\infty$.

However,

$$L_n(f) \geq \int_n^{n+1} \frac{n}{m_n} \cdot \frac{m_n}{e} \, dx = \frac{n}{e} \to \infty \quad \text{as} \quad n \to \infty.$$

Therefore $L_n(f)$ diverges to ∞, which concludes.

(b) ~~We recall that Banach-Alaoglu only holds for reflexive spaces, i.e. spaces X s.t. $X^{**} = X$.~~
~~Since L^∞ is not reflexive, Banach-Alaoglu does not apply here~~

$\longrightarrow$

(b) We recall that Banach-Alaoglu implies that the closed ball in $(L^\infty)^*$ is weak* compact.
Since L^∞ is not separable, weak* compact does not imply weak* sequentially compact. Therefore ~~the~~ a ~~weak~~ closed ~~unit~~ ball in $(L^\infty)^*$ is not necessarily sequentially weak* compact.

To see that L^∞ is not separable, consider

$$\mathcal{F} = \{ \chi_{[0,t)} : t > 0 \}$$

Clearly $\mathcal{F}$ is uncountable and $\forall\ t' > t$,

$$\| \chi_{[0,t')} - \chi_{[0,t)} \|_{L^\infty} = \| \chi_{[t,t')} \|_{L^\infty} = 1$$

Therefore L^∞ is not separable. $\square$

7 405 346 515

By Montel's theorem, it suffices to show that $\mathcal{F}_M$ is ~~locally~~ uniformly bounded on compact subsets of $\mathbb{D}$. Since any compact subset of $\mathbb{D}$ can be contained in a closed disk of radius $0<R<1$, it suffices to consider ~~$\overline{D(0,R)}$~~, $\{|z|\le R\}$ for $0<R<1$.

Fix $0<R<1$ and $z\in\{|z|\le R\}$. By the Cauchy integral formula, $\forall\ R<r<1$, $\forall f\in\mathcal{F}_M$,

$$|f(z)| = \left|\int_{\{|z|=r\}} \frac{f(w)}{w-z}\,dw\right| \le \int_{\{|z|=r\}} \frac{|f(w)|}{|w-z|}|dw| \le \frac{1}{r-R}\int_{\{|z|=r\}} |f(w)|\,|dw|$$

~~For all $f\in\mathcal{F}_M$, f is uniformly continuous on~~

Parameterizing $\{|z|=r\}$, this implies

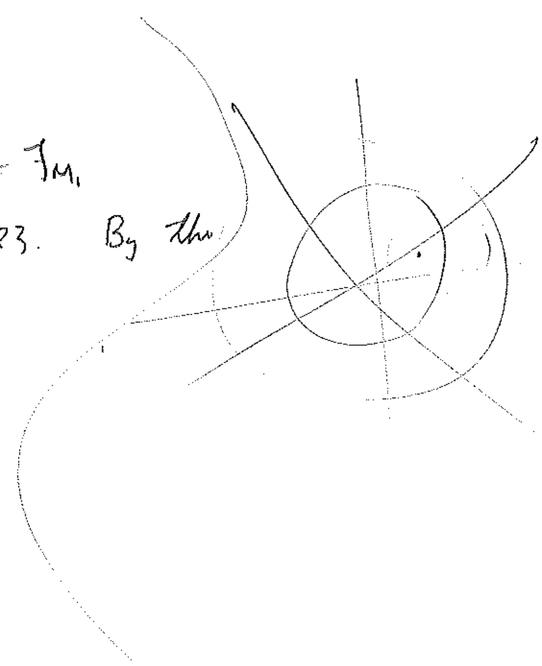
$$|f(z)| \le \frac{r}{r-R}\int_0^{2\pi} |f(re^{i\theta})|\,d\theta$$

~~$w=re^{i\theta}$~~ ~~$|dw|=|rie^{i\theta}d\theta|$~~ $=r$

For all $f\in\mathcal{F}_M$, f is uniformly continuous on $\overline{\mathbb{D}}$. Therefore, taking the limit as $r\to1$,

$$\begin{aligned}|f(z)| &\le \limsup_{r\to1}\frac{r}{r-R}\int_0^{2\pi}|f(re^{i\theta})|\,d\theta\\ &= \frac{1}{1-R}\int_0^{2\pi}|f(e^{i\theta})|\,d\theta\\ &\le \frac{M}{1-R}\end{aligned}$$

As this holds $\forall\ z\in\{|z|\le R\}$ and $\forall f\in\mathcal{F}_M$, this implies $\mathcal{F}_M$ is uniformly bounded on $\{|z|\le R\}$. By the earlier reasoning, this concludes. $\square$



⑧ 405 846 515

(a) To show that F is entire, it suffices to show that the series converges uniformly on all compact subsets of $\mathbb{C}$.

Fix a compact $K \subset \mathbb{C}$. To show the series converges uniformly on K, it suffices to show that it is uniformly Cauchy.

Let $R = \max\{|z| : z \in K\}$. Then $\forall\, n \le m$,

$$\left| \sum_{j \geq n} (-1)^n \frac{(z/2)^{2j}}{(j!)^2} - \sum_{j \geq m} (-1)^n \frac{(z/2)^{2j}}{(j!)^2} \right| \le \sum_{j=n}^{m-1} \frac{|z|^{2j}}{2^{2j}(j!)^2} \le \sum_{j=n}^{m-1} \frac{R^{2j}}{2^{2j}(j!)^2}$$

which goes to 0 as $n, m \to \infty$ uniformly for $z \in K$. Therefore the series converges uniformly on K and so F is entire.

By direct computation, $\forall z$

$$|F(z)| \le \sum_{n \geq 0} \frac{|z|^{2n}}{2^{2n}(n!)^2}$$

Consider $2^{2n}(n!)^2$. By definition,

$$2^{2n}(n!)^2 = \left(2^n \cdot n!\right)^2 = \left((2n)!!\right)^2 \geq ((2n)!!)((2n-1)!!) = (2n)!$$

Therefore,

$$|F(z)| \le \sum_{n \geq 0} \frac{|z|^{2n}}{(2n)!} \le \sum_{k \geq 0} \frac{|z|^k}{k!} = e^{|z|}$$

as desired.

(b) Since $F(z)$ is a power series w/ only even terms, F is an even function. Therefore if $F(z) = 0$ then $F(-z) = 0$.

Moreover, if z is a zero of order k, then $-z$ is a zero of order k.

Let $a_1, a_2, \ldots$ enumerate the zeros of F in $\{z : \operatorname{Im}(z) \geq 0,\ z \notin (-\infty, 0)\}$, repeated according to multiplicity. Since F is not identically 0, $\{a_n\}$ is at most countably infinite.

Moreover, since $F(0) = 1$, $0 \notin (a_n)$.

Let $a_{-n} = -a_n$. $\longrightarrow$

~~By the Hadamard factorization theorem, since~~
~~$(a_{\pm n})$ are all zeros of F, repeated according to multiplicity~~
~~and $F(z)$ is of order at most 1,~~

$$\cancel{F(z) = e^{az+b} \prod}$$

Suppose that $(a_n) = (a_1, a_2, \dots, a_k)$ for $k \in \mathbb{N} \cup \{\infty\}$ w/ the obvious notation. Then by the Hadamard factorization theorem, since F is of order at most 1,

$$F(z) = e^{az+b} \prod_{\substack{n=-k \\ n \neq 0}}^{k} E_1(z/a_n)$$

$$= e^{az+b} \prod_{\substack{n=-k \\ n \neq 0}}^{k} e^{z/a_n} (1 - z/a_n)$$

where the product converges locally uniformly.

Rearranging and using the fact that $a_{-n} = -a_n$, this implies

$$F(z) = e^{az+b} \prod_{n=1}^{k} e^{z/a_n} e^{-z/a_n} (1 - z/a_n)(1 + z/a_n)$$

$$= e^{az+b} \prod_{n=1}^{k} (1 - z^2/a_n^2)$$

Since F is even, $a = 0$. Since $F(0) = 1$, $b = 0$. Therefore

$$F(z) = \prod_{n=1}^{k} (1 - z^2/a_n^2)$$

If $k < \infty$, then F is a polynomial. However, this would ~~+~~ imply that the power series expansion of F at 0 is finite, ~~contra~~ contradicting the earlier expansion. Therefore $k = \infty$ and

$$F(z) = \prod_{n=1}^{\infty} (1 - z^2/a_n^2)$$

as desired.

$\square$

(11) 405 846 515

(a) Fix $K \subset S^1$ compact w/ $K \neq S^1$. Suppose that $e^{i\theta} \notin K$.

Let $\mathbb{C}_\theta = \mathbb{C} \setminus \{re^{i\theta} : r \geq 0\}$. Then $K \subset \mathbb{C}_\theta$.

Since $\mathbb{C}_\theta$ is an open simply connected subset of $\mathbb{C}$, the Riemann mapping theorem implies that $\exists$ a conformal map φ s.t. φ maps $\mathbb{C}_\theta$ to $\mathbb{D}$.

We note that on S^1, $\bar{z} = 1/z$. Therefore it suffices to find a sequence of polynomials which uniformly approximate $f(z) = 1/z$. We note that f is holomorphic on $\mathbb{C}_\theta$. Consider $f \circ \varphi^{-1}$. Then $f \circ \varphi^{-1}$ is holomorphic on $\mathbb{D}$. In particular, $f \circ \varphi^{-1}$ can be expressed as a power series which converges uniformly on compact subsets of $\mathbb{D}$. Then $\exists$ a sequence of polynomials $\tilde{p}_n$ which converge uniformly to $f \circ \varphi^{-1}$ on $\varphi(K)$. ~~Con~~ Since φ is uniformly continuous on $\varphi(K)$, this implies that $\tilde{p}_n \circ \varphi$ converge uniformly to f on K. Since $\tilde{p}_n$ are entire,

$\tilde{p}_n \circ$ …

<u>idea</u>: It suffices to approximate $1/z$, since $\bar{z} = 1/z$ on S^1.

Since $1/z$ is holomorphic on $\mathbb{C}_\theta$, this should be possible.

(b) Suppose on the contrary. Then since S^1 is compact,

$$\int_{S^1} \bar{z}\, dz = \lim_{n \to \infty} \int_{S^1} P_n(z)\, dz = 0$$

However, $\int_{S^1} \bar{z}\, dz = \int_{S^1} \frac{1}{z}\, dz = 2\pi i \neq 0.$

Therefore no such polynomials exist.

(12) 405 846 515

(a) Suppose on the contrary that $f \circ f$ has no fixed points.
Consider the function
$$g(z) = \frac{f(f(z)) - z}{f(z) - z}$$
If $f(z) - z = 0$ then $f(f(z)) = f(z) = z$ which contradicts the fact that $f \circ f$ has no fixed points. Therefore $g(z)$ is entire.
Since $f \circ f$ has no fixed points, g is nonvanishing.
If $g(z) = 1$ then $f(f(z)) - z = f(z) - z \Rightarrow f(f(z)) = f(z)$ and so f has a fixed point. As above, this would imply $f \circ f$ has a fixed point. Therefore $g(z) \neq 1$.
B/c g avoids both 0 and 1, Little Picard's theorem implies $g \equiv a$ for some $a \in \mathbb{C}$.
Then $f(f(z)) - z = a f(z) - az \Rightarrow f(f(z)) - af(z) = (1-a) z$

ANALYSIS

FALL 2019

① 405 846 515

We aim to show that $\forall$ measurable g on $X \times X$,

$$\int g(x,y)\, d\mu_1 \otimes \mu_2(x,y) = \int g(x,y) \frac{d\mu_1}{d\nu_1}(x) \frac{d\mu_2}{d\nu_2}(y)\, d\nu_1 \otimes \nu_2(x,y) \qquad (*)$$

Doing so will show that $\mu_1 \otimes \mu_2 \ll \nu_1 \otimes \nu_2$ and then that $\frac{d\mu_1 \otimes \mu_2}{d\nu_1 \otimes \nu_2}$ exists and is equal a.e. to $\frac{d\mu_1}{d\nu_1}(x) \frac{d\mu_2}{d\nu_2}(y)$. We note that we have not assumed the existence of $\frac{d\mu_1 \otimes \mu_2}{d\nu_1 \otimes \nu_2}$ in this reasoning, ~~and we do it so~~ in order to avoid it becoming circular.

Define

$$\mathcal{F} = \{\text{measurable } g \text{ s.t. } (*) \text{ holds}\}$$

We claim that $\mathcal{F}$ is all measurable functions. To show this, b/c $\mathcal{X} \times \mathcal{X}$ is generated by product sets, the monotone class theorem implies that it suffices to show that $\mathbb{1}_{A \times B} \in \mathcal{F}$ $\forall A, B \in \mathcal{X}$, that $\mathcal{F}$ is closed under linear combination, and that $\mathcal{F}$ is closed under monotone increasing sequences. Linearity of the integral and MCT imply that it suffices to show $\mathbb{1}_{A \times B} \in \mathcal{F}$ $\forall A, B \in \mathcal{X}$. By direct computation,

$$\begin{aligned}
\int \mathbb{1}_{A \times B}(x,y)\, d\nu_1 \otimes \nu_2 &= \int_B \int_A d\nu_1(x)\, d\nu_2(y) \\
&= \int_B \int_A \frac{d\mu_1}{d\nu_1}\, d\mu_1(x)\, d\nu_2(y) \\
&= \int_B \int_A \frac{d\mu_1}{d\nu_1}(x) \frac{d\mu_2}{d\nu_2}(y)\, d\mu_1(x)\, d\mu_2(y) \\
&= \int \mathbb{1}_{A \times B} \frac{d\mu_1}{d\nu_1}(x) \frac{d\mu_2}{d\nu_2}(y)\, d\mu_1 \otimes \mu_2(x,y)
\end{aligned}$$

~~which concludes~~ as desired. Therefore $(*)$ holds $\forall$ measurable functions. ~~Therefore~~ By the earlier reasoning, this concludes.

□

③ 405 846 515

We first show $\|f\|_{p,\infty} \gtrsim [f]_p$ uniformly in f.

~~Fix some $E \in \mathcal{X}$ s.t. $\mu(E) \in (0,\infty)$. Then by Hölder's,~~

$$\cancel{\frac{1}{\mu(E)^{1-1/p}} \int_E |f| \le \frac{1}{\mu(E)^{1-1/p}} \mu(E)^{1-1/p} \|f\|_{L^p}}$$

Fix some $f \in L^{p,\infty}$. Then $\forall t>0$, $\mu(|f|>t) \in [0,\infty)$. If $\mu(|f|>t) = 0$, then trivially $t\mu(|f|>t)^{1/p} = 0 \le \|f\|_{p,\infty}$. If $\mu(|f|>t) > 0$, then by definition

$$\|f\|_{p,\infty} \ge \frac{1}{\mu(|f|>t)^{1-1/p}} \int_{|f|>t} |f|\, d\mu \ge \frac{1}{\mu(|f|>t)^{1-1/p}}\, t\,\mu(|f|>t) = t\mu(|f|>t)^{1/p}$$

Therefore $\forall\, t>0$, $\|f\|_{p,\infty} \ge t\mu(|f|>t)^{1/p} \Rightarrow \|f\|_{p,\infty} \ge [f]_p$ as desired.

WRONG but original attempt

We now show $\|f\|_{p,\infty} \lesssim [f]_p$ uniformly in f.

To show $[f]_p \lesssim \|f\|_{p,\infty}$, we note that by Jensen's inequality, $\forall E$ w/ $0 < \mu(E) < \infty$, since $p > 1$

$$\left(\frac{1}{\mu(E)} \int_E |f|\, d\mu\right)^p \le \frac{1}{\mu(E)} \int_E |f|^p\, d\mu$$

$$\Rightarrow \frac{1}{\mu(E)^{1-1/p}} \int_E |f|\, d\mu \le \left(\int_E |f|^p\, d\mu\right)^{1/p}$$

By layer cake, we can then write

$$\begin{aligned} \frac{1}{\mu(E)^{1-1/p}} \int_E |f|\, d\mu &\le \left(\int_0^\infty p\, t^{p-1} \mu(\{|f|>t\} \cap E)\, dt\right)^{1/p} \\ &\le [f]_p \left(\int_0^\infty p\, t^{-1}\, dt\right)^{1/p} \end{aligned} \quad (*)$$

~~Choosing E s.t. $\frac{1}{\mu(E)^{1-1/p}} \int_E |f|\, d\mu \ge \frac{1}{2}\|f\|_{p,\infty}$~~ and

The step $(*)$ doesn't work because the resulting integral is ∞, but a cleverer use of σ-finiteness should resolve this.

Choosing E s.t. $\frac{1}{\mu(E)^{1-1/p}} \int_E |f|\, d\mu \ge \frac{1}{2}\|f\|_{p,\infty}$ would then conclude the desired ~~so~~ inequality.

□

③ cont.

Failed solution - at (*)

$$\frac{1}{\mu(E)^{1-1/p}}\int_E |f|\,d\mu = \left(\int_0^\infty p\, t^{p-1}\mu(\{|f|>t\}\cap E)\,dt\right)^{1/p}$$

$$\le \left(\int_0^\infty p\, t^{p-1}\min\{\mu(|f|>t),\mu(E)\}\,dt\right)^{1/p}$$

$$\le \left(\int_0^\infty p\, t^{p-1}\min\left\{\frac{[f]_p^p}{t^p},\mu(E)\right\}dt\right)^{1/p}$$

$$= \left(\int_{\left\{\frac{[f]_p^p}{t^p}\le\mu(E)\right\}} p\, t^{-1}[f]_p^p\,dt + \int_{\left\{\mu(E)\le\frac{[f]_p^p}{t^p}\right\}} p\, t^{p-1}\mu(E)\,dt\right)^{1/p}$$

$\left(\lambda = \sqrt[p]{\frac{\mu(E)}{[f]_p^p}}\right)$

$$= p\left(\int_\lambda^\infty t^{-1}[f]_p^p\,dt + \int_0^\lambda t^{p-1}\mu(E)\,dt\right)^{1/p}$$

$$\int_E |f|\,d\mu = \int_0^\infty \mu(\{|f|>t\}\cap E)\,dt$$

$$\le \int_0^\infty \min\{\mu(|f|>t),\mu E\}\,dt$$

$$\le \int_0^\infty \min\left\{\frac{[f]_p^p}{t^p},\mu E\right\}dt$$

$$= \int_{\left\{\frac{[f]_p^p}{t^p}\le\mu E\right\}} \frac{[f]_p^p}{t^p}\,dt + \int_{\left\{\mu E\le\frac{[f]_p^p}{t^p}\right\}} \mu E\,dt$$

$\left(\lambda^p = \frac{[f]_p^p}{\mu E}\right)$
$$= \int_\lambda^\infty \frac{[f]_p^p}{t^p}\,dt + \int_0^\lambda \mu E\,dt$$

$(p>1)$
$$= [f]_p^p\,\frac{\lambda^{1-p}}{p-1} + \mu(E)\,\lambda$$

$$= [f]_p\left(\frac{\mu(E)^{1-1/p}}{p-1} + \mu(E)^{1-1/p}\right)$$

$$= [f]_p\,\mu(E)^{1-1/p}\left(\frac{p}{p-1}\right)$$

$\Rightarrow \dfrac{1}{\mu(E)^{1-1/p}}\displaystyle\int_E |f|\,d\mu \lesssim [f]_p$ uniformly in f, as desired. $\square$

(4) 405 846 515

Suppose $\exists A \subset \mathbb{R}$ w/ ~~positive~~ $|A|>0$. We aim to show that $A-A = \{z-y : z, y \in A\}$ has non-empty interior. By ~~intersecting~~ A w/ a sufficiently large ball, we may assume WLOG that $0<|A|<\infty$.

Per the hint, we define $\varphi(x) = \int \chi_A(x+y)\chi_A(y)dy$.
We claim that φ is continuous at 0. By definition,

$$|\varphi(x)-\varphi(0)| = \left| \int \chi_A(x+y)\chi_A(y)dy - \int \chi_A(y)^2 dy \right|$$
$$\le \int_A |\chi_A(x+y) - \chi_A(y)|dy$$

Let $\tau_x : L^1 \to L^1$ be the translation operator $\tau_x f(y) = f(x+y)$.
We recall that τ_x is continuous in L^1 since Lebesgue measure is translation invariant. Then

$$|\varphi(x)-\varphi(0)| \le \|\tau_x\chi_A - \chi_A\|_{L^1} \to 0 \text{ as } x\to 0 \quad \text{b/c } \chi_A \in L^1$$

and so φ is continuous at 0.
By direct calculation, we find that

$$\varphi(0) = \int \chi_A(y)^2 dy = \int \chi_A(y)dy = |A| > 0$$

Therefore by ~~continuity~~, φ is ~~contin~~ positive on a ball centered at 0.
By definition, we note that if $\varphi(x)>0$, then $\int \chi_A(x+y)\chi_A(y)dy>0$ and so $\exists$ some y s.t. $x+y, y\in A \Rightarrow x \in A-A$. Therefore, since $\varphi>0$ on a ball at 0, ~~A-A conta~~ $\exists$ a ball at 0 contained in $A-A$. Then $A-A$ has non-empty interior. $\square$

5) 405 846 515

We recall the uniform boundedness principle, which states

$\boxed{\begin{array}{l}\text{Let } X \text{ be a Banach space and } Y \text{ a normed vector space.} \\ \text{Suppose } \mathcal{F} \text{ is a collection of continuous linear operators } X \to Y. \\ \text{If } \quad \sup_{T \in \mathcal{F}} \|T(x)\|_Y < \infty \text{ for all } x \in X \\ \text{then } \quad \sup_{T \in \mathcal{F}} \|T\| < \infty.\end{array}}$

$$\cancel{\begin{array}{l} T_x : \mathcal{H} \to \mathbb{R} \qquad \mathcal{F} = \{T_x : \|x\| \le 1\} \\ T_x(y) = (Bx, y) = (x, Ay) \\ |T_x(y)| = |(Bx, y)| = |(x, Ay)| \le \begin{cases} \|Bx\| \|y\| \le \|Ay\| < \infty \\ \|Ay\| \|x\| \end{cases} \end{array}}$$

Define $T_x : \mathcal{H} \to \mathbb{R} : y \mapsto (Bx, y) = (x, Ay)$ and $\mathcal{F} = \{T_x : \|x\| \le 1\}$.

Then T_x is a continuous linear operator $\forall x$ b/c

$$\|T_x(y)\| = |(Ax, y)| \le \underbrace{\|Ax\|}_{<\infty} \|y\| \quad \text{and} \quad \|Ax\| < \infty.$$

so $\mathcal{F}$ is a family of continuous linear operators $\mathcal{H} \to \mathbb{R}$.

Moreover, for fixed $y \in \mathcal{H}$;

$$\sup_{T_x \in \mathcal{F}} |T_x(y)| = \sup_{T_x \in \mathcal{F}} |(x, Ay)| \le \sup_{T_x \in \mathcal{F}} \|Ay\| \|x\| = \|Ay\| < \infty$$

Therefore the uniform boundedness principle implies

$$\sup_{T_x \in \mathcal{F}} \|T_x\| = M < \infty$$

hence $T_x(Bx) = |Bx|^2 \quad \forall \|x\| \le 1$, it then follows that $\longrightarrow$

$\|B_x|$

for all $\|x\| \leq 1$ s.t. $Bx \neq 0$,

$$|Bx| = T_x\left(\frac{Bx}{|Bx|}\right) \leq M\left|\frac{Bx}{|Bx|}\right| = M < \infty$$

As this holds for all $\|x\| \leq 1$ s.t. $Bx \neq 0$, this implies that

$$\|B\| \leq M < \infty$$

and so B is a bounded linear operator.

A symmetric argument thus implies A is a bounded linear operator. $\square$

(6) 405 846 515

(1) Define $L \subset \ell^\infty(\mathbb{N})$ as
$$L = \{ x \in \ell^\infty(\mathbb{N}) : \lim_{n\to\infty} x_n \text{ exists} \}$$
By the linearity of the limit, L is a linear subspace of $\ell^\infty(\mathbb{N})$.
Define $\tilde{\varphi} : L \to \mathbb{N}$ by
$$\tilde{\varphi}(x) = \lim_{n\to\infty} x_n$$
Then $\tilde{\varphi}$ is ~~a~~ linear by the linearity of the limit.
Moreover, $|\tilde{\varphi}(x)| = \lim_{n\to\infty} |x_n| \le \lim_{n\to\infty} \|x\|_\infty = \|x\|_\infty$, and so $\tilde{\varphi}$ is bounded. B/c L is a linear subspace and $\tilde{\varphi}$ is a continuous linear functional defined on L, Hahn-Banach implies that $\exists$ a continuous linear functional φ on $\ell^\infty(\mathbb{N})$ s.t. $\varphi|_L = \tilde{\varphi}$ and $\|\varphi\| \le \|\tilde{\varphi}\| \le 1$. Therefore φ is a continuous linear functional on $\ell^\infty(\mathbb{N})$ s.t. $\varphi(x) = \lim_{n\to\infty} x_n$ whenever the limit exists.

(2) Define $L_e = \{ x \in \ell^\infty(\mathbb{N}) : \lim_{n\to\infty} x_{2n} \text{ exists} \}$
$$L_o = \{ x \in \ell^\infty(\mathbb{N}) : \lim_{n\to\infty} x_{2n+1} \text{ exists} \}$$
Define $\tilde{\varphi}_e : L_e \to \mathbb{N} : x \mapsto \lim_{n\to\infty} x_{2n}$
$$\tilde{\varphi}_o : L_o \to \mathbb{N} : x \mapsto \lim_{n\to\infty} x_{2n+1}$$
By the same reasoning as above, $\tilde{\varphi}_e, \tilde{\varphi}_o$ extend to continuous linear functionals φ_e, φ_o on $\ell^\infty(\mathbb{N})$.
Since $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+1}$ provided $\lim_{n\to\infty} x_n$ exists, it follows that $\varphi_e, \varphi_o : x \mapsto \lim_{n\to\infty} x_n$ provided the limit exists. However, let $\exists y \in \ell^\infty(\mathbb{N})$ s.t. $y_n = \begin{cases} 1 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$. Then $\varphi_e(y) = 2 \ne 1 = \varphi_o(y)$ so $\varphi_e \ne \varphi_o$. Therefore φ in part 1 is not unique. $\square$

(7) 405 846 515

Apply Stone-Weierstrass to closure of linear span of

$$t \longmapsto \frac{1}{z-t}$$

to show that linear combinations are dense in $C(J)$.
To show it is an algebra, you need to use closure for squaring elements.

Then $F_\mu = F_\nu$ iff $\int f\,d\mu = \int f\,d\nu \;\; \forall f \in C(J) \Rightarrow \mu = \nu$. $\square$

(8) 405, 846 515

Let $a_1, \dots, a_n$ denote the nonzero zeros of f in $\overline{\mathbb{D}}$ repeated according to mult. and note that $a_1, \dots, a_n \in \mathbb{D}$ b/c $|f|=1$ on $\partial\mathbb{D}$. Suppose that f has a zero of order m at 0.

We recall the Blaschke factor
$$\varphi_a(z) = \frac{z-a}{1-\bar{a}z} \quad \text{for } a \in \mathbb{D}$$
We recall that $|\varphi_a|=1$ on $\partial\mathbb{D}$, that φ_a is holomorphic on an open neighborhood of $\overline{\mathbb{D}}$ and that φ_a only has a simple zero at a.

Then $g(z) = f(z) / z^{m-1}\prod_{i=1}^{n} \varphi_{a_i}(z)$ is holomorphic on an open neighborhood of $\overline{\mathbb{D}}$, and $g(0)=0$. Additionally, $\forall z \in \partial\mathbb{D}$
$$|g(z)| = |f(z)| / |z|^{m-1} \prod_{i=1}^{n} |\varphi_{a_i}(z)| = 1$$
By the maximum modulus principle, $|g| \le 1$ on $\mathbb{D}$. The Schwarz lemma then implies that $|g(z)| \le |z|$ on $\mathbb{D}$ and $|g'(0)| \le 1$.

