UCLA Analysis Qualifying Exam Solutions

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Problem 1. Let f and g be real-valued integrable functions on a measure space $(X, \mathcal{B}, \mu)$ and define

$$F_t = \{x \in X : f(x) > t\}, \quad G_t = \{x \in X : g(x) > t\}.$$

Prove

$$\int |f - g| \, d\mu = \int_{-\infty}^{\infty} \mu \left((F_t \backslash G_t) \cup (G_t \backslash F_t) \right).$$

Solution. First assume that X is σ-finite. Then we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \mu \left((F_t \backslash G_t) \cup (G_t \backslash F_t) \right) &= \int_{-\infty}^{\infty} \int_X \chi_{\{x \in X : \min(f(x), g(x)) \leqslant t < \max(f(x), g(x))\}}(x) \, d\mu(x) \, dt \\
&= \int_X \int_{-\infty}^{\infty} \chi_{\{x \in X : \min(f(x), g(x)) \leqslant t < \max(f(x), g(x))\}}(x) \, dt \, d\mu(x) \quad \text{by Tonelli} \\
&= \int_X |f(x) - g(x)| \, d\mu(x),
\end{aligned}$$

which is the desired result. Now drop the assumption that X is σ-finite. Let $Y = \{x \in X : |f(x) - g(x)| \neq 0\}$ and let $\nu = \mu|_Y$. Note that $Y = \bigcup_{n=1}^{\infty} \{x \in X : |f(x) - g(x)| > 1/n\}$, and since f and g are both integrable, each of those sets must have finite measure. Thus (Y, ν) is a σ-finite measure space. Thus by the work above we have

$$\int_Y |f - g| \, d\nu = \int_{-\infty}^{\infty} \nu \left((F_t \cap Y \backslash G_t \cap Y) \cup (G_t \cap Y \backslash F_t \cap Y) \right).$$

But note that $\int_X |f - g| \, d\mu = \int_Y |f - g| \, d\mu + \int_{Y^c} |f - g| \, d\mu = \int_Y |f - g| \, d\nu$ by definition of Y and ν. Also note that $F_t \backslash G_t, G_t \backslash F_t \subseteq Y$ for every t, so $(F_t \cap Y \backslash G_t \cap Y) \cup (G_t \cap Y \backslash F_t \cap Y) = (F_t \backslash G_t) \cup (G_t \backslash F_t)$, and $\nu \left((F_t \backslash G_t) \cup (G_t \backslash F_t) \right) = \mu \left((F_t \backslash G_t) \cup (G_t \backslash F_t) \right)$. Substituting all of this into the above equation gives the desired result. $\square$

Problem 2. Let H be an infinite dimensional real Hilbert space.
(a) Prove the unit sphere $S = \{x \in H : ||x|| = 1\}$ is weakly dense in the unit ball $B = \{x \in H : ||x|| \leqslant 1\}$.
(b) Prove there is a sequence T_n of bounded linear operators from H to H such that $||T_n|| = 1$ for all n but $\lim_{n \to \infty} T_n(x) = 0$ for all $x \in H$.

Solution. (a) Fix $x \in B$. We may assume $||x|| < 1$ because if $x \in S$ the result is obvious. Using a standard Zorn's Lemma/Gram-Schmidt argument, together with the fact that H is infinite-dimensional, we can construct an orthonormal set $\{x / ||x|| , e_1, e_2, \ldots\}$. Let $x_n = x + \sqrt{1 - ||x||^2} e_n$. By the Pythagorean theorem we have $||x_n||^2 = ||x||^2 + (1 - ||x||^2) ||e_n||^2 = 1$, so $x_n \in S$. Now we claim that $\{x_n\}$ converges weakly to x. For $y \in H$ fixed, we have

$$\langle x_n - x, y \rangle = \sqrt{1 - ||x||^2} \langle e_n, y \rangle.$$

This goes to 0 as $n \to \infty$ because since $\{e_n\}$ is an orthonormal set, Bessel's inequality gives $\sum_{n=1}^{\infty} |\langle e_n, y \rangle|^2 \leqslant ||y||^2$ and the terms of a convergent series must go to 0. $\square$

(b) Fix an infinite orthonormal set $\{e_1, e_2, \ldots\}$. Define $T_n(x) := \langle x, e_n \rangle e_n$. It's clear that T_n is a linear operator $H \to H$. We have $||T_n(x)|| = |\langle x, e_n \rangle| \, ||e_n|| \leqslant ||x||$ by Cauchy-Schwarz, so $||T_n|| \leqslant 1$. Also it's clear that $T_n(e_n) = e_n$, so $||T_n|| = 1$. Finally, for any $x \in H$ we have $\lim_{n \to \infty} ||T_n(x)|| = \lim_{n \to \infty} |\langle x, e_n \rangle| = 0$ by the same Bessel's inequality argument in part (a). $\square$

Problem 3. Let X be a Banach space. Prove that if X^* is separable then X is separable.

Solution. See Fall 2014 # 6.

Problem 4. Let $f(x)$ be a non-decreasing function on $[0,1]$.
(a) Prove that $\int_0^1 f'(x)\,dx \leqslant f(1) - f(0)$.
(b) Let $\{f_n\}$ be a sequence of non-decreasing functions on $[0,1]$ such that the series $F(x) = \sum_{n=1}^\infty f_n(x)$ converges for all $x \in [0,1]$. Prove that $F'(x) = \sum_{n=1}^\infty f_n'(x)$ almost everywhere.

Solution. (a) First we extend the definition of f by setting $f(x) = f(1)$ for $x > 1$. Note that f is differentiable almost everywhere because it is non-decreasing. So for almost every x, the representation

$$f'(x) \;=\; \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$$

is valid. Since f is non-decreasing, the difference quotient is non-negative for every x and every h. Thus by Fatou's lemma we have

$$\begin{aligned}\int_0^1 f'(x)\,dx \;&=\; \int_0^1 \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}\,dx \;\leqslant\; \liminf_{h\to 0^+} \int_0^1 \frac{f(x+h)-f(x)}{h}\,dx \\ &=\; \liminf_{h\to 0^+} \frac{1}{h}\int_1^{1+h} f(x)\,dx - \frac{1}{h}\int_0^h f(x)\,dx \;\leqslant\; f(1)-f(0)\end{aligned}$$

where we used the fact that f is non-decreasing again in the last inequality. $\square$

(b) First note that since each f_n is non-decreasing, F also is, so F is differentiable almost everywhere. Let $r_N(x) = \sum_{n=N+1}^\infty f_n(x)$ and write $F(x) = \sum_{n=1}^N f_n(x) + r_N(x)$. Since r_N is also non-decreasing, we can write $F'(x) = \sum_{n=1}^N f_n'(x) + r_N'(x)$ for all x at which all three of those functions are differentiable, which is still almost everywhere. Thus to show the desired result it's enough to show that $r_N'(x) \to 0$ almost everywhere as $N \to \infty$. First note that for almost every x, $r_N'(x) - r_{N+1}'(x) = (r_N - r_{N+1})'(x) = f_N'(x) \geqslant 0$ because f_N is non-decreasing so its derivative is non-negative wherever it exists. So $\{r_N'(x)\}$ is monotonically decreasing in N for almost every x. So the limit $\lim_{N\to\infty} r_N'(x)$ exists almost everywhere and is non-negative (as a limit of non-negative terms). Thus by the monotone convergence theorem we have

$$\int_0^1 \lim_{N\to\infty} r_N'(x)\,dx \;=\; \lim_{N\to\infty}\int_0^1 r_N'(x)\,dx \;\leqslant\; \lim_{N\to\infty} r_N(1) - r_N(0) \;=\; 0$$

where the second to last inequality uses part (a) because each r_N is non-decreasing and the last equality is by the hypothesis that the series defining F converges everywhere. Thus $\lim_{N\to\infty} r_N'(x)$ is a non-negative function which integrates to 0, so it must be zero almost everywhere. $\square$

Problem 5. Let $I_{0,0} = [0,1]$ and for $n \geqslant 0$, $0 \leqslant j \leqslant 2^n - 1$, let

$$I_{n,j} \;=\; [j2^{-n}, (j+1)2^{-n}].$$

For $f \in L^1([0,1])$ define $E_n f = \sum_{j=0}^{2^n-1}\left(2^n \int_{I_{n,j}} f(t)\,dt\right)\chi_{I_{n,j}}$. Prove that $E_n f \to f$ almost everywhere on $[0,1]$.

Solution. For a fixed $x \in [0,1]$, $E_n f(x)$ is simply the average value of f over the interval $I_{n,j(n,x)}$ that x lies in. It's clear that the family of intervals $\{I_{n,j(n,x)}\}_{n=1}^\infty$ shrinks nicely to x, so it's a direct consequence of the Lebesgue differentation theorem that $E_n f(x) \to f(x)$ for all Lebesgue points of f, which is almost everywhere. $\square$

Problem 6. For $I_{n,j}$ as in Problem 5, define the Haar function $h_{n,j} = 2^{n/2}\left(\chi_{I_{n+1,2j}} - \chi_{I_{n+1,2j+1}}\right)$.
(a) Draw $I_{2,1}$ and graph $h_{2,1}$.
(b) Prove that if $f \in L^2([0,1])$ and $\int_0^1 f(t)\,dt = 0$, then

$$\int_0^1 |f(x)|^2\,dx \;=\; \sum_{n\geqslant 0, 0\leqslant j\leqslant 2^n-1} \left|\int_0^1 f(t)h_{n,j}(t)\,dt\right|^2.$$

(c) Prove that if $f \in L^1([0,1])$ and $\int_0^1 f(t)\,dt = 0$, then almost everywhere on $[0,1]$,

$$f(x) \;=\; \sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \left(\int_0^1 f(t) h_{n,j}(t)\,dt \right) h_{n,j}(x).$$

Solution. (a)

(b) Let $M = \left\{ f \in L^2([0,1]) : \int_0^1 f = 0 \right\}$. First note that M is a closed subspace of L^2: if $f_n \in M$ and $f_n \to f$ in L^2, then by Cauchy-Schwarz we also have $f_n \to f$ in L^1, so in particular $\int f_n \to \int f$, so $\int_f = 0$ as well. Thus we can consider M as a Hilbert space. Next note that $\{h_{n,j}\}_{n,j}$ form an orthonormal set in M: It's clear that $\int h_{n,j}^2 = 1$ for each n,j. Now consider $\int h_{n,j} h_{m,k}$. Suppose without loss of generality that $m \geqslant n$. There are only two possibilities, either $h_{n,j}$ and $h_{m,k}$ have disjoint supports, in which case the integral is clearly zero, or the support of $h_{m,k}$ is contained in a set on which $h_{n,j}$ is constant, in which case the integral is just a constant multiple of $\int h_{m,k}$, which is 0. Thus they form an orthonormal set. We want to show they form an orthonormal basis for M. If we show this, then the desired conclusion is just the statement of Parseval's identity and we will be done. Let $f \in M$ and suppose that $\int f h_{n,j} = 0$ for all n,j. It's enough to show this implies $f = 0$. First note that we have $\int_0^1 f = \int_0^{1/2} f + \int_{1/2}^1 f = 0$. We also have by assumption $\int f h_{0,0} = \int_0^{1/2} f - \int_{1/2}^1 f = 0$. Combining these two yields $\int_0^{1/2} f = \int_{1/2}^1 f = 0$. Continuing, we have $0 = \int_0^{1/2} f = \int_0^{1/4} f + \int_{1/4}^{1/2}$, and by assumption, $0 = \int f h_{1,0} = \int_0^{1/4} f - \int_{1/4}^{1/2} f$, and combining these gives $\int_0^{1/4} f = \int_{1/4}^{1/2} f = 0$. Continuing in this way inductively shows that $\int_{I_{n,j}} f = 0$ for all n,j. Any closed interval can be written as a countable disjoint union of the $I_{n,j}$, so the integral of f over any closed interval vanishes, which implies $f = 0$. $\square$