We recall that $\mathbb{D}$ is conformally equivalent to $\mathbb{H}$, the upper half plane, $\partial\mathbb{D}$ going to $\mathbb{R}$. Therefore, since g is a holomorphic map on a neighborhood of $\overline{\mathbb{D}}$ that fixes $\partial\mathbb{D}$, we can apply the Cayley transform φ that takes $\mathbb{H} \to \mathbb{D}$ to get a holomorphic map $\varphi^{-1} \circ g \circ \varphi$ that is holomorphic on a neighborhood of $\overline{\mathbb{H}}$ and fixes $\mathbb{R}$. The Schwartz reflection principle then implies that $\varphi^{-1} \circ g \circ \varphi$ extends to an entire function and so g extends to an entire function. B/c $|f(z)| \ne 0$, $f / z^{m-1} \prod \varphi_{a_i}$ has no poles, is entire and agrees w/ g on $\mathbb{D}$, this implies that $g = f / z^{m-1} \prod \varphi_{a_i}$ on all of $\mathbb{C}$.

→

$\frac{1}{\bar{a}_i} = \frac{z-i}{z+i} \qquad z =$

$z+i = \bar{a}_i(z-i)$

$\cancel{-i-\bar{a}_i i} \qquad \frac{-i(\bar{a}_i+1)}{1-\bar{a}_i}$

In particular,

$$f(z) = g(z)\, z^{m-1} \prod_{i=1}^{n} \varphi_{a_i}(z)$$

on all of $\mathbb{C}$. Since f is entire, this implies $g(z) z^{m-1} \prod_{i=1}^{n} \varphi_{a_i}(z)$ is entire. and hence g has a zero at $1/\bar{a}_i$ for all i, if $a_i \neq 0$.

By the Schwartz reflection principle,

$$\varphi^{-1} \circ g \circ \varphi(z) = \overline{\varphi^{-1} \circ g \circ \varphi(\bar{z})}$$

We recall that φ takes reflection across the $\mathbb{R}$-axis and turns it into reflection across the unit disk $z \mapsto 1/\bar{z}$.

Therefore $g(1/\bar{z}) = g(z)$ $\forall$ z, and so $g(1/\bar{a}_i) = g(a_i) \neq 0$.

This implies that $a_i = 0$ $\forall i$, but a_i were chosen to be nonzero, so no a_i exist. Then $g(z) = \frac{f(z)}{z^{m-1}}$

By the same reflection symmetry, we note that $\sup_{z\in\mathbb{C}} |g(z)| = \max_{z \in \bar{\mathbb{D}}} |g(z)| < \infty$.

Therefore g is an entire bounded function and hence is a constant. Since $|g| = 1$ on $\partial\mathbb{D}$, it follows that $g = a$ for some $|a| = 1$.

Then $f(z) = a z^{m-1}$ for some $|a| = 1$, as desired. $\square$

(8) 405 846 515

Cleaner proof.

Let $\{a_1, \ldots\}$ denote the zeros of f in $\overline{\mathbb{D}}$ (repeated according to multiplicity). Since $|f|=1$ on $\partial\mathbb{D}$, $a_1, \ldots, a_n \in \mathbb{D}$ and we know that $f \not\equiv 0$. Therefore the zeros of f cannot accumulate and so we can enumerate them $a_1, \ldots, a_n$.

Let φ_a be the Blaschke factor as before and note all properties. Define $g = f / \prod_{i=1}^{n} \varphi_{a_i}$. Then g has no zeros on $\overline{\mathbb{D}}$ and $|g|=1$ on $\partial\mathbb{D}$. Then $1/g$ is holomorphic on $\mathbb{D}$ w/ $|1/g| = 1$ on $\partial\mathbb{D}$. Maximum modulus then implies that $|g|, 1/|g| \leq 1$ on $\mathbb{D}$

$\Rightarrow 1 \leq |g| \leq 1$ on $\mathbb{D}$ $\Rightarrow |g| = 1$ on $\mathbb{D}$.

~~By the Cauchy-Riemann~~ Therefore $g = a$ on $\mathbb{D}$ for some $|a|=1$. Then $f = a \prod_{i=1}^{n} \varphi_{a_i}$. Since f is entire and φ_{a_i} has a pole if $a_i \neq 0$, this implies $a_i = 0$ $\forall i$ and so $f = az^n$ as desired. $\square$

(9) 405 846 515

We first claim that P has 6 zeros inside $2\mathbb{D} = \mathbb{D}(0,2)$.
To show this, we use Rouché's theorem to show that P and z^6 have the same number of zeros inside $2\mathbb{D}$.
To use Rouché's, it must be shown that $|6z^2 - 10z - 2| < |z^6|$ on $\partial(2\mathbb{D})$.

Fix some $z \in \partial 2\mathbb{D}$. Then $|z| = 2$ and so

$$\begin{aligned} |6z^2 - 10z - 2| &\le 6|z|^2 + 10|z| + 2 \\ &= 46 \\ &< 64 = 2^6 = |z^6| \end{aligned}$$

Therefore Rouché's theorem implies z^6 and $z^6 - (6z^2 - 10z - 2) = P(z)$ have the same # of zeros in $2\mathbb{D}$ and hence P has 6 zeros in $2\mathbb{D}$.

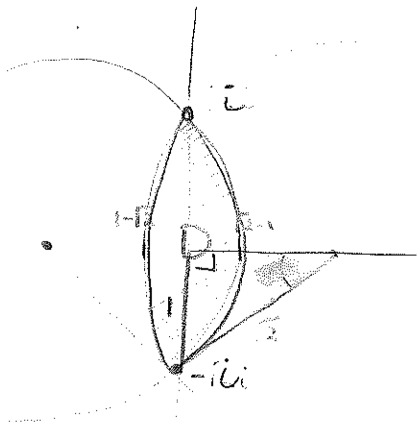
We now claim that P has 1 zero inside $\overline{\mathbb{D}}$. To do so, we use Rouché's again to ~~find~~ show that P and $10z$ have the same number of zeros inside $\mathbb{D}$. Along the way, we will find that P has no zeros on $\partial\mathbb{D}$.
Fix some $z \in \partial\mathbb{D} \Rightarrow |z| = 1$. Then

$$\begin{aligned} |z^6 - 6z^2 + 2| &\le |z|^6 + 6|z|^2 + 2 \\ &= 9 \\ &< 10 = |10z| \end{aligned}$$

This implies that $10z$ and $10z + z^6 - 6z^2 + 2 = P(z)$ have the same # of zeros inside $\mathbb{D}$ and hence P has 1 zero inside $\mathbb{D}$. Further, this shows that $|P(z)| \ge |10z| - |z^6 - 6z^2 + 2| \ge 1 > 0$ on ∂z and so P has no zeros on $\partial\mathbb{D}$.
Since $A = \{1 < |z| < 2\} = 2\mathbb{D} \setminus (\overline{\mathbb{D}})$, this implies that P has $6 - 1 = 5$ zeros inside A.

$\square$

(11) 405 846 515



We aim to find a conformal map that takes D to the unit disk.

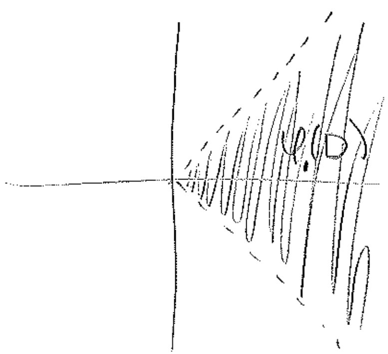
First, we find a ~~to~~ Mobius transform $\varphi_1(z) = \frac{az+b}{cz+d}$ that takes

$$-i \mapsto 0$$
$$i \mapsto \infty$$
$$0 \mapsto 1$$

Then $\varphi_1(z) = \frac{z+i}{i-z}$ suffices.

φ_1

Since Mobius transforms preserve circles on the Riemann sphere, the edges of D will be mapped to straight lines through the origin. Geometric reasoning yields that the edges of D meet at right angles at $\pm i$ and so they will still meet at right angles after φ.



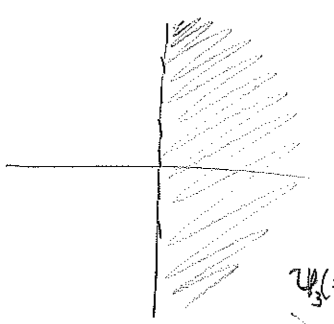
Finally, $\varphi_1(-a+bi) = \frac{-a+bi+i}{i+a-bi} = \overline{\left(\frac{-a-bi-i}{a+bi-i}\right)} = \overline{\varphi(a+bi)}$

So φ takes reflection across $i\mathbb{R}$ to reflection across $\mathbb{R}$. This implies that $\partial\varphi(D)$ reflected across $\mathbb{R}$ is still $\partial\varphi(D)$. Therefore, combining all these

$$\varphi(D) = \{z = re^{i\theta} : -\pi/4 < \theta < \pi/4\}$$

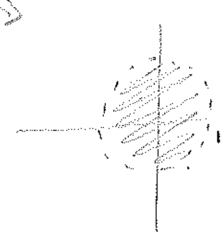
$\varphi_2 : z \mapsto z^2$

Define $\varphi_2(z) = z^2$. Then φ_2 is conformal on $\varphi_1(D)$ and $\varphi_2(\varphi_1(D)) = \{z : \mathrm{Re}(z) > 0\}$.



Finally, take $\varphi_3 = \frac{z-1}{z+1} = \frac{iz-i}{iz+i} = \Psi(iz)$ where Ψ is the Cayley transform. Since the Cayley transform takes $\mathbb{H}$ to $\mathbb{D}$ conformally, φ_3 conformally maps $\varphi_2(\varphi_1(D))$ to $\mathbb{D}$.

$\varphi_3(z) = \frac{z-1}{z+1}$



Then $\varphi = \varphi_3 \circ \varphi_2 \circ \varphi_1$ conformally maps D to $\mathbb{D}$.

(12) 405 846 515

To show that $F(z) = \int_1^\infty \frac{t^z}{\sqrt{1+t^3}}\,dt$ is well-defined and analytic on $A = \{z \in \mathbb{C} : Re(z) < 1/2\}$, it suffices to show that F is well-defined and analytic on any compact subset of A.

Fix a compact subset $K \subset A$. Let $R = \max\{Re(z) : z \in K\} < 1/2$.
Then $\forall\, z \in K$,

$$\left|\frac{t^z}{\sqrt{1+t^3}}\right| = \left|\frac{e^{z\log t}}{\sqrt{1+t^3}}\right|$$
$$= \frac{1}{\sqrt{1+t^3}} e^{Re(z)\log t}$$
$$\le \frac{t^R}{\sqrt{1+t^3}}$$

It then follows that

$$\left|\frac{t^z}{\sqrt{1+t^3}}\right| \le t^{R-3/2}$$

for all $t \in [1,\infty)$ and uniformly in $z \in K$. Since $R < 1/2$, $R - 3/2 < -1$ and so $t^{R-3/2}$ is absolutely integrable over $[1,\infty)$. Then $t^z/\sqrt{1+t^3}$ is absolutely integrable over $[1,\infty)$ $\forall\, z \in K$, so $F(z)$ is well-defined on K.
Moreover, $\forall\, z \in K$, $|F(z)| \le \int_1^\infty \left|\frac{t^z}{\sqrt{1+t^3}}\right| dt \le \int_1^\infty t^{R-3/2}\,dt \sim 1$

We now show F is analytic on K by Morera's theorem. We first show continuity.
Suppose $\exists\, z_n \in K$ s.t. $z_n \to z \in K$. Then since $\left|\frac{t^z}{\sqrt{1+t^3}}\right| \le t^{R-3/2} \in L^1[1,\infty)$ on K, DCT implies $\lim_{n\to\infty} F(z_n) = \lim_{n\to\infty} \int_1^\infty \frac{t^{z_n}}{\sqrt{1+t^3}}\,dt = \int_1^\infty \frac{t^z}{\sqrt{1+t^3}} = F(z)$ so F is continuous.

Now let γ be a closed loop in K such that $|\gamma| < \infty$ (i.e. a triangle).
Then since $\int_1^\infty \left|\frac{t^z}{\sqrt{1+t^3}}\right| dt \lesssim 1$ on K, $\int_\gamma \int_1^\infty \left|\frac{t^z}{\sqrt{1+t^3}}\right| dt\,dz \lesssim |\gamma| < \infty$.
Fubini's then implies that

$$\int_\gamma F(z)\,dz = \int_1^\infty \int_\gamma \frac{t^z}{\sqrt{1+t^3}}\,dz\,dt = \int_1^\infty 0\,dt = 0$$

b/c $\frac{t^z}{\sqrt{1+t^3}}$ is holomorphic. As F is cont and $\int_\gamma F\,dz = 0$ $\forall$ triangles γ, Morera's theorem implies F is holomorphic on K.

As this holds $\forall$ compact $K \subset A$, F is holomorphic on A.

Meromorphic extension will follow from considering

$$(z-1/2)F(z)$$

applying integration by parts in a clever way. $\square$

After

We would like to pick up a factor of $\frac{1}{z-1/2}$ in our expression for $F(z)$. To do so, we apply integration by parts.

$$F(z) = \int_1^\infty \frac{t^z}{\sqrt{1+t^3}}\,dt = \int_1^\infty t^{z-3/2}\,\frac{t^{3/2}}{\sqrt{1+t^3}}\,dt$$

$$dv = t^{z-3/2}\,dt \quad \Rightarrow \quad v = \frac{1}{z-1/2}\,t^{z-1/2}$$

$$u = \frac{t^{3/2}}{\sqrt{1+t^3}} \quad \Rightarrow \quad du = \frac{\sqrt{1+t^3}\,\frac{3}{2}t^{1/2} - t^{3/2}\,\frac{1}{2}\frac{3t^2}{\sqrt{1+t^3}}}{1+t^3}\,dt$$

$$= \frac{3}{2}\,\frac{t^{1/2}}{(1+t^3)^{3/2}}\,dt$$

$$\Rightarrow F(z) = \frac{1}{z-1/2}\,\frac{t^{z+1}}{\sqrt{1+t^3}}\Bigg|_{t=1}^{\infty} - \frac{1}{z-1/2}\int_1^\infty \frac{3t^z}{2(1+t^3)^{3/2}}\,dt$$

$$\left|\frac{t^{z+1}}{\sqrt{1+t^3}}\right| \sim \frac{t^{Re(z)+1}}{\sqrt{1+t^3}} \sim t^{Re(z)+1-3/2}$$

$$Re(z)-1/2<0$$

Given $Re(z) < 3/2$, this yields

$$F(z) = \frac{-1}{\sqrt{2}(z-1/2)} - \frac{3}{2(z-1/2)}\int_1^\infty \frac{t^z}{(1+t^3)^{3/2}}\,dt \quad (*)$$

By the same reasoning as earlier, we find that

$\int_1^\infty \frac{t^z}{(1+t^3)^{3/2}}\,dt$ is holomorphic provided

$Re(z) - 9/2 < -1 \iff Re(z) < 3/2$. Therefore $F(z)$ extends to a meromorphic function on $\{Re(z) < 3/2\}$ with a simple pole at $1/2$, because $(*)$ agrees w/ F on $\{Re(z) < 1/2\}$ and extends meromorphically to $\{Re(z) < 1/2\}$. $\square$

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① 405 840 515

Suppose for the sake of contradiction that $f''(c) \neq 0 \ \forall c \in \mathbb{R}$.
Since f'' is cont., this implies that either $f''>0$ everywhere or $f''<0$ everywhere. WLOG, assume $f''>0$ since otherwise we can replace f w/ $-f$.

Since $f''>0$, f is ~~strictly~~ convex.

Suppose $\exists$ $a \in \mathbb{R}$ s.t. $f'(a) \neq 0$. Since f is convex,

$$f(x) \geq f'(a)x + f(a)$$

B/c $f'(a)x + f(a)$ is unbounded, this implies that f is unbounded.

Suppose instead that $f'=0$. Then $f''=0$.

In either case, we reach a contradiction. Therefore $\exists c \in \mathbb{R}$ s.t. $f''(c)=0$.

$\square$

② 405 846 515

We first show that $F_n \to F$ pointwise.

Fix some $x \in [0,1]$. Let $\exists$ continuous $g_k, h_m \in C[0,1]$ s.t.

$g_k \uparrow \chi_{[0,x)}$, $h_m \downarrow \chi_{[0,x]}$. Then $\forall\, k, m$

$$\int g_k \, d\mu_n \le \mu_n[0,x] \le \int h_m \, d\mu_n$$

Since $g_k, h_m \in C[0,1]$, weak* convergence implies

$$\int g_k \, d\mu = \lim_{n\to\infty} \int g_k \, d\mu_n \le \liminf_{n\to\infty} \mu_n[0,x] \le \limsup_{n\to\infty} \mu_n[0,x] \le \int h_m \, d\mu$$

MCT then implies

$$\mu[0,x) = \liminf_{n\to\infty} \mu_n[0,x] \le \limsup_{n\to\infty} \mu_n[0,x] \le \mu[0,x]$$

Since $\mu\{x\}=0$, $\Rightarrow \lim_{n\to\infty} \mu_n[0,x] = \mu[0,x]$.

Therefore $F_n(x) \to F(x)$ pointwise.

We now upgrade this to uniform convergence.

Since μ is atom-less, F is continuous and hence equicontinuous.

Fix $\varepsilon > 0$. Then $\exists\ \delta > 0$ s.t. $|F(x)-F(y)| < \varepsilon\ \forall\ |x-y|<\delta$.

B/c $[0,1]$ is compact, $\exists\, 0 = t_1 < \dots < t_k = 1$ s.t. $[0,1] \subset \bigcup_{i=1}^{k} (t_i - \delta/2, t_i + \delta/2)$

Since $F_n \to F$ pointwise, $\forall\, i\ \exists\ N_i$ s.t. $|F_n(t_i) - F(t_i)| < \varepsilon\ \forall\, n \ge N_i$.

Let $N = \max(N_1, \dots, N_k)$. Then $\forall\, n \ge N$, $\forall i$ $|F_n(t_i) - F(t_i)| < \varepsilon$

Pick some $x \in [0,1]$. By construction $\exists$ some i s.t. $t_i \le x \le t_{i+1}$

and $|x-t_i|, |x-t_{i+1}| < \delta$. Then $\forall n \ge N$, b/c F_n, F are monotonically increasing

$$\begin{aligned} F_n(x) - F(x) &\le F_n(t_{i+1}) - F(t_i) \\ &\le \varepsilon + F(t_{i+1}) - F(t_i) \\ &\le 2\varepsilon \end{aligned}$$

and

$$\begin{aligned} F(x) - F_n(x) &\le F(t_{i+1}) - F_n(t_i) \\ &\le \varepsilon + F(t_{i+1}) - F(t_i) \\ &\le 2\varepsilon \end{aligned}$$

So $|F_n(x) - F(x)| \le 2\varepsilon$. ~~bec~~ As this holds $\forall\, x \in [0,1]$, $\|F_n - F\|_\infty \le 2\varepsilon\ \forall\, n \ge N$.

As such an N can be found $\forall\, \varepsilon > 0$, this implies $F_n \to F$ uniformly. $\square$

③ 405 846 515

(a) Since f is cont. and $\lim_{|t|\to\infty} f(t)=0$, it follows that $\|f\|_{L^\infty} = M < \infty$. ~~Note~~

Let $\mathcal{H} = \{hf : h\in L^1(\mathbb{R},m),\ \|h\|_{L^1}\le K\}$. We first show $\mathcal{H}$ is closed in L^1.
Suppose $\exists \{h_n f\}\subset \mathcal{H}$ s.t. $h_n f \to hf \in L^1$.
Then $\exists$ a subsequence $h_{n_k} f$ s.t. $h_{n_k} f \to hf$ pointwise a.e.
Since f is positive, this implies that $h_{n_k} \to h$ pointwise a.e.
Fatou's lemma then implies

$$\|h\|_{L^1} = \int |h| = \int \lim_{k\to\infty} |h_{n_k}| \le \liminf_{k\to\infty} \int |h_{n_k}| \le K$$

and so $hf \in \mathcal{H}$. As this holds $\forall$ such $\{h_n f\}\subset \mathcal{H}$, this implies $\mathcal{H}$ is closed.

We now aim to show that $\mathcal{H}$ is nowhere dense. Since H is closed, it suffices to show $\mathcal{H}$ has empty interior. Fix some $hf\in\mathcal{H}$.
Since $\lim_{t\to\infty} f(t) = 0$, $\exists$ $t_1 < t_2 < \dots \in \mathbb{N}$ s.t. $f(t) \le 2^{-n}$ $\forall$ $t\in[t_n, t_{n+1})$.
Define $g = \sum_{n=1}^{\infty} \chi_{[t_n, t_{n+1})}$. Then

$$\|gf\|_{L^1} \le \sum_{n=1}^{\infty} 2^{-n} = 1 < \infty$$

Fix $\varepsilon > 0$ and consider $(h+\varepsilon g)f \in L^1$. Then by direct computation,

$$\|(h+\varepsilon g)f - hf\|_{L^1} = \|\varepsilon g f\|_{L^1} \le \varepsilon$$

$$\|h+\varepsilon g\|_{L^1} \ge \varepsilon\|g\|_{L^1} - \underset{\forall hf\in H}{\|h\|_{L^1}} \ge \varepsilon \sum_{n=1}^{\infty} |[t_n, t_{n+1})| - K = \infty$$

Therefore $(h+\varepsilon g)f \notin \mathcal{H}$ and so $\forall \varepsilon>0$ $\exists$ $\varphi = (h-\varepsilon g)f \in L^1$ s.t. $\|\varphi - hf\|_{L^1} \le \varepsilon$ but $\varphi \notin \mathcal{H}$. This implies that $\mathcal{H}$ has empty interior and hence is nowhere dense since it is closed. □

(b) We ~~construct~~ g similarly to in part 6.

~~Since $f_1(t) \to 0$ as $t \to \infty$, $\exists\ t_1 \in \mathbb{N}$ s.t.~~

~~$f_1(t) \le 2^{-1}\ \forall\ t \in [t_1, t_1+1)$~~

We ~~now~~ aim to construct $t_1 < t_2 < \ldots \in \mathbb{N}$ ~~iterated~~ s.t. $\forall\ n$,

$$f_k(t) \le 2^{-n}\ \forall\ k \le n\ \forall\ t \in [t_n, t_{n+1})$$

We construct these t_n's ~~iterated~~ iteratively.

For $n=1$, since $f_1(t) \to 0$ as $t \to \infty$, $\exists\ t_1 \in \mathbb{N}$ s.t.

$$f_1(t) \le 2^{-1}\ \forall t \in [t_1, t_1+1)$$

Suppose $t_1 < \ldots < t_{n-1}$ have been constructed for $n > 1$.

For all $k \le n$, since $f_k(t) \to 0$ as $t \to \infty$, $\exists$ some $t_n^k \in \mathbb{N}$

s.t. $f_k(t) \le 2^{-n}\ \forall\ t \ge t_n^k$

Define $t_n = \max(t_{n-1}+1, t_n^1, \ldots, t_n^{n-1}, t_n^n)$

Then $t_n > t_{n-1}$, $t_n \in \mathbb{N}$ and $\forall\ k \le n$, $\forall t \in [t_n, t_{n+1})$, $f_k(t) \le 2^{-n}$.

Define g as

$$g = \sum_{n=1}^{\infty} 2^{-n} \chi_{[t_n, t_{n+1})}$$

Then

$$\|g\|_{L^1} \le \sum_{n \ge 1} 2^{-n} = 1 < \infty$$

By construction, $\forall k$, since $g, f_k \ge 0$,

$$g/f_k \ge \sum_{n \ge k} \frac{1}{f_k} 2^{-n} \chi_{[t_n, t_{n+1})}$$

b/c $f_k \le 2^{-n}$ on $[t_n, t_{n+1})$ $\forall\ n \ge k$,

$$g/f_k \ge \sum_{n \ge k} 2^n 2^{-n} \chi_{[t_n, t_{n+1})}$$

$$\Rightarrow \|g/f_k\|_{L^1} \ge \Big\| \sum_{n \ge k} \chi_{[t_n, t_{n+1})} \Big\|_{L^1} = \sum_{n \ge k} 1 = \infty$$

as desired. $\square$

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(a) Define $\tilde{\varphi} : V \to \mathbb{R}$ by

$$\tilde{\varphi} : f \longmapsto \lim_{n\to\infty} \int_{[0,1/n]} f \, d\mu$$

By definition of V, $\tilde{\varphi}$ is well-defined. By the linearity of the limit and integrals, $\tilde{\varphi}$ is linear. We claim that $\tilde{\varphi}$ is continuous. Suppose $\exists \{f_k\} \subset V$ s.t. $f_k \to f$ in L^∞. Then by linearity

$$|\tilde{\varphi}(f_k) - \tilde{\varphi}(f)| = \left| \lim_{n\to\infty} n \int_{[0,1/n]} f_k \, d\mu - \lim_{n\to\infty} n \int_{[0,1/n]} f \, d\mu \right|$$

$$\leq \lim_{n\to\infty} n \int_{[0,1/n]} |f_k - f| \, d\mu$$

$$\leq \lim_{n\to\infty} n \cdot \frac{1}{n} \cdot \|f_k - f\|_{L^\infty}$$

$$= \|f_k - f\|_{L^\infty}$$

Therefore $\tilde{\varphi}(f_k) \to \tilde{\varphi}(f)$ as $f_k \to f$ in L^∞.

Therefore $\tilde{\varphi}$ is a cont (bounded) linear functional on V, which is a linear subspace of L^∞ by the linearity of the limit.

Can be shortened by showing $\tilde{\varphi}$ is bounded instead

Hahn-Banach then implies that $\exists$ an extension $\varphi \in (L^\infty)^*$ of $\tilde{\varphi}$ s.t. $\varphi|_V = \tilde{\varphi}$ as desired.

(b) Suppose for the sake of contradiction that such a $g \in L^1$ exists. Consider $(a,b) \subset [0,1]$ w/ $a \neq 0$. ~~If $a = 0$, then for sufficiently large n, $(a,b) \cap [0,1/n] = (0,1/n]$~~

hence ~~$\Rightarrow \lim_{n\to\infty} n \int_{(0,1/n]} \chi_{(a,b)} \, d\mu = \lim_{n\to\infty} n \, \mu(0,1/n] = \lim_{n\to\infty} 1 = 1$~~

If $a \neq 0$, then for sufficiently large n, $(a,b) \cap [0,1/n] = \emptyset$. Then

$$\lim_{n\to\infty} n \int_{[0,1/n]} \chi_{(a,b)} \, d\mu = 0$$

~~In either case,~~ Therefore $\chi_{(a,b)} \in V$. $\longrightarrow$

Therefore, $\forall\ a,b>0$,

$$\int_{(a,b)} g\,d\mu = \int g\,\chi_{(a,b)}\,d\mu = \lim_{n\to\infty} n\int_{[0,1/n]} \chi_{(a,b)}\,d\mu = 0$$

hence $g \in L^1$ and any open interval in $[0,1]$ can be written as the countable union of intervals (a,b) w/ $a>0$, this implies that $\int_{(a,b)} g\,d\mu = 0\ \ \forall\ a,b \in [0,1]$. The Lebesgue differentiation theorem then implies that $g=0$ a.e.

However, then $\varphi(f)=0\ \ \forall\ f\in L^\infty$, while

$$\varphi(1) = \lim_{n\to\infty} n\int_{[0,1/n]} d\mu = 1 \neq 0.$$

Therefore our supposition is incorrect and such a g DNE. $\square$

(5) 905 840 515

(a) We first show $L^\infty([0,1],\mu)$ is not separable. To do so, it suffices to construct an uncountable collection of functions $\{f_x\}$ w/ $\|f_x - f_y\|_{L^\infty} \geq 1$ $\forall x \neq y$.