(c) Let

$$S_N f(x) \;=\; \sum_{n=0}^{N} \sum_{j=0}^{2^n-1} \left(\int_0^1 f(t) h_{n,j}(t)\,dt \right) h_{n,j}(x).$$

In light of problem 5 above, it's enough to show that $S_N f(x) = E_{N+1} f(x)$ for almost every x. We show this holds for any x which is not an endpoint of any $I_{n,j}$. Fix such an x. Define $j(n)$ to be the unique j such that $x \in I_{n,j}$ and define $j(n)^c$ to be the unique $j \neq j(n)$ such that $I_{n,j(n)} \cup I_{n,j(n)^c} = I_{n-1,j(n-1)}$. Then we have

$$\begin{aligned}
S_N f(x) \;&=\; \sum_{n=0}^{N} \left(\int_0^1 f(t) h_{n,j(n)}(t)\,dt \right) h_{n,j(n)}(x) \\
&=\; \sum_{n=0}^{N} 2^n \left(\int_{I_{n+1,j(n+1)}} f - \int_{I_{n+1,j(n+1)^c}} f \right) \\
&=\; \sum_{n=0}^{N} 2^n \left(2 \int_{I_{n+1,j(n+1)}} f - \int_{I_{n,j(n)}} f \right) \\
&=\; \sum_{n=0}^{N} 2^{n+1} \int_{I_{n+1,j(n+1)}} f - 2^n \int_{I_{n,j(n)}} f \\
&=\; 2^{N+1} \int_{I_{N+1,j(N+1)}} f - \int_{I_{0,0}} f \\
&=\; 2^{N+1} \int_{I_{N+1,j(N+1)}} f \;=\; E_{N+1} f(x). \quad \square
\end{aligned}$$

Problem 7. Let μ be a finite positive Borel measure on $\mathbb{C}$.

(a) Prove that $F(z) = \int_{\mathbb{C}} \frac{1}{z-w}\, d\mu(w)$ exists for almost all $z \in \mathbb{C}$ and that $\int_K |F(z)|\, dx\, dy < \infty$ for every compact $K \subseteq \mathbb{C}$.
(b) Prove that for almost every horizontal line L and all compact $K \subseteq L$, $\int_K |F(x+iy)|\, dx < \infty$.
(c) Prove that for almost all open squares S with sides parallel to the axes,

$$\mu(S) \;=\; \frac{1}{2\pi i}\int_{\partial S} F(z)\, dz.$$

Solution. (a) The second half of the assertion implies the first half, so we focus on the second. It's enough to show that $\int_{|z|\leqslant R} |F(z)|\, dA(z) < \infty$ for each R. We estimate

$$\begin{aligned}
\int_{|z|\leqslant R} |F(z)|\, dA(z) \;&\leqslant\; \int_{|z|\leqslant R}\int_{w\in\mathbb{C}} \frac{1}{|z-w|}\, d\mu(w)\, dA(z) \;=\; \int_{w\in\mathbb{C}}\int_{|z|\leqslant R} \frac{1}{|z-w|}\, dA(z)\, d\mu(w) \quad \text{by Tonelli}\\
&=\; \int_{|w|\leqslant 2R}\int_{|z|\leqslant R} \frac{1}{|z-w|}\, dA(z)\, d\mu(w) + \int_{|w|>2R}\int_{|z|\leqslant R} \frac{1}{|z-w|}\, dA(z)\, d\mu(w)\\
&\leqslant\; \int_{|w|\leqslant 2R}\int_{|z-w|\leqslant 3R} \frac{1}{|z-w|}\, dA(z)\, d\mu(w) + \int_{|w|>2R}\int_{|z|\leqslant R} \frac{1}{R}\, dA(z)\, d\mu(w)\\
&\leqslant\; \int_{|w|\leqslant 2R} C_R\, d\mu(w) + \int_{|w|>2R} \pi R\, d\mu(w) \quad \text{where } C_R \text{ is some constant depending on } R\\
&<\; \infty
\end{aligned}$$

because μ is a finite measure. $\square$

(b) As in part (a), it's enough to prove the assertion with any compact set K replaced by any interval of the form $[-R, R]$. Fix some R and an integer m. Then by part (a) and Tonelli's theorem, we know $\int_m^{m+1}\int_R^R |F(x+iy)|\, dx\, dy < \infty$. This implies that there is a set $Y_{m,R}$ of full measure in $[m, m+1]$ such that $\int_R^R |F(x+iy)|\, dx < \infty$ for each $y \in Y_{m,R}$. By setting $Y_m = \bigcap_{R=1}^{\infty} Y_{m,R}$, we see that Y still has full measure in $[m, m+1]$ and now for any $y \in Y_m$, $\int_R^R |F(x+iy)|\, dx < \infty$ for every R. Thus we have shown that almost every horizontal line with y-intercept in $[m, m+1]$ satisfies the desired property. Now setting $Y = \bigcup_{m=-\infty}^{\infty} Y_m$, we see that Y is an almost everywhere subset of $\mathbb{R}$ with the property that $y \in Y$ implies $\int_R^R |F(x+iy)|\, dx < \infty$ for every R, which is the desired conclusion. In fact, by examining the proof of part (a) it's clear that we actually proved something a bit stronger, which is that $y \in Y$ implies $\int_K \int_{w\in\mathbb{C}} \frac{1}{|x+iy-w|}\, d\mu(w)\, dx < \infty$ for all compact sets K (we'll need this version in part (c)). $\square$

(c) The same argument as in part (b) shows that the analogous result to part (b) for vertical lines also holds. Let $\mathcal{S}$ be the collection of squares S in $\mathbb{C}$ such that all four sides of S lie on lines for which the conclusion of part (b) holds. It's clear that $\mathcal{S}$ is almost every square in $\mathbb{C}$. Thus for $S \in \mathcal{S}$, we have

$$\begin{aligned}
\int_{\partial S} F(z)\, dz \;&=\; \int_{\partial S}\int_{\mathbb{C}} \frac{1}{z-w}\, d\mu(w)\, dz \;=\; \int_{\mathbb{C}}\int_{\partial S} \frac{1}{z-w}\, dz\, d\mu(w)\\
&=\; \int_{\mathbb{C}} 2\pi i \chi_S(w)\, d\mu(w) \;=\; 2\pi i \mu(S),
\end{aligned}$$

which is the desired result. We just need to justify switching the order of integration in the first line. Note that by definition of $\mathcal{S}$,

$$\int_{\partial S}\int_{\mathbb{C}} \frac{1}{|z-w|}\, d\mu(w)\, dz$$

is simply a sum of four integrals along horizontal or vertical lines which are known to be finite by the comment at the end of part (b). Thus Fubini-Tonelli applies, so the switch is justified. $\square$

Problem 8. Let f be an entire non-constant function that satisfies the functional equation

$$f(1-z) \;=\; 1 - f(z)$$

for all $z \in \mathbb{C}$. Show that $f(\mathbb{C}) = \mathbb{C}$.

Solution. The functional equation implies that $w \in \text{Im}(f)$ if and only if $1 - w \in \text{Im}(f)$. Thus suppose that there were some $w \notin \text{Im}(f)$, then $1-w \notin \text{Im}(f)$ either, so f misses two points (if $w \neq 1/2$). But Picard's little theorem says that an entire function that misses two points is constant, a contradiction. Thus f hits everything except possibly $1/2$. But putting $z = 1/2$ into the functional equation gives $f(1/2) = 1 - f(1/2)$, so $f(1/2) = 1/2$. Thus f is surjective. $\square$

Problem 9. Let $f(z)$ be an analytic function on the entire complex plane $\mathbb{C}$ and assume $f(0) \neq 0$. Let $\{a_n\}$ be the zeros of f, counted with multiplicity.
(a) Let $R > 0$ be such that $|f(z)| > 0$ on $|z| = R$. Prove

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(Re^{i\theta})\right| \, d\theta \;=\; \log|f(0)| + \sum_{|a_n|<R} \log\left(\frac{R}{|a_n|}\right).$$

(b) Assume $|f(z)| \leqslant Ce^{|z|^\lambda}$ for positive constants C and λ. Prove that

$$\sum_n \left(\frac{1}{|a_n|}\right)^{\lambda+\epsilon} \;<\; \infty$$

for all $\epsilon > 0$.

Solution. See Spring 2017 # 9.

Problem 10. Let μ be Lebesgue measure on $\mathbb{D}$. Let H be the subspace of $L^2(\mathbb{D}, \mu)$ consisting of holomorphic functions. Show that H is complete.

Solution. See Fall 2014 #10 (not exactly the same problem, but a similar idea).

Problem 11. Suppose that $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and injective in some annulus $\{z : r < |z| < 1\}$. Show that f is injective in $\mathbb{D}$.

Solution. Suppose there are $z_1, z_2 \in \mathbb{D}$ with $f(z_1) = f(z_2) = w$. Then there is a circle C of radius $s \in (r, 1)$ containing both z_1 and z_2 in its interior. Then the function $f - w$ has at least two zeros inside C, so the argument principle tells us that the curve $f(C)$ has winding number at least 2 around zero. But a curve of winding number at least 2 has to intersect itself, meaning that there are two different points on the curve C at which $f - w$ takes the same value. But since S lies in the annulus $r < |z| < 1$, this contradicts the fact that f is injective on the annulus. $\square$

Problem 12. Let Q be the closed unit square in $\mathbb{C}$ and let R be the closed rectangle in $\mathbb{C}$ with vertices $\{0, 2, i, 2+i\}$. Prove there does not exists a surjective homeomorphism $f : Q \to R$ that is conformal on the interior of Q and maps corners to corners.

Solution. Suppose $f : Q \to R$ satisfies the given conditions. By continuity, it must preserve the order of the vertices, so by precomposing with rotations and flips if necessary, we may assume that f fixes the vertical line segment $[0, i]$. By the Schwarz reflection principle, applied iteratively and reflecting over the vertical lines, we can extend f to a map from the strip $0 \leqslant \text{Im}(z) \leqslant 1$ to itself. We can then reflect over the two horizontal lines to extend f to a map from the strip $-1 \leqslant \text{Im}(z) \leqslant 2$ to itself. This strip is simply connected and so is conformally equivalent to $\mathbb{D}$. So f has been extended to a conformal automorphism of a region conformally equivalent to $\mathbb{D}$, and f has two fixed points, which implies f is the identity, a contradiction. $\square$

2 Fall 2009

Problem 1. Find a non-empty closed set in the Hilbert space $L^2([0,1])$ that does not contain an element of smallest norm.

Solution. Let $f_n = n \cdot \chi_{[0,1/n^2+1/n^3]}$. We claim $\{f_n\}_{n=2}^{\infty}$ is such a set. First note that

$$\int |f_n|^2 = \left(\frac{1}{n^2} + \frac{1}{n^3}\right) \cdot n^2 \;=\; 1 + \frac{1}{n},$$

so we see that the set has no element of smallest norm. To show it's closed, suppose $g \in L^2$ is a limit point. Then there is a subsequence f_{n_k} converging to g in L^2. But this implies there is a further subsequence $f_{n_{k_\ell}}$ converging almost everywhere to g. But it's clear that $f_n \to 0$ almost everywhere, so $g = 0$. But 0 is clearly not a limit point of $\{f_n\}$ because $||f_n||_{L^2} > 1$ for each n. Thus $\{f_n\}$ has no limit points so it's closed. $\square$

Problem 2. Let v be a trigonometric polynomial in two variables, i.e.

$$v(x,y) \;=\; \sum_{n,m\in\mathbb{Z}} a_{n,m} e^{2\pi i(nx+my)}$$

with only finitely many nonzero $a_{n,m}$. If $u = v - \Delta v$ where $\Delta = \partial_x^2 + \partial_y^2$ is the Laplacian, prove that

$$||v||_{L^\infty([0,1]^2)} \;\leqslant\; C\,||u||_{L^2([0,1]^2)}$$

for some constant C independent of v.

Solution. A straightforward computation shows that

$$u(x,y) \;=\; \sum_{n,m} a_{n,m}(1+4\pi^2(n^2+m^2))e^{2\pi i(nx+my)}.$$

Thus, using orthonormality and the fact that only finitely many coefficients are nonzero, we have

$$\begin{aligned}
\int_0^1\int_0^1 |u(x,y)|^2\,dx\,dy \;&=\; \int_0^1\int_0^1 \sum_{n,m,k,\ell} a_{n,m}\overline{a_{k,\ell}}(1+4\pi^2(n^2+m^2))(1+4\pi^2(k^2+\ell^2))e^{2\pi i(nx+my)}e^{-2\pi i(kx+\ell y)}\,dx\,dy \\
&=\; \sum_{n,m,k,\ell} a_{n,m}\overline{a_{k,\ell}}(1+4\pi^2(n^2+m^2))(1+4\pi^2(k^2+\ell^2))\int_0^1 e^{2\pi i(n-k)x}\,dx \int_0^1 e^{2\pi i(m-\ell)y}\,dy \\
&=\; \sum_{n,m} |a_{n,m}|^2\,(1+4\pi^2(n^2+m^2))^2.
\end{aligned}$$

Now we simply estimate v using the triangle inequality and Cauchy-Schwarz:

$$\begin{aligned}
|v(x,y)|^2 \;&\leqslant\; \left(\sum_{n,m}|a_{n,m}|\right)^2 \;=\; \left(\sum_{n,m}|a_{n,m}|(1+4\pi^2(n^2+m^2))\cdot\frac{1}{(1+4\pi^2(n^2+m^2))}\right)^2 \\
&\leqslant\; \left(\sum_{n,m}|a_{n,m}|^2(1+4\pi^2(n^2+m^2))^2\right)\left(\sum_{n,m}\frac{1}{(1+4\pi^2(n^2+m^2))^2}\right) \\
&= C\cdot ||u||^2_{L^2([0,1]^2)}
\end{aligned}$$

because $\sum_{n,m}\frac{1}{(1+4\pi^2(n^2+m^2))^2}$ converges. Thus we have established $||v||^2_{L^\infty([0,1]^2)} \leqslant C\,||u||^2_{L^2([0,1]^2)}$ which implies the desired result. $\square$

Problem 3. Let $f:[0,1]\to\mathbb{R}$ be continuous with

$$\min_{x\in[0,1]} f(x) \;=\; 0.$$

Assume that for all $0 \leqslant a < b \leqslant 1$ we have

$$\int_a^b (f(x) - \min_{y \in [a,b]} f(y))\, dx \ \leqslant\ \frac{1}{2}(b-a).$$

(a) Prove that for all $\lambda \geqslant 0$,

$$|\{x : f(x) > \lambda + 1\}| \ \leqslant\ \frac{1}{2} |\{x : f(x) > \lambda\}| .$$

(b) Prove that for all $1 \leqslant c < 2$,

$$\int_0^1 c^{f(x)}\, dx \ \leqslant\ \frac{100}{2-c}.$$

Solution. (a) Fix $\lambda \geqslant 0$. Since f is continuous, $\{x : f(x) > \lambda\}$ is open, and thus it can be written as a countable union of disjoint open intervals (a_j, b_j) (the set is only open relative to $[0,1]$, so it's possible that one of the intervals is closed on the left at 0 and another is closed on the right at 1, but that doesn't change any of the following work, so we ignore it). Also by continuity, we must have $\min_{y \in [a_j,b_j]} f(y) = \lambda$ for each j. Thus using the hypothesis on f, for each j we have

$$\frac{1}{2}(b_j - a_j) \ \geqslant\ \int_{a_j}^{b_j} (f(x) - \lambda)\, dx \ =\ \int_{a_j}^{b_j} f(x)\, dx - \lambda(b_j - a_j).$$

Summing both sides from $j = 1$ to ∞ gives

$$\left(\frac{1}{2} + \lambda\right) |\{x : f(x) > \lambda\}| \ \geqslant\ \int_{\{f > \lambda\}} f(x)\, dx.$$

We also have

$$\begin{aligned}
\int_{\{f>\lambda\}} f(x)\, dx \ &=\ \int_{\{f>\lambda+1\}} f(x)\, dx + \int_{\{\lambda < f \leqslant \lambda+1\}} f(x)\, dx \\
&\geqslant\ (\lambda+1)\, |\{x : f(x) > \lambda+1\}| + \lambda\, |\{x : f(x) > \lambda+1\} \backslash \{x : f(x) > \lambda\}| \\
&=\ (\lambda+1)\, |\{x : f(x) > \lambda+1\}| + \lambda(|\{x : f(x) > \lambda\}| - |\{x : f(x) > \lambda+1\}|) \\
&=\ |\{x : f(x) > \lambda+1\}| + \lambda\, |\{x : f(x) > \lambda\}| .
\end{aligned}$$

Combining this with the above inequality and rearranging gives the desired result. $\square$

(b) Fix $1 \leqslant c < 2$. We can write

$$\int_0^1 c^{f(x)}\, dx \ =\ c^0 \cdot |\{f = 0\}| + \sum_{j=0}^{\infty} \int_{\{j < f \leqslant j+1\}} c^{f(x)}\, dx \ \leqslant\ 1 + \sum_{j=0}^{\infty} c^{j+1}\, |\{j < f \leqslant j+1\}| \ \leqslant\ 1 + \sum_{j=0}^{\infty} c^{j+1}\, |\{f > j\}| .$$

We know that $|\{x : f(x) > 0\}| \leqslant 1$, so by inductively applying the conclusion of part (a) we see that $|\{x : f(x) > j\}| \leqslant 2^{-j}$. Thus we have

$$\int_0^1 c^{f(x)}\, dx \ \leqslant\ 1 + \sum_{j=0}^{\infty} c^{j+1} 2^{-j} \ =\ 1 + c \sum_{j=0}^{\infty} (c/2)^j \ =\ 1 + \frac{c}{1 - c/2} \ =\ \frac{2+c}{2-c} \ \leqslant\ \frac{100}{2-c}$$

where the geometric series converges because $c < 2$. $\square$

Problem 4. Prove the following variant of the Lebesgue differentiation theorem: Let μ be a finite Borel measure on $\mathbb{R}$, singular with respect to Lebesgue measure. Then for Lebesgue almost every $x \in \mathbb{R}$,

$$\lim_{\epsilon \to 0} \frac{\mu([x-\epsilon, x+\epsilon)}{2\epsilon} \ =\ 0.$$

Solution. See Fall 2016 #2.

Problem 5. Construct a Borel subset E of the real line $\mathbb{R}$ such that for all intervals $[a, b]$ we have

$$0 \;<\; m(E \cap [a,b]) \;<\; b-a$$

where m denotes Lebesgue measure.

Solution.

Problem 6. The Poisson kernel for $0 \leqslant \rho < 1$ is the 2π-periodic function on $\mathbb{R}$ defined by

$$P_\rho(\theta) \;=\; \operatorname{Re}\left(\frac{1+\rho e^{i\theta}}{1-\rho e^{i\theta}}\right).$$

For functions h continuous on and harmonic inside the closed disc of radius R about the origin one has

$$h(re^{i\eta}) \;=\; \frac{1}{2\pi}\int_0^{2\pi} P_{r/R}(\eta-\theta)h(Re^{i\theta})\,d\theta.$$

Assume that h is harmonic and positive on $\mathbb{D}$. Prove that there exists a positive Borel measure μ on $[0, 2\pi]$ such that for all $re^{i\eta} \in \mathbb{D}$ one has

$$h(re^{i\nu}) \;=\; \int_0^{2\pi} P_r(\eta-\theta)\,d\mu(\theta).$$

Solution. For each $0 < R < 1$, define the measure μ_R by $d\mu_R(\theta) = h(Re^{i\theta})\,d\theta$. By scaling we may assume $h(0) = 1$. Since h is positive and continuous, each μ_R is a positive Borel measure on $[0, 2\pi]$. By the Riesz representation theorem, we may view each μ_R as a bounded linear functional on the Banach space $C([0, 2\pi])$. Note that by the special case of the given formula with $r = 0$ (i.e. the mean value property), we have

$$||\mu_R|| \;=\; \mu_R([0,2\pi]) \;=\; \frac{1}{2\pi}\int_0^{2\pi} h(Re^{i\theta})\,d\theta \;=\; h(0).$$

Thus each μ_R is in the unit ball of the dual space $C([0, 2\pi])^*$. By Banach-Alaoglu and the fact that $C([0, 2\pi])$ is separable, this implies that we have a subsequence of Rs converging to 1 and some measure μ in the unit ball of $C([0, 2\pi])$ with $\mu_R \to \mu$ in the weak-$*$ topology. A standard approximation argument shows that μ must also be a positive measure since each μ_R is. We claim that μ is the desired measure. Fix $re^{i\eta} \in \mathbb{D}$. Note that each P_ρ is continuous on $[0, 2\pi]$ and $P_{r/R} \to P_r$ uniformly on $[0, 2\pi]$ as $R \to 1$. For each $R < 1$ the given formula tells us

$$h(re^{i\eta}) \;=\; \int_0^{2\pi} P_{r/R}(\eta-\theta)\,d\mu_R(\theta).$$

Taking the limit as $R \to 1$ on both sides gives the desired result, where we have assumed the following lemma: if f_n are continuous and $f_n \to f$ uniformly on $[0, 2\pi]$ and $\mu_n \to \mu$ in weak-$*$, then $\int f_n\,d\mu_n \to \int f\,d\mu$. The proof of this just follows by writing

$$\left|\int f_n\,d\mu_n - \int f\,d\mu\right| \;\leqslant\; \left|\int f_n\,d\mu_n - \int f_n\,d\mu\right| + \left|\int f_n\,d\mu - \int f\,d\mu\right|$$

and noting that the first term goes to 0 by weak-$*$ convergence and the second term goes to zero by uniform convergence. $\square$

Problem 7. (a) Define *unitary operator* on a complex Hilbert space.
(b) Let S be a unitary operator on a complex Hilbert space. Prove that for every complex number $|\lambda| < 1$

the operator $S - \lambda I$ is invertible.
(c) For a fixed vector v in the Hilbert space and all $|\lambda| < 1$, define

$$h(\lambda) \;=\; \big\langle (S + \lambda I)(S - \lambda I)^{-1} v, v \big\rangle.$$

Show $\mathrm{Re}(h)$ is a positive harmonic function (you may not use the spectral theorem).

Solution. (a) $S : H \to H$ is unitary if $\langle Sx, Sy\rangle = \langle x, y\rangle$ for all $x, y \in H$.

(b) Suppose $(S - \lambda I)x = 0$ but $x \neq 0$. Then we have

$$\begin{aligned} 0 \;&=\; \langle (S-\lambda I)x, (S-\lambda I)x\rangle \;=\; \langle Sx - \lambda x, Sx - \lambda x\rangle \;=\; ||Sx||^2 + |\lambda|^2\, ||x||^2 - 2\,\mathrm{Re}(\lambda\langle x, Sx\rangle) \\ &=\; (1 + |\lambda|^2)\, ||x||^2 - 2\,\mathrm{Re}(\lambda\langle x, Sx\rangle). \end{aligned}$$

Thus we have

$$(1 + |\lambda|^2)\, ||x||^2 \;=\; 2\,\mathrm{Re}(\lambda\langle x, Sx\rangle) \;\leqslant\; 2|\lambda||\langle x, Sx\rangle| \;\leqslant\; 2|\lambda|\, ||x||\, ||Sx|| \;=\; 2|\lambda|\, ||x||^2.$$

Since we are assuming $x \neq 0$ this implies $(1 + |\lambda|^2) \leqslant 2|\lambda|$, which is impossible for $|\lambda| < 1$. Thus $S - \lambda I$ is injective and therefore invertible. $\square$

(c)

Problem 8. Let Ω be an open convex region in the complex plane. Assume f is a holomorphic function on Ω and the $\mathrm{Re}(f'(z)) > 0$ for all $z \in \Omega$.
(a) Prove that f is one-to-one.
(b) Show by example that the word "convex" cannot be replaced by "connected and simply connected".