Define $f_x = \chi_{[0,x]}$ and consider $\mathcal{F} = \{f_x : x \in [0,1]\}$. Then $\mathcal{F}$ is uncountable and $\forall x<y$, $\|f_x - f_y\|_{L^\infty} = \|\chi_{(x,y]}\|_{L^\infty} = 1$. Therefore $L^\infty[0,1]$ is not separable

We now show $L^p[0,1]$ for $1 \leq p < \infty$ is separable.
Recalling the definition of integration, we note that step functions are dense in $L^p[0,1]$ $\forall$ $1 \leq p < \infty$. It then suffices to show that step functions are ~~sep~~ separable.
Consider $\mathcal{F} = \left\{ \sum_{j=1}^{n} c_j \chi_{(a_j, b_j)} : a_j, b_j \in \mathbb{Q} \cap [0,1],\ c_j \in \mathbb{Q},\ n \geq 0 \right\}$.
Then $\mathcal{F}$ is countable. To show $\mathcal{F}$ is dense in the simple functions, linearity implies that it suffices to approximate χ_A for measurable $A \subset [0,1]$.* By outer regularity, $\exists$ open $\{U_n\}$ s.t. $A \subset U_n \subset (0,1)$ and $\lim_{n \to \infty} |U_n| = |A|$. Since U_n is open, it can be expressed as the countable union of disjoint open intervals $U_n = \bigcup_{i=1}^{\infty} (a_i^n, b_i^n)$.
~~Since $\mathbb{Q}$ is dense in $\mathbb{R}$, each interval (a_i^n, b_i^n) can be written as the countable union of intervals w/ rational endpoints.~~
Fix $\varepsilon > 0$. Choose n sufficiently large so that $|U_n \setminus A| < \varepsilon$.
Since $U_n = \bigcup_{i=1}^{\infty} (a_i^n, b_i^n)$, upwards monotone convergence implies $\exists$ k s.t. $|U_n \setminus \bigcup_{i=1}^{k} (a_i^n, b_i^n)| < \varepsilon$. Finally, since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\forall i = 1, \ldots, k$ $\exists$ $\tilde{a}_i, \tilde{b}_i \in \mathbb{Q}$ s.t. $(\tilde{a}_i, \tilde{b}_i) \subset (a_i^n, b_i^n)$ w/ $|(a_i^n, b_i^n)| < \varepsilon/k$. Then

---

* Since $|\{0\}| = |\{1\}| = 0$, we assume WLOG that $A \subset (0,1)$ to simplify notation

Let $g = \sum_{i=1}^{k} \chi_{(\bar{a}_i, \bar{b}_i)} \in \mathcal{F}$. Then

$$\|\chi_A - g\|_{L^p}^p \leq \|\chi_A - \chi_{U_n}\|_{L^p}^p + \|\chi_{U_n} - \chi_{\bigcup_{i=1}^{n}(a_i^n, b_i^n)}\|_{L^p}^p + \sum_{i=1}^{k} \|\chi_{(a_i^n, b_i^n)} - \chi_{(\bar{a}_i, \bar{b}_i)}\|_{L^p}^p$$

$$= |A \setminus U_n|^p + |U_n \setminus \bigcup_{i=1}^{n}(a_i^n, b_i^n)|^p + \sum_{i=1}^{k} |(a_i^n, b_i^n) \setminus (\bar{a}_i, \bar{b}_i)|^p$$

$$\leq 3\varepsilon^p$$

Since $g \in \mathcal{F}$, taking $\varepsilon \to 0$ then concludes that $\mathcal{F}$ is dense in the step functions and hence dense in $L^p[0,1]$. Therefore $L^p[0,1]$ is separable.

(b) Suppose on the contrary that $\exists$ a bounded surjective linear map $T: L^p[0,1] \to L^1[0,1]$ for $p > 1$. Here we assume that $p \neq \infty$ was intended.

Consider the adjoint (or pullback) map

$$T^* : (L^1[0,1])^* \to (L^p[0,1])^* : \varphi \longmapsto \varphi \circ T$$

Since T is linear, T^* is linear. Moreover, $\forall \varphi \in (L^1[0,1])^*$,

$$\|T^*\varphi\| = \|\varphi \circ T\|$$

$$= \sup_{\|f\|_{L^p} \leq 1} \|\varphi \circ T(f)\|$$

$$\leq \sup_{\|f\|_{L^p} \leq 1} \|\varphi\| \|T\| \|f\|_{L^p}$$

$$= \|\varphi\| \|T\|$$

Therefore $\|T^*\| \leq \|T\| < \infty$ and so T^* is a bounded linear map.

We claim that T^* is injective. Suppose $\exists\, \varphi \in (L^1[0,1])^*$ s.t. $T^*\varphi = 0$. Then $\forall f \in L^p$, $\varphi(T(f)) = 0$. Since T is surjective, this implies $\varphi(g) = 0\ \forall g \in L^1$ and so $\varphi = 0$. Therefore T has trivial kernel and hence is injective by linearity.

By Riesz representation, $p < \infty$ T^* gives an injective ~~bounded~~ linear map

$$T^* : L^\infty[0,1] \to L^{p'}[0,1]$$

Since $1 < p < \infty$, $p' < \infty$ and so $L^{p'}$ is separable. $\longrightarrow$

We claim that this is a contradiction.

Consider the map

$$T^*: L^\infty[0,1] \to T^*(L^\infty[0,1])$$

which is a bijection. The open mapping theorem implies that T^* is open and hence has a continuous (bounded) inverse T^{*-1}.

Consider the set $\{f_x\} \subset L^\infty$ from part a. Then $\forall x \neq y$

$$1 = \|f_x - f_y\|_{L^\infty} = \|T^{*-1}T^* f_x - T^{*-1}T^* f_y\|_{L^\infty}$$

$$\leq \|T^{*-1}\| \, \|T^* f_x - T^* f_y\|_{L^{p'}}$$

However, then $\{T^* f_x\}$ would be an uncountable subset of $L^{p'}[0,1]$ ~~that~~ s.t. $\|T^* f_x - T^* f_y\|_{L^{p'}} \gtrsim 1$ $\forall x \neq y$. Since $L^{p'}$ is separable, this is a contradiction and no such map can ~~exist~~ exist. $\square$

(6) 405 846 515

(a) Suppose that $\xi_n \rightharpoonup \xi$ and $\|\xi\| = 1$.

By direct computation, $\forall n$

$$\|\xi_n - \xi\|^2 = \langle \xi_n - \xi, \xi_n - \xi \rangle$$
$$= \|\xi_n\|^2 + \|\xi\|^2 - \langle \xi, \xi_n \rangle - \overline{\langle \xi, \xi_n \rangle}$$
$$= 2 - \langle \xi, \xi_n \rangle - \overline{\langle \xi, \xi_n \rangle}$$

Since $\xi_n \rightharpoonup \xi$ and $\langle \xi, \cdot \rangle$ is a continuous linear functional on H, it follows that $2 - \langle \xi, \xi_n \rangle - \overline{\langle \xi, \xi_n \rangle} \to 2 - 2\|\xi\|^2 = 0$ as $n \to \infty$.

Therefore $\|\xi_n - \xi\|^2 \to 0$ and so $\lim_{n\to\infty} \|\xi_n - \xi\| = 0$.

(b) Suppose that $\lim_{n,m\to\infty} \|\xi_n + \xi_m\| = 2$. By direct computation,

$$\|\xi_n - \xi_m\|^2 + \|\xi_n + \xi_m\|^2 = \langle \xi_n - \xi_m, \xi_n - \xi_m \rangle + \langle \xi_n + \xi_m, \xi_n + \xi_m \rangle$$
$$= \|\xi_n\|^2 + \|\xi_m\|^2 - \langle \xi_m, \xi_n \rangle - \langle \xi_n, \xi_m \rangle$$
$$+ \|\xi_n\|^2 + \|\xi_m\|^2 + \langle \xi_m, \xi_n \rangle + \langle \xi_n, \xi_m \rangle$$
$$= 4$$

Then

$$\|\xi_n - \xi_m\|^2 = 4 - \|\xi_n + \xi_m\|^2$$

Therefore

$$\lim_{n,m\to\infty} \|\xi_n - \xi_m\|^2 = 4 - \left(\lim_{n,m\to\infty} \|\xi_n + \xi_m\|\right)^2 = 4 - 4 = 0$$

and so $\{\xi_n\}_n$ is a Cauchy sequence. Then $\exists\, \xi \in \mathcal{H}$ s.t. $\lim_{n\to\infty} \|\xi_n - \xi\| = 0$, since $\mathcal{H}$ is complete.

$\square$

S19.7 405 846 515

We recall Jensen's formula which states that if f is entire and $f(0)\neq 0$, then

$$\log|f(0)| + \sum_{|a_n|<r} \log\left(r/|a_n|\right) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta$$

where $\{a_n\}$ are the zeros of f repeated according to multiplicity.

~~Taking the convention that $\log(0) = -\infty$, this can be extended to all entire f by~~

$$\cancel{\log|f(0)| + \sum_{0<|a_n|<r} \log(r/|a_n|) \le \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta}$$

In particular,

$$\log|f(0)| + \sum_{|a_n|<r} \log\left(r/|a_n|\right) \le \frac{1}{2\pi}\int_0^{2\pi} \log_+|f(re^{i\theta})|\,d\theta = \frac{1}{2\pi}T_f(r) \quad (1)$$

Suppose first that f has a zero $a_n\neq 0$ and $f(0)\neq 0$.

Then by (1), for sufficiently large r,

$$T_f(r) \geq \log|f(0)| + \log\left(r/|a_n|\right)$$

As $r\to\infty$, this implies that $T(r)\to\infty$ as desired.

Now suppose that f has a zero of order $k>0$ at 0.

Then $g(z)=f(z)/z^k$ is ~~non-vanis~~ nonzero at 0 and entire, with the same zeros $a_n\neq 0$ as f.

Applying (1) to f/z^k then yields

$$\log|g(0)| + \sum_{0<|a_n|<r} \log\left(r/|a_n|\right) \le \frac{1}{2\pi}\int_0^{2\pi} \log_+ \frac{|f(re^{i\theta})|}{r^k}\,d\theta$$

$$= \frac{T_f(r)}{2\pi} - k\log(r)$$

Therefore $\log|g(0)| + k\log(r) + \sum_{0<|a_n|<r} \log\left(r/|a_n|\right) \le \frac{T_f(r)}{2\pi}$. Since $\log(r)\to\infty$ as $r\to\infty$, this implies that $T_f(r)\to\infty$ as $r\to\infty$.

$\longrightarrow$

It remains to ~~that~~ check the case where f is non-vanishing.

$|f|$

$(f-\varepsilon)$ has a zero

$f = e^{g(z)}$

$\log_+ |e^{g(z)}|$

$= \log_+ \{ e^{Re(g)} \}$

$=$ ~~max(log~~

$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})| \leq$

$\max(Re(g), 0)$

$= Re(g) \vee 0 \leq Re(g)$

$\forall n \ \forall R$

then ~~$\exists R$~~ $\exists\, r > R \ \exists$

s.t.

$\int_0^{2\pi} T_f(r) < n$

$\log_+$ is uniformly continuous

if $f_n \rightrightarrows f$

$\log_+ f_n \rightrightarrows f$

$T(r) \leq$

Then $\exists$ an entire function g s.t. $f(z) = e^{g(z)}$ and so $\log|f(z)| = Re(g)$ which is harmonic. Let $u(z) = Re(g(z))$.

~~Suppose that~~ By adding a constant, which shifts

log

⑧ 405 846 515 Let $f(z) = \sin z - z\cos z$ *

We first claim that f is of order 1. By definition of sin/cosine,

$$f(z) = \frac{e^{iz} - e^{-iz}}{2i} - z\,\frac{e^{iz} + e^{-iz}}{2}$$

$$\Rightarrow |f(z)| \leq \frac{|e^{iz}| + |e^{-iz}|}{2} + |z|\,\frac{|e^{iz}| + |e^{-iz}|}{2}$$

$$\leq e^{|z|} + |z|e^{|z|} = (1+|z|)e^{|z|}$$

$$\leq 2e^{|z|}$$

b/c $1+|z| \leq e^{|z|}\ \forall z$. Therefore f is of order 1.

~~Let $\{\lambda_n\}$ denote the zeros of f. Since f is entire, these zeros are countable.~~

We now claim that f has a zero of order 3 at 0. By direct calculation,

$$f(0) = \sin 0 - 0\cos 0 = 1$$

$$f'(0) = (\cos z - \cos z + z\sin z)|_{z=0}$$

$$= (+z\sin z)|_{z=0} = 0$$

$$f''(0) = (+\sin z + z\cos z)|_{z=0} = 0$$

$$f'''(0) = (+\cos z + \cos z + z\sin z)|_{z=0}$$

$$= 2 \neq 0$$

Therefore f has a zero of order 3 at 0. the Hadamard's factorization theorem then implies that

$$f(z) = z^3 e^{az+b} \prod_n E_1(z/\lambda_n) = z^3 e^{az+b}\left(\prod_n (1 - z/\lambda_n)\, e^{z/\lambda_n}\right)$$

where $\{\lambda_n \neq 0\}$ are the nonzero zeros of f, repeated according to multiplicity.

* We aim to use the Hadamard factorization theorem to show the formula for f

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(⇐) Suppose that

$$f(z) = \lambda \prod_{j=1}^{N} \frac{z-a_j}{1-\bar{a}_j z}$$

for $a_j \in \mathbb{D}$. Then f is holomorphic on a neighborhood of $\mathbb{D}$ since $1/\bar{a}_j \notin \overline{\mathbb{D}}$ $\forall j$ and hence f is continuous on $\overline{\mathbb{D}}$. It remains to show that $|f|=1$ on $\partial\mathbb{D}$. By direct computation, for $|z|=1$, we note that since $|z|=1$, $1/z = \bar{z}$. Thus

$$|f(z)| = |\lambda| \prod_{j=1}^{N} \left|\frac{z-a_j}{1-\bar{a}_j z}\right|$$

$z; a_j$

$$= \prod_{j=1}^{N} \left|\frac{z-a_j}{1/z-\bar{a}_j}\right|$$

$$= \prod_{j=1}^{N} \frac{|z-a_j|}{|\overline{z-a_j}|} \qquad (1)$$

$$= 1$$

as desired. Therefore $f \in \mathcal{U}$.

(⇒) Suppose that $f \in \mathcal{U}$. Let $\{a_n\}$ denote the zeros of f in $\mathbb{D}$, repeated. Since f is holomorphic and not identically 0, it follows that $\{a_n\}$ is finite as otherwise $\{a_n\} \subset \overline{\mathbb{D}}$ would have a limit point in $\overline{\mathbb{D}}$ which would imply $f \equiv 0$. or $f=0$ somewhere on $\partial\mathbb{D}$.

Then let $a_1 \ldots a_N$ denote the zeros, repeated according to multiplicity.

~~Then $g = f/\prod_{i=1}^{N}(z-a_j)$ is a holomorphic function on $\mathbb{D}$, cont on $\overline{\mathbb{D}}$ that vanishes nowhere.~~

Define $g = f \prod_{i=1}^{N} \frac{1-\bar{a}_j z}{z-a_j}$. Then g is holomorphic on $\mathbb{D}$, cont on $\overline{\mathbb{D}}$ and satisfies $|g|=1$ on $\partial\mathbb{D}$ by the computation in 1.

Let $\varphi : \mathbb{C}_\infty \to \mathbb{C}_\infty : z \mapsto \frac{z-i}{z+i}$ be the Cayley transform which conformally maps $\mathbb{H} \to \mathbb{D}$. Then $\varphi^{-1} \circ g \circ \varphi : \mathbb{H} \to \mathbb{H}$. Since $g : \partial\mathbb{D} \to \partial\mathbb{D}$, and $\varphi : \partial\mathbb{D} \to \mathbb{R}$, it follows that $\varphi^{-1} \circ g \circ \varphi$ fixes the real line. The Schwarz reflection principle then implies that $\varphi^{-1} \circ g \circ \varphi$ extends to an entire function w/ $\varphi^{-1} \circ g \circ \varphi(\bar{z}) = \overline{\varphi^{-1} \circ g \circ \varphi(z)}$.

Since φ, φ^{-1} are conformal, this implies that $g = \varphi \circ \varphi^{-1} \circ g \circ \varphi \circ \varphi^{-1}$ extends to an entire function.

Moreover, since the value of $\varphi^{-1} \circ g \circ \varphi$ on $-\mathbb{H}$ was determined by its value on $\mathbb{H}$ and $\varphi : \mathbb{H} \to \mathbb{D}$, it follows that that value of g on $\mathbb{C} \setminus \overline{\mathbb{D}}$ will be determined by the value of g on $\overline{\mathbb{D}}$.

In particular,

$$\sup_{z \in \mathbb{C}} |g(z)| = \sup_{z \in \overline{\mathbb{D}}} |g(z)| \leq \sup_{z \in \partial\mathbb{D}} |g(z)| = 1$$

by the maximum modulus principle. Therefore g is bounded so Liouville's implies $g \equiv \lambda$ for some $\lambda \in \mathbb{C}$. Since $|g| = 1$ on $\partial\mathbb{D}$, $|\lambda| = 1$. Therefore

$$\lambda = f \prod_{j=1}^{N} \frac{1 - \overline{a_j} z}{z - a_j} \Rightarrow f = \lambda \prod_{j=1}^{N} \frac{z - a_j}{1 - \overline{a_j} z} \quad \text{w/ } |\lambda| = 1$$

and so f is a Blaschke product.

As both directions have been shown, this concludes. $\square$

~~The inverse Cayley transform is found via~~

$$\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}^{-1} = \frac{1}{2i} \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

$$\Rightarrow \quad \varphi^{-1} = \frac{z+1}{i(z-1)}$$

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Because u is 2π-periodic, the map

$$\partial\mathbb{D} \to \mathbb{R} : z \longmapsto u\left(\frac{\log z}{i}\right)$$

is well-defined and smooth. This is b/c $\log : \partial\mathbb{D} \to i\mathbb{R}$ and log is a multivalued smooth function w/ $\log(z) = \{ \mathrm{Log}(z) + 2\pi i k : k \in \mathbb{Z} \}$.

B/c $u(\frac{\log z}{i})$ is smooth on $\partial\mathbb{D}$, we can solve the Dirichlet problem on $\mathbb{D}$ to find a harmonic function $\tilde{u}$ on $\mathbb{D}$ that extends continuously to $u(\frac{\log(z)}{i})$ on $\partial\mathbb{D}$. Since $\mathbb{D}$ is simply connected, $\exists$ a holomorphic function $\hat{f}$ on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$ s.t. $\mathrm{Re}(\hat{f}) = \tilde{u}$. Note that $\hat{f}$ is bounded on $\mathbb{D}$ by the maximum modulus principle since $u(\frac{\log z}{i})$ is bounded on $\mathbb{D}$.

Define $f : \mathbb{H} \to \mathbb{C} : z \longmapsto \hat{f}(e^{iz})$. Then f is holomorphic on $\mathbb{H}$, bounded on $\mathbb{H}$, and (heuristically) $\mathrm{Re}(f(z)) = \tilde{u}(e^{iz}) \to u(\frac{\log e^{iz}}{i}) = u(z)$ as z approaches the real line. In particular,

$$\begin{aligned} \lim_{\varepsilon \to 0^+} \mathrm{Re}(f(x + i\varepsilon)) &= \lim_{\varepsilon \to 0^+} \tilde{u}(e^{ix - \varepsilon}) \\ &= \lim_{\varepsilon \to 0^+} \tilde{u}(e^{-\varepsilon}(e^{ix})) \\ &= u\left(\frac{\log e^{ix}}{i}\right) \\ &= u(x) \end{aligned}$$

~~Since f extends continuously to $\mathbb{R}$ w/ $f(x) \in \mathbb{R}$ $\forall x \in \mathbb{R}$, the Schwarz reflection principle extends f to $\mathbb{C}$ w/ $f(\bar{z}) = \overline{f(z)}$.~~

Define $f_- : -\mathbb{H} \to \mathbb{C} : z \longmapsto -\frac{1}{2}\overline{f(\bar{z})}$. Then f_- is holomorphic and bounded. Define $f_+ : \mathbb{H} \to \mathbb{C}$ as $f_+ = \frac{1}{2} f$. Then $\forall x \in \mathbb{R}$,

$$\begin{aligned} \lim_{\varepsilon \to 0^+} \left(f_+(x + i\varepsilon) - f_-(x - i\varepsilon)\right) &= \lim_{\varepsilon \to 0^+} \tfrac{1}{2}\left(f(x + i\varepsilon) + \overline{f(x + i\varepsilon)} \right) \\ &= \lim_{\varepsilon \to 0^+} \mathrm{Re}(f(x + i\varepsilon)) \\ &= u(x) \end{aligned}$$

~~which is what~~ as desired. $\square$

$\longrightarrow$

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(a) ~~Suppose $\exists \{f_n\} \subset \mathcal{H}$ s.t. $f_n \to f$ in $L^2(d\mu)$. To show that $f \in \mathcal{H}$, it suffices to show that f is entire. To do so, we~~ aim to apply Montel's theorem ~~to $\{f_n\}$ to show~~ that $f_n \to f$ uniformly on compact subsets of $\mathbb{C}$. ~~By Montel's theorem, it then suffices to~~ since uniform convergence on compact subsets preserves holomorphicity, Montel's

~~To show $\mathcal{H}$ is~~

Suppose $\exists \{f_n\} \subset \mathcal{H}$ s.t. $f_n \to f \in L^2(d\mu)$. To show $f \in \mathcal{H}$, it must be shown that f is entire. Since holomorphicity is preserved by local uniform convergence, it suffices to show $f_{n_k} \to f$ locally uniformly for some subsequence $\{f_{n_k}\}$. Montel's theorem then implies that it suffices to show that $\{f_n\}$ is locally uniformly bounded. To do so, it suffices to show that $\{f_n\}$ is bounded on $\overline{D(0,R)}$ $\forall$ $R>0$.

Fix some $R>0$. By the mean value theorem, $\forall$ n $\forall z \in \overline{D(0,R)}$,

$$f_n(z) = \frac{1}{|D(z,R)|}\int_{D(z,R)} f_n(w)\,d\lambda(w)$$

B/c $D(z,R) \subset D(0,2R)$ $\forall$ $z \in D(0,R)$, this implies that

$$|f_n(z)| \le \frac{1}{\pi R^2}\int_{D(0,2R)} |f_n|\,d\lambda \lesssim \int_{D(0,2R)} |f_n|\,d\lambda$$

B/c $e^{-|z|^2/2} \gtrsim 1$ on $D(0,2R)$, this implies

$$\begin{aligned} |f_n(z)| &\lesssim \int_{D(0,2R)} |f_n|\,e^{-|w|^2/2}\,d\lambda \\ &= \int \chi_{D(0,2R)}|f_n|e^{-|w|^2/2}\,d\lambda \\ \text{(Hölder's)} \quad &\le \sqrt{|D(0,2R)|}\; \big\| |f_n| e^{-|w|^2/2} \big\|_{L^2(d\lambda)} \\ &\lesssim \|f_n\|_{L^2(d\mu)} \end{aligned}$$

Since $f_n \to f$ in $L^2(d\mu)$, it follows that $\sup_n\{\|f_n\|_{L^2(d\mu)}\} < \infty$. Since this bound is independent of $z \in \overline{D(0,R)}$, this concludes. $\square$

(b) We start by showing that the normalized monomials

$$e_n = \frac{1}{\sqrt{\pi n!}} z^n$$

form an orthonormal basis for $\mathcal{H}$.

First, we calculate that $\forall\, n \geq 1$

$$\begin{aligned}
\|e_n\|_{L^2(d\mu)} &= \frac{1}{\pi n!} \int_{\mathbb{C}} |z|^{2n} e^{-|z|^2} d\lambda(z) \\
&= \frac{2}{n!} \int_0^\infty r^{2n+1} e^{-r^2} dr \\
\text{(integration by parts)} \quad &= \frac{2}{n!}\left(-\frac{r^{2n}e^{-r^2}}{2}\Big|_0^\infty + n\int_0^\infty r^{2n-1} e^{-r^2} dr \right) \\
&= \frac{2}{(n-1)!} \int_0^\infty r^{2(n-1)+1} e^{-r^2} dr \\
&= \|e_{n-1}\|_{L^2(d\mu)}
\end{aligned}$$

$$\left[\begin{aligned} u &= r^{2n} & dv &= re^{-r^2} \\ du &= 2nr^{2n-1} & v &= -\tfrac{1}{2}e^{-r^2} \end{aligned}\right.$$

Finally, for e_0 we have $\|e_0\|_{L^2(d\mu)} = \frac{1}{\pi}\int_{\mathbb{C}} e^{-|z|^2} d\lambda = 1$
and so $\|e_n\|_{L^2(d\mu)} = 1 \quad \forall n$.

We now check orthogonality. $\forall\, n \neq m$,

$$\begin{aligned}
\langle e_n, e_m \rangle &\sim \int z^n \bar{z}^m e^{-|z|^2} d\lambda \\
&= \int_0^{2\pi}\int_0^\infty r^{n+m} e^{-r^2} e^{i(n-m)\theta}\, r\,dr\,d\theta \\
&= \Bigg(\underbrace{\int_0^{2\pi} e^{i(n-m)\theta} d\theta}_{=0 \text{ b/c } n\neq m}\Bigg)\Bigg(\underbrace{\int_0^\infty r^{n+m+1} e^{-r^2} dr}_{<\infty}\Bigg)
\end{aligned}$$

Therefore $\{e_n\}$ ~~are~~ is an orthonormal subset of $\mathcal{H}$.

To show $\{e_n\}$ is a basis for $\mathcal{H}$, it suffices to show that for $f \in \mathcal{H}$

$$\langle f, e_n\rangle_{d\mu} = 0 \ \forall n \implies f = 0$$

Suppose $\exists\, f$ s.t. $\langle f, e_n\rangle_{d\mu} = 0 \ \forall n$.

$\longrightarrow$

Since f is entire, $\exists \{a_m\}$ s.t. $f(z)=\sum_{n\geq 0} a_m z^m$ ~~and~~ where the series converges uniformly. Then

$$\langle f, e_n\rangle \sim \int_{\mathbb{C}} f(z)\bar{z}^n e^{-|z|^2}\,d\lambda$$

$$= \sum_{m\geq 0} a_m \int_{\mathbb{C}} z^m \bar{z}^n e^{-|z|^2}\,d\lambda \quad \text{(uniform convergence} + \mu(\mathbb{C})<\infty)$$

$$\sim \sum_{m\geq 0} a_m \langle e_m, e_n\rangle$$

$$= a_n$$

and so $a_n=0$ $\forall n$. Therefore $f=0$. As this holds $\forall$ such f, this concludes that $\{e_n\}$ is an orthonormal basis for H. Therefore $\forall f\in H$,

$$f(z) = \sum_{n\geq 0} \langle f, e_n\rangle e_n(z)$$

$$= \sum_{n\geq 0} \frac{z^n}{\pi n!}\int f(w)\bar{w}^n e^{-|w|^2}\,d\lambda$$

To obtain the desired formula for f, we wish to interchange the integral and the sum. Fubini's theorem implies that it suffices to show that the sum/integral are absolutely integrable. Checking this,

$$\int_{\mathbb{C}} \sum_{n\geq 0}\frac{1}{\pi n!}|f(w)|\,|z\bar{w}|^n e^{-|w|^2}\,d\lambda = \int_{\mathbb{C}} |f(w)|\left(\frac{1}{\pi}\sum_{n\geq 0}\frac{1}{n!}|z\bar{w}|^n\right)d\mu$$

$$= \frac{1}{\pi}\int_{\mathbb{C}} |f(w)|\, e^{|z\bar{w}|}\,d\mu$$

$$\leq \frac{1}{\pi}\underbrace{\|f\|_{L^2(d\mu)}}_{<\infty}\underbrace{\left(\int_{\mathbb{C}} e^{2|z\bar{w}|-|w|^2}\,d\lambda\right)}_{<\infty}$$

$$<\infty$$

Therefore Fubini-Tonelli implies that

$$f(z)=\sum_{n\geq 0}\frac{z^n}{\pi n!}\int f(w)\bar{w}^n e^{-|w|^2}\,d\lambda = \int_{\mathbb{C}} f(w)\left(\sum_{n\geq 0}\frac{1}{\pi n!}(z\bar{w})^n\right)d\mu$$

$$= \frac{1}{\pi}\int_{\mathbb{C}} f(w)\, e^{z\bar{w}}\,d\mu$$

which is what was to be shown. $\boxtimes$

ANALYSIS

Fall 2018

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We first show that properties (i), (ii), (iii) extend to f.
By Fatou's lemma, since $f_n \to f$ a.e.,

$$\int |x||f(x)|\,dx \leq \liminf_{n\to\infty} \int |x||f_n(x)|\,dx \leq 100$$

$$\int |f(x)|^2\,dx \leq \liminf_{n\to\infty} \int |f_n(x)|^2\,dx \leq 100$$

as desired. We now show that $f_n, f \in L^1$. By direct computation, since $|x||f(x)| \geq |f(x)| \ \forall\ |x| \geq 1$,

$$\|f\|_{L^1} = \int_{|x|<1} |f(x)|\,dx + \int_{|x|\geq 1} |f(x)|\,dx$$

$$\leq \int_{|x|<1} |f| + \int_{|x|\geq 1} |x||f(x)|$$

$$\text{(Hölder + monotonicity)} \leq |\{|x|<1\}|^{1/2} \|f\|_{L^2} + \int_{\mathbb{R}} |x||f(x)|$$

$$\leq \underbrace{10\sqrt{2\pi} + 100}_{M} < \infty$$

The same reasoning holds to show $f_n \in L^1$, w/ $\|f_n\|_{L^1} \leq M$.
We now aim to bound $\int_{|x|\geq R} |f|$, $\int_{|x|\geq R} |f_n|$ for $R \geq 0$. Direct computation yields.

$$\int_{|x|\geq R} |f| \leq \int_{|x|\geq R} \frac{|x|}{R} |f|$$

$$\leq \frac{1}{R} \int_{\mathbb{R}} |x||f|$$

$$\leq \frac{100}{R}$$

and similarly for f_n.

We now show $f_n \to f$ in L^1. Fix $\varepsilon > 0$ and choose $R > 0$ s.t. $1/R < \varepsilon$.
Since $\{|x| \leq R\}$ is compact, Egorov's theorem implies $\exists$ E w/ $|E| < \varepsilon$ s.t. $f_n \rightrightarrows f$ on $\{|x| \leq R\} \setminus E$. Then $\exists$ N s.t. $\forall\ n \geq N$, $\|f_n - f\|_{L^\infty(\{|x|\leq R\}\setminus E)} \leq \frac{\varepsilon}{R}$.
Then $\forall\ n \geq N$

$$\cancel{\|f_n - f\|_{L^1} = \left(\int_{\{|x|\leq R\}\setminus E} + \int_E + \int_{\{|x|>R\}}\right) |f_n - f| \leq \frac{\varepsilon}{R} |\{|x|\leq R\}\setminus E|}$$

$\longrightarrow$

$$\|f_n - f\|_{L^1} \le \int_{\{|x|\le R\}\setminus E} |f_n - f| + \int_E |f_n - f| + \int_{\{|x|>R\}} (|f_n| + |f|)$$

(Hölder) $$\le \frac{\varepsilon}{R} \underbrace{\left|\{|x|\le R\}\setminus E\right|}_{\le 2R} + |E| \, \|f_n - f\|_{L^1} + 200\varepsilon$$

$$\lesssim \varepsilon + \varepsilon \|f_n - f\|_{L^1}$$

$\Rightarrow \|f_n - f\|_{L^1} \lesssim \frac{\varepsilon}{1-\varepsilon} \lesssim \varepsilon$ for sufficiently small ε.

~~Thus~~ As this holds $\forall n \ge N$ and such an N can be chosen $\forall \varepsilon > 0$, this implies $\|f_n - f\|_{L^1} \to 0$, as desired.

Suppose that (ii) was omitted. Define $f_n = f = \chi_{\{|x|\ge 1\}} |x|^{-1}$.
Then (i) holds since $f_n \equiv f \Rightarrow f_n \to f$ a.e., and (iii) holds since
$$\|f_n\|_{L^2} = \|f\|_{L^2} = \int_{|x|\ge 1} |x|^{-2} \, dx = -2x^{-1} \Big|_{x=1}^{\infty} = 2$$
Moreover ~~however,~~ (ii) does not hold since $|x||f_n| = |x||f| = \chi_{\{|x|\ge 1\}}$ is not integrable.
~~Thus~~ $f_n, f \notin L^1$ since $|x|^{-1}$ is not integrable near ∞.
~~However~~ Therefore (ii) cannot be omitted.

Suppose that (iii) was omitted. Define $f_n = f = \chi_{\{|x|\le 1\}} |x|^{-1}$.
Then (i) holds since $f_n = f \Rightarrow f_n \to f$ a.e. and (ii) holds since
$$\|x f_n\|_{L^1} = \|x f\|_{L^1} = \int_{\mathbb{R}} \chi_{\{|x|\le 1\}} = 2 < \infty$$
Moreover, (iii) does not hold since $(|x|^{-1})^2 = |x|^{-2}$ is not integrable near 0.
However, $f_n, f \notin L^1$ since $|x|^{-1}$ is not integrable near 0.
Therefore (iii) cannot be omitted.

$\square$

$|x-y| \geq ||x|-|y||$

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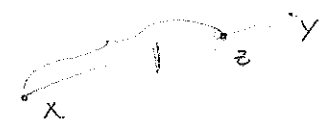
(a) Since X is compact, ~~A~~ Stone-Weierstrass implies that it suffices to show that A separates points and vanishes nowhere.

Since X contains ≥ 2 points, $\forall x \in X \ \exists y \in X$ w/ $y \neq x$.
Then $\rho(x,y) > 0$. Since D is dense in X, $\exists z \in D$ s.t. $\rho(z,y) < \rho(x,y)$. In particular, this implies $z \neq x$ and so $\rho(z,x) > 0$. Then $f_z(x) > 0$ w/ $f_z \in A$ and ~~so~~ so A vanishes nowhere.

We now show A separates points. Fix $x, y \in X$. As above, $\exists$ $z \in D$ s.t. $\rho(z,y) < 1/2\, \rho(x,y)$. Then
$\rho(z,x) \geq \rho(x,y) - \rho(z,y) > 1/2\, \rho(x,y)$.
In particular, $f_z \in A$ satisfies

$$f_z(y) < \tfrac{1}{2}\rho(x,y) < f_z(x)$$
$$\Rightarrow f_z(x) \neq f_z(y)$$



Therefore A separates points. Stone-Weierstrass then implies ~~that~~ A is dense in $C(X)$.

(b) B/c A is dense in $C(X)$, it suffices to find a countable dense subset of A that is dense in A. By definition, all elements $g \in A$ can be written as

$$g = \sum_{i=1}^{n} \alpha_i f_{z_{i1}} \cdots f_{z_{ik_i}} \quad \text{for } \alpha_i \in \mathbb{R},\ z_{ij} \in D.$$

~~By linearity, it then suffices to show that~~ We claim that $\{\sum_{i=1}^{n} q_i f_{z_{i1}} \cdots f_{z_{ik_i}} : q \in \mathbb{Q}\} = Q$ is dense in A. By linearity, it suffices to approximate $\alpha f_{z_1} \cdots f_{z_n}$. Fix $\varepsilon > 0$. ~~and~~ Then $\exists\, q \in \mathbb{Q}$ s.t. $|q - \alpha| < \varepsilon$. Direct computation yields that $\forall x$,

$$|\alpha f_{z_1}(x) \cdots f_{z_n}(x) - q f_{z_1}(x) \cdots f_{z_n}(x)| \leq \varepsilon |f_{z_1}(x) \cdots f_{z_n}(x)|$$

Since X is separable, we recall that compact $\Rightarrow$ bounded.
Therefore since X is compact, $\exists\, M < \infty$ s.t. $\rho(x,y) \leq M \ \forall x, y \in X$.

Therefore $f_z(x) \le M$ $\forall$ $z \in D, x \in X$. Then

$$|\alpha f_{z_1}(x)\cdots f_{z_n}(x) - q f_{z_1}(x)\cdots f_{z_n}(x)| \le |\alpha - q|\, M^n \le \varepsilon$$

As ~~[illegible]~~ such a $q f_{z_1}\cdots f_{z_n} \in \mathcal{Q}$ can be found $\forall \varepsilon$, this implies that $\mathcal{Q}$ is dense in A. Since $\mathbb{Q}, D$ are countable, $\mathcal{Q}$ is countable. Therefore A and hence $C(X)$ are separable. $\square$

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Suppose that E satisfies $\mu(E^\circ)=\mu(E)=\mu(\bar{E})$.

Since E° is ~~ob~~ open, $\mathbb{1}_{E^\circ}$ is lower semi-continuous and so $\exists\, f_k \in C(X)$ s.t. $f_k \uparrow \mathbb{1}_{E^\circ}$. These f_k can be constructed explicitly as

$$f_k(x) = \min\left(k\,d(x, X\setminus E^\circ), 1\right)$$

where d is the metric on X. The fact that $f_k \uparrow \mathbb{1}_{E^\circ}$ everywhere follows from the fact that $X\setminus E^\circ$ is closed so $d(x, X\setminus E^\circ) > 0 \ \forall\, x \in E^\circ$.

~~Since E~~ Similarly, since $\bar{E}$ is closed, $\mathbb{1}_{\bar{E}}$ is upper semi-continuous and hence $\exists\, g_l \in C(X)$ s.t. $g_l \downarrow \mathbb{1}_{\bar{E}}$. These g_k can similarly be explicitly constructed as $g_l(x) = \max(1 - l\,d(x, \bar{E}), 0)$ w/ $g_l \downarrow \mathbb{1}_{\bar{E}}$ following b/c $\bar{E}$ is closed.

By monotonicity, $\forall\, k, l, n$

$$\int f_k \, d\mu_n \le \mu_n(E^\circ) \le \mu_n(E)$$

$$\mu_n(E) \le \mu_n(\bar{E}) \le \int g_l \, d\mu_n$$

Since $f_k, g_l \in C(X)$, this implies that

$$\int f_k \, d\mu \le \liminf_{n\to\infty} \mu_n(E) \le \limsup_{n\to\infty} \mu_n(E) \le \int g_l \, d\mu$$

Since μ is finite, the MCT then implies that

$$\mu(E^\circ) \le \liminf_{n\to\infty} \mu_n(E) \le \limsup_{n\to\infty} \mu_n(E) \le \mu(\bar{E})$$

As $\mu(E^\circ) = \mu(E) = \mu(\bar{E})$, this concludes that

$$\lim_{n\to\infty} \mu_n(E) = \mu(E)$$

as desired. $\square$

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~~To show that $\{f_y\}$ is norm dense in $L^2(\mathbb{R})$, it suffices to show that $\{\hat{f}_y\}$ is norm dense in L^2 b/c~~

Let $\mathcal{F}$ be the L^2-closure of the set of finite linear combination of translates $f_y(x) = f(x-y)$. Suppose for the sake of contradiction that $\mathcal{F} \neq L^2(\mathbb{R})$. Since $L^2(\mathbb{R})$ is a Hilbert space, this implies $\exists\ g \in \mathcal{F}^\perp$ s.t. $\|g\|_{L^2} = 1$.

Define $L : \mathcal{F} + \text{span}\{g\} \to \mathbb{R} : f + \alpha g \mapsto \alpha$. Then L is linear and bounded as
$$|L(f+\alpha g)| = |\alpha| = \|\alpha g\|_{L^2} \leq \|\alpha g\|_{L^2} + \|f\|_{L^2} = \|f + \alpha g\|_{L^2}$$ since $g \perp f$.
Therefore L is a bounded linear functional on $\mathcal{F} + \text{span}\{g\}$ which is a linear subspace of $L^2(\mathbb{R})$. Hahn-Banach then extends L to a bounded linear functional L on $L^2(\mathbb{R})$ s.t. $L|_{\mathcal{F}} = 0$ and $|L(g)| = 1$.
By Riesz representation, $\exists\ h \in L^2$ s.t. $L(u) = \int uh\ \forall\ u \in L^2(\mathbb{R})$.
By definition $\forall\ y$,
$$0 = L(f_y) = \int f(x-y)h(x)dx = \tilde{f} * h(y) \qquad (1)$$
where $\tilde{f}(x) = f(-x)$. Direct computation yields
$$\hat{\tilde{f}}(\xi) = \int e^{-2\pi i x\xi} f(-x)dx = \hat{f}(-\xi)$$
Taking the Fourier transform of (1) implies that
$$0 = \hat{f}(-\xi)\hat{h}(\xi) \quad \text{a.e}$$
Since $|\hat{f}(\xi)| > 0$ a.e, this implies that $\hat{h}(\xi) = 0$ a.e and hence $h = 0$. However, this contradicts $1 = L(g) = \int gh$ and so our supposition must be incorrect. Therefore $\mathcal{F} = L^2(\mathbb{R})$ and so the finite linear combination of translates of f are norm-dense in L^2. $\square$

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Let f be entire such that $\int_{\mathbb{C}} |\log|f(z)|| \, dx\,dy < \infty$.

Since f is entire, $\log|f(z)|$ is sub-harmonic. In particular, $\log|f(z)|$ satisfies the sub-mean value property which states that

$$
\begin{aligned}
\log|f(z)| &\leq \frac{1}{|D(z,1)|} \int_{D(z,1)} \log|f(w)| \, dx\,dy \qquad (*) \\
&\leq \frac{1}{\pi} \int_{D(z,1)} |\log|f(w)|| \, dx\,dy \\
&\leq \frac{1}{\pi} \int_{\mathbb{C}} |u(w)| \, dx\,dy < \infty
\end{aligned}
$$

Therefore $\log|f(z)|$ is uniformly bounded. Since $\log$ is monotonically increasing, this implies that $|f(z)|$ is uniformly bounded. Liouville's theorem then concludes that f is constant, as desired.

We note that $(*)$ also follows from Jensen's formula. $\square$

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(a) As given, $\mathcal{D} = \{\text{analytic } f : \|f'\|_{L^2(\mathbb{D})} < \infty\}$. We equip $\mathcal{D}$ w/ the L^2 norm on the derivative, i.e. $f \mapsto \|f'\|_{L^2}$.

Suppose $\exists$ a Cauchy sequence $\{f_n\} \subset \mathcal{D}$. Then by construction $\{f_n'\}$ is Cauchy in $L^2(\mathbb{D})$. Since L^2 is complete, $\exists$ some $g \in L^2$ s.t. $f_n' \to g'$ in $L^2(\mathbb{D})$. We claim that g is analytic.

~~Passing to a subsequence, we may assume that $f_n' \to g$ pointwise a.e.~~

B/c f_n' is analytic $\forall n$, it ~~then~~ suffices to show that $f_n' \to g$ uniformly on compact subsets of $\mathbb{D}$. By Montel's theorem, it then suffices to show that $\{f_n'\}$, is uniformly bounded on compact subsets of $\mathbb{D}$.

Let K be a compact subset of $\mathbb{D}$. Let $r = d(K, \mathbb{C}\setminus\mathbb{D}) > 0$.

Then $\forall n\ \forall z \in K$, the mean value theorem implies, since $D(z,r) \subset \mathbb{D}$

$$|f_n'(z)| = \left|\int_{D(z,r)} f_n'(w)\,dx\,dy\right|$$

$$\leq \int_{D(z,r)} |f_n'(w)|\,dx\,dy$$

(Hölder) $$\leq \|f_n'\|_{L^2(\mathbb{D})}\,|D(z,r)|^{1/2}$$

$$= \|f_n'\|_{L^2(\mathbb{D})}\, r\sqrt{\pi}$$

Since f_n' is Cauchy in L^2, $\|f_n'\|_{L^2}$ is bounded by some M uniformly in n. Then $\forall z \in K\ \forall n$, $|f_n'(z)| \leq Mr\sqrt{\pi} < \infty$ and so $\{f_n'\}$ is uniformly bounded on compact subsets of $\mathbb{D}$. Montel's theorem then implies that f_n' converges ~~locall~~ uniformly on compact subsets to some analytic limit. Since $f_n' \to g$ in L^2 and L^2 convergence admits pointwise convergence along a subsequence, it then follows that $f_n' \to g$ uniformly ~~co~~ on compact subsets of $\mathbb{D}$. Therefore g is analytic on $\mathbb{D}$.

B/c g is analytic, g admits an analytic principal f s.t. $f'=g$, where f is unique up to a constant. Fixing $f(0)=0$ then yields a unique $f \in D$ s.t. $f'=g$. Then $f_n \to f$ in D.

As this holds $\forall$ Cauchy $\{f_n\}$, this implies that D is complete.

(b) We recall that $f(z) = \sum_{n \geq p} a_n z^n$ is analytic on $\mathbb{D}$ iff $\sum a_n z^n$ has a radius of convergence $R \geq 1$.

By the root test, this occurs iff

$$\liminf_{n \to \infty} |a_n|^{-1/n} \geq 1 \qquad (1)$$

Therefore (1) is necessary.

Suppose that (1) holds. Then $\sum_{n \geq 1} a_n z^n = f(z)$ is analytic on $\mathbb{D}$ w/ $f'(z) = \sum_{n \geq 1} n a_n z^{n-1}$.

We recall that $\{\sqrt{\frac{n+1}{\pi}} z^n\}$ is an orthonormal set in L^2.

This can be shown by direct calculation as $\forall k > 0$ $\forall n$,

$$\int_{\mathbb{D}} \sqrt{\tfrac{n+1}{\pi}} z^n \, \overline{\sqrt{\tfrac{n+1}{\pi}} z^n} \, dx\,dy = \frac{n+1}{\pi} \int_{\mathbb{D}} |z|^{2n} \, dx\,dy$$

$$= (2n+2) \int_0^1 r^{2n+1} \, dr$$

$$= 1$$

$$\int_{\mathbb{D}} \sqrt{\tfrac{n+1}{\pi}} z^n \, \overline{\sqrt{\tfrac{n+k+1}{\pi}} z^{n+k}} \, dx\,dy = \frac{\sqrt{(n+1)(n+k+1)}}{\pi} \int_{\mathbb{D}} |z|^n \bar{z}^k \, dx\,dy = 0$$

w/ the final equality following b/c $\mathbb{D}$ is symmetric under rotations, but $|z|^n \bar{z}^k$ is odd under multiplication by $e^{i\pi/k}$ (rotation by π/k).

Therefore, $\{\sqrt{\frac{n+1}{\pi}} z^n\}$ is orthonormal in $L^2(\mathbb{D})$ and hence

$$\|f'\|_{L^2} = \Big\| \sum_{n\geq 1} n a_n \sqrt{\tfrac{\pi}{n}} \sqrt{\tfrac{n}{\pi}}\, z^{n-1} \Big\|_{L^2}$$
$$= \sum_{n\geq 1} n |a_n| \sqrt{\tfrac{\pi}{n}}$$

and so $f' \in L^2$ iff $\sum_{n\geq 1} n|a_n| \sqrt{\frac{\pi}{n}} < \infty$.

Therefore $f = \sum_{n\geq 1} a_n z^n \in D$ iff $\sum_{n\geq 1} n|a_n|\sqrt{\frac{\pi}{n}} < \infty$ and $\liminf_{n\to\infty} |a_n|^{-1/n} \geq 1$. $\square$

⑨ 405 846 515

Consider the meromorphic function $g(z) = -\pi z \cot(z) = -\pi z \frac{\cos(z)}{\sin(z)}$.

(a) ~~We are~~ As given, $g(z) = -\pi z \frac{\cos(z)}{\sin(z)}$. Since $\sin(z)$ has a simple zero at πk for all $k \in \mathbb{Z}$ and no other poles, it follows that g has a simple pole at πk for $k \in \mathbb{Z} \setminus \{0\}$.

We now aim to determine the residue of g at each pole.

B/c $-\pi z \cos(z)$ is entire, we aim to compute the residue of $\frac{1}{\sin(z)}$ at πk for $k \neq 0$. Since $\sin(z)$ is 2π-periodic, it suffices to calculate the residue at $z = 0, \pi$

We recall that

$$\mathrm{Res}_0\left(\frac{1}{\sin(z)}\right) = \lim_{z \to 0} \frac{z}{\sin(z)} = 1 \qquad *$$

Therefore $\mathrm{Res}_{\pi k}\left(\frac{1}{\sin(z)}\right) = (-1)^k \ \forall \ k \neq 0$ and

hence

$$\mathrm{Res}_{2\pi k}\, g(= \ \pi k \cos(\pi k)\, \mathrm{Res}_{2\pi k}\left(\frac{1}{\sin(z)}\right)$$
$$= \pi k \quad \forall \ k \neq 0$$

Which is what was to be found.

$*$ and

$$\mathrm{Res}_{\pi}\left(\frac{1}{\sin z}\right) = \lim_{z \to \pi} \frac{(z-\pi)}{\sin(z)} = \lim_{w \to 0} \frac{w}{\sin(w+\pi)} = \lim_{w \to 0} \frac{w}{-\sin(w)} = -1$$

(b) This follows from a contour integral centered at 0 expanding outwards to capture all poles.

$\longrightarrow$

By definition and the Cauchy Integral Formula,

$$a_{2k} = \frac{1}{(2k)!} g^{(2k)}(0)$$
$$= \frac{1}{2\pi i} \int_{\partial([-\pi/2,\pi/2]^2)} \frac{g(z)}{z^{2k}}\, dz$$

hence

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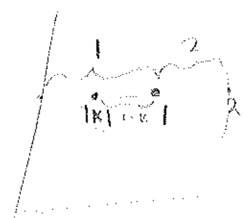
Let $f(z) = e^z (z-1)^n - \alpha$. We aim to count the zeros of f in $H = \{z : Re(z) > 0\}$, to show that it is n.

$$|e^z (z-1)^n| = \underbrace{e^{Re(z)}}_{>0} |z-1|^n$$
$$\geq |z-1|^n$$
$$> |\alpha| \text{ provided } |z-1| > |\alpha|^{1/n}$$



Let $D = D(1,1) \subseteq H$. We claim that f is non vanishing on $H \setminus D$. Consider some $z \in H \setminus D$. Then $Re(z) > 0$ and $|z-1| \geq 1$. Direct computation then implies

$$|f(z)| \geq |e^z (z-1)^n| - |\alpha|$$
$$= e^{Re(z)} |z-1|^n - |\alpha|$$
$$\geq |z-1|^n - |\alpha|$$
$$\geq 1 - |\alpha|$$
$$(0 < |\alpha| < 1) \qquad > 0$$

Therefore f is non-vanishing on $H \setminus D$. Also, by the same computation, $|e^z (z-1)^n| > |\alpha|$ on ∂D and so Rouché's theorem implies that $f(z) = e^z (z-1)^n - \alpha$ and $e^z (z-1)^n$ have the same # of zeros inside D. Since $e^z (z-1)^n$ has n zeros inside D, (counting multiplicities), this implies that f has n zeros inside D. ~~and hence~~ Therefore f has n zeros inside H.

$\square$

ANALYSIS

SPRING 2018

① 405 846 515

Define $F(x) = \int_{-\infty}^{x} f(y)\,dy$. Note that since $f \in L^1$, F is well-defined.

Then $\forall\, x, h \in \mathbb{R}$,

$$\left| \frac{F(x+h) - F(x)}{h} \right| = \left| \int_{-\infty}^{x} \frac{(f(t+h) - f(t))}{h} dt \right|$$

$$\leq \int_{\mathbb{R}} \left| \frac{f(t+h) - f(t)}{h} \right| dt$$

Therefore $\limsup_{h \to 0} \frac{F(x+h) - F(x)}{h} = 0$ and so F is differentiable w/ derivative 0. Then $\forall$ intervals $[a,b]$,

$$\int_a^b f = F(b) - F(a) = 0$$

Lebesgue differentiation then implies $f = 0$ a.e.

□

(2) 405 346 515

(a) Define Q s.t. $\forall f \in L^2(\mathbb{R})$ and $h > 0$

$$Q(f,h) = \int_{\mathbb{R}} \frac{2f(x) - f(x+h) - f(x-h)}{h^2} f(x)\,dx$$
$$= \left\langle \frac{2f(x) - f(x+h) - f(x-h)}{h^2}, f \right\rangle_{L^2}$$

Since $f \in L^2$ and the Fourier transform is an isometry on L^2,

$$Q(f,h) = \left\langle \left(\frac{2f(x) - f(x+h) - f(x-h)}{h}\right)^{\wedge}, \hat{f} \right\rangle$$
$$= \left\langle \frac{\left(2 - e^{2\pi i h\xi} - e^{-2\pi i h\xi}\right)\hat{f}(\xi)}{h^2}, \hat{f}(\xi) \right\rangle$$
$$= \frac{2}{h^2}\int_{\mathbb{R}} (1 - \cos(2\pi h\xi))\,|\hat{f}(\xi)|^2\,d\xi$$

Since $1 - \cos(2\pi h\xi) \geq 0$, this implies that $Q(f,h) \geq 0$ $\forall f$, $\forall h > 0$.

(b) Define $E = \{ f \in L^2 : \limsup_{h \to 0} Q(f,h) \leq 1 \}$.

Suppose $\exists$ $f_n \in E$ s.t. $f_n \to f$ in L^2. We claim that $\xi \hat{f}_n, \xi \hat{f} \in L^2$.

We recall that $\cos(\theta) = 1 - \frac{1}{2}\theta^2 + \ldots$ so

$$\lim_{h \to 0} \frac{1 - \cos(2\pi h\xi)}{h^2} = 2\pi^2\xi^2$$

Therefore $\forall n$, Fatou's gives

$$\int_{\mathbb{R}} \xi^2 |\hat{f}_n(\xi)|^2 d\xi \leq \liminf_{h \to 0} \int_{\mathbb{R}} \frac{1 - \cos(2\pi h\xi)}{h^2} |\hat{f}_n(\xi)|^2 d\xi \leq 1$$

and so $\xi \hat{f}_n \in L^2$. Passing to a subsequence, we may assume $\hat{f}_n \to \hat{f}$ pointwise, Fatou's then yields $\int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)| d\xi \leq \liminf_{n \to \infty} \int_{\mathbb{R}} \xi^2 |\hat{f}_n(\xi)|^2 d\xi \leq 1$ and so $\xi \hat{f}(\xi) \in L^2$.

Finally, we note that $1 - \cos\theta \leq \theta^2$ $\forall \theta$ and so

$$\limsup_{h \to 0} \int_{\mathbb{R}} \frac{2(1 - \cos(2\pi h\xi))}{h^2} |\hat{f}(\xi)|^2 d\xi \leq \limsup_{h \to 0} \int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 d\xi \leq 1$$

Therefore $f \in E$ as desired. $\square$

③ 405 846 515

$$\cancel{\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)f(y)|}{|x-y|^2+\varepsilon^2}\,dx\,dy \qquad f=\chi_E}$$

$$\cancel{z=x-y \quad dz=dx}$$

$$\cancel{\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(z+y)f(y)|}{|z|^2+\varepsilon^2}\,dz\,dy}}$$

~~Consider a simple function $\varphi = c_1\chi_{[a_1,b_1]}+\dots+c_n\chi_{[a_n,b_n]} \le |f|$ w/ $c_j \ge 0$ $\forall j$.~~
~~Then $\forall \varepsilon>0$,~~

$$\cancel{\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)f(y)|}{|x-y|^2+\varepsilon^2}\,dx\,dy \ge \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\varphi(x)\varphi(y)|}{|x-y|^2+\varepsilon^2}\,dx\,dy = \sum_{i,j} c_i c_j \int_{a_i}^{b_i}\int_{a_j}^{b_j} \frac{1}{|x-y|^2+\varepsilon^2}\,dx\,dy}$$

By Fatou's lemma,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)f(y)|}{|x-y|^2}\,dx\,dy \le \limsup_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)f(y)|}{|x-y|^2+\varepsilon^2}\,dx\,dy < \infty$$

Suppose that x_0 is a Lebesgue point of $|f|$. Then $\forall h>0$,

$$\left(\int_{x_0-h}^{x_0+h} \frac{|f(x)|}{2h}\,dx\right)^2 \approx \int_{x_0-h}^{x_0+h}\int_{x_0-h}^{x_0+h} \frac{|f(x)f(y)|}{h^2}\,dx\,dy \le \int_{x_0-h}^{x_0+h}\int_{x_0-h}^{x_0+h} \frac{|f(x)f(y)|}{|x-y|^2}\,dx\,dy$$

since $|x-y| \le h$ for $x,y \in [x_0-h, x_0+h]$.

B/c $\left\|\frac{|f(x)f(y)|}{|x-y|^2}\right\|_{L^1} < \infty$, $\limsup_{h\to 0}\int_{x_0-h}^{x_0+h}\int_{x_0-h}^{x_0+h} \frac{|f(x)f(y)|}{|x-y|^2}\,dx\,dy = 0$ by uniform integrability.

Therefore $f(x_0)^2 \le \lim_{h\to 0}\int_{x_0-h}^{x_0+h}\int_{x_0-h}^{x_0+h} \frac{|f(x)f(y)|}{|x-y|^2}\,dx\,dy = 0 \Rightarrow f(x_0)=0$ $\forall$ Lebesgue points x_0. As a.e. x is a Lebesgue point, this implies $f=0$ a.e.