Solution. (a) Let $a \neq b \in \Omega$. Let γ be a straight line from a to b, parameterized by $\gamma(t) = (1-t)b + ta$. By convexity, γ lies in Ω. So we can write $\int_\gamma f'(z)\,dz \;=\; f(b) - f(a)$. Write $f = u + iv$, then $f' = u_x + iv_x$. Examining the integral above, we have

$$f(b) - f(a) \;=\; \int_\gamma f'(z)\,dz \;=\; \int_0^1 (u_x(\gamma(t)) + iv_x(\gamma(t)))(b-a)\,dt \;=\; (b-a)\int_0^1 (u_x(\gamma(t)) + iv_x(\gamma(t)))\,dt.$$

Note that the integral on the right side has nonzero real part because u_x is always positive. Thus the whole right side is just some nonzero complex number since $b - a$ is a nonzero constant, so $f(b) \neq f(a)$. $\square$

Problem 9. Let f be a non-constant meromorphic function on $\mathbb{C}$ that obeys

$$f(z) \;=\; f(z + \sqrt{2}) \;=\; f(z + i\sqrt{2}).$$

Assume f has at most one pole in the closed unit disc $\overline{\mathbb{D}}$.
(a) Prove that f has exactly one pole in $\overline{\mathbb{D}}$.
(b) Prove that this is not a simple pole.

Solution. (a) We just need to show f has at least one pole in $\mathbb{D}$. Let $\Lambda = [0, \sqrt{2}] \times [0, i\sqrt{2}]$ be a fundamental domain for f and let M be the discrete lattice generated by $\sqrt{2}$ and $i\sqrt{2}$. Simple geometry shows that every point of Λ is at most 1 away from one of the vertices. Thus every point of Λ is equivalent mod M to some point of $\overline{\mathbb{D}}$. Since f is non-constant and doubly periodic, it must have a pole somewhere (otherwise it would be holomorphic and bounded and therefore constant), so it must have a pole in Λ, and thus must have a pole in $\overline{\mathbb{D}}$.

(b) The work in part (a) shows that every point of $\mathbb{C}$ is equivalent mod M to some point of $\overline{\mathbb{D}}$, so the fact that f has exactly one pole in $\overline{\mathbb{D}}$ implies that f has exactly one distinct pole mod M. The desired result now follows from the general fact that a doubly periodic function can't have only a single simple pole (mod

M), a proof of which is reproduced here (see e.g. Ahlfors Complex Analysis). Since the zeros and poles of f are discrete, we can find a fundamental domain Λ of M such that f has no zeros or poles on $\partial\Lambda$. Thus by double periodicity, it is clear that $\int_{\partial\Lambda} f(z)\,dz = 0$ because the integrals over opposite sides of Λ going in opposite directions cancel each other out. So by the residue theorem, the sums of residues of all the poles inside Λ is 0, implying there can't only be one simple pole. $\square$

3 Spring 2010

Problem 1. (a) Let $1 \leqslant p < \infty$. Show that if a sequence of real-valued functions $\{f_n\}$ converges in $L^p(\mathbb{R})$, then it contains a subsequence that converges almost everywhere.
(b) Give an example of a sequence of functions converging to 0 in $L^2(\mathbb{R})$ that does not converge almost everywhere.

Solution.

Problem 2. Let $p_1, \ldots, p_n$ be distinct points in $\mathbb{C}$ and let U be the domain $C \backslash \{p_1, \ldots, p_n\}$. Let A be the vector space of real harmonic functions on U and let $B \subseteq A$ be the subspace of real parts of complex analytic functions on U. Find the dimension of the quotient space A/B and give a basis.

Solution. See Spring 2017 #10.

Problem 3. For $f : \mathbb{R} \to \mathbb{R}$ in $L^1(\mathbb{R})$, let Mf be the (centered) Hardy-Littlewood maximal function. Prove there is a constant A such that for any $\lambda > 0$,

$$m\{x \in \mathbb{R} : Mf(x) > \lambda\} \ \leqslant \ \frac{A}{\lambda} \, ||f||_{L^1}$$

where m is Lebesgue measure. If you use a covering lemma, you should prove it.

Solution. See Fall 2011 #5.

Problem 4. Let $f(z)$ be a continuous function on $\overline{\mathbb{D}}$ such that f is analytic on $\mathbb{D}$ and $f(0) \neq 0$.
(a) Prove that if $0 < r < 1$ and if $\inf_{|z|=r} |f(z)| > 0$, then

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left| f(re^{i\theta}) \right| \, d\theta \ \geqslant \ \log |f(0)| \, .$$

(b) Prove that $m\{\theta \in [0, 2\pi] : f(e^{i\theta}) = 0\} = 0$ where m is Lebesgue measure.

Solution. See Fall 2016 #8.

Problem 5. (a) For $f \in L^2(\mathbb{R})$ and a sequence $\{x_n\} \subseteq \mathbb{R}$ which converges to zero, define $f_n(x) := f(x + x_n)$. Show that $\{f_n\}$ converges to f in L^2.
(b) Let $W \subseteq \mathbb{R}$ be a Lebesgue measurable set of positive Lebesgue measure. Show that the set of differences $W - W = \{x - y : x, y \in W\}$ contains an open neighborhood of the origin.

Solution. (a) See Fall 2011 #3.

(b) Let $f(x) = \chi_W(x)$ and $f_y(x) = \chi_W(x + y)$. We calculate

$$\begin{aligned} ||f - f_y||_{L^2}^2 &= \int (\chi_W(x) - \chi_W(x + y))^2 \, dx \\ &= \int \chi_W(x)^2 + \chi_W(x + y)^2 - 2\chi_W(x)\chi_W(x + y) \, dx \\ &= 2m(W) - 2 \int \chi_W(x)\chi_W(x + y) \, dx. \end{aligned}$$

By part (a), this quantity goes to 0 as $y \to 0$. Thus for all y sufficiently small,

$$\int \chi_W(x)\chi_W(x + y) \, dx \ > \ \frac{1}{2} m(W) \ > \ 0.$$

In particular, there is at least one x such that $\chi_W(x)\chi_W(x+y) = 1$, i.e. $x \in W$ and $x+y \in W$, so $y \in W-W$. Thus $W - W$ contains all sufficiently small y, as desired. $\square$

Problem 6. Let μ be a finite, positive, regular Borel measure supported on a compact subset of $\mathbb{C}$ and define the Newtonian potential

$$U_\mu(z) \;=\; \int_{\mathbb{C}} \left| \frac{1}{z-w} \right| d\mu(w).$$

(a) Prove that U_μ exists at Lebesgue almost all $z \in \mathbb{C}$ and that

$$\iint_K U_\mu(z)\, dx\, dy \;<\; \infty$$

for every compact $K \subseteq \mathbb{C}$.
(b) Prove that for almost every horizontal or vertical line $L \subseteq \mathbb{C}$, $\mu(L) = 0$ and $\int_K U_\mu(z)\, ds < \infty$ for every compact subset $K \subseteq L$, where ds denotes Lebesgue linear measure on L.
(c) Define the Cauchy potential of μ to be

$$\int_{\mathbb{C}} \frac{1}{z-w}\, d\mu(w).$$

Let R be a rectangle in $\mathbb{C}$ whose four sides are contained in lines L having the conclusions of (b). Prove that

$$\frac{1}{2\pi i} \int_{\partial R} S_\mu(z)\, dz \;=\; \mu(R).$$

Solution. See Spring 2009 #7.

Problem 7. Let H be a Hilbert space and let E be a closed convex subset of H. Prove that there exists a unique element $x \in E$ such that

$$||x|| \;=\; \int_{y \in E} ||y||\,.$$

Solution. See Fall 2012 #3

Problem 8. Let $F(z)$ be a non-constant meromorphic function on the complex plane $\mathbb{C}$ such that $F(z+1) = F(z) = F(z+i)$ for all z. Let Q be a square with vertices z, $z+1$, $z+i$, and $z+1+i$ such that F has no zeros and no poles on ∂Q. Prove that inside Q the function F has the same number of zeros as poles (counting multiplicities).

Solution.

Problem 9. Let

$$A \;=\; \{x \in \ell^2 : \sum_{n \geqslant 1} n|x_n|^2 \;\leqslant\; 1\}.$$

(a) Show that A is compact in the ℓ^2 topology.
(b) Show that the mapping from A to $\mathbb{R}$ defined by

$$x \mapsto \int_0^{2\pi} \left| \sum_{n \geqslant 1} x_n e^{in\theta} \right| \frac{d\theta}{2\pi}$$

achieves its maximum on A.

Solution.

Problem 10. Let $\Omega \subseteq \mathbb{C}$ be a connected open set, let $z_0 \in \Omega$, and let $\mathcal{U}$ be the set of positive harmonic functions U on Ω such that $U(z_0) = 1$. Prove that for every compact set $K \subseteq \Omega$ there is a finite constant M such that

$$\sup_{U \in \mathcal{U}} \sup_{z \in K} U(z) \leqslant M.$$

Solution.

Problem 11. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support.
(a) Prove there is a constant A such that

$$||f * \phi||_{L^q} \leqslant A\,||f||_{L^p} \quad \text{for all } 1 \leqslant p \leqslant q \leqslant \infty \quad \text{and all } f \in L^p.$$

If you use Young's convolution inequality you should prove it.
(b) Show by example that such a general inequality cannot hold for $p > q$.

Solution. (a) Define α to be the number $\geqslant 1$ so that $1/\alpha = 1/q - 1/p + 1$ (if $q = \infty$ and $p = 1$ then $\alpha = \infty$). Then $1/q + 1 = 1/p + 1/\alpha$, so by Young's convolution inequality we have

$$||f * \phi||_{L^q} \leqslant ||f||_{L^p}\,||\phi||_{L^\alpha} \leqslant \sup_{x \in \mathbb{R}} |\phi(x)| \cdot ||f||_{L^p}$$

as desired. Now we prove Young's convolution inequality: the statement is that if $1/p + 1/q = 1/r + 1$, and $f \in L^p$ and $g \in L^q$, then $||f * g||_{L^r} \leqslant ||f||_{L^p}\,||g||_{L^q}$. Proof: note that the condition on p, q, r implies that $1/p, 1/q \geqslant 1/r$. We have

$$1 = \frac{1}{p} + \frac{1}{q} - \frac{1}{r} = \left(\frac{1}{p} - \frac{1}{r}\right) + \left(\frac{1}{q} - \frac{1}{r}\right) + \frac{1}{r} = \frac{r-p}{pr} + \frac{r-q}{qr} + \frac{1}{r}.$$

By Hölder using the three conjugate exponents above, we have

$$\begin{aligned}
|(f * g)(x)| &\leqslant \int |f(x-y)g(y)|\,dy \\
&\leqslant \int |f(x-y)|^{(r-p)/r}|g(y)|^{(r-q)/r}|f(x-y)^{p/r}g(y)^{q/r}|\,dy \\
&\leqslant \left(\int |f(x-y)|^p\,dy\right)^{(r-p)/pr}\left(\int |g(y)|^q\,dy\right)^{(r-q)/pr}\left(\int |f(x-y)^p g(y)^q|\,dy\right)^{1/r} \\
&= ||f||_{L^p}^{(r-p)/r}\,||g||_{L^q}^{(r-q)/r}\left(\int |f(x-y)^p g(y)^q|\,dy\right)^{1/r}.
\end{aligned}$$

Thus

$$\begin{aligned}
||f * g||_{L^r}^r &= \int |(f * g)(x)|^r\,dx \leqslant ||f||_{L^p}^{r-p}\,||g||_{L^q}^{r-q}\int\int |f(x-y)^p g(y)^q|\,dy\,dx \\
&= ||f||_{L^p}^{r-p}\,||g||_{L^q}^{r-q}\int\int |f(x-y)^p g(y)^q|\,dx\,dy \quad \text{by Tonelli} \\
&= ||f||_{L^p}^r\,||g||_{L^q}^r. \quad \square
\end{aligned}$$

(b) Fix $p > q$. Let ϕ be equal to 1 on $[0, 1]$, have support contained in $[-1, 2]$, and have $0 \leqslant \phi \leqslant 1$ everywhere. Fix $1/\alpha \in (q, p)$ and let $f(y) = 1/y^\alpha$ for $y \in [10, \infty)$ and 0 otherwise. Note that $f \in L^p$ but $f \notin L^q$. We have, for all $x > 100$,

$$(f * \phi)(x) = \int f(x-y)\phi(y)\,dy \geqslant \int_0^1 f(x-y)\,dy = \int_{x-1}^x f(y)\,dy = \int_{x-1}^x \frac{1}{y^\alpha}\,dy \geqslant \frac{1}{x^\alpha}.$$

Thus $f * \phi \notin L^q$, so the inequality fails. $\square$

Problem 12. Let F be a function from $\mathbb{D}$ to $\mathbb{D}$ such that whenever z_1, z_2, z_3 are distinct points of $\mathbb{D}$ there exists an analytic function f_{z_1,z_2,z_3} from $\mathbb{D}$ into $\mathbb{D}$ such that $F(z_j) = f_{z_1,z_2,z_3}(z_j)$. Prove that F is analytic at every point of $\mathbb{D}$.