□

⑤ 405 846 515

We claim that $\int_0^1 f\,d\mu = 0 \ \forall\ f \in C[0,1]$.
We note that by linearity, $\int_0^1 g\,d\mu = 0\ \forall$ finite linear combinations g of functions of the form $t \mapsto \frac{1}{x+t}$ for $x > 0$. ~~since~~ $\cancel{[0,1]}$.

~~We claim that~~ $\cancel{|\mu|[0,1] < \infty}$. ~~Let~~ $\cancel{\mu = \mu^+ - \mu^-}$ ~~be the Hahn-decomposition of~~ $\cancel{\mu}$. ~~Then,~~ $\cancel{\forall\ x > 1,}$

$$\cancel{\int_0^1 \frac{1}{t+x}\,d\mu^+ = \int \frac{1}{t+x}\,d\mu}$$

As given, μ is real-valued. Therefore $|\mu|[0,1] < \infty$. It then follows that if $f_n \to f$ uniformly and $\int f_n\,d\mu = 0\ \forall n$, then

$$\left|\int f\,d\mu\right| \le \int |f_n - f|\,d|\mu| \le \|f_n - f\|_{L^\infty} |\mu|[0,1] \to 0$$

and so $\int f\,d\mu = 0$. Therefore to show $\int_0^1 f\,d\mu = 0\ \forall\ f \in C[0,1]$, it suffices to show it for a uniformly dense subset of $C[0,1]$.
We claim the finite linear combination of $t \mapsto \frac{1}{x+t}$ for $x > 1$ are uniformly dense in $C[0,1]$.

Let $\mathcal{F}$ denote the set of all finite linear combinations of maps $t \to \frac{1}{t+x}$, $x > 0$, and let $\overline{\mathcal{F}}$ denote its uniform closure.
We aim to apply Stone-Weierstrass to $\overline{\mathcal{F}}$ to show it is uniformly dense in $C[0,1]$. We note that

$$t \mapsto \frac{1}{t+2}$$

is non-vanishing and injective on $[0,1]$. Therefore $\overline{\mathcal{F}}$ is non-vanishing and separates points. By Stone-Weierstrass, it then suffices to show $\overline{\mathcal{F}}$ is a subalgebra. By construction, it follows that $\overline{\mathcal{F}}$ is linear, therefore it must only be shown that $\overline{\mathcal{F}}$ is closed under pointwise multiplication.

$\longrightarrow$

To do so, it suffices to consider $x_1, x_2 > 1$ and the product of the maps $t \mapsto \frac{1}{t+x_1}$ and $t \mapsto \frac{1}{t+x_2}$.

If $x_1 \neq x_2$, then by partial fractions

$$\frac{1}{(t+x_1)} * \frac{1}{(t+x_2)} = \frac{(x_1-x_2)^{-1}}{t+x_1} + \frac{(x_2-x_1)^{-1}}{t+x_2} \in \overline{\mathcal{F}}$$

Now consider $x = x_1 = x_2 > 1$.

Choose $\{x_n\}$ s.t. $x_n > x$, $x_n \downarrow x$. Then $\forall t \in [0,1]$, $\forall n$

$$\left|\frac{1}{t+x_n} - \frac{1}{t+x}\right| = |x-x_n| \left|\frac{1}{(t+x_n)(t+x)}\right| \leq |x-x_n| \left|\frac{1}{(t+x)^2}\right| \leq |x-x_n| \left|\frac{1}{(1+x)^2}\right|$$

Since this bound goes to 0 uniformly in t, $\frac{1}{t+x_n} \rightrightarrows \frac{1}{t+x}$ on $[0,1]$. Therefore

$$\left\|\frac{1}{(t+x_n)(t+x)} - \frac{1}{(t+x)^2}\right\|_\infty \leq \frac{1}{\|1+x\|} \left\|\frac{1}{t+x_n} - \frac{1}{t+x}\right\|_\infty \to 0$$

B/c $\frac{1}{(t+x_n)(t+x)} \in \overline{\mathcal{F}}$, this implies that $\frac{1}{(t+x)^2} \in \overline{\mathcal{F}}$ as desired. Therefore $\overline{\mathcal{F}}$ is a sub-algebra and so Stone-Weierstrass implies that $\overline{\mathcal{F}}$ is dense in $C[0,1]$ and hence $\mathcal{F}$ is dense in $C[0,1]$.

$$\frac{1}{(t+x_1)(t+x_2)} = \frac{a}{t+x_1} + \frac{b}{t+x_2}$$

$$at + bt + ax_1 + bx_2 = 1$$

$$a+b = 0 \Rightarrow a = -b$$

$$ax_1 + bx_2 = 1$$

$$\Rightarrow a = 1/x_1 - x_2$$

$$b = 1/x_2 - x_1$$

$$\frac{1}{(t+x)^2} =$$

$$\left|\frac{1}{t+x_n} - \frac{1}{t+x}\right| = \left\|\frac{|x-x_n|}{(t+x_n)(t+x)}\right|$$

As noted earlier, this implies that $\forall f \in C[0,1]$,

$$\int f \, d\mu = 0$$

Fix an interval $[a,b] \subset [0,1]$. Then

$$f_n = \max(1 - d(x, [a,b])n, 0) \in C[0,1]$$

and $f_n \downarrow \chi_{[a,b]}$. DCT implies that

$$\mu[a,b] = \int_0^1 \chi_{[a,b]} \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu = 0$$

By monotonicity, $\mu(a,b) = 0$ as well. Since $\mathcal{B}(\mathbb{R})$ is generated by intervals, this concludes $\mu = 0$. $\square$

⑥ 405 846 515

Let $\mathcal{M}(T\times T)$ denote the space of signed Borel measures on $T\times T$. We ~~no~~ recall that $\mathcal{M}(T\times T) \cong C(T\times T)^*$ ~~and so by Banach~~ by the Riesz representation theorem. Therefore by Banach-Alaoglu, the ~~unit ball~~ closed unit ball $\bar{B} \subset \mathcal{M}(T\times T)$ is weak*-compact.

Consider $M \subset \bar{B} \subset \mathcal{M}(T\times T)$ defined by

$$M = \left\{ \gamma \in P(T\times T) : \iint f(x)g(y)\,d\gamma(x,y) = \int f(x)\,d\mu(x) \int g(y)\,d\nu(y) \quad \forall f,g \in C(T) \right\}$$

We claim that M is ~~weak close~~ weak* closed.

Suppose $\exists\ \gamma_n \in M$ s.t. $\gamma_n \xrightarrow{*} \gamma$. Then $\forall f,g \in C(T)$, $f(x)g(y) \in C(T\times T)$ and so

$$\int f(x)\,d\mu(x) \int g(y)\,d\nu(y) = \iint f(x)g(y)\,d\gamma_n \to \iint f(x)g(y)\,d\gamma \qquad (1)$$

Additionally, since $1 \in C(T\times T)$,

$$\gamma(T\times T) = \iint d\gamma = \lim_{n\to\infty} \iint d\gamma_n = 1 \qquad (2)$$

To show $\gamma \in M$, it then remains to show that γ is non-negative. T.o. show non-negative, it suffices to show $\iint f\,d\gamma \geq 0$ $\forall f \geq 0$ w/ $f \in C(T\times T)$. By definition, $\forall f \in C(T\times T)$ w/ $f \geq 0$,

$$\iint f\,d\gamma = \lim_{n\to\infty} \iint f\,d\gamma_n \geq 0 \qquad (3)$$

since γ_n is non-negative $\forall n$. Therefore γ is non-negative. Combining (1), (2), (3), we find that $\gamma \in M$. Therefore M is weak* closed. In particular, since $\bar{B} \supset M$ is compact, M is weak* compact.

Now let $\gamma_n \in M$ be a minimizing sequence of F. Since M is weak* compact, passing to a ~~sub~~ subsequence implies $\exists\ \gamma \in M$ s.t. $\gamma_n \xrightarrow{*} M$. B/c $\sin^2\left(\frac{\theta-\phi}{2}\right) \in C(T\times T)$, this implies that F achieves its minimum at γ.

$\square$

Since F is continuous, $F(z,z)$ is continuous.
Therefore to show that $F(z,z)$ is entire, it suffices to show that $\int_\gamma F(z,z)dz = 0$ $\forall$ triangles γ.

Suppose that γ is a triangle. Let γ' be a triangle sufficiently large so that $\gamma \subset \text{int}\,\gamma'$ and $d(\gamma,\gamma') \geq 1$.
Then $\forall$ $z \in \gamma$, the Cauchy integral formula implies that

$$2\pi i\, F(z,z) = \int_{\gamma'} \frac{F(w,z)}{w-z}\, dw$$

and so

$$2\pi i \int_\gamma F(z,z)dz = \int_\gamma \int_{\gamma'} \frac{F(w,z)}{w-z}\, dw\, dz$$

Since γ,γ' are compact, F is uniformly bounded on $\gamma' \times \gamma$. By construction, $|w-z| \geq 1$ on $\gamma' \times \gamma$. Therefore

$$\int_\gamma \int_{\gamma'} \left|\frac{F(w,z)}{w-z}\right| |dw||dz| \lesssim |\gamma'||\gamma| < \infty$$

and so Fubini's implies that

$$2\pi i \int_\gamma F(z,z)dz = \int_{\gamma'} \int_\gamma \frac{F(w,z)}{w-z}\, dz\, dw$$

By construction, $\forall$ $w \in \gamma'$, $w \notin \text{int}(\gamma)$. Therefore $\frac{F(w,z)}{w-z}$ is holomorphic in z on the interior of γ. Therefore

$$\int_\gamma F(z,z)dz = \int_{\gamma'} 0\, dw = 0$$

Morera's theorem then concludes $F(z,z)$ is holomorphic. $\square$

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Suppose that $u:\mathbb{D}\to[0,1]$ is harmonic. Then $\forall$ $0<R<1$, the Poisson kernel implies that for $r<R$, $\varphi\in[0,2\pi]$

$$u(re^{i\varphi})=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\left(\frac{R+re^{i\theta-\varphi}}{R-re^{i\theta-\varphi}}\right)u(Re^{i\theta})\,d\theta$$

In particular, for $\varphi=0$

$$u(r)=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\left(\frac{R+re^{i\theta}}{R-re^{i\theta}}\right)u(Re^{i\theta})\,d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|R-re^{i\theta}|^2}\mathrm{Re}\left((R+re^{i\theta})(R-\overline{re^{i\theta}})\right)u(Re^{i\theta})\,d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{(R-r\cos\theta)^2+r^2\sin^2\theta}\left(R^2-r^2\cos^2\theta-r^2\sin^2\theta\right)u(Re^{i\theta})\,d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2r\cos\theta+r^2}u(Re^{i\theta})\,d\theta$$

Then by DCT,

$$u_x(r)=\frac{1}{2\pi}\int_0^{2\pi}\frac{-(R^2-2r\cos\theta+r^2)2r-(R^2-r^2)(-2\cos\theta+2r)}{(R^2-2r\cos\theta+r^2)^2}u(Re^{i\theta})\,d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{-2rR^2+2r^2\cos^2\theta\cancel{-2r^3}+2R^2\cos\theta-2rR^2-2r^2\cos\theta\cancel{+2r^3}}{(R^2-2r\cos\theta+r^2)^2}u(Re^{i\theta})\,d\theta$$

$$=\frac{1}{\pi}\int_0^{2\pi}\frac{2r^2\cos^2\theta+R^2\cos\theta-2rR^2-r^2\cos\theta}{(R^2-2r\cos\theta+r^2)^2}u(Re^{i\theta})\,d\theta$$

In particular,

$$u_x(0)=\frac{1}{\pi}\int_0^{2\pi}\frac{\cos\theta}{R^2}u(Re^{i\theta})\,d\theta$$

Taking $R\to1$, DCT implies

$$u_x(0)=\frac{1}{\pi}\int_0^{2\pi}\cos\theta\,\limsup_{R\to1}u(Re^{i\theta})\,d\theta \qquad (1)$$

~~hence this is~~ $0\le u\le1$, this is maximized or minimized when $\limsup_{R\to1}u(Re^{i\theta})=\begin{cases}\pm1 & 0\le\theta\le\pi/2\\ \pm1 & \frac{3\pi}{2}\le\theta\le2\pi\\ 0 & \text{else}\end{cases}$

~~in which case $u_x(0)$~~

Therefore $|u_x(0)|\le\frac{2}{\pi}$.

The prior computation implies that ~~the~~ a maximizer is likely of the form u s.t. $u|_{\partial\mathbb{D}} = \chi_{\{-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\}}$.

Define $u:\mathbb{D}\to[0,1]$ via

$$u(re^{i\varphi}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Re}\left(\frac{1+re^{i(\theta-\varphi)}}{1-re^{i(\theta-\varphi)}}\right)\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\theta)\,d\theta$$

Then u maximizes (1) by definition of the Poisson kernel and so achieves the maximum. $\square$

Therefore $\sup\{|u_x(0)|\} = \frac{2}{\pi}$.

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we do both parts at once.

Fix some compact $K \subseteq \mathbb{C}$, $\sup\{|z| : z \in K\} := R < \infty$.

Consider $z \in K$ and define $a_n(z) = (1+\frac{1}{n})^z(1-\frac{z}{n})$.

By a Taylor expansion at 0, $(1+x)^z = 1 + zx + O_{Re(z)}(x^2)$

Then $a_n(z) = 1 - \frac{z}{n} + \frac{z}{n} - \frac{z^2}{n^2} + O_R(\frac{1}{n^2}) = 1 - \frac{z^2}{n^2} + O_R(n^{-2})$.

In particular, $|1 - a_n(z)| \leq \frac{R^2}{n^2} + O_R(n^{-2})$ so $a_n \to 1$ uniformly on K. Then for sufficiently large m, $a_n(z) \in D(1, 1/2)$ $\forall z \in K, n \geq m$ and so a complex logarithm can be defined $\forall n \geq m$.

Then

$$\prod_{n \geq 1} a_n(z) \text{ converges} \iff \prod_{n \geq m} a_n(z) \text{ converges}$$
$$\iff \sum_{n \geq m} \mathrm{Log}(a_n(z)) \text{ converges}$$

Expanding the logarithm as a Taylor series about 1,

$$\Big|\sum_{n \geq m} \mathrm{Log}(a_n(z))\Big| = \Big|\sum_{n \geq m} (a_n(z) - 1) + O((a_n(z)-1)^2)\Big|$$
$$= \Big|\sum_{n \geq m} (a_n(z) - 1) + O\Big(\frac{|z|^4}{n^4} + O_R(n^{-4})\Big)\Big|$$
$$= \Big|\sum_{n \geq m} \frac{-z^2}{n^2} + O_R(n^{-2})\Big|$$
$$\leq R \sum_{n \geq m} \frac{1}{n^2} + O_R(n^{-2})$$

which converges uniformly for $z \in K$.

Therefore $\prod_{n \geq 1} a_n(z)$ converges uniformly on K.

Since $a_n(z)$ is holomorphic $\forall z$, this implies that $\prod_{n \geq 1} a_n(z)$ is entire, concluding part (b).

Taking $K = \{z\}$ for $z \in (-\infty, 0)$ concludes part (a). $\square$

(11) 405 846515

Suppose that $\varphi : A_{R_1} \to A_{R_2}$ is conformal. Since $\partial A_{R_1}, \partial A_{R_2}$ are smooth, φ extends continuously to a map $\overline{A_{R_1}} \to \overline{A_{R_2}}$

i.e. $\partial A_{R_1} \to \partial A_{R}$. By composing w/ an inversion if necessary, we may assume WLOG that $\varphi : \partial \mathbb{D} \to \partial \mathbb{D}$.

~~we first show a brief lemma~~

We claim that φ extends to an entire function.

To show this, we use Schwarz reflection to extend φ to all of $\mathbb{C}$.

Let $\Psi(z) = \frac{z-i}{z+i}$ be the Cayley transform, a conformal map which sends $\mathbb{H} \to \mathbb{D}$ and takes $\mathbb{R} \to \partial \mathbb{D}$.

Consider $g = \Psi^{-1} \circ \varphi \circ \Psi^{-1}$. By construction, Ψ takes A_{R_1}, A_{R_2} to subsets of the lower half plane. Therefore g is a holomorphic map between regions of the lower half plane. Since φ fixes $\partial \mathbb{D}$, g fixes $\mathbb{R}$ and so the Schwarz reflection principle extends g to $\Psi(A_{R_1}) \cup \overline{\Psi(A_{R_1})} \cup \mathbb{R}$ via $g(z) = \overline{g(\bar{z})}$. Let $\tilde{g}$ denote this extension.

We note that Ψ takes reflection across $\mathbb{R}$ to inversion across $\partial \mathbb{D}$.

Therefore $\Psi^{-1} \circ \tilde{g} \circ \Psi$ is an extension of φ to a holomorphic map $\{1/R_1 < |z| < R_1\} \to \{1/R_2 < |z| < R_2\}$ via the symmetry $\varphi(1/\bar{z}) = 1/\overline{\varphi(z)}$.

~~Rep~~ Iterating this process, we find that φ extends to a holomorphic map $\{0 < |z| < R_1\} \to \{0 < |z| < R_2\}$. Since φ is bounded, the singularity at 0 is removable. By the maximum modulus principle, since φ is non-constant, $\varphi(0) = 0$.

Therefore $\varphi : D(0,R_1) \to D(0,R_2)$ is holomorphic i.e. $\varphi : \partial \mathbb{D} \to \partial \mathbb{D}$ and $\varphi(0) = 0$. B.

By maximum modulus, $\varphi : \mathbb{D} \to \mathbb{D}$. ~~Therefore b~~

Therefore by the Schwarz lemma,

$$|\varphi(z)| \leq |z| \quad \forall \quad z \in \mathbb{D}$$

We recall that by construction, φ maps the circle of radius $1/R_1$ to the circle of radius $1/R_2$.

Therefore

$$1/R_2 \leq 1/R_1$$

$$\Rightarrow \quad R_1 \leq R_2$$

The symmetric argument w/ φ^{-1} then implies $R_2 \leq R_1$ and so $R_1 = R_2$.

$\square$

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We recall that for any $a \in \mathbb{D}$,

$$\frac{1}{(1-a)^2} = \sum_{n \geq 1} n a^{n-1}$$

$$\cancel{\left(\frac{1}{1-x}\right)' = \frac{+1}{(1-x)^2}} \qquad \cancel{\frac{1}{1-x} = \sum_{n\geq 0} x^n}$$

Therefore, since $\bar{z}w \in \mathbb{D}$ if $z, w \in \mathbb{D}$, $\forall\, w, z \in \mathbb{D}$

$$\frac{f(z)}{(1-\bar{z}w)^2} = f(z) \sum_{n \geq 0} (n+1)(\bar{z}w)^n$$

In particular,

$$\frac{f(z)}{(1-\bar{z}w)^2} = \lim_{k \to \infty} \sum_{n=0}^{k} (n+1)(\bar{z}w)^n$$

For $z, w \in \mathbb{D}$,

$$|1-\bar{z}w| \geq 1 - |\bar{z}||w| \geq 1-|w|$$

Therefore since f is bounded, $\frac{f(z)}{(1-\bar{z}w)^2}$ is uniformly bounded on $\mathbb{D}$ for fixed w. The dominated convergence theorem then implies

$$\frac{1}{\pi} \int \frac{f(z)}{(1-\bar{z}w)^2} dz = \sum_{n \geq 0} \frac{(n+1)}{\pi} w^n \int_{\mathbb{D}} f(z) \bar{z}^n \, dA(z)$$

$$= \sum_{n \geq 0} \left(\sqrt{\frac{n+1}{\pi}}\, w^n \right) \left\langle f(z), \sqrt{\frac{n+1}{\pi}}\, z^n \right\rangle$$

where $\langle \cdot, \cdot \rangle$ is the ~~on~~ L^2 inner product over $\mathbb{D}$.

We claim that $\mathcal{F} = \left\{ \sqrt{\frac{n+1}{\pi}}\, z^n \right\}_{n \geq 0}$ is orthonormal.

By direct computation,

$$\left\| \sqrt{\frac{n+1}{\pi}}\, z^n \right\|_{L^2}^2 = \frac{n+1}{\pi} \int_{\mathbb{D}} |z|^{2n} \, dA(z) = \frac{n+1}{\pi}\, 2\pi \, \frac{1}{2n+2} = 1$$

For $n \neq m$, we make the change of variables $z = e^{i\frac{\pi}{n-m}} w$ and use rotational invariance to see

$$\langle z^n, z^m \rangle = \int_{\mathbb{D}} z^n \bar{z}^m \, dA(z) = \int_{\mathbb{D}} z^{n-m} |z|^{2m} \, dA(z)$$

$$= -\int_{\mathbb{D}} w^{n-m} |w|^{2m} \, dA(w) = -\langle z^n, z^m \rangle$$

Therefore $\langle z^n, z^m \rangle = 0$, and so $\mathcal{F}$ is orthonormal.

We claim that $\mathcal{F}$ forms a basis for L^2 holomorphic functions on $\mathbb{D}$, which we recall is a Hilbert space.

FROM HERE, PROOF GETS REDUNDANT

<u>alternate</u>

Since f is holomorphic on $\mathbb{D}$, we may write

$$f(z) = \sum_{n \geq 0} a_n z^n$$

where the sum converges uniformly on compact subsets of $\mathbb{D}$.

Similarly, we recall that since $|\bar{z}w| < 1$ for all $w, z \in \mathbb{D}$,

$$\frac{1}{(1-\bar{z}w)^2} = \sum_{m \geq 0} (m+1)(\bar{z}w)^m$$

where the sum converges uniformly on compact subsets of $\mathbb{D}$.

Then $\forall \; 0 < R < 1$,

$$\int_{|z| \leq R} \frac{f(z)}{(1-w\bar{z})^2}\, dA(z) = \sum_{n \geq 0} \sum_{m \geq 0} a_n (m+1) w^m \int_{|z| \leq R} z^n \bar{z}^m \, dA(z)$$

$$= \sum_{n,m \geq 0} a_n (m+1) w^m \int_0^R \int_0^{2\pi} r^{n+m+1} e^{i\theta(n-m)}\, dr\, d\theta$$

$$= \sum_{n,m \geq 0} a_n (m+1) w^m \, 2\pi \, \delta_{nm} \int_0^R r^{n+m+1}\, dr$$

$$= \sum_{n \geq 0} a_n (n+1) w^n \frac{1}{2n+2} R^{2n+2}\, 2\pi$$

$$= \pi R^2 \sum_{n \geq 0} a_n (R^2 w)^n$$

$$= \pi R^2 f(R^2 w)$$

Since f is bounded and $|1-\bar{z}w| \geq 1 - |\bar{z}||w| \geq 1 - |w|$, $\frac{f(z)}{(1-\bar{z}w)^2}$ is uniformly bounded for fixed w. DCT and ~~continuity~~ thus imply that $\int_{\mathbb{D}} \frac{f(z)}{(1-\bar{z}w)^2}\, dA(z) = \pi f(w)$ as desired. $\square$

Analysis Fall 2017

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Let $\{f_n\} \subset L^2[0,1]$ be bounded by some $M>0$. Suppose $\exists$ f s.t. $f_n \to f$ a.e. We claim that $f_n \to f$ in the weak topology on L^2. By Fatou's lemma, $\|f\|_{L^2} \leq \liminf_{n\to\infty} \|f_n\|_{L^2} \leq M$.

~~Fix some $g \in L^2[0,1]$.~~ We first claim that $\forall\, g \in C[0,1]$

$$\int f_n g\, dx \to \int f g\, dx$$

Fix some $\varepsilon > 0$. By ~~Egorov~~ Egorov's theorem, since $f_n g \to fg$ a.e., $\exists$ some $E \subset [0,1]$ s.t. $g f_n \to fg$ uniformly on $[0,1]\setminus E$ and $|E| < \varepsilon^2$. Then

$$\left| \int f_n g\, dx - \int f g\, dx \right| \leq \int_E |f_n - f||g|\, dx + \int_{[0,1]\setminus E} |f_n - f||g|\, dx$$

$$\text{(Hölder)} \qquad \leq \|f_n - f\|_{L^2} \, \||g|\chi_E\|_{L^2} + \|f_n g - fg\|_{L^\infty([0,1]\setminus E)}$$

$$\leq 2M \|g\|_{L^\infty}^{1/2} |E|^{1/2} + \|f_n g - fg\|_{L^\infty([0,1]\setminus E)}$$

Since $g \in C[0,1]$ and $f_n g \to fg$ uniformly on $[0,1]\setminus E$, this implies that $\limsup_{n\to\infty} \left| \int f_n g\, dx - \int f g\, dx \right| \lesssim \varepsilon$.

Taking $\varepsilon \to 0$ then implies that $\int f_n g\, dx \to \int f g\, dx$ $\forall\, g \in C[0,1]$.

We now extend this to all $g \in L^2[0,1]$.

We recall that $C[0,1]$ is dense in $L^2[0,1]$.

Therefore $\forall\, g \in L^2[0,1]$, $\exists$ $g_k \in C[0,1]$ s.t. $g_k \to g$ in L^2.

Then $\forall k$,

$$\left| \int f_n g\, dx - \int f g\, dx \right| \leq \left| \int f_n (g - g_k)\, dx \right| + \left| \int f (g - g_k)\, dx \right| + \left| \int (f_n - f) g_k\, dx \right|$$

$$\text{(Hölder)} \qquad \leq \|f_n\|_{L^2} \|g - g_k\|_{L^2} + \|f\|_{L^2} \|g - g_k\|_{L^2} + \left| \int (f_n - f) g_k\, dx \right|$$

$$\leq 2M \|g - g_k\|_{L^2} + \left| \int (f_n - f) g_k\, dx \right|$$

Since the claim holds $\forall\, k$ and $g_k \to g$ in L^2, the result ~~then~~ follows from taking $n \to \infty$ and then $k \to \infty$. $\square$

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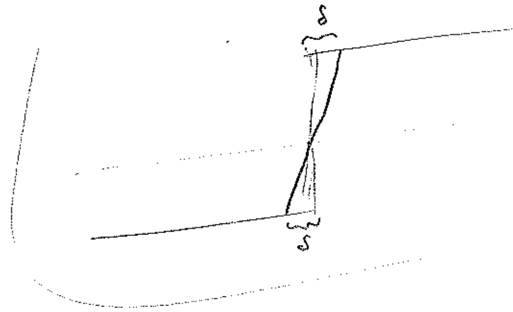
Define $\Lambda : V \to \mathbb{R}$ by

$$\Lambda(f) = \int_0^1 f\,dx - \int_{-1}^0 f\,dx$$



We claim that $\Lambda(B) = (-2,2)$.

~~Consider some $f \in B$. Then $\forall \varepsilon > 0\ \exists \delta > 0$ s.t.~~
~~if $|x|<\delta$ then $|f(x)-f(0)|<\varepsilon$. Then $\forall \varepsilon>0$~~

$$\cancel{|\Lambda(f)| = \left|\int_0^1 f\,dx - \int_{-1}^0 f\,dx\right|}$$

Suppose $\exists$ $f \in B$.
Let $E = \{x \in [0,1] : f(x) < 1\}$; $F = \{x \in [-1,0] : f(x) > -1\}$.
Then

$$\begin{aligned}
\Lambda(f) &= |[0,1]\setminus E| + |[-1,0]\setminus F| + \int_E f\,dx - \int_F f\,dy \\
&= 2 - |E| - |F| + \int_E f\,dx - \int_F f\,dy \\
&\leqslant 2 - |E| - |F| + |E| + |F| \\
&= 2
\end{aligned}$$

w/ equality iff $|E| = |F| = 0$. Therefore $\Lambda(f) \le 2$ w/ equality iff $f = 1$ a.e on $[0,1]$ and $f = -1$ a.e on $[-1,0]$. Since f is continuous, this is impossible and hence $\Lambda(f) < 2$ $\forall f \in B$.
A symmetric argument implies that $\Lambda(f) > -2$ $\forall f \in B$.
Therefore ~~$A(B)$~~ $\Lambda(B) \subset (-2,2)$.

Define $f_r = \begin{cases} rx & x \in [-1/r, 1/r] \\ 1 & x \geqslant 1/r \\ -1 & x \le -1/r \end{cases}$ for $r \geqslant 1$

Then

$$\begin{aligned}
\Lambda(f_r) &= 1(1-1/r) + \int_0^{1/r} rx\,dx - \int_{-1/r}^0 rx\,dx + 1(1-1/r) \\
&= 2\frac{r-1}{r} + \frac{r}{2}x^2\Big|_0^{1/r} - \frac{r}{2}x^2\Big|_{-1/r}^0 \\
&= 2\frac{r-1}{r} + \frac{1}{2r} - \left(0 - \frac{r}{2}\frac{1}{r^2}\right) = 2\frac{r-1}{r} + \frac{1}{r} = \frac{2r-1}{r}
\end{aligned}$$

Varying r from 1 to ∞ implies that

$\Lambda(f_r)$ takes all values in $[1, 2)$

Considering $-f_r$ implies that $\Lambda(-f_r)$ takes all ~~val~~ values in $(-2, 1]$.

Now consider, for $k \in [-1, 1]$

$$\Lambda(kx) = \int_0^1 kx\,dx - \int_{-1}^0 kx\,dx$$

$$= \frac{k}{2} + \frac{k}{2}$$

$$= k$$

Therefore $\Lambda(kx)$ takes all values in $[-1, 1]$.

Combining these w/ the fact that $\Lambda(B) \subset (-2, 2)$,

this implies that $\Lambda(B) = (-2, 2)$ which is open. $\square$

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By combining $f(x+1)=f(x)$ and $f(x)=f(2x)$, it follows that $f(x+q)=f(x)$ $\forall$ dyadic rationals q.

Let x_0 be a Lebesgue point of f. Let $I_n(x_0)$ be an interval $[2^{-n}k, 2^{-n}(k+1))$ for some $k\in\mathbb{Z}$ s.t. $x\in I_n(x_0)$.
Then $\forall n$

$$\left| f(x_0) - \frac{1}{|I_n(x_0)|}\int_{I_n(x_0)} f(x)\,dx \right| = \left| 2^n \int_{I_n(x_0)} (f(x_0)-f(x))\,dx \right|$$

$$\leq 2^n \int_{[x_0-2^{-n},\, x_0+2^{-n}]} |f(x_0)-f(x)|\,dx$$

$$= \frac{2}{|[x_0-2^{-n},\, x_0+2^{-n}]|}\int_{[x_0-2^{-n},\, x_0+2^{-n}]} |f(x_0)-f(x)|\,dx$$

By the stronger corollary to the Lebesgue differentiation theorem, this implies that

$$\limsup_{n\to\infty}\left| f(x_0) - \frac{1}{|I_n(x_0)|}\int_{I_n(x_0)} f(x)\,dx \right| \leq 2\limsup_{n\to\infty}\frac{1}{|[x_0-2^{-n},x_0+2^{-n}]|}\int_{[x_0-2^{-n},x_0+2^{-n}]} |f(x_0)-f(x)|\,dx = 0$$

b/c x_0 is a Lebesgue point of f. Therefore $\frac{1}{|I_n(x_0)|}\int_{I_n(x_0)} f(x)\,dx \to f(x_0)$ $\forall$ Lebesgue points x_0 of f.

Now let y be any Lebesgue point of f. By the symmetry of f and the reasoning above,

$$f(y) = \lim_{n\to\infty}\frac{1}{|I_n(y)|}\int_{I_n(y)} f(x)\,dx = \lim_{n\to\infty}\frac{1}{|I_n(x_0)|}\int_{I_n(x_0)} f(x)\,dx = f(x_0).$$

Therefore f is ~~constant~~ on all Lebesgue points and hence ~~constant~~ a.e.

$\square$

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To show $|g|<\infty$ a.e., it suffices to show that $g \in L^1_{loc}$.
Fix $R>0$. Then by Tonelli's and direct computation

$$\int_{|z|\le R} |g(z)|\,dA(z) = \int_{|z|\le R}\int_{|w-z|\le 1} \frac{|f(w)|}{|w-z|}\,dA(w)\,dA(z)$$

$$= \int_{|z|\le R}\int_{|w|\le R+1} \chi_{\{|w-z|\le 1\}}(w)\,\frac{|f(w)|}{|w-z|}\,dA(w)\,dA(z)$$

(Tonelli's) $$= \int_{|w|\le R+1} |f(w)| \int_{|z|\le R} \chi_{\{|w-z|\le 1\}}(z)\,\frac{1}{|w-z|}\,dA(z)\,dA(w)$$

$$\le \int_{|w|\le R+1} |f(w)| \int_{|z-w|\le 1} \frac{1}{|w-z|}\,dA(z)\,dA(w)$$

$$= \int_{|w|\le R+1} |f(w)|\, 2\pi\, dA(w)$$

(Hölder) $$\le 2\pi \sqrt{\pi(R+1)^2}\, \|f\|_{L^2} < \infty$$

~~Therefore~~ Since this holds $\forall R$, this implies $g \in L^1_{loc}$ and hence $|g|<\infty$ a.e.

We now show $g \in L^2$.
By definition,

$$\|g\|_{L^2} = \left(\int_{\mathbb{C}} \left|\int_{\mathbb{C}} \chi_{\{|z-w|\le 1\}} \frac{|f(w)|}{|w-z|}\,dA(w)\right|^2 dA(z)\right)^{1/2}$$

By Hölder,

$$\|g\|_{L^2}^2 \le \int_{\mathbb{C}} \left| \|\chi_{\{|z-w|\le 1\}} |f(w)|/|w-z|^{1/2}\|_{L^2(w)} \|\chi_{\{|z-w|\le 1\}} 1/|w-z|^{1/2}\|_{L^2(w)}\right|^2 dA(z)$$

$\longrightarrow$

Then

$$\|g\|_{L^2}^2 \leq \int_{\mathbb{C}} \left(\left(\int_{\mathbb{C}} \chi_{\{|w-z|\leq 1\}} \frac{|f(w)|^2}{|w-z|} \, dA(w) \right) \underbrace{\left(\int_{\mathbb{C}} \chi_{\{|w-z|\leq 1\}} \frac{1}{|w-z|} \, dA(w) \right)}_{=2\pi} \right) dA(z)$$

$$= 2\pi \int_{\mathbb{C}} \int_{\mathbb{C}} \chi_{\{|w-z|\leq 1\}} \frac{|f(w)|^2}{|w-z|} \, dA(w)\, dA(z)$$

By Tonelli's,

$$\|g\|_{L^2}^2 \leq 2\pi \int_{\mathbb{C}} |f(w)|^2 \int_{\mathbb{C}} \chi_{\{|w-z|\leq 1\}} \frac{1}{|w-z|} \, dA(z)\, dA(w)$$

$$= 4\pi^2 \int_{\mathbb{C}} |f(w)|^2 \, dA(w)$$

$$= 4\pi^2 \|f\|_{L^2}^2 < \infty$$

~~Therefore~~ Therefore $g \in L^2$ as desired.

$\square$

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Define $f(z) = \dfrac{\sin(2\pi z)}{\sin(2\pi(z-1/3))}$. We recall that $\sin(w)$ is entire w/ simple zeros only at $\mathbb{Z}$. Therefore f is meromorphic w/ $f(z)=0$ iff $z\in\mathbb{Z}$ and $f(z)=\infty$ iff $z-1/3\in\mathbb{Z}$.

We aim to find a bound on $|f(x+iy)|$ for $|y|\geq 1$.

We recall that $\sin(w) = \dfrac{e^{iw}-e^{-iw}}{2i}$. Therefore

$$f(z) = \frac{e^{2\pi i z} - e^{-2\pi i z}}{e^{2\pi i(z-1/3)} - e^{-2\pi i(z-1/3)}}$$

By the triangle and reverse triangle inequalities, this implies that

$$|f(z)| \leq \frac{|e^{2\pi i z}| + |e^{-2\pi i z}|}{\left||e^{2\pi i(z-1/3)}| - |e^{-2\pi i(z-1/3)}|\right|}$$

$$= \frac{e^{-2\pi y} + e^{2\pi y}}{|e^{-2\pi y} - e^{2\pi y}|}$$

$$= \frac{\cancel{e^{-2\pi y}}(1+e^{4\pi y})}{\cancel{e^{-2\pi y}}|1-e^{4\pi y}|} = \frac{1+e^{4\pi y}}{|1-e^{4\pi y}|}$$

As $y\to-\infty$, both the numerator and denominator $\to 1$ and so $\limsup_{y\to-\infty}|f(x+iy)|\leq 1$.

For $y>0$, ~~then~~ the denominator is $e^{4\pi y}-1$ and hence

$$\limsup_{y\to\infty}|f(x+iy)| \leq \limsup_{y\to\infty}\frac{e^{4\pi y}+1}{e^{4\pi y}-1} = 1$$

since $\dfrac{1+e^{4\pi y}}{|e^{4\pi y}-1|}$ is continuous on $\{|y|\geq 1\}$.

Therefore $\dfrac{1+e^{4\pi y}}{|1-e^{4\pi y}|}$ is bounded by some M on $\{|y|\geq 1\}$,

Taking $g(z) = \dfrac{f(z)}{M}$ is thus the desired function. $\square$

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We note that this is precisely the Poisson kernel.

That is to say that if $f: \partial\mathbb{D} \to \mathbb{R}$ is continuous then

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) f(e^{i\theta})\,d\theta$$

is the unique harmonic function on $\mathbb{D}$ which extends continuously ~~on~~ to f on $\partial\mathbb{D}$.

Suppose first that u is uniformly continuous on $\mathbb{D}$. Then u extends continuously to $\partial\mathbb{D}$. Let $f = u|_{\partial\mathbb{D}}$. Then by the Poisson kernel and uniqueness

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) f(e^{i\theta})\,d\theta \quad \forall z \in \mathbb{D}$$

as desired.

Similarly, if u admits the representation, then u extends continuously to f on $\partial\mathbb{D}$ and hence u is continuous on $\overline{\mathbb{D}}$ and so uniformly continuous.

(9) 405 346 516

Suppose that F is as given.

Consider

$$G(z,w) = \frac{F(z,w)-F(z,0)}{F(z,1)-F(z,0)}$$

Since $w \mapsto F(z,w)$ is injective, $F(z,1) \neq F(z,0)$ $\forall z \in \mathbb{C}$.

Therefore $z \mapsto G(z,w)$ is holomorphic $\forall w$.

As given,

$$G(0,w) = \frac{F(0,w)-F(0,0)}{F(0,1)-F(0,0)} = \frac{w-0}{1-0} = w \qquad (1)$$

Suppose that it has been shown that $z \mapsto G(z,w)$ is constant for fixed w. Then by (1), $G(z,w) = w$ $\forall w, z$ and so

$$F(z,w) = w\underbrace{\left(F(z,1)-F(z,0)\right)}_{a(z)} + \underbrace{F(z,0)}_{b(z)}.$$

Since $w \mapsto F(z,w)$ is injective, $a(z)$ is non-vanishing and entire.

Since $F(0,w)=0$, $a(0)=1$ and $b(0)=0$. This would conclude the problem.

Therefore, it only remains to show that $G(z,w)$ is constant for fixed w.

By direct calculation, ~~later~~

$$G(z,0) = \frac{F(z,0)-F(z,0)}{F(z,1)-F(z,0)} = 0$$

$$G(z,1) = \frac{F(z,1)-F(z,0)}{F(z,1)-F(z,0)} = 1$$

Now consider $w \neq 0,1$. Then by injectivity, $G(z,w)$ avoids 0 since $F(z,w) \neq F(z,0)$ $\forall w$. Similarly, $G(z,w)$ avoids 1 since $F(z,1) \neq F(z,w)$ $\forall w$. Therefore, by little Picard's theorem, G is constant. As above $G(z,w)=w$.

In all cases, $G(z,w)=w$ which concludes. $\square$

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We recall that the modulus squared of holomorphic functions are subharmonic. Therefore $\forall\ k \geq 1$,

$$\sum_{n=1}^{k} |f_n|^2$$

is subharmonic. Since subharmonicity is preserved through uniform convergence, this implies that $F(z)$ is subharmonic on all compact subsets of $\mathbb{D}$ and hence subharmonic on $\mathbb{D}$. It thus remains to show that F converges uniformly on compact subsets of $\mathbb{D}$.

To show that F converges uniformly ~~cor~~ on compact sets, it suffices to show that F converges uniformly on $\overline{D(0,R)}$ $\forall\ R \in (0,1)$.

To show uniform convergence on $\overline{D(0,R)}$, we must show that the tail $\sum_{n \geq k} |f_n(z)|^2 \to 0$ uniformly as $k \to \infty$ $\forall\ z \in \overline{D(0,R)}$.

By the mean value theorem, since f_n^2 is holomorphic $\forall\ n$, $\forall\ z \in \overline{D(0,R)}$

$$f_n^2(z) = \frac{1}{\pi(1-R)^2} \int_{D(z,1-R)} f_n^2(w)\, dA(w)$$

$$\Rightarrow \sum_{n \geq k} |f_n^2(z)| \leq \frac{1}{\pi(1-R)^2} \sum_{n \geq k} \int_{D(z,1-R)} |f_n(w)|^2 dA(w)$$

$$\leq \frac{1}{\pi(1-R)^2} \sum_{n \geq k} \int_{\mathbb{D}} |f_n(w)|^2 dA(w)$$

By Tonelli's theorem, this implies

$$\sum_{n \geq k} |f_n(z)|^2 \leq \frac{1}{\pi(1-R)^2} \int_{\mathbb{D}} \sum_{n \geq k} |f_n(w)|^2\, dA(w)$$

Since $\sum_n |f_n(z)|^2 \leq 1$ $\forall\ z \in \mathbb{D}$, we know that $\sum_{n \geq k} ||f_n(w)|^2 \to 0$ monotonically as $k \to 0$ $\forall\ w \in \mathbb{D}$. Therefore by the MCT, $\forall\ z \in \overline{D(0,R)}$

$$\limsup_{k \to \infty} \sum_{n \geq k} |f_n(z)|^2 \leq \frac{1}{\pi(1-R)^2} \lim_{k \to \infty} \int_{\mathbb{D}} \sum_{n \geq k} |f_n(w)|^2 dA(w) = 0$$

Since the right hand side is independent of $z \in \overline{D(0,R)}$ $\Rightarrow$

this implies that $\sum_{n \geq k} |f_n(z)|^2 \to 0$ uniformly as $k \to \infty$ on $\overline{D(0,R)}$. Therefore $F(z)$ converges uniformly on compact subsets of $\mathbb{D}$.

(11) 405 846 515

Suppose ~~for the sake of contradiction~~ that
$$\inf\{|w| : w \notin f(\mathbb{D})\} \geq 1.$$
Then $\mathbb{D} \subset f(\mathbb{D})$.

Since f is injective, $f : \mathbb{D} \to f(\mathbb{D})$ is a bijection and f' is nonvanishing. Therefore f has a holomorphic inverse $f^{-1} : f(\mathbb{D}) \to \mathbb{D}$.

Since $f(0) = 0$, $f'(0) = 1$, $f^{-1}(0) = 0$ and $(f^{-1})'(0) = 1/1 = 1$.
Since $f(\mathbb{D}) \supset \mathbb{D}$, we can restrict f^{-1} to a map $f^{-1} : \mathbb{D} \to \mathbb{D}$.
The Schwarz lemma then implies that
$$f^{-1}(z) = z$$
and so $f(z) = z$.

Therefore, if $f(z) \neq z$ then $\inf\{|w| : w \notin f(\mathbb{D})\} < 1$.
If $f(z) = z$ then $\inf\{|w| : w \notin f(\mathbb{D})\} = 1$ trivially, and if $\inf\{|w| : w \notin f(\mathbb{D})\} = 1$ then $f(z) = z$.
This completes the claim. $\square$

(12) 405 846 515

Suppose $\exists$ $z \in \mathbb{C}$ s.t. $g'(z) \neq 0$. Then by the inverse function theorem, $\exists$ a neighborhood U of z s.t. $g|_U$ is conformal onto $(g|_U(U))$.
Then on U,

$$h = (g|_U)^{-1} \circ f$$

since $g|_U$ is conformal and f is holomorphic, h is holomorphic on U.
Since holomorphicity is a local property, this implies that h is holomorphic on $\{z : g' \neq 0\}$.

Since g is non-constant, $g' \not\equiv 0$. Therefore $g'(z) = 0$ for at most finitely many discrete z. Let $a_1, a_2, \dots$ denote these discrete points.
Then h is meromorphic on $\mathbb{C}$ w/ poles at $\{a_n\}$.
We claim that these poles are removable.
~~By the Riemann removable singularity theorem, it suffices to show that g is bounded near a_n $\forall n$.~~
Fix some a_n. Suppose for the sake of contradiction that h has an essential singularity at a_n.
Then by Picard, $\forall$ neighborhoods U of a_n, $h(U)$ is dense in $\mathbb{C}$. Since g is non-constant, the image of g excludes at most one point in $\mathbb{C}$. Therefore,
$\forall$ neighborhoods U of a_n, $g \circ h(U)$ is dense in $\mathbb{C}$.
However, this contradicts the continuity of f at a_n. $\longrightarrow$

Therefore h cannot have an essential singularity at a_n.
~~#~~ Suppose for the sake of contradiction that h has a pole of order $k \geq 1$ at a_n.
Then on a neighbourhood U of a_n, $|h| \gtrsim |z|^{-k}$.
By continuity of f, this implies that

$$|f(a_n)| = \lim_{z \to a_n} |g(h(z))|$$
$$\geq \limsup_{|z| \to \infty} |g(z)|$$

and so $\limsup_{|z|\to\infty} |g(z)| < \infty$. However, this would imply that g is bounded and hence constant, which is a contradiction.
Therefore h has a removable singularity at a_n.
Since h is ~~constant~~ continuous, this implies that h is holomorphic.

□

Bounded also just follows from continuity, dumbass.

Harmonic Prep

Analysis

S11.8

(a) A function $f: \mathbb{C} \to [-\infty, \infty)$ is called upper-semicontinuous if $z_n \to z$ implies $\limsup_{n\to\infty} f(z_n) \le f(z)$.

Equivalently, $f^{-1}[-\infty, a)$ is open $\forall\ a \in \mathbb{R}$.

(b) An upper-semicontinuous function is subharmonic if it satisfies the sub-mean value property,

$$f(z) \le \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})\,d\theta$$

$\forall\ r>0,\ z\in\mathbb{C}$. It suffices to check only sufficiently small r.

Equivalently, f is subharmonic if $\forall$ discs $D = D(z,r) \subset \mathbb{C}$, if h is a harmonic function on D, continuous on $\bar{D}$ s.t. $f \le h$ on ∂D then $f \le h$ on D.

(c) (i) Suppose $\mathcal{F}$ is a bounded family of subharmonic functions. Define $f(z) = \sup_{g\in\mathcal{F}} g(z)$. Then $\forall\ g \in \mathcal{F}$, $\forall z \in \mathbb{C}$, $r>0$

$$g(z) \le \frac{1}{2\pi}\int_0^{2\pi} g(z+re^{i\theta})\,d\theta \le \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})\,d\theta$$

In particular, taking the supremum over $\mathcal{F}$,

$$f(z) = \sup_{g\in\mathcal{F}} g(z) \le \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})\,d\theta$$

Therefore f satisfies the sub-mean value ~~prop~~ property.

~~It remains to show that f is upper semi-continuous.~~

Similarly, for all $g \in \mathcal{F}$ and $z_n \to z$

$$\limsup_{n\to\infty} g(z_n) \le g(z) \le f(z)$$

Then

$$\limsup_{n\to\infty} f(z_n) \le \sup_n \limsup_{n\to\infty} g(z_n) \le f(z)$$

so f is upper-semicontinuous and hence subharmonic.

It is not possible to determine whether the supremum of a bounded ~~upper~~ collection of subharmonic functions is upper-semicontinuous

~~(ii) Suppose (ii) is true.~~

Take for example $\mathcal{F} = \{\chi_{(1/n,\infty)}\}_{n\geq 1}$. Then

$f = \sup_n \chi_{(1/n,\infty)} = \chi_{(0,\infty)}$ ~~which~~ which is not ~~too~~ upper semicontinuous since $\limsup_{x\to 0} f(x) = 1 > f(0) = 0$.

Therefore the pointwise supremum need not be subharmonic.

(ii) Take $\mathcal{F}$ as above. Then

$f = \inf_n \chi_{(1/n,\infty)}$

S12.7

We recall that a function is harmonic iff it is continuous and satisfies the mean value property.

Define $u(z) = \inf_n u_n(z) = \lim_{n\to\infty} u_n(z)$. This is well-defined since u_n is non-increasing and non-negative.

We first show u is continuous on $\mathbb{D}$.

Fix some $z \in \mathbb{D}$ and $R>0$ s.t. $\overline{D(z,R)} \subset \mathbb{D}$.

Then by Harnack's inequality, for all $w \in D(z,R)$ and $\forall n$,

$$u_n(z)\frac{R-|z-w|}{R+|z-w|} \le u_n(w) \le \frac{R+|z-w|}{R-|z-w|} u_n(z)$$

Taking the limit as $n\to\infty$, this implies that $\forall w \in D(z,R)$,

$$u(z)\frac{R-|z-w|}{R+|z-w|} \le u(w) \le \frac{R+|z-w|}{R-|z-w|} u(z)$$

Upon rearranging, this yields

$$u(w) \le \frac{R+|z-w|}{R-|z-w|} u(z) \Rightarrow u(w)-u(z) \le \frac{|z-w|(u(z)-u(w))}{R}$$

$$u(w) \ge \frac{R-|z-w|}{R+|z-w|} u(z) \Rightarrow u(w)-u(z) \ge -\frac{|z-w|(u(z)-u(w))}{R}$$

Since $u \le u_1$, this yields

$$|u(w)-u(z)| \le \frac{|z-w| 2\max_{x\in \overline{D(z,R)}} |u_1(x)|}{R} \lesssim_R |z-w|$$

Therefore u is continuous at z, and hence continuous on $\mathbb{D}$.

We now show u satisfies the mean value property.

Fix $z \in \mathbb{D}$ and $r>0$ s.t. $\partial D(z,r) \subset \mathbb{D}$. Since u_1 is continuous, u_1 is bounded and hence integrable over $\partial D(z,r)$. Therefore by DCT since $u_n \le u_1 \; \forall n$

$$u(z) = \lim_{n\to\infty} u_n(z) = \lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} u_n(z+re^{i\theta})\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \lim_{n\to\infty} u_n(z+re^{i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})\,d\theta$$

As this holds $\forall z, r$, this concludes that u is harmonic. $\square$

S12.8

Map Ω to a horizontal strip via $z \mapsto z^2$.

Then define $v = \mathrm{Im}(e^{\pi z})$ on the strip and map back to Ω.

F9.6

For $R \in (0,1)$, define μ_R to be the measure defined by

$$d\mu_R(\theta) = \frac{h(Re^{i\theta})}{2\pi} d\theta$$

Then $\forall$ $r \in [0,R)$, $\forall \theta$

$$h(re^{i\eta}) = \frac{1}{2\pi}\int_0^{2\pi} P_{r/R}(\eta-\theta) h(Re^{i\theta}) d\theta$$

$$= \int_0^{2\pi} P_{r/R}(\eta-\theta)\, d\mu_R(\theta)$$

Note that since h is non-negative and continuous, μ_R is non-negative and Borel.
This implies that $\forall R$, since $P_0 \equiv 1$,

$$\mu_R[0,2\pi] = \int_0^{2\pi} \frac{h(Re^{i\theta})}{2\pi} d\theta$$

$$= \int_0^{2\pi} \frac{P_0(\eta-\theta)}{2\pi} h(Re^{i\theta}) d\theta$$

$$= h(0)$$

$P_{r/R}(\theta) = \mathrm{Re}\left(\frac{R+re^{i\theta}}{R-re^{i\theta}}\right)$

& $\lim_{R\to 1} P_{r/R}(\theta) = P_r$

$\mu_R \xrightarrow{*} \mu$ Then

$\int P_r(\eta-\theta)\, d\mu = \lim_{R\to 1} \int$

$h(Re^{i\theta}) \leq \frac{R+R+\frac{1-R}{2}}{\frac{1-R}{2}} h(0)$

$\frac{4R}{1-R} h(0) + h(0)$

As this holds $\forall R$, $\{\mu_R\}$ is contained in the closed ball of radius $h(0)$. Viewing $\{\mu_R\}$ as a subset of $(\mathcal{C}[0,2\pi])^*$, Banach-Alaoglu implies that $\exists$ a subsequence μ_{R_K} s.t. μ_{R_K} converges weak* to some $\mu \in (C[0,2\pi])^*$.
We claim that $P_{r/R} \rightrightarrows P_r$ as $R \to 1$. By direct computation, for fixed $r<R$

$$|P_{r/R}(\theta) - P_r(\theta)| = \left|\mathrm{Re}\left(\frac{R+re^{i\theta}}{R-re^{i\theta}} - \frac{1+re^{i\theta}}{1-re^{i\theta}}\right)\right|$$

$$= \left|\mathrm{Re}\left(\frac{\cancel{R}+\cancel{re^{i\theta}} - Rre^{i\theta} - r^2e^{\cancel{2i\theta}} - \cancel{R} - Rre^{i\theta} + r^2e^{\cancel{2i\theta}} - \cancel{re^{i\theta}}}{(R-re^{i\theta})(1-re^{i\theta})}\right)\right|$$

$$= \left|\mathrm{Re}\left(\frac{-2Rre^{i\theta}}{(R-re^{i\theta})(1-re^{i\theta})}\right)\right|$$

$$\leq \left|\frac{2Rr}{(R-r)(1-r)}\right| \to 0$$

Since this bound is uniform in θ, $P_{r/R} \rightrightarrows P_r$ as $R \to 1$.

Then, for fixed r, η, for sufficiently large k

$$\left| h(re^{i\eta}) - \int_0^{2\pi} P_r(\eta-\theta)\,d\mu(\theta) \right| \leq \left| h(re^{i\eta}) - \int_0^{2\pi} P_{r/R_k}(\eta-\theta)\,d\mu_{R_k}(\theta) \right|$$

$$+ \left| \int_0^{2\pi} \left(P_{r/R_k}(\eta-\theta) - P_r(\eta-\theta)\right) d\mu_{R_k}(\theta) \right|$$

$$+ \left| \int_0^{2\pi} P_r(\eta-\theta)\,d\mu_{R_k}(\theta) - \int_0^{2\pi} P_r(\eta-\theta)\,d\mu(\theta) \right|$$

$$\leq \|P_{r/R_k} - P_r\|_{L^\infty} \underbrace{\mu_{R_k}[0,2\pi]}_{\to h(0)} + \left| \int_0^{2\pi} P_r(\eta-\theta)\,d\mu_{R_k}(\theta) - \int_0^{2\pi} P_r(\eta-\theta)\,d\mu(\theta) \right|$$

Since $P_{r/R_k} \rightrightarrows P_r$ and $\mu_{R_k} \overset{*}{\rightharpoonup} \mu$, taking $k \to \infty$ implies

$$h(re^{i\eta}) = \int_0^{2\pi} P_r(\eta-\theta)\,d\mu(\theta)$$

as desired.

$\square$

S10.10

We recall Harnack's inequality, which states that if u is non-negative harmonic on Ω and $K \subset \Omega$ is compact then

$$\sup_{z \in K} u(z) \leq C_{K,\Omega} \inf_{z \in K} u(z)$$

where $C_{K,\Omega}$ depends only on K, Ω.

Assuming Harnack's inequality,

$$\sup_{u \in \mathcal{U}} \sup_{z \in K} u(z) \leq C_{K,\Omega} \sup_{u \in \mathcal{U}} \inf_{z \in K} u(z)$$

We recall Harnack's inequality which states:

Suppose u is a non-negative harmonic function on a neighborhood of $\overline{D(z_0,R)}$. Then $\forall z \in D(z_0,R)$,

$$\frac{R-|z-z_0|}{R+|z-z_0|} u(z_0) \leq u(z) \leq \frac{R+|z-z_0|}{R-|z-z_0|} u(z_0)$$

Define $C_R \subset \Omega$ via

$$C_R = \{z : |z-z_0| \leq R,\ d(z, \mathbb{C}\setminus\Omega) \geq 1/R\}.$$

Then C_R is closed $\forall R$. Since $C_R \subset \overline{D(z_0,R)}$, C_R is bounded and hence compact. Since Ω is connected, C_R is connected for sufficiently large R.