Solution.

Problem 13. Let X and Y be Banach spaces. A bounded linear transformation $A : X \to Y$ is *compact* if for every bounded sequence $\{x_n\} \subseteq X$, the sequence $\{Ax_n\}$ has a convergent subsequence in Y. Suppose X is reflexive ($X^{**} = X$) and X^* is separable. Show that $A : X \to Y$ is compact if and only if for every bounded sequence $\{x_n\} \subseteq X$, there exists a subsequence $\{x_{n_j}\}$ and a vector $\phi \in X$ such that $x_{n_j} = \phi + r_{n_j}$ and $Ar_{n_j} \to 0$ in Y.

Solution.

4 Fall 2010

Problem 1. Consider just Lebesgue measurable functiions $f : [0,1] \to \mathbb{R}$ together with Lebesgue measure.
(a) State Fatou's lemma,
(b) State and prove the Dominated Convergence Theorem.
(c) Give an example where $f_n(x) \to 0$ a.e. but $\int f_n(x)\,dx \to 1$.

Solution. (a) If f_n are non-negative, then $\int \liminf_{n\to\infty} f_n \leqslant \liminf_{n\to\infty} \int f_n$.

(b) If $f_n \to f$ almost everywhere and $|f_n| \leqslant g$ for some integrable function g and all f_n, then $\int |f - f_n| \to 0$.
Proof: Since $|f_n| \leqslant g$ and $f_n \to f$ almost everywhere, we also have $|f| \leqslant g$ almost everywhere, so the functions $2g - |f - f_n|$ are non-negative. Thus we can apply Fatou's lemma to get

$$\int \liminf_{n\to\infty} 2g - |f - f_n| \;\leqslant\; \liminf_{n\to\infty} \int (2g - |f - f_n|).$$

The left side simplifies to $\int 2g$ and the right side simplifies to $\int 2g - \limsup_{n\to\infty} \int |f - f_n|$. Thus by canceling and rearranging we get $\limsup \int |f - f_n| \leqslant 0$, and since it's a limsup of non-negative quantities this implies the limit exists and equals 0. $\square$

(c) Let $f_n = n \cdot \chi_{[0,1/n]}$. $f_n \to 0$ almost everywhere but $\int f_n = 1$ for all n.

Problem 2. Prove the following form of Jensen's inequality: if $f : [0,1] \to \mathbb{R}$ is continuous, then

$$\int_0^1 e^{f(x)}\,dx \;\geqslant\; \exp\left(\int_0^1 f(x)\,dx\right).$$

Moreover, if equality occurs then f is a constant function.

Solution. Let $u = \int_0^1 f(x)\,dx$. Let L be the tangent line to the graph of $y = e^x$ at $x = u$. Say L has the equation $y = ax + b$. Since exp is convex, we know that $au + b = e^u$ and $at + b < e^t$ for all $t \neq u$. So we have

$$au + b \;=\; a\int_0^1 f(x)\,dx + b \;=\; \int_0^1 (af(x) + b)\,dx \;\leqslant\; \int_0^1 e^{f(x)}\,dx$$

by definition of the line $y = ax + b$. Furthermore, if equality holds in the last step, we must have $f(x) = u$ for all x. This is because f is continuous, so if $f(x) \neq u$ somewhere, then $f \neq u$ on some open interval, and for all x in that interval we would have $af(x) + b < e^{f(x)}$, leading to a strict inequality above. $\square$

Problem 3. Consider the following sequence of functions:

$$f_n : [0,1] \to \mathbb{R} \quad \text{by} \quad f_n(x) \;=\; \exp(\sin(2\pi n x)).$$

(a) Prove that f_n converges weakly in $L^1([0,1])$.
(b) Prove that f_n converges weak-$*$ in $L^\infty([0,1])$, viewed as the dual of $L^1([0,1])$.

Solution. (a) This requires showing the existence of some $f \in L^1$ with $\int f_n g \to \int fg$ for all $g \in L^\infty$. Since $L^\infty([0,1]) \subseteq L^1([0,1])$, this conclusion is implied by part (b) below.

(b) We need to find some $f \in L^\infty$ such that $\int f_n g \to \int fg$ for all $g \in L^1$. First note that each f_n is $1/n$-periodic, so we have

$$\int_0^1 f_n(x)\,dx \;=\; \int_0^1 \exp(\sin(2\pi n x))\,dx \;=\; n\int_0^{1/n} \exp(\sin(2\pi n x)) \;=\; \int_0^1 \exp(\sin(2\pi u))\,du \;=\; \int_0^1 f_1(u)\,du.$$

Thus the quantity $\int_0^1 f_n(x)\,dx$ is independent of n. By viewing this as the dual pairing with the constant function 1, we see that if the weak limit f exists it must be equal to the constant $C := \int_0^1 \exp(\sin(2\pi u))\,du$.

So we need to show that $\int_0^1 f_n g \to C\int_0^1 g$ for any $g \in L^1$. We do this with a standard density argument. Suppose we knew the desired conclusion for all ϕ in some family $\mathcal{F}$ dense in L^1. Then for any $g \in L^1$, let ϕ_k be a sequence in $\mathcal{F}$ converging to g, then we have

$$\left|\int f_n g - C\int g\right| \leqslant \left|\int f_n g - \int f_n\phi_k\right| + \left|\int f_n\phi_k - \int C\phi_k\right| \leqslant e\cdot ||g-\phi_k||_{L^1} + \left|\int f_n\phi_k - \int C\phi_k\right|$$

because each f_n is bounded uniformly by e. For a fixed k, take $n \to \infty$ and the second term on the right goes to zero by assumption on the ϕ_k. Then take $k \to \infty$ and the first term also goes to zero by construction, so the desired result follows. Now we just need to prove the desired result for a dense family $\mathcal{F}$. We take $\mathcal{F}$ to be the set of linear combinations of characteristic functions of closed intervals. Since the desired property is linear, it's enough to verify for the characteristic function $g = \chi_{[a,b]}$. We need to show that $\int_a^b \exp(\sin(2\pi nx))\,dx \to C(b-a)$ as $n \to \infty$. Let a_n be the least number of the form $q/n > a$ and b_n be the greatest number of the form $q/n < b$. Then we write, using the periodicity,

$$\begin{aligned}
\int_a^b \exp(\sin(2\pi nx))\,dx &= \left(\int_a^{a_n} + \int_{b_n}^b + (\lfloor (b-a)n\rfloor - 2)\int_{a_n}^{a_n+1/n}\right)\exp(\sin(2\pi nx))\,dx \\
&= e(a_n - a) + e(b-b_n) + (\lfloor (b-a)n\rfloor - 2)\int_0^{1/n}\exp(\sin(2\pi nx))\,dx \\
&= e(a_n-a) + e(b-b_n) + \frac{\lfloor (b-a)n\rfloor - 2}{n}C
\end{aligned}$$

which tends to $(b-a)C$ as $n \to \infty$, so we're done. $\square$

Problem 4. Let T be a linear transformation on $C_c(\mathbb{R})$ (continuous functions with compact support) that has the following two properties:

$$||Tf||_{L^\infty} \leqslant ||f||_{L^\infty} \quad \text{and} \quad m\{x \in \mathbb{R} : |Tf(x)| > \lambda\} \leqslant \frac{||f||_{L^1}}{\lambda}$$

where m denotes Lebesgue measure. Prove that

$$\int |Tf(x)|^2\,dx \leqslant C\int |f(x)|^2\,dx$$

for all $f \in C_c(\mathbb{R})$ and some fixed number C.

Solution. We mimic the proof of the Hardy-Littlewood maximal theorem, with a few annoying things changed because T is only defined for C_c functions. First we will establish the result when f is a real-valued, non-negative function, and extend it at the end. We use the identity

$$\int |Tf|^2 = 2\int_0^\infty \lambda\cdot m\{x : |Tf(x)| > \lambda\}\,d\lambda.$$

For each fixed λ, we have the decomposition $f = g+h$ where $h := \min(f, \lambda/2)$ and $g := f - h = 0$ if $f < \lambda/2$ and $f - \lambda/2$ if $f > \lambda/2$. Note that both g and h are continuous and non-negative with compact support. Then we have $Tf = Tg + Th$, so $|Tf| \leqslant |Tg| + |Th|$, which implies that

$$\{x : |Tf(x)| > \lambda\} \subseteq \{x : |Tg(x)| > \lambda/2\} \cup \{x : |Th(x)| > \lambda/2\}.$$

But we have $||Th||_{L^\infty} \leqslant ||h||_{L^\infty} \leqslant \lambda/2$ by construction, so the second set has measure zero and we just have (up to measure zero sets)

$$\{x : |Tf(x)| > \lambda\} \subseteq \{x : |Tg(x)| > \lambda/2\}.$$

Thus we have

$$\begin{aligned}
\int |Tf|^2 \;&\leqslant\; 2\int_0^\infty \lambda\cdot m\{x : |Tg(x)| > \lambda/2\}\,d\lambda \\
&\lesssim\; \int_0^\infty \lambda \frac{2\,||g||_{L^1}}{\lambda}\,d\lambda \quad \text{by the weak-type hypothesis} \\
&\lesssim\; \int_0^\infty \int_{\mathbb{R}} |g(x)|\,dx\,d\lambda \;=\; \int_0^\infty \int_{\{x:f(x)>\lambda/2\}} (f(x)-\lambda/2)\,dx\,d\lambda \;\leqslant\; \int_0^\infty \int_{\{x:f(x)>\lambda/2\}} f(x)\,dx\,d\lambda \\
&=\; \int_{\mathbb{R}} |f(x)| \int_0^{2|f(x)|} d\lambda\,dx \quad \text{by Tonelli} \\
&\lesssim\; \int_{\mathbb{R}} |f(x)|^2\,dx.
\end{aligned}$$

This establishes the result for positive real-valued f. For general real-valued f, write $f = f_+ - f_-$. Then we have

$$\begin{aligned}
\int |Tf|^2 \;&=\; \int |Tf_+ - Tf_-|^2 \;=\; \int |Tf_+|^2 + |Tf_-|^2 + |Tf_+||Tf_-| \\
&\leqslant\; \int |Tf_+|^2 + \int |Tf_-|^2 \;\lesssim\; ||f_+||_{L^2}^2 + ||f_-||_{L^2}^2 \;=\; ||f||_{L^2}^2
\end{aligned}$$

where the last equality is valid by the Pythagorean theorem because since $f_+(x)f_-(x) = 0$ for all x, f_+ and f_- are orthogonal. This establishes the result for general real-valued f. For complex-valued f, write $f = \mathrm{Re}(f) + i\,\mathrm{Im}(f)$, then we have

$$\int |Tf|^2 \;=\; \int |T\,\mathrm{Re}(f) + iT\,\mathrm{Im}(f)|^2 \;=\; \int |T\,\mathrm{Re}(f)|^2 + |T\,\mathrm{Im}(f)|^2 \;\lesssim \int |\,\mathrm{Re}(f)|^2 + |\,\mathrm{Im}(f)|^2 \;=\; \int |f|^2,$$

so we're done. □

Problem 5. Let $\mathbb{R}/\mathbb{Z}$ denote the torus (whose elements we write as cosets) and fix an irrational $\alpha > 0$.
(a) Show that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(n\alpha + \mathbb{Z}) \;=\; \int_0^1 f(x+\mathbb{Z})\,dx$$

for all continuous functions $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$.
(b) Show that the conclusion is also true when f is the characteristic function of a closed interval.