Fix some compact $K \subset \Omega$. Then $d(K, \mathbb{C}\setminus\Omega) > 0$. Since K is bounded, $\exists \tilde{R}$ sufficiently large so that $K \subset \overline{D(z_0,\tilde{R})}$.

Define $R \geq \max(\tilde{R}, 1/d(K, \mathbb{C}\setminus\Omega))$. Then $K \subset C_R$. Taking R larger if necessary, we may assume that C_R is connected.

$\longrightarrow$

We now work ~~at~~ on C_R.

Since $C_R \subset \Omega$ is compact, $d(C_R, \overline{\mathbb{C} \setminus \Omega}) = 2r > 0$. Then $\forall\, z \in C_R$, $\forall\, u \in \mathcal{U}$, u is harmonic on $\overline{D(z,r)}$.

Consider the cover $\{D(z, r/2)\}_{z \in C_R}$ of C_R. Since C_R is compact $\exists$ a finite subcover centered at $z_1, \ldots, z_n \in C_R$.

Since C_R is connected, $\forall i$ $\exists\, j \neq i$ s.t. $D(z_i, r/2) \cap D(z_j, r/2) \neq \emptyset$.

In particular, $\forall i$ $\exists\, j \neq i$ s.t. $z_j \in D(z_i, r/2)$.

Then since C_R is connected, $\forall\, z \in C_R$ $\exists$ a sequence $i_1, \ldots, i_k$ w/ $k \leq n$ s.t. $z \in D(z_{i_1}, r/2)$, $z_0 \in D(z_{i_k}, r/2)$ and $z_{i_j} \in D(z_{i_{j+1}}, r/2)$ $\forall j$. Therefore by Harnack's inequality, since u is harmonic on $\overline{D(w,r)}$ $\forall\, w \in C_R$, then for z as above

$$u(z) \leq \frac{r + |z - z_{i_1}|}{r - |z - z_{i_1}|} u(z_{i_1}) \leq \frac{r + r/2}{r - r/2} u(z_{i_1}) = 3u(z_{i_1})$$

Repeating this, this implies

$$u(z) \leq 3u(z_{i_1}) \leq 3^2 u(z_{i_2}) \leq \ldots \leq 3^k u(z_{i_k}) \leq 3^{k+1} u(z_0)$$

Since $k \leq n$, this implies $u(z) \leq 3^{n+1} u(z_0) = 3^{n+1}$ $\forall\, u \in \mathcal{U}$.

We note that n depends only on K, Ω. Therefore

$$\sup_{u \in \mathcal{U}} \sup_{z \in K} u(z) \leq \sup_{u \in \mathcal{U}} \sup_{z \in C_R} u(z) \leq 3^{n+1}$$

as desired.

$\square$

F13.4

Fix some $\varepsilon > 0$. Then for sufficiently small $r > 0$,

$$\frac{1}{r^2 \log(1/r)} \int_{\{0<|z|<r\}} u(z)\,d\lambda \leq \pi\varepsilon$$

$$\Rightarrow \quad \frac{1}{\pi r^2} \int_{\{0<|z|<r\}} u(z)\,d\lambda \leq \varepsilon \log(1/r) \qquad (1)$$

In particular, for $|z| = r/2$, $D(z, r/2) \subset \{0<|z|<r\}$ and so by the sub mean value theorem,

$$u(z) \leq \frac{4}{\pi r^2} \int_{D(z,r/2)} u(z)\,d\lambda \leq \frac{4}{\pi r^2} \int_{\{0<|z|<r\}} u(z)\,d\lambda \leq 4\varepsilon \log(1/r) \leq 4\varepsilon \log(1/|z|) \quad (2)$$

Since (1) holds $\forall$ sufficiently small r, (2) holds $\forall$ sufficiently small $|z|$.

Fix some $\rho > 0$ s.t. $u(z) \leq 4\varepsilon \log(1/|z|)$ for $|z| = \rho$.

Then $u(z) - 4\varepsilon \log(1/|z|) \leq 0$ on $\{|z| = \rho\}$.

As given $u(z) - 4\varepsilon \log(1/|z|) = 0$ on $\{|z|=1\}$. Therefore since u and $\log(1/|z|)$ are subharmonic on $A = \{\rho < |z| < 1\}$ and 0 is harmonic A, it follows that $u(z) - 4\varepsilon \log(1/|z|) \leq 0$ on $\{\rho < |z| < 1\}$.

Taking $\rho \to 0$ implies $u(z) \leq 4\varepsilon \log(1/|z|)$ on $\{0<|z|<1\}$ $\forall \varepsilon > 0$.

Taking $\varepsilon \to 0$ thus yields $u \leq 0$ on $\{0<|z|<1\}$.

Since u is nonnegative, this gives $u \equiv 0$ as desired. $\square$

F14.11

Cover K in finitely many closed balls. Then connect the balls via paths to find compact connected $\tilde{K} \supset K$.

~~Choose $z_0 \in K$ s.t. $u(z_0) \leq 2 \inf_{z \in K} u(z)$.~~

For any $z \in K$ we can apply Harnack's inequality n times to get $u(z) \lesssim_K u(w)$ $\forall z, w \in K$, which concludes $\sup \lesssim \inf$ as desired.

For details see S14.10

Alternate: ~~Pick $z_0 \in K$ s.t. $u(z_0) \leq 2 \inf$~~

By the Riemann mapping theorem, $\exists$ a conformal map $\varphi : \mathbb{D} \to \Omega$

Define $v = u \circ \varphi$. For all compact $K \subset \Omega$, $\varphi^{-1}(K)$ is compact.

Then $\exists$ $0 < R < 1$ s.t. $\varphi^{-1}(K) \subset \{|z| \leq R\}$. Note v harmonic on $\{|z| \leq R + \frac{1-R}{2}\}$.

Harnack's inequality then implies that $\forall$ $z, w \in \varphi^{-1}(K)$,

$$\begin{aligned} v(z) &\leq \frac{R + \frac{1-R}{2} + |z|}{R + \frac{1-R}{2} - |z|} v(0) \\ &\leq \frac{3R+1}{1-R} v(0) \\ &\leq \left(\frac{3R+1}{1-R}\right)^2 v(w) \end{aligned}$$

Then $\forall$ $z, w \in K$, $u(z) \leq \left(\frac{3R+1}{1-R}\right)^2 u(w)$. In particular,

$$\sup_{z \in K} u(z) \leq \left(\frac{3R+1}{1-R}\right)^2 \inf_{z \in K} u(z)$$

where R depends only on K, Ω.

We note that the Ω dependence is necessary. $\square$

F10.11

Let $u=Re(f)$ and $v=Im(f)$. Then
$f=u+iv$

$$(\partial_{xx}+\partial_{yy})(u+iv)=0$$
$$(\partial_{xx}+\partial_{yy})u=0$$
$$(\partial_{xx}+\partial_{yy})v=0$$

$$f^2 = u^2+v^2 \quad 2i(uv)$$

$\partial_{xx}+\partial_{yy}$

$$2uu_x - 2vv_x + 2iuv_x + 2iu_xv$$

$$2u_x^2 + \cancel{2uu_{xx}} - 2v_x^2 - \cancel{2vv_{xx}} + 2i\,u_xv_x + \underline{2iuv_{xx}} + \underline{2ivu_{xx}}$$

$$2u_x^2+2u_y^2-2v_x^2-2v_y^2 + 4iu_xv_x + 4iu_yv_y = 0$$

$$u_x = \frac{-u_yv_y}{v_x}$$
$$u_xv_x = -u_yv_y$$

$$u_x^2+u_y^2 = v_x^2+v_y^2$$

$$u_x^2-v_x^2 = v_y^2-u_y^2 \cancel{+u_x^2-v_x^2} = u_y^2-v_y^2$$

$$u_x^2-2u_xv_x-v_x^2 = u_y^2+2u_yv_y-u_y^2$$

$$(u_x-v_x)^2$$

$$(u_x-v_x)(u_x+v_x)$$

F12.13

(a) Let $M' = \sup_{z\in\Omega} u(z)$. Then $\exists (z_n) \subset \Omega$ s.t. $u(z_n) \to M'$.

Since Ω is bounded, $\overline{\Omega}$ is compact. Therefore by passing to a subsequence, $z_n \to z \in \overline{\Omega}$. If $z \in \Omega$ then by cont. $u(z) = M'$ and so u achieves its maximum on the interior and hence $u \equiv M'$. The limsup condition then immediately implies that $u = M' \le M$ on Ω as desired.

Suppose instead that $z \in \partial\Omega$. Then as given,

$$\sup_{z\in\Omega} u(z) = M' = \lim_{n\to\infty} u(z_n) = \limsup_{n\to\infty} u(z_n) \le M$$

as desired.

Alternate: Since Ω is bounded, $\partial\Omega$ is compact. Then we may cover $\partial\Omega$ by finitely many balls $B_1, \ldots, B_n$ s.t. $u(z) \le M+\varepsilon$ in $\cup B_i$. Taking $d = d(\partial\Omega, \Omega\setminus\cup B_i) > 0$, $K_n = \{z : d(z, \partial\Omega) \ge d/n\}$ is compact, $\Omega = \bigcup_n K_n$ and $\forall$ $z \in K_n$, $u(z) \le \sup_{w\in\partial K_n} u(w) \le M+\varepsilon$. Then $u \le M+\varepsilon$ on Ω $\forall \varepsilon > 0$ which concludes.

(b) ~~No~~ No.

F14.12

Define $v(z) = u(1/z)$. Then v is harmonic on $\{0<|z|<1\}$, continuous on $\{0<|z|\le 1\}$, bounded, and satisfies $v \le 0$ on $\partial\mathbb{D}$.
Let $M>0$ denote a bound of u, v.
Fix some $\varepsilon>0$ and consider the function $\varepsilon \log|1/z|$. Since $1/z$ is holomorphic non-vanishing on $\mathbb{C}\setminus\{0\}$, $\varepsilon\log|1/z|$ is harmonic on $\mathbb{C}\setminus\{0\}$.
Therefore $v - \varepsilon\log|1/z|$ is harmonic on $\{0<|z|<1\}$ and continuous on $\{0<|z|\le 1\}$. For $|z| \le e^{-M/\varepsilon}$, $\varepsilon\log|1/z| \ge M \ge v$ and so

$$v(z) - \varepsilon\log|1/z| \le 0 \quad \text{for } |z| \le e^{-M/\varepsilon}$$

In particular, since $\varepsilon\log(1/1) = 0$, $\forall$ sufficiently small $r>0$,

$$v(z) - \varepsilon\log|1/z| \le 0 \text{ on } \partial\{r<|z|<1\}.$$

Therefore by the maximum modulus principle, $v(z) - \varepsilon\log|1/z| \le 0$ on $\{r<|z|<1\}$ for all sufficiently small $r>0$.
Taking $r \to 0$ and then $\varepsilon \to 0$ then yields $v \le 0$ on $\{0<|z|<1\}$.
~~and~~ Therefore $u \le 0$ on Ω.

$\square$

S15.11

We claim that u extends harmonically to 0.

Let $\tilde{u} : \{|z| < 1/2\} \to \mathbb{R}$ be the harmonic function which solve the Dirichlet problem $\tilde{u}|_{\{|z|=1/2\}} = u|_{\{|z|=1/2\}}$. To be explicit, $\tilde{u}$ can be expressed by the Poisson kernel as

$$\tilde{u}(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{1/2 + re^{i\theta - i\eta}}{1/2 - re^{i(\theta-\eta)}}\right) u(\tfrac{1}{2}e^{i\eta})\,d\eta$$

Consider $u - \tilde{u}$. By construction $u - \tilde{u}$ is harmonic on $\{0<|z|<1/2\}$ and bounded since u is bounded and $\tilde{u}$ is cont. on $\{|z| \le 1/2\}$. Additionally, $u - \tilde{u} = 0$ on $\{|z| = 1/2\}$. We claim $u - \tilde{u} = 0$ on $\{0<|z|<1/2\}$.

Let $M > 0$ be a bound on $u - \tilde{u}$. Fix $\varepsilon > 0$. Then $\forall$ $|z| \le e^{-M/\varepsilon}$,

$$u(z) - \tilde{u}(z) \le M \le \varepsilon \log|1/z| \qquad (1)$$

Since $\varepsilon \log|1/z|$ is harmonic on $\mathbb{C}\setminus\{0\}$, $u(z) - \tilde{u}(z) - \varepsilon\log|1/z|$ is harmonic on $\{0<|z|<1/2\}$. By (1), it follows that $u(z) - \tilde{u}(z) - \varepsilon\log|1/z| \le 0$ on $\partial\{r<|z|<1/2\}$ $\forall$ $r \in (0, e^{-M/\varepsilon})$. Then by the maximum modulus principle, $u(z) - \tilde{u}(z) - \varepsilon\log|1/z| \le 0$ on $\{r<|z|<1/2\}$.

Taking $r \to 0$ and $\varepsilon \to 0$ then implies $u(z) - \tilde{u}(z) \le 0$ on $\{0<|z|<1/2\}$. Repeating this argument for $\tilde{u} - u$, this implies $\tilde{u} = u$ on $\{0<|z|<1/2\}$. Since $\tilde{u}$ is harmonic on $\{|z|<1/2\}$, this implies u extends harmonically to $\mathbb{D}$. ~~Since~~

B/c $\mathbb{D}$ is simply connected, $\exists$ a harmonic conjugate v of u on $\mathbb{D}$, as claimed.

$\square$

S20.13

We note that subharmonicity, the limsup condition, and the ~~state constant~~ of being constant are all preserved by adding a constant to u. Therefore we may assume WLOG that

$$\max_{|z|\le 1} u(z) = 0$$

Since u is continuous, $u(z_0)=0$ for some $|z_0|\le 1$, i.e. the maximum is attained.

Fix some $\varepsilon>0$. Then for sufficiently large R,

$$\frac{u(z)}{\log|z|} \le \varepsilon \qquad \text{for} \quad |z|\ge R$$

$$\Rightarrow u(z) - \varepsilon\log|z| \le 0 \qquad \text{for} \quad |z|\ge R$$

Since $\log|z|$ is harmonic away from 0, $u(z)-\varepsilon\log|z|$ is ~~subharmo~~ subharmonic on $\{1<|z|<R\}$. ~~$\forall$ sufficiently lar~~

Since $\log 1 = 0$, $u(z)-\varepsilon\log|z|\le 0$ on $\partial\{1<|z|<R\}$ for sufficiently large $R>0$. Then by the maximum principle (ok b/c 0 is harmonic on $\{1<|z|<R\}$), $u(z)\le\varepsilon\log|z|$ on $\{1<|z|<R\}$.

Taking $R\to\infty$, $u(z)\le\varepsilon\log|z|$ on $\{1<|z|\}$.

Then taking $\varepsilon\to 0$, $u(z)\le 0$ on $\{1<|z|\}$, and so $u\le 0$ on $\mathbb{C}$.

Since $u(z_0)=0$, the maximum principle implies $u\equiv 0$ on $\mathbb{C}$ as desired.

$\square$

Analysis

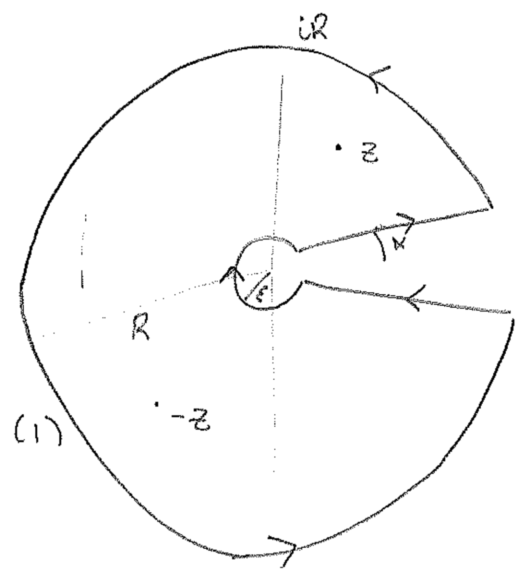
Contour Integrals

S16.7

Define $f(w) = \frac{w^{a-1}}{w+z}$ w/ branch cut along the positive real axis.

Then f has a simple pole at $-z$ w/ residue
$$\mathrm{Res}(f,-z) = (-z)^{a-1} = z^{a-1}\, e^{\pi i(a-1)}$$

Now consider the contour γ shown to the right w/ $0<\varepsilon<|z|<R$ and $0<\alpha<<\mathrm{Arg}\, z$. Let $C_R^\alpha, C_\varepsilon^\alpha$ denote the outer and inner arcs, oriented positively, ~~and~~ let u_α, l_α denote the line segments above and below $\mathbb{R}$ respectively, oriented outwards. Then by the residue theorem,



$$\left(\int_{C_R^\alpha} - \int_{C_\varepsilon^\alpha} + \int_{u_\alpha} - \int_{l_\alpha}\right) f(w)\,dw = 2\pi i\, z^{a-1} e^{\pi i(a-1)} \quad (1)$$

Now consider C_R^α. By direct calculation, $\forall \alpha$
$$\left|\int_{C_R^\alpha} f\,dw\right| \le \int_\alpha^{2\pi-\alpha} |f(Re^{i\theta})|\, R\,d\theta \le \frac{R^a}{R-|z|}\int_\alpha^{2\pi-\alpha} |e^{i(a-1)\theta}|\,d\theta \le \frac{2\pi R^a}{R-|z|} \to 0$$

as $R\to\infty$ since $a<1$.

~~Similarly~~, By the same calculation, $\forall \alpha$
$$\left|\int_{C_\varepsilon^\alpha} f\,dw\right| \le \frac{\varepsilon^a 2\pi}{|z|-\varepsilon} \to 0$$

as $\varepsilon \to 0$ since $a>0$.

Now consider u_α. Parametrizing u_α,
$$\int_{u_\alpha} f(w)\,dw = \int_\varepsilon^R f(re^{i\alpha})\, e^{i\alpha}\,dr = \int_\varepsilon^R \frac{r^{a-1} e^{i\alpha a}}{re^{i\alpha}+z}\,dr$$

For fixed ε, R,
$$\left|\frac{r^{a-1}e^{i\alpha a}}{re^{i\alpha}+z}\right| \le \frac{r^{a-1}}{|r-|z||} \lesssim_{R,\varepsilon,z} 1$$

$\longrightarrow$

Therefore since $[\varepsilon, R]$ is compact, DCT implies

$$\lim_{\kappa\to 0}\int_{u_\kappa} f(w)\,dw = \int_\varepsilon^R \frac{r^{a-1}}{r+z}\,dr$$

Similarly, for l_κ

$$\int_{l_\kappa} f(w)\,dw = \int_\varepsilon^R \frac{r^{a-1}e^{i(a-1)(2\pi-\kappa)}}{re^{i(2\pi-\kappa)}+z}\,e^{i(2\pi-\kappa)}\,dr$$

$$= e^{2\pi i(a-1)}\int_\varepsilon^R \frac{r^{a-1}e^{-i\kappa a}}{re^{-i\kappa}+z}\,e^{-i\kappa}\,dr$$

$$\to e^{2\pi i(a-1)}\int_\varepsilon^R \frac{r^{a-1}}{r+z}\,dr \quad \text{as } \kappa\to 0$$

Therefore by (1), taking $\kappa\to 0$

$$2\pi i\, z^{a-1}e^{\pi i(a-1)} = \left(1-e^{2\pi i(a-1)}\right)\int_\varepsilon^R \frac{r^{a-1}}{r+z}\,dr + \lim_{\kappa\to 0}\left(\int_{C_R^\kappa} - \int_{C_\varepsilon^\kappa}\right) f\,dw$$

Since $\int_{C_R^\kappa} f\,dw, \int_{C_\varepsilon^\kappa} f\,dw \to 0$ as $R\to\infty$, $\varepsilon\to 0$ uniformly in κ,

~~thus~~ taking $R\to\infty$ and $\varepsilon\to 0$ thus implies

$$\int_0^\infty \frac{r^{a-1}}{r+z}\,dr = \frac{2\pi i\, z^{a-1}e^{\pi i(a-1)}}{1-e^{2\pi i(a-1)}}$$

which is what was to be calculated. $\square$

S16.7

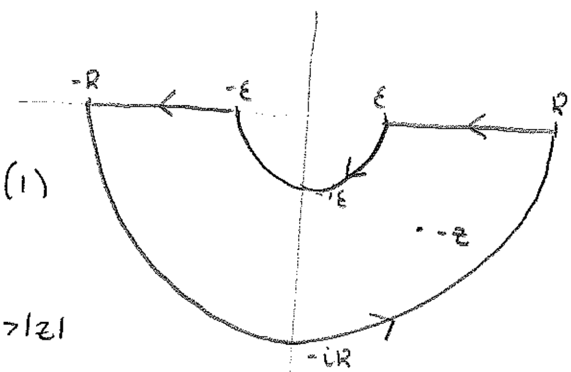
Fix $z \in \mathbb{H}$ and $0<a<1$. We aim to compute

$$\int_0^\infty \frac{x^{a-1}}{x+z}\,dx$$

Define $f(w) = \frac{w^{a-1}}{w+z}$ for $w \in \mathbb{C}$ where we choose a branch cut along the positive imaginary axis for z^{a-1}.
Then f has a simple pole at $-z$ w/ residue

$$\operatorname{Res}(f,z) = \lim_{w\to z}\left((w+z)f(w)\right)$$
$$= (-z)^{a-1}$$

Consider the contour to the right w/ $0<\varepsilon<|z|<R$.
Let C_R, C_ε denote the lower and upper semicircles w/ positive orientation.



Then by the residue theorem,

$$\left(\int_{C_R} - \int_{C_\varepsilon} - \int_{-R}^{-\varepsilon} - \int_{\varepsilon}^{R}\right) f(w)\,dw = 2\pi i(-z)^{a-1} \quad (1)$$

Consider C_R. By direct calculation, since $R>|z|$

$$\left|\int_{C_R} f(w)\,dw\right| \le \int_{C_R} |f|\,|dw|$$
$$= \int_0^\pi \left|\frac{R^{a-1}e^{i(a-1)\theta}}{Re^{i\theta}+z}\right| R\,d\theta$$
$$\le \int_0^\pi \frac{R^a}{R-|z|}\,d\theta = \frac{R^a\pi}{R-|z|}$$

Since $a\in(0,1)$, this implies $\int_{C_R} f\,dw \to 0$ as $R\to\infty$.
Similarly, for $\varepsilon<|z|$, since $a\in(0,1)$

$$\left|\int_{C_\varepsilon} f\,dw\right| \le \frac{\varepsilon^a}{|z|-\varepsilon} \to 0 \quad \text{as } \varepsilon\to 0.$$

$\longrightarrow$

Therefore by taking $R \to \infty$ and $\varepsilon \to 0$, (1) yields

$$\int_{-\infty}^{\infty} f(w)\,dw = -2\pi i(-z)^{a-1}$$

For $x \in \mathbb{R}$, $(-x)^{a-1} = x^{a-1} e^{i(a-1)\pi}$. Then

$$\int_{-\infty}^{0} \frac{x^{a-1}}{x+z}\,dx = e^{i(a-1)\pi} \int_{0}^{\infty} \frac{x^{a-1}}{-x+z}\,dx$$

This implies

$$\int_{0}^{\infty} \frac{x^{a-1}}{x+z}\,dx = \frac{1}{1+e^{i(a-1)\pi}} \int_{-\infty}^{\infty} f(w)\,dw = \frac{-2\pi i z^{a-1} e^{i(a-1)\pi}}{1+e^{i(a-1)\pi}}$$

$$= \frac{-2\pi i z^{a-1}}{}$$

S19.10

Define $f(z) = \dfrac{\operatorname{Log} z^2}{(z+a)^2+b^2}$ where the branch of Log is along the negative imaginary axis.

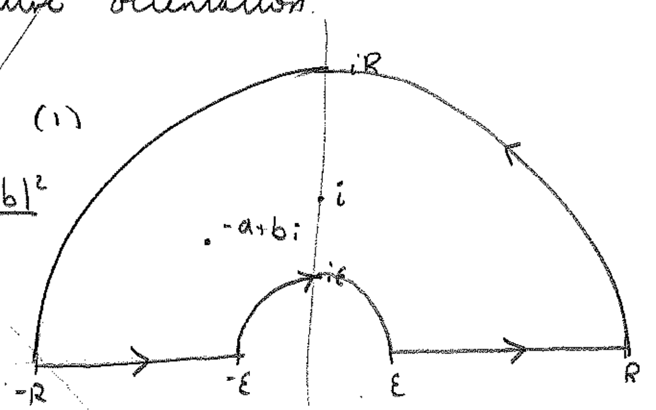
Then besides the negative imaginary axis, f only has simple poles at $-a \pm ib$ w/

$$\operatorname{Res}(f, -a+ib) = \frac{\operatorname{Log}(-a+ib)^2}{(-a+ib)+a+ib} = \frac{\operatorname{Log}(-a+ib)^2}{2ib}$$

Consider the contour shown to the right w/ $0 < \varepsilon < |a+ib|, 1 < R$

Let C_R, C_ε denote the semicircles w/ positive orientation.

Then by the residue theorem,



$$\left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} + \int_{C_R} - \int_{C_\varepsilon}\right) f(z)\,dz = \frac{\pi \operatorname{Log}(-a+ib)^2}{b} \quad (1)$$

Consider C_R. By direct calculation,

$$\left|\int_{C_R} f\,dz\right| \le \int_0^{\pi} |f(Re^{i\theta})|\, R\,d\theta$$

$$= R\int_0^{\pi} \frac{|2\log R + i\operatorname{Arg}(e^{2i\theta})|}{|(Re^{i\theta}+a)^2+b^2|}\,d\theta$$

$$(R > |-a+ib|) \le \frac{R}{(R-a)^2-b^2}\int_0^{\pi} |2\log R + i\operatorname{Arg}(e^{2i\theta})|\,d\theta$$

$$\left(\operatorname{Arg} z \le \tfrac{3\pi}{2} \le 2\pi\right) \le \frac{R}{(R-a)^2-b^2}\int_0^{\pi} (2\log R + 2\pi)\,d\theta$$

$$= \frac{2\pi R(\log R + \pi)}{(R-a)^2-b^2} \to 0 \text{ as } R \to \infty$$

Similarly, for C_ε,

$$\left|\int_{C_\varepsilon} f\,dz\right| \le \frac{2\pi\varepsilon(\log\varepsilon + \pi)}{b^2-(a-\varepsilon)^2} \to 0 \text{ as } \varepsilon \to 0.$$

$\longrightarrow$

Therefore by taking $R \to \infty$ and $\varepsilon \to 0$, (1) yields

$$\int_{-\infty}^{\infty} f(z)\,dz = \frac{\pi \operatorname{Log}(-a+ib)^2}{b}$$

Since $\dfrac{\operatorname{Log} x^2}{(x+a)^2+b^2}$ is even, this implies

$$\int_0^{\infty} \frac{\log x}{(x+a)^2+b^2} = \frac{1}{2} \int_0^{\infty} \frac{\log x^2}{(x+a)^2+b^2}$$

$$= \frac{1}{4} \int_{\infty}^{\infty} \frac{\log x^2}{(x+a)^2+b^2}$$

$$= \frac{\pi \operatorname{Log}(-a+ib)^2}{4b}$$

F11.7

Compute $\displaystyle\int_0^\infty \frac{\cos(x)}{(1+x^2)^2}\,dx$.

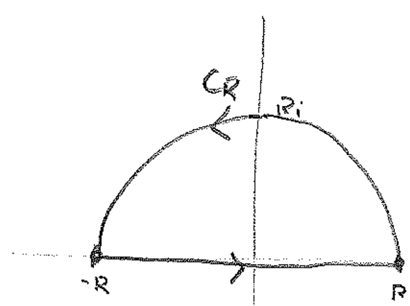
Define $f(z) = \dfrac{e^{iz}}{(1+z^2)^2}$. Then f is meromorphic w/ double poles at $\pm i$. By direct calculation,

$$\begin{aligned}
\operatorname{Res}(f,i) &= \lim_{z\to i} \frac{d}{dz}\left((z-i)^2 f(z)\right) \\
&= \lim_{z\to i} \frac{d}{dz}\left(\frac{e^{iz}}{(z+i)^2}\right) \\
&= \lim_{z\to i} \left(\frac{(z+i)^2 ie^{iz} - e^{iz}2(z+i)}{(z+i)^4}\right) \\
&= \frac{-4ie^{-1} - 4ie^{-1}}{16} \\
&= -i/2e
\end{aligned}$$

Let $C_R = \{Re^{i\theta} : 0 \le \theta \le \pi\}$. Then by the residue theorem, for $R>1$,

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \operatorname{Res}(f,i) = \pi/e$$

Consider C_R. By direct calculation,



$$\begin{aligned}
\left|\int_{C_R} f(z)\,dz\right| &\le \int_0^\pi \left|\frac{e^{iRe^{i\theta}}}{(R^2e^{2i\theta}+1)^2}\right| R\,d\theta \\
&\le \frac{R}{(R^2-1)^2}\int_0^\pi e^{-R\sin\theta}\,d\theta \\
&\le \frac{R}{(R^2-1)^2}\int_0^{\pi/2} e^{-R\theta}\,d\theta \\
&= \frac{1}{(R^2-1)^2}\left(1 - e^{-R\pi/2}\right) \to 0 \text{ as } R\to\infty.
\end{aligned}$$

Therefore, taking $R\to\infty$, $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \pi/e$.

Then by symmetry

$$\int_0^\infty \frac{\cos(x)}{(x^2+1)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos(x)}{(x^2+1)^2}\,dx = \frac{1}{2}\operatorname{Re}\left(\int_{-\infty}^{\infty} f(x)\,dx\right) = \pi/2e.$$

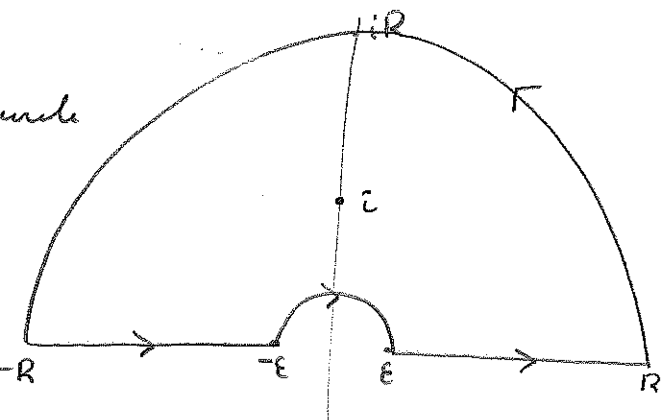
□

S14.11

For $p \in (-1,1)$, calculate $\int_0^\infty \frac{x^p}{x^2+1}\,dx$.

Define $f(z) = \frac{z^p}{z^2+1}$ where we have chosen z^p to have a branch cut along the negative imaginary axis.

Consider the contour shown to the right w/ $0 < \varepsilon < 1 < R$. Let C_R denote the upper semicircle and C_ε the lower semicircle w/ their respective orientations.



Then by the Residue theorem, since f has a simple pole at i,

$$\left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} + \int_{C_R} + \int_{C_\varepsilon}\right) f(z)\,dz = 2\pi i \,\mathrm{Res}(f, i)$$

$$= 2\pi i \lim_{z\to i} (z-i) f(z)$$

$$= 2\pi i \left(\frac{i^p}{2i}\right)$$

$$= \pi e^{p i\pi/2}$$

Consider C_R. By direct calculation,

$$\left|\int_{C_R} f(z)\,dz\right| \le \int_{C_R} |f|\,|dz|$$

$$= \int_0^\pi \left|\frac{R^p e^{ip\theta}}{R^2 e^{2i\theta}+1}\right| R\,d\theta$$

$$\le \int_0^\pi \frac{R^{p+1}}{R^2-1}\,d\theta = \frac{R^{p+1}\pi}{R^2-1}$$

Since $p<1$, this implies $\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$.

By the same calculation, since $p > -1$

$$\left|\int_{C_\varepsilon} f(z)\,dz\right| \le \frac{\varepsilon^{p+1}\pi}{\varepsilon^2-1} \to 0 \quad \text{as} \quad \varepsilon \to 0$$

Therefore, taking the limit as $R \to \infty$ and $\varepsilon \to 0$,

$$\int_{-\infty}^{\infty} f(x)\,dx = \pi e^{pi\pi/2}$$

For $x \in (0,\infty)$, $(-x)^p = x^p e^{p\log(-1)} = x^p e^{\pi i p}$. Therefore

$$\int_{-\infty}^{\infty} \frac{x^p}{x^2+1}\,dx = \left(1+e^{\pi i p}\right)\int_0^{\infty} \frac{x^p}{x^2+1}\,dx$$

This implies

$$\begin{aligned}
\int_0^{\infty} \frac{x^p}{x^2+1}\,dx &= \frac{1}{(1+e^{\pi i p})}\int_{-\infty}^{\infty} f(x)\,dx \\
&= \frac{\pi\, e^{pi\pi/2}}{1+e^{\pi i p}} \\
&= \frac{\pi}{e^{\pi i p/2}+e^{-\pi i p/2}} \\
&= \frac{\pi}{2\cos\left(\frac{\pi p}{2}\right)}
\end{aligned}$$

Which is what was to be found. $\square$

S15.10

Fix $x \in \mathbb{R}$. Compute $\displaystyle\int_{-\infty}^{\infty} \frac{dy}{(1+y^2)(1+(x-y)^2)}$.

Define $\displaystyle f(z) = \frac{1}{(1+z^2)(1+(x-z)^2)}$.

If $x=0$, f has a double pole at $\pm i$, w/ residue

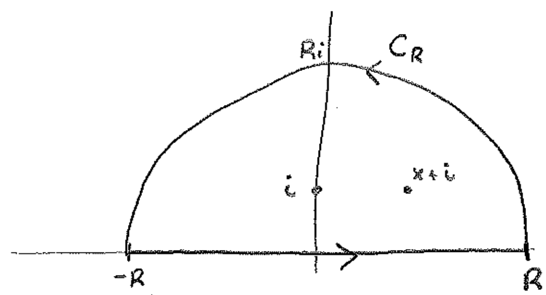
$$\mathrm{Res}(f,i) = \lim_{z\to i} \frac{d}{dz}\left((z-i)^2 f(z)\right)$$
$$= \lim_{z\to i} \frac{d}{dz}\left(\frac{1}{(z+i)^2}\right)$$
$$= \lim_{z\to i}\left(\frac{-2}{(z+i)^3}\right)$$
$$= \frac{-1}{-4i} = \frac{-i}{4}$$

If $x \neq 0$ then f has single poles at $\pm i$, $x \pm i$ w/ residues

$$\mathrm{Res}(f,i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i} \frac{1}{(z+i)(1+(x-z)^2)} = \frac{1}{2i(1+(x-i)^2)}$$

$$\mathrm{Res}(f,x+i) = \lim_{z\to x+i}(z-x-i)f(z) = \lim_{z\to x+i}\frac{-1}{(z^2+1)(x-z-i)} = \frac{1}{2i(1+(x+i)^2)}$$

Consider the contour to the right where $R \gg |x|$, $R \gg 1$. Let C_R denote the upper semicircle. Then



$$\left|\int_{C_R} f(z)\,dz\right| \le \int_{C_R} |f|\,|dz|$$
$$= \int_0^{\pi} \left|\frac{1}{(1+R^2e^{2i\theta})(1+(x-Re^{i\theta})^2)}\right| R\,d\theta$$
$$\le \int_0^{\pi} \frac{R}{(R^2-1)((R-x)^2-1)}\,d\theta$$
$$= \frac{R\pi}{(R^2-1)((R-x)^2-1)} \to 0 \quad \text{as } R\to\infty.$$

By the residue theorem, taking $R\to\infty$ this implies

$$\int_{-\infty}^{\infty} f(x)\,dx = \begin{cases} 2\pi i\,\mathrm{Res}(f,i) & x=0 \\ 2\pi i\left(\mathrm{Res}(f,i)+\mathrm{Res}(f,x+i)\right) & x\neq 0 \end{cases} = \begin{cases} \pi/2 & x=0 \\ \dfrac{\pi}{(1+(x+i)^2)} + \dfrac{\pi}{(1+(x-i)^2)} & x \neq 0 \end{cases}$$

By direct calculation,

$$\pi \left(\frac{1}{1+(x+i)^2} + \frac{1}{1+(x-i)^2} \right) = \pi \left(\frac{1}{x^2+2ix} + \frac{1}{x^2-2ix} \right)$$

$$= \pi \left(\frac{2x^2}{x^4+4x^2} \right)$$

$$= \frac{2\pi}{x^2+4}$$

We note that $\frac{2\pi}{0^2+4} = \pi/2$. Then $\forall x \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} f(y)\,dy = \frac{\pi}{x^2+4}$$

$\square$

F15.10

Define $f(z) = \dfrac{e^{(e^{iz})}}{z}$. Note that for $x \in \mathbb{R}$,

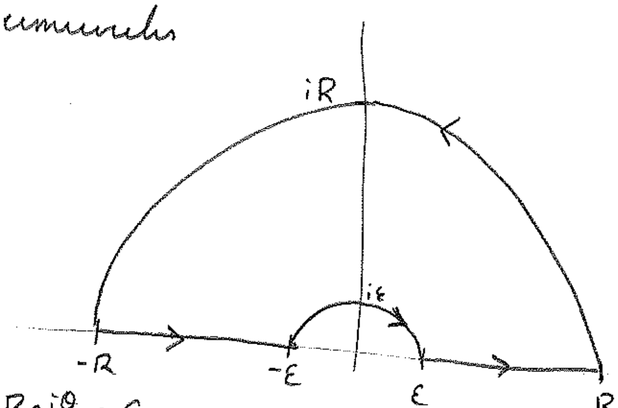
$$f(x) = \frac{\exp(e^{ix})}{x} = \frac{\exp(\cos x + i\sin x)}{x} = \frac{e^{\cos x}\left(\cos(\sin x) + i\sin(\sin(x))\right)}{x}$$

and so $\dfrac{e^{\cos x}\sin(\sin x)}{x} = \mathrm{Im}(f(x))$ for $x \in \mathbb{R}$.

By construction, f has a simple pole at 0 w/ $\mathrm{Res}(f,0) = e$.

Consider the contour to the right for $0 < \varepsilon < 1 < R$

Let C_R, C_ε denote the upper and lower semicircles w/ their indicated orientations. Since f ~~has~~ only has a pole at 0, the residue theorem implies



$$\left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} + \int_{C_\varepsilon} + \int_{C_R} \right) f(z)\,dz = 0 \qquad (1)$$

Consider C_R. By direct calculation, for $z = Re^{i\theta} \in C_R$

$$\begin{aligned} |f(z)| &= \frac{1}{R}\left|\exp(\exp(iRe^{i\theta}))\right| \\ &= \frac{1}{R}\exp\left(\mathrm{Re}(\exp(iRe^{i\theta}))\right) \\ &= \frac{1}{R}\exp\left(e^{-R\sin\theta}\cos(R\cos\theta)\right) \\ &\le \frac{1}{R}\exp(e^{-R}) \le \frac{1}{R}e^{1/e} \quad \text{for } R > 1. \end{aligned}$$

Therefore $|Rf|$ is dominated by $e^{1/e}$. *

DCT thus implies

$$\int_{C_R} f\,dz = \int_0^{\pi} Rie^{i\theta} f(Re^{i\theta})\,d\theta \to \pi i \text{ as } R \to \infty.$$

Now consider C_ε. B/c f has a simple pole at 0, and C_ε is negatively oriented,

$$\int_{C_\varepsilon} f\,dz \to -\pi i\,\mathrm{Res}(f,0) = -\pi i e \quad \text{as } \varepsilon \to 0.$$

Then by (1), taking $R \to \infty$ and $\varepsilon \to 0$ yields

$$\int_{-\infty}^{\infty} f(x)\,dx + \pi i - \pi i e = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \pi i(e-1)$$

Therefore

$$\int_{-\infty}^{\infty} \frac{e^{\cos(x)} \sin(\sin(x))}{x}\,dx = \operatorname{Im}(\pi i(e-1)) = \pi(e-1).$$

Since $\dfrac{e^{\cos(x)} \sin(\sin(x))}{x}$ is even, this implies

$$\int_{0}^{\infty} \frac{e^{\cos x} \sin(\sin x)}{x}\,dx = \frac{\pi}{2}(e-1) \quad \text{as desired.} \qquad \square$$

---

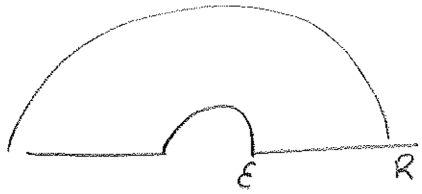
* By the same sort of calculation,

$$\begin{aligned} Re^{i\theta} f(Re^{i\theta}) &= \exp(\exp(iRe^{i\theta})) \\ &= \exp\Big(\underbrace{e^{-R\sin\theta}(\cos(R\cos\theta) + i\sin(R\cos\theta))}_{\to 0 \text{ as } R\to\infty}\Big) \\ &\Rightarrow e^0 = 1 \quad \text{as } R \to \infty. \end{aligned}$$

F18.10

Define $f(z) = \frac{z^{\beta}}{z^2+1}$ w/ branch cut along negative imaginary.

Contour



Outer boundary goes to 0, inner boundary goes to 0.

Then residue gives

$$\int_{-\infty}^{\infty} f(z)\,dz = 2\pi i \operatorname{Res}(f, i)$$

A little calculation then concludes. $\square$

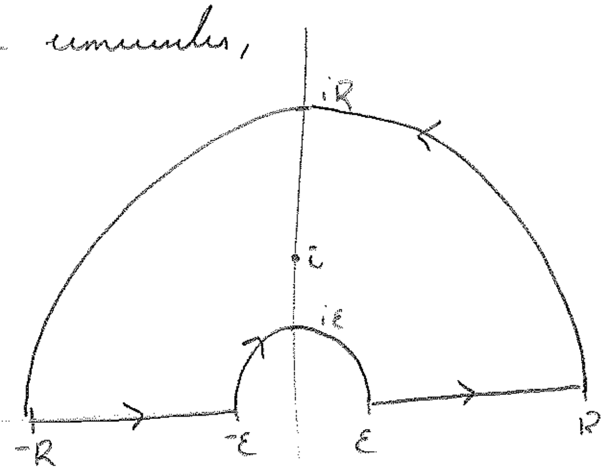
S20.10

Define $f(z) = \frac{z^2-1}{z^2+1}\frac{e^{iz}}{z}$. Then for $x \in \mathbb{R}$,

$$\operatorname{Im}(f(x)) = \frac{x^2-1}{x^2+1}\frac{\sin x}{x}$$

Consider the contour to the right for $0 < \varepsilon < 1 < R$

Let C_R, C_ε denote the upper and lower semicircles, oriented in the positive direction.



By construction, f has a simple pole at $\pm i$ and 0 w/ residues

$$\operatorname{Res}(f,0) = \lim_{z\to 0} \frac{z^2-1}{z^2+1} e^{iz} = -1$$

$$\operatorname{Res}(f,i) = \lim_{z\to i} \frac{z^2-1}{z+i}\frac{e^{iz}}{z} = {}^{1}/e$$

Then by the residue theorem, since only i is contained in the contour,

$$\left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} + \int_{C_R} - \int_{C_\varepsilon}\right) f(z)\,dz = \frac{2\pi i}{e} \qquad (1)$$

Consider C_R. By direct calculation, $\forall\ z = Re^{i\theta}$

$$|f(z)| = \left|\frac{R^2e^{2i\theta}-1}{R^2e^{2i\theta}+1}\frac{\exp(iRe^{i\theta})}{Re^{i\theta}}\right|$$

$$\leq \frac{R^2+1}{R^2-1}\frac{e^{-R\sin\theta}}{R}$$

For sufficiently large R, $\frac{R^2+1}{R^2-1} \leq 2$ and so $|f(z)| \leq 2$. In particular, $f(Re^{i\theta})$ is dominated by an L^1 function in θ.

Since $\frac{R^2+1}{R^2-1} \to 1$ as $R\to\infty$, the above calculation implies $R|f(Re^{i\theta})| \to 0$ $\forall\ \theta \neq 0, \pi$ as $R\to\infty$. Then $R|f(Re^{i\theta})| \to 0$ a.e. and so DCT implies

$$\left|\int_{C_R} f\,dz\right| \leq \int_0^{\pi} |f(Re^{i\theta})|\,R\,d\theta \to 0 \quad \text{as } R\to\infty.$$

$\longrightarrow$

Now consider C_ε. Since f has a pole at 0,

$$\int_{C_\varepsilon} f\,dz \to \frac{1}{2} 2\pi i \operatorname{Res}(f,0) = -\pi i \quad \text{as } \varepsilon \to 0.$$

Therefore, taking $R \to \infty$ and $\varepsilon \to 0$, (1) implies that

$$\int_{-\infty}^{\infty} f(x)\,dx + \pi i = \frac{2\pi i}{e}$$

Taking the imaginary part, this implies

$$\int_{-\infty}^{\infty} \frac{x^2-1}{x^2+1} \frac{\sin x}{x}\,dx = \frac{+2\pi}{e} - \pi$$

$$\cancel{\Rightarrow \int_{-\infty}^{0} \frac{x^2-1}{x^2+1} \frac{\sin x}{x}\,dx + \int_{0}^{\infty} \frac{x^2-1}{x^2+1} \frac{\sin x}{x}\,dx = -\pi(1+2/e)}$$

Since; $\frac{x^2-1}{x^2+1} \frac{\sin x}{x}$ is an even function, this implies

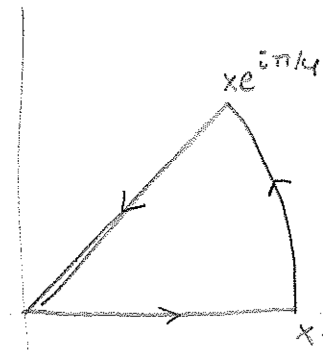
$$\int_{0}^{\infty} \frac{x^2-1}{x^2+1} \frac{\sin x}{x}\,dx = \boxed{\frac{\pi}{e} - \frac{\pi}{2}}$$

$\square$

19F.10

Define $f(z) = e^{iz^2}$. Then for $t \in \mathbb{R}$, $\mathrm{Im}(f(t)) = \sin(t^2)$.

Consider the contour to the right



Let C_x denote the outermost arc. Then since f is entire,

$$\int_0^x f(t)\,dt + \int_{C_x} f(z)\,dz - \int_{[0, xe^{i\pi/4}]} f(z)\,dz = 0 \qquad (1)$$

where $[0, xe^{i\pi/4}]$ is the line segment from 0 to $xe^{i\pi/4}$.

First consider C_x. By direct calculation,

$$\left|\int_{C_x} f\,dz\right| \le \int_0^{\pi/4} |f(xe^{i\theta})| \cdot x\,d\theta$$

$$= \int_0^{\pi/4} x e^{-x^2 \sin 2\theta}\,d\theta$$

Now recall that for $t \in [0, \pi/2]$, $\sin t \geq \frac{2}{\pi} t$. Therefore

$$\left|\int_{C_x} f\,dz\right| \le \int_0^{\pi/4} x\, e^{-x^2 \frac{4}{\pi}\theta}\,d\theta$$

$$= \frac{-x\pi}{x^2 4} e^{-x^2 \frac{4}{\pi}\theta}\Big|_0^{\pi/4} = \frac{\pi}{4x}\left(1 - e^{-x^2}\right) \to 0 \text{ as } x \to \infty.$$

Now consider $[0, xe^{i\pi/4}]$. Parametrizing this segment

$$\int_{[0, xe^{i\pi/4}]} f(z)\,dz = \int_0^x f(re^{i\pi/4})\, e^{i\pi/4}\,dr$$

$$= \int_0^x e^{-r^2} e^{i\pi/4}\,dr$$

$$\to \int_0^\infty e^{i\pi/4} e^{-r^2}\,dr = \frac{\sqrt{\pi}}{2} e^{i\pi/4} \text{ as } x \to \infty$$

$\longrightarrow$

Therefore by (1)

$$\int_0^X e^{it^2}\,dt = \int_0^X e^{-r^2} e^{i\pi/4}\,dr - \int_{C_X} f(z)\,dz$$

$$\Rightarrow \lim_{X\to\infty} \int_0^X e^{it^2}\,dt = \frac{\sqrt{\pi}}{2} e^{i\pi/4} - 0$$

Then

$$\lim_{X\to\infty} \int_0^X \sin(t^2)\,dt = \operatorname{Im}\left(\lim_{X\to\infty} \int_0^X e^{it^2}\,dt\right)$$

$$= \sin(\pi/4)\frac{\sqrt{\pi}}{2}$$

$$= \frac{\sqrt{2\pi}}{4}$$

F22. 8

We first show uniform convergence on compact subsets. To show this, it suffices to show uniform convergence on $\{|z|\le R\}$ for $R>0$.

Fix some $R>0$ and consider $z\in\{|z|\le R\}$. Then for $|n|\geq 2R\geq 2|z|$

$$\left|\frac{(-1)^n}{z^2-n^2}\right| \le \frac{1}{n^2-|z|^2} \le \frac{4}{3n^2} \le \frac{1}{n^2}$$

~~and so~~ As this holds $\forall$ sufficiently large n and is independent of $z\in\{|z|\le R\}$, this implies absolute uniform convergence on $\{|z|\le R\}$ $\forall R>0$.

Define $f(z)=\dfrac{\pi}{(z-w)\sin(\pi z)}$ for $w\in\mathbb{C}\setminus\mathbb{Z}$. Then f has simple poles at w, $\mathbb{Z}$ w/ residues

$$\operatorname{Res}(f,w)=\frac{\pi}{\sin(\pi w)}$$

$$\operatorname{Res}(f,n)=\frac{\pi}{(n-w)}\lim_{z\to n}\frac{z-n}{\sin(\pi z)}=\frac{(-1)^n}{n-w}\lim_{z\to n}\frac{\pi z-n\pi}{\sin(\pi z-n\pi)}=\frac{(-1)^n}{n-w}$$

Let C_R denote the contour $\{|z|=R\}$ for $R>0$.

Then $\forall$ $R>|w|$,

$$\left|\int_{C_R} f(z)\,dz\right| \le \int_0^{2\pi}|f(Re^{i\theta})|\,R\,d\theta$$

$$=\int_0^{2\pi}\frac{\pi R}{|\sin(Re^{i\theta}\pi)(Re^{i\theta}-w)|}\,d\theta$$

$$\le \frac{2\pi R}{R-|w|}\int_0^{2\pi}\frac{1}{|\sin(R\pi e^{i\theta})|}\,d\theta.$$

Consider $R\in\mathbb{N}+1/2$. Then $|\sin(R\pi e^{i\theta})|=1$ for $\theta=0,\pi$. Moreover, $|\sin(R\pi e^{i\theta})|\geq 1/2$ for θ near $0,\pi$ uniformly in R.

For θ away from $0,\pi$, $|\sin(R\pi e^{i\theta})|=\frac{1}{2}\left|e^{i\pi Re^{i\theta}}+e^{-i\pi Re^{i\theta}}\right|$ and so $|\sin(R\pi e^{i\theta})|\geq\frac{1}{2}\left(e^{R\pi\sin\theta}-e^{-R\pi\sin\theta}\right)\to\infty$ as $R\to\infty$. Therefore $|\sin(R\pi e^{i\theta})|$ is bounded below and so $\dfrac{1}{|\sin(R\pi e^{i\theta})|}$ is dominated

$\longrightarrow$

Since $|\sin(R\pi e^{i\theta})| \to \infty$ for $\theta \neq 0, \pi$, DCT then implies that

$$\left|\int_{C_R} f\,dz\right| \to 0 \quad \text{as} \quad R = k + 1/2 \to \infty.$$

By the residue theorem, we also have, for $R = k + 1/2 > |w|$

$$\int_{C_R} f\,dz = 2\pi i\left(\frac{\pi}{\sin(\pi w)} + \sum_{|n| \le k} \frac{(-1)^n}{n-w}\right)$$

$$= 2\pi i\left(\frac{\pi}{\sin(\pi w)} + \frac{-1}{w} + \sum_{1 \le n \le k} (-1)^n \left(\frac{1}{n-w} + \frac{1}{-n-w}\right)\right)$$

$$= 2\pi i\left(\frac{\pi}{\sin(\pi w)} - \left(\frac{1}{w} + \sum_{1 \le n \le k} (-1)^n \frac{2w}{w^2-n^2}\right)\right)$$

Taking $k \to \infty$ then yields

$$\frac{\pi}{\sin(\pi w)} = \frac{1}{w} + 2w \sum_{1 \le n} \frac{(-1)^n}{w^2-n^2}$$

as desired.

$\square$

S22.13

(a) For $u \in \mathbb{C}\setminus\mathbb{Z}$, define

$$f(z) = \frac{\pi\cot(\pi z)}{(z-u)^2}$$

Then f has a double pole at u and simple poles at $\mathbb{Z}$.
We calculate

$$\mathrm{Res}(f,u) = \lim_{z\to u}\frac{d}{dz}\pi\cot(\pi z) = \lim_{z\to u}\frac{-\sin^2(\pi z)\pi^2 - \pi^2\cos^2\pi z}{\sin^2(\pi z)} = \frac{-\pi^2}{\sin^2(\pi u)}$$

$$\mathrm{Res}(f,n) = \frac{\pi\cos(\pi n)}{(n-u)^2}\lim_{z\to n}\frac{(z-n)}{\sin(\pi z)} = \frac{\pi(-1)^n}{(n-u)^2}\lim_{z\to n}\frac{z-n}{\sin(\pi z - n\pi)(-1)^n}$$

$$= \frac{1}{(n-u)^2}$$

Consider the contour $C_R = \{|z|=R\}$ for $R \in \mathbb{N}+\frac{1}{2}$, $R > |u|$. Then by the ~~residue~~ residue theorem,

$$\int_{C_R} f(z)\,dz = 2\pi i\left(\frac{-\pi^2}{\sin^2(\pi u)} + \sum_{|n|<R}\frac{1}{(n-u)^2}\right) \qquad (1)$$

Now consider $\left|\int_{C_R} f\,dz\right|$. Parametrizing C_R yields

$$\left|\int_{C_R} f(z)\,dz\right| \le \int_0^{2\pi} |f(Re^{i\theta})|\,R\,d\theta$$

$$\le \pi\int_0^{2\pi}\frac{|\cot(\pi Re^{i\theta})|}{(R-|u|)^2}R\,d\theta$$

$$= \frac{\pi R}{(R-|u|)^2}\int_0^{2\pi}|\cot(\pi Re^{i\theta})|\,d\theta$$

Consider $|\cot(\pi Re^{i\theta})|$. Since $R \in \mathbb{N}+1/2$, $\cot(\pi Re^{i\theta})$ has a zero at $\theta = 0, \pi$.
Then $|\cot(\pi Re^{i\theta})| \le K$ on a neighborhood of $\theta = 0, \pi$, uniformly in R.
For θ away from $0, \pi$, we have

$$|\cot(\pi Re^{i\theta})| = \left|\frac{e^{iRe^{i\theta}} + e^{-iRe^{i\theta}}}{e^{iRe^{i\theta}} - e^{-iRe^{i\theta}}}\right| \to 1 \quad \text{as } R\to\infty$$

and hence is bounded. Then

$$\left|\int_{C_R} f(z)\,dz\right| \lesssim \frac{R}{(R-|u|)^2} \to 0 \quad \text{as } R\to\infty$$

Taking the limit as $R \to \infty$ in (1) then yields

$$\frac{\pi^2}{\sin^2(\pi u)} = \sum_{n \in \mathbb{Z}} \frac{1}{(n-u)^2} \qquad (2)$$

$\forall \; u \in \mathbb{C} \setminus \mathbb{Z}$ as desired.

(b) Plugging in $u = 1/2$ into (2),

$$\begin{aligned} \pi^2 &= \sum_{n \in \mathbb{Z}} \frac{1}{(n - 1/2)^2} \\ &= 4 \sum_{n \in \mathbb{Z}} \frac{1}{(2n-1)^2} \\ &= 4 \sum_{\substack{n \in \mathbb{Z} \\ n \text{ odd}}} \frac{1}{n^2} \\ &= 8 \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{1}{n^2} \end{aligned}$$

Then

$$\sum_{n \geq 1} \frac{1}{n^2} =$$

This is a stupid problem.