Solution. (a) Define $A_N(f) = \frac{1}{N}\sum_{n=0}^{N-1} f(n\alpha+\mathbb{Z})$ and $I(f) = \int_0^1 f(x+\mathbb{Z})\,dx$. First we show the conclusion when f is a trig polynomial. By linearity, it's enough to assume $f(x) = e^{2\pi ikx}$ for some $k \in \mathbb{Z}$. If $k = 0$ then both sides are clearly equal to 1 so assume $k \neq 0$. Then we have

$$\begin{aligned}
A_N(f) \;&=\; \frac{1}{N}\sum_{n=0}^{N-1} (e^{2\pi ik\alpha})^n \;=\; \frac{1}{N}\frac{1-e^{2\pi ik\alpha N}}{1-e^{2\pi ik\alpha}} \;\to\; 0 \quad \text{as } N\to\infty \\
I(f) \;&=\; \int_0^1 e^{2\pi ikx}\,dx \;=\; 0.
\end{aligned}$$

So the result is verified for trig polynomials. Now for general $f \in C(\mathbb{R}/\mathbb{Z})$, fix $\epsilon > 0$ and let P be a trig polynomial with $||f-P||_{L^\infty} < \epsilon$. Then we have

$$\begin{aligned}
|A_N(f) - I(f)| \;&\leqslant\; |A_N(f) - A_N(P)| + |A_N(P) - I(P)| + |I(P) - I(f)| \\
&\leqslant\; 2\epsilon + |A_N(P) - I(P)|.
\end{aligned}$$

First take $N \to \infty$, then we see that $|\lim_{N\to\infty} A_N(f) - I(f)| < 2\epsilon$, and since this holds for arbitrary ϵ, the desired result follows. □

(b) Let $f = \chi_{[a,b]}$. Let g_k and h_k be sequences of continuous functions satisfying $0 \leqslant g_k \leqslant f \leqslant h_k \leqslant 1$ for all k, and g_k and h_k both converge almost everywhere to f as $k \to \infty$ (it's clear that such sequences exist by just taking the graph of f and smoothing it out a bit). Then for each N and k we have

$$A_N(g_k) \;\leqslant\; A_N(f) \;\leqslant\; A_N(h_k), \quad I(g_k) \;\leqslant\; I(f) \;\leqslant\; I(h_k).$$

For k fixed, take $N \to \infty$. Since g_k and h_k are continuous, this implies that

$$I(g_k) \;\leqslant\; \liminf_{N\to\infty} A_N(f) \;\leqslant\; \limsup_{N\to\infty} A_N(f) \;\leqslant\; I(h_k).$$

Since everything is dominated by 1 and we have pointwise convergence almost everywhere, by the dominated convergence theorem we can take $k \to \infty$ and get

$$I(f) \;\leqslant\; \liminf_{N\to\infty} A_N(f) \;\leqslant\; \limsup_{N\to\infty} A_N(f) \;\leqslant\; I(f),$$

which implies the desired result. $\square$

Problem 6. Consider the complex Hilbert space

$$H \;:=\; \left\{ f : \overline{\mathbb{D}} \to \mathbb{C} : f(z) = \sum_{k=0}^{\infty} \widehat{f}(k) z^k \quad \text{with} \quad ||f||^2 := \sum_{k=0}^{\infty} (1+k^2)|\widehat{f}(k)|^2 < \infty \right\}.$$

(a) Prove that the linear function $L : f \mapsto f(1)$ is bounded.
(b) Find the element $g \in H$ representing L.
(c) Show that $f \mapsto \operatorname{Re} L(f)$ achieves its maximal value on the set

$$B \;:=\; \{f \in H : ||f|| \leqslant 1 \quad \text{and} \quad f(0) = 0\},$$

that this maximum occurs at a unique point, and determine this maximal value.

Solution. (a) We have

$$|f(1)| \;\leqslant\; \sum_{k=0}^{\infty} |\widehat{f}(k)| \;=\; \sum_{k=0}^{\infty} |\widehat{f}(k)|\sqrt{1+k^2}\frac{1}{\sqrt{1+k^2}} \;\leqslant\; \left(\sum_{k=0}^{\infty} |\widehat{f}(k)|^2(1+k^2)\right)^{1/2} \left(\sum_{k=0}^{\infty} \frac{1}{1+k^2}\right)^{1/2} \;=\; C\,||f||$$

where $C^2 = \sum_{k=0}^{\infty} \frac{1}{1+k^2} < \infty$.

(b) We are implicitly assuming the inner product in H is given by

$$\langle f, g\rangle \;=\; \sum_{k=0}^{\infty} \widehat{f}(k)\overline{\widehat{g}(k)}(1+k^2).$$

If g represents L then we must have

$$\langle f, g\rangle \;=\; \sum_{k=0}^{\infty} \widehat{f}(k)\overline{\widehat{g}(k)}(1+k^2) \;=\; f(1) \;=\; \sum_{k=0}^{\infty} \widehat{f}(k).$$

It's clear that if $\widehat{g}(k) = \frac{1}{1+k^2}$ then this would be satisfied. So we can just define

$$g(z) \;=\; \sum_{k=0}^{\infty} \frac{1}{1+k^2} z^k.$$

The series converges uniformly on $\overline{\mathbb{D}}$ so this definition actually makes sense (and in fact is holomorphic, but that's not necessary). $\square$

(c) First we note that the maximum value of $\mathrm{Re}(L(f))$ on B must happen when $||f|| = 1$, otherwise we could normalize f and increase the value of $\mathrm{Re}(L(f))$. The condition that $f(0) = 0$ corresponds to having $\widehat{f}(0) = 0$. So the problem is reduced to maximizing $\sum_{k=1}^{\infty} \mathrm{Re}(\widehat{f}(k))$ subject to the condition that $\sum_{k=1}^{\infty}(1+k^2)|\widehat{f}(k)|^2 = 1$. Note that the constraint only depends on $|\widehat{f}(k)|$. Thus we can always increase $\mathrm{Re}(f(1))$ while keeping the norm constant if we assume that each $\widehat{f}(k)$ is real and positive. So without loss of generality we can assume each $\widehat{f}(k) \geqslant 0$. Using the same Cauchy-Schwarz argument from part (a), we have

$$\sum_{k=1}^{\infty} \widehat{f}(k) \;\leqslant\; \left(\sum_{k=1}^{\infty} |\widehat{f}(k)|^2(1+k^2)\right)^{1/2} \left(\sum_{k=1}^{\infty} \frac{1}{1+k^2}\right)^{1/2} \;=\; \left(\sum_{k=1}^{\infty} \frac{1}{1+k^2}\right)^{1/2}$$

and equality holds if and only if $\widehat{f}(k)\sqrt{1+k^2} = \frac{\alpha}{\sqrt{1+k^2}}$ for some $\alpha \in \mathbb{R}$. This shows that that maximum on B is achieved at a unique point, i.e.

$$f(z) \;=\; \sum_{k=1}^{\infty} \frac{\alpha}{1+k^2} z^k.$$

Also, this α is determined by the condition that f has norm 1:

$$1 \;=\; \sum_{k=1}^{\infty}(1+k^2)|\widehat{f}(k)|^2 \;=\; \sum_{k=1}^{\infty} \frac{\alpha^2}{1+k^2},$$

so $\alpha = \left(\sum_{k=1}^{\infty} \frac{1}{1+k^2}\right)^{-1/2}$. Thus the maximum value achieved is

$$\sum_{k=1}^{\infty} \frac{\alpha}{1+k^2} \;=\; \left(\sum_{k=1}^{\infty} \frac{1}{1+k^2}\right)^{1/2}. \quad \square$$

Problem 7. Suppopse that $f : \mathbb{C} \to \mathbb{C}$ is continuous and holomorphic on $\mathbb{C}\backslash\mathbb{R}$. Prove that f is entire.

Solution. By Morera's theorem it's enough to show that the integral around any rectangle with sides parallel to the axes is zero. Let R be any rectangle. If R doesn't intersect the real axis, the integral is obviously zero by hypothesis. If R does intersect the real axis, break up R into two pieces, one in the upper half plane and one in the lower, and by continuity the integral over R is equal to limit of the integrals as the two pieces approach the real axis, so you still get zero (this is a really standard argument). $\square$

Problem 8. Let $A(\mathbb{D})$ be the $\mathbb{C}$-vector space of all holomorphic functions on $\mathbb{D}$ and suppose that $L : A(\mathbb{D}) \to \mathbb{C}$ is a multiplicative linear functional. If L is not identically zero, show that there is a $z_0 \in \mathbb{D}$ so that $L(f) = f(z_0)$ for all $f \in A(\mathbb{D})$.

Solution. Note that if this were true, then we would have to have $L(z) = z_0$. So define $z_0 := L(z)$ and we want to show that $L(f) = f(z_0)$ for any $f \in A(\mathbb{D})$. Since we are assuming that L is not identically zero, let f be such that $L(f) \neq 0$. Then because L is multiplicative we can write $L(f) = L(f \cdot 1) = L(f)L(1)$, so $L(1) = 1$. This, combined with the linear and multiplicative hypotheses again, imply that $L(P) = P(z_0)$ for any polynomial P. Now let f be any element of $A(\mathbb{D})$. We can write $f(z) - f(z_0) = (z - z_0)g(z)$ for some other $g \in A(\mathbb{D})$. Therefore we have

$$L(f) - f(z_0) \;=\; L((z-z_0)g(z)) \;=\; (L(z) - z_0)L(g) \;=\; 0,$$

which establishes the desired result. The only thing left to check is that we actually have $z_0 \in \mathbb{D}$. If not, then $1/(z - z_0)$ would be in $A(\mathbb{D})$, and so we would have

$$L(1/(z-z_0)) \;=\; 1/L(z-z_0) \;=\; 1/(z_0 - z_0),$$

a contradiction. $\square$

Problem 9. Let

$$f(z) \;=\; \sum_{n=0}^{\infty} a_n z^n$$

be a holomorphic function in $\mathbb{D}$. Show that if

$$\sum_{n=2}^{\infty} n|a_n| \;\leqslant\; |a_1|$$

with $a_1 \neq 0$ then f is injective.

Solution. We have $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$. Thus for any fixed $z \in \mathbb{D}$ we have

$$|f'(z)| \;=\; \left|\sum_{n=1}^{\infty} n a_n z^{n-1}\right| \;\geqslant\; |a_1| - \sum_{n=2}^{\infty} n|a_n||z|^n \;>\; |a_1| - \sum_{n=2}^{\infty} a|a_n| \;\geqslant\; 0,$$

so f' is nonvanishing in $\mathbb{D}$.

Problem 10. Prove that the punctured disc $\{z : 0 < |z| < 1\}$ and the annulus $\{z : 1 < |z| < 2\}$ are not conformally equivalent.

Solution. Let P be the punctured disc and A be the annulus. Suppose $f : P \to A$ is conformal. Then, since A is bounded, the singularity of f at 0 must be removable. So we extend f to a holomorphic function $f : \mathbb{D} \to A$. If we knew that f were still conformal, this would be a contradiction because $\mathbb{D}$ is simply connected but A is not. We already know f is holomorphic and surjective, so to show f is conformal we just need to show that f is still injective when we extend it to be defined at 0. Suppose $f(0) = f(z)$ with $z \in P$ (this is the only possibility because f is injective on P). Let U and V be disjoint open balls around 0 and z respectively. By the open mapping theorem, $f(U)$ and $f(V)$ are open. They interset at $f(0) = f(z)$, so their intersection is open and non-empty, and therefore in particular there is some other point $w \in f(U) \cap f(V)$. So we have $z_1 \in U$, $z_2 \in V$ with $f(z_1) = f(z_2)$. But $z_1 \neq 0$ because $w \neq f(0)$, so this contradicts the fact that f is injective on P. $\square$

Problem 11. Let $\Omega \subseteq \mathbb{C}$ be a non-empty open connected set. If $f : \Omega \to \mathbb{C}$ is harmonic and f^2 is also harmonic, show that either f or $\overline{f}$ is holomorphic on Ω.

Solution. Recall the Wirtinger derivates $\partial_z = (1/2)(\partial_x - i\partial_y)$ and $\partial_{\overline{z}} = (1/2)(\partial_x + i\partial_y)$. A straightforward computation verifies the identity $\Delta = 4\partial_z\partial_{\overline{z}}$. By hypothesis, f^2 is harmonic, so $\Delta f^2 = 0$. Putting this into the above identity and using the chain and product rules and the hypothesis that f is also harmonic, this reduces to $(\partial_z f)(\partial_{\overline{z}} f) = 0$. Suppose $\overline{f}$ is not holomorphic. Then there is a point in Ω at which $\partial_z f \neq 0$. By continuity, $\partial_z f$ is nonzero on an open ball, so $\partial_{\overline{z}} f = 0$ on an open ball. Since f is harmonic, $\partial_{\overline{z}}$ also is (because ∂_x and ∂_y both are). But then we have a harmonic function on all of Ω which vanishes on an open ball. In particular it has a local maximum on that open ball, so the maximum principle implies $\partial_{\overline{z}} f$ is constant and therefore identically zero, so f is holomorphic. $\square$

Problem 12. Let $\mathcal{F}$ be the family of functions f holomorphic on $\mathbb{D}$ with

$$\iint\limits_{x^2+y^2<1} |f(x+iy)|^2\,dx\,dy \;<\; 1.$$

Prove that for each compact subset $K \subseteq \mathbb{D}$ there is a constant A so that $|f(z)| < A$ for all $z \in K$ and all $f \in \mathcal{F}$.

Solution. See e.g. the first half of Fall 2014 #10.

5 Spring 2011

Problem 1.

(a) Define what it means to say that $f_n \to f$ weakly in $L^2([0,1])$.

(b) Suppose $f_n \in L^2([0,1])$ converge weakly to $f \in L^2([0,1])$ and define 'primitive' functions

$$F_n(x) \ := \ \int_0^x f_n(t)\,dt \quad \text{and} \quad F(x) \ := \ \int_0^x f(t)\,dt.$$

Show that $F_n, F \in C([0,1])$ and that $F_n \to F$ uniformly on $[0,1]$.

Solution.

(a) For every $g \in L^2([0,1])$, $\lim_{n\to\infty} \int_0^1 f_n(x)g(x)\,dx = \int_0^1 f(x)g(x)\,dx$.

(b) First, we know that weakly convergent sequences are bounded, so we can say $||f_n||_{L^2} \leqslant M$ for all n. To show that F_n and F are continuous, note that

$$|F_n(x+h) - F_n(x)| \ \leqslant \ \int_x^{x+h} |f_n(t)|\,dt \ \leqslant \ \left(\int_x^{x+h} |f_n(t)|^2\,dt\right)^{1/2} \left(\int_x^{x+h} 1\,dt\right)^{1/2} \ \leqslant \ M|h|^{1/2}.$$

Note that the above estimate for $|F_n(x+h) - F_n(x)|$ is independent of both n and x, so we have actually shown that $\{F_n\}$ is an equicontinuous family of functions. A similar estimate shows $|F(x+h) - F(x)| \leqslant ||f||_{L^2}\,|h|^{1/2}$, so F is also continuous. Now we show $F_n \to F$ uniformly. First note that

$$|F_n(x)| \ \leqslant \ \int_0^x |f_n(t)|\,dt \ \leqslant \ \left(\int_0^x |f_n(t)|^2\,dt\right)^{1/2} x^{1/2} \ \leqslant \ M,$$

so F_n is also a uniformly bounded family. To show that $F_n \to F$ uniformly, it's enough to show that any subsequence of F_n has a further subsequence converging uniformly to F. Let F_{n_k} be any subsequence. We have shown it is a uniformly bounded and equicontinuous family, so by Arzela-Ascoli it has a further subsequence converging uniformly to some function g. But note that for each x,

$$\lim_{n\to\infty} F_n(x) \ = \ \lim_{n\to\infty} \int_0^x f_n(t)\,dt \ = \ \lim_{n\to\infty} \int_0^1 f_n(t)\chi_{[0,x]}(t)\,dt \ = \ \int_0^1 f(t)\chi_{[0,x]}(t)\,dt \ = \ \int_0^x f(t)\,dt \ = \ F(x)$$

by weak convergence because $\chi_{[0,x]} \in L^2([0,1])$. Thus, since F_n converges pointwise to F, and F_{n_k} has a subsequence converging uniformly to some g, we must in fact have $g = F$. Thus every subsequence F_{n_k} has a further subsequence converging uniformly to F, so $F_n \to F$ uniformly. $\square$

Problem 2. Let $f \in L^3(\mathbb{R})$ and $\phi(x) = \sin(\pi x) \cdot \chi_{[-1,1]}(x)$. Show that

$$f_n(x) \ := \ n\int f(x-y)\phi(ny)\,dy \to 0$$

Lebesgue almost everywhere.

Solution. Let $\phi_n(x) = n\phi(nx)$. Let $g(x) = -\phi(x)\chi_{[-1,0]}$ be the negative part of ϕ and let $h(x) = \phi(x)\chi_{[0,1]}$ be the positive part. Also define g_n and h_n similarly to ϕ_n. Note that $\phi_n = h_n - g_n$ so to show that $f * \phi_n \to 0$ a.e. it's enough to show that $f * g_n, f * h_n \to (\pi/2)f$ a.e. We show it for h_n and the argument for g_n is exactly the same. First note that $\int h_n(x)\,dx = \int_0^{1/n} \sin(n\pi x)\,dx = 2/\pi$. We have

$$\begin{aligned}\left|(f*h_n)(x) - \frac{\pi}{2}f(x)\right| &= \left|\int_0^{1/n} f(x-y)n\sin(n\pi y)\,dy - \int_0^{1/n} f(x)n\sin(n\pi y)\,dy\right| \\ &\leqslant n\int_0^{1/n} |f(x-y)-f(x)|\,|\sin(n\pi y)|\,dy \\ &\leqslant n\int_0^{1/n} |f(x-y)-f(x)|\,dy,\end{aligned}$$

which goes to 0 almost everywhere by the Lebesgue differentiation theorem ($f \in L^1_{loc}$ because $f \in L^3$). $\square$

Problem 3. Let μ be a Borel probability measure on $\mathbb{R}$ and define $f(t) = \int e^{itx}\,d\mu(x)$. Suppose that

$$\lim_{t\to 0} \frac{f(0)-f(t)}{t^2} = 0.$$

Show that μ is supported at 0.

Solution. Rewrite the limit condition as

$$\lim_{t\to 0}\int \frac{1-e^{itx}}{t^2}\,d\mu(x) = 0.$$

Just looking at the real part of the above gives

$$\lim_{t\to 0}\int \frac{1-\cos(tx)}{t^2}\,d\mu(x) = 0.$$

Since the integrand is positive for all t, x, by Fatou's lemma we have

$$0 = \lim_{t\to 0}\int \frac{1-\cos(tx)}{t^2}\,d\mu(x) \geqslant \int \lim_{t\to 0}\frac{1-\cos(tx)}{t^2}\,d\mu(x) = \int \frac{1}{2}x^2\,d\mu(x),$$

and since the last term on the right is also non-negative, we have $\int x^2\,d\mu(x) = 0$. This immediately implies that μ is supported at 0 because if μ gave nonzero measure to $\mathbb{R}\backslash\{0\}$, it would have to give positive measure to some set of the form $(-\infty,-\delta]\cap[\delta,\infty)$ for some $\delta > 0$, and then we would have $\int x^2\,d\mu(x) > \delta^2\mu((-\infty,-\delta]\cap[\delta,\infty)) > 0$, a contradiction. $\square$

Problem 4. Let $f_n : [0,1] \to [0,\infty)$ be Borel functions with

$$\sup_n \int_0^1 f_n(x)\log(2+f_n(x))\,dx \leqslant M < \infty.$$

Suppose $f_n \to f$ Lebesgue almost everywhere. Show that $f \in L^1$ and $f_n \to f$ in L^1.

Solution. By Fatou's lemma (since everything is positive) we have

$$M \geqslant \liminf_{n\to\infty}\int_0^1 f_n(x)\log(2+f_n(x))\,dx \geqslant \int_0^1 f(x)\log(2+f(x))\,dx \geqslant \log(2)\int_0^1 f(x)\,dx,$$

so $f \in L^1$. Now to show $f_n \to f$ in L^1, we first want to establish the following claim: for all $\epsilon > 0$ there is $\delta > 0$ such that for any n and any $E \subseteq [0,1]$, $m(E) < \delta$ implies $\int_E f(x)\,dx < \epsilon$. Suppose this were not true, then there would be a sequence of sets E_k and functions f_{n_k} with $m(E_k) < 1/k$ and $\int_{E_k} f_{n_k} \geqslant \epsilon$. Then by Jensen's inequality, since $t \mapsto t\log(2+t)$ is convex, we would have

$$\left(\frac{1}{m(E_k)}\int_{E_k} f_{n_k}\right)\log\left(2+\frac{1}{m(E_k)}\int_{E_k} f_{n_k}\right) \leqslant \frac{1}{m(E_k)}\int_{E_k} f_{n_k}\log(2+f_{n_k}) \leqslant \frac{1}{m(E_k)}M.$$

Cancelling terms on both sides and using the fact that $t \mapsto t\log(2+t)$ is also increasing, we get

$$M \;\geqslant\; \epsilon \log(2 + k\epsilon),$$

which is a contradiction for k large enough. Thus the claim is established. Now to finish the problem, fix $\epsilon > 0$. By the previous claim we can pick $\delta > 0$ so that $m(E) < \delta$ implies $\int_E f_n < \epsilon$ for all n and $\int_E f < \epsilon$. By Egorov's theorem, we can find a set $E \subseteq [0,1]$ with $f_n \to f$ uniformly on E^c and $m(E) < \delta$. Then

$$\int |f_n - f| \;\leqslant\; \int_{E^c} |f_n - f| + \int_E |f_n| + \int_E |f| \;\leqslant\; \int_{E^c} |f_n - f| + 2\epsilon.$$

First take $n \to \infty$, then take $\epsilon \to 0$, and we get the desired result. □

Problem 5. (a) Show that $\ell^\infty(\mathbb{Z})$ contains continuum many functions $x_\alpha : \mathbb{Z} \to \mathbb{R}$ obeying $||x_\alpha||_{\ell^\infty} = 1$ and $||x_\alpha - x_\beta||_{\ell^\infty} \geqslant 1$ whenever $\alpha \neq \beta$.
(b) Deduce (assuming the axiom of choice) that the Banach space dual of $\ell^\infty(\mathbb{Z})$ cannot contain a countable dense subset.
(c) Deduce that $\ell^1(\mathbb{Z})$ is not reflexive.

Solution. (a) For each subset $\alpha \subseteq \mathbb{Z}$, let $x_\alpha(j) = 1$ if $j \in \alpha$ and 0 otherwise. Then each $||x_\alpha||_{\ell^\infty} = 1$ and for any two distinct subsets $\alpha \neq \beta$, there is a point at which x_α and x_β disagree, so $||x_\alpha - x_\beta||_{\ell^\infty} \geqslant 1$. It's standard that there are continuum many subsets of $\mathbb{Z}$. □

(b) Part (a) shows that the dual of ℓ^∞ is not separable. So it just follows from the general fact that if X is a Banach space and X^* is separable, then X is also separable (see Fall 2014 #6). □

(c) Recall that the dual of ℓ^1 is ℓ^∞. If ℓ^1 is separable, then $(\ell^1)^{**} = (\ell^\infty)^* = \ell^1$, which is separable, so by part (b) ℓ^∞ is also separable, a contradiction. □

Problem 6. Suppose μ and ν are finite positive (regular) Borel measures on $\mathbb{R}^n$. Prove the existence and uniqueness of the Lebesgue decomposition: there are a unique pair of positive Borel measures μ_a and μ_s such that

$$\mu \;=\; \mu_a + \mu_s, \quad \mu_a \ll \nu, \quad \mu_s \perp \nu.$$

Solution. First we show uniqueness. Suppose that $\mu = \mu_a + \mu_s = \mu_a' + \mu_s'$ are two decompositions. It's enough to show that $\mu_s = \mu_s'$. Write $\mathbb{R}^n = X \cup Y = X' \cup Y'$ where $\nu(Y) = \nu(Y') = 0$ and $\mu_s(X) = \mu_s'(X')0$. By the absolute continuity of μ_a and μ_a', we see that $\mu_s(A) = \mu_s'(A)$ for any A satisfying $\nu(A) = 0$. For a general set E, write

$$E \;=\; (E \cap X \cap X') \cup (E \cap Y \cap X') \cup (E \cap X \cap Y') \cup (E \cap Y \cap Y') \;=:\; (E \cap X \cap X') \cup \widetilde{E}.$$

Note that since $\nu(\widetilde{E}) = 0$ and $E \cap X \cap X'$ is contained in both X and X' we have

$$\mu_s(E) \;=\; \mu_s(E \cap X \cap X') + \mu_s(\widetilde{E}) \;=\; \mu_s'(E \cap X \cap X') + \mu_s'(\widetilde{E}) \;=\; \mu_s'(E).$$

Thus the decomposition is unique. Now we show existence. Let $\lambda = \mu + \nu$ and note that since all of the measures involved are positive, ν is clearly absolutely continuous with respect to λ. Let $f = \frac{d\nu}{d\lambda}$ be the Radon-Nikodym derivative, and note that $f \geqslant 0$ because the measures are positive. Define $X = \{x : f(x) \neq 0\}$ and $Y = \{x : f(x) = 0\}$. We define $\mu_s(E) := \mu(E \cap Y)$ and $\mu_a(E) := \mu(E \cap X)$. It's clear that $\mu_s + \mu_a = \mu$. We need to show that μ_s is singular to ν and μ_a is absolutely continuous with respect to ν. For the singular part, note that X,Y are disjoint, $\mathbb{R}^n = X \cup Y$, $\mu_s(X) = 0$ by definition, and

$$\nu(Y) \;=\; \int_Y f \, d\lambda \;=\; 0$$

by definition of X. This shows $\mu_s \perp \nu$. For absolute continuity, suppose $\nu(E) = 0$. Then we have

$$0 \;=\; \nu(E) \;=\; \int_E f\, d\lambda \;=\; \int_E f\, d\mu + \int_E f\, d\nu \;=\; \int_E f\, d\mu \;=\; \int_{E\cap X} f\, d\mu \;=\; \int_{E\cap X} f\, d\mu_a$$

because μ_s vanishes on X. But since f is strictly positive on $E \cap X$, the fact that $\int_{E\cap X} f\, d\mu_a = 0$ implies that $\mu_a(E \cap X) = 0$, which is the same as saying $\mu_a(E) = 0$ by definition. Thus $\mu_a \ll \nu$. $\square$

Problem 7. Prove Goursat's theorem: if $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable, then for every triangle $T \subseteq \mathbb{C}$

$$\oint_{\partial T} f(z)\, dz \;=\; 0.$$

Solution.

Problem 10. Evaluate

$$\sup \left\{ \operatorname{Re} f'(i/2) : f : \mathbb{H} \to \mathbb{D} \text{ is holomorphic} \right\}.$$

Solution. We can freely post-compose f with a rotation, so it's equivalent to find $|f'(i/2)|$ instead of the real part. Let f be any holmorphic function $\mathbb{H} \to \mathbb{D}$. Let $\psi : \mathbb{D} \to \mathbb{D}$ be an automorphism sending $f(i/2)$ to 0. Concretely, $\psi(z) = \frac{z - f(i/2)}{1 - \overline{f(i/2)}z}$. An easy calculation shows that

$$\psi'(f(i/2)) \;=\; \frac{1}{1 - |f(i/2)|^2}.$$

Let $\phi : \mathbb{D} \to \mathbb{H}$ be a conformal map sending 0 to $i/2$. Concretely we can take $\phi(z) = \frac{1}{2} \cdot \frac{-i(z+1)}{z-1}$. Another easy calculation shows that $\phi'(0) = i$. Now $\psi \circ f \circ \phi$ is a holomorphic function $\mathbb{D}$ to $\mathbb{D}$ sending 0 to 0, so by the Schwartz lemma we have

$$1 \;\geqslant\; \left|(\psi \circ f \circ \phi)'(0)\right| \;=\; \left|\psi'(f(\phi(0)))f'(\phi(0))\phi'(0)\right| \;=\; \frac{1}{1 - |f(i/2)|^2}|f'(i/2)| \;\geqslant\; |f'(i/2)|.$$

Thus the supremum in question is at most 1. Finally note that taking $f(z) = \phi^{-1}(z) = \frac{2z-i}{2z+i}$, a calculation shows that $f'(i/2) = -i$. So 1 is achieved and therefore is the desired supremum. $\square$

6 Fall 2011

Problem 1. Prove Egorov's theorem, that is:
Consider a sequence of measurable functions $f_n : [0,1] \to \mathbb{R}$ that converges Lebesgue almost everywhere to a measurable function $f : [0,1] \to \mathbb{R}$. Then for any $\epsilon > 0$ there exists a measurable set $E \subseteq [0,1]$ with measure $\lambda(E) < \epsilon$ such that f_n converges uniformly on $[0,1]\backslash E$.

Solution. Let Z be the measure zero set of x for which $f_n(x) \nrightarrow f(x)$ and set $I = [0,1]\backslash Z$. Define

$$E_n(k) \ := \ \{x \in I : |f_j(x) - f(x)| < 1/k \text{ for all } j \geqslant n\}.$$

Fix $\epsilon > 0$. First we show a lemma: For each k there is an N_k such that $\lambda(E_{N_k}(k)) > 1 - \epsilon 2^{-k}$. To see this, fix a k and note that by definition of pointwise convergence, we have $\bigcup_{n=1}^{\infty} E_n(k) = I$. So by continuity of measure from below we can pick N_k large enough so that $\lambda(E_{N_k}(k)) > \lambda(I) - \epsilon 2^{-k} = 1 - \epsilon 2^{-k}$. This proves the lemma.

Now we upgrade to the full result. Define $E := \bigcup_{k=1}^{\infty} E_{N_k}(k)^c$. We have

$$\lambda(E) \ \leqslant \ \sum_{k=1}^{\infty} \lambda(E_{N_k}(k)^c) \ < \ \sum_{k=1}^{\infty} \epsilon 2^{-k} \ = \ \epsilon.$$

We claim that $f_n \to f$ uniformly on E^c. Fix $\alpha > 0$. Pick k big enough so that $1/k < \alpha$. Then for any $x \in E^c$, we have $x \in E_{N_k}(k)$, so $n \geqslant N_k$ implies that $|f_n(x) - f(x)| < 1/k < \alpha$ for all $x \in E^c$. Thus $f_n \to f$ uniformly on E^c. $\square$

Problem 2.

(a) Let $d\sigma$ denote surface measure on the unit sphere $S^2 \subset \mathbb{R}^3$. Note $\int d\sigma(x) = 4\pi$. For $\xi \in \mathbb{R}^3$, compute

$$\int_{S^2} e^{ix\cdot\xi}\, d\sigma(x),$$

where $\cdot$ denotes the usual inner product on $\mathbb{R}^3$.

(b) Using this, or otherwise, show that the mapping

$$f \mapsto \int_{S^2}\int_{S^2} f(x+y)\, d\sigma(x)\, d\sigma(y)$$

extends uniquely from the space of all C^∞ functions on $\mathbb{R}^3$ with compact support to a bounded linear functional on $L^2(\mathbb{R}^3)$.

Solution.

(a) It is clear that the integral in question depends only on $|\xi|$ (a simple proof could be given if necessary, using an orthogonal transformation and the change of variables formula). Therefore, given the magnitude $c = |\xi|$ of ξ, we are free to choose ξ so that the integral is as easy as possible to evaluate. We choose $\xi = (0,0,c)$. Then

$$\int_{S^2} e^{ix\cdot\xi}\, d\sigma(x) = \int_{S^2} \cos(cx_3)\, d\sigma(x) + i\int_{S^2} \sin(cx_3)\, d\sigma(x) = \int_{S^2} \cos(cx_3)\, d\sigma(x),$$

since sin is odd and S^2 is symmetric about the origin. Using spherical coordinates, the last integral equals

$$\begin{aligned}\int_{S^2} \cos(cx_3)\, d\sigma(x) &= \int_0^{2\pi}\int_0^{\pi} \cos(c\cos\phi)\cdot\sin\phi\, d\phi\, d\theta \\ &= -\frac{2\pi}{c}\sin(c\cos\phi)\Big|_0^{\pi} \\ &= \frac{4\pi\sin c}{c}. \\ &= \frac{4\pi\sin|\xi|}{|\xi|}.\end{aligned}$$

(b) For $f \in C_c^\infty(\mathbb{R}^3)$, define

$$L(f) = \int_{S^2}\int_{S^2} f(x+y)\,d\sigma(x)\,d\sigma(y).$$

Since $C_c^\infty(\mathbb{R}^3)$ is dense in $L^2(\mathbb{R}^3)$, to show that L extends uniquely to a bounded linear functional on $L^2(\mathbb{R}^3)$ it will be enough to prove a bound of the form $|L(f)| \leqslant C||f||_2$ for all $f \in C_c^\infty(\mathbb{R}^3)$ (where C is independent of f). Since f is smooth with compact support, it lies in the Schwartz space, and therefore Fourier inversion applies and gives

$$f(x) = \int_{\mathbb{R}^3} e^{2\pi i\xi\cdot x}\hat{f}(\xi)\,d\xi = \int_0^\infty r^2 \int_{S^2} e^{2\pi i r x\cdot\xi}\hat{f}(r\xi)\,d\sigma(\xi)\,dr$$

for all $x \in \mathbb{R}^3$. (Note that since $\hat{f}$ is in the Schwartz space as well, $||\hat{f}||_{L^\infty(rS^2)}$ decays faster than any power of r, so the integral on the right is convergent.) Therefore, by Fubini's theorem and the calculation in (a),

$$\begin{aligned}
L(f) &= \int_{S^2}\int_{S^2} f(x+y)\,d\sigma(x)\,d\sigma(y)\\
&= \int_{S^2}\int_{S^2}\int_0^\infty r^2\int_{S^2} e^{2\pi i r(x+y)\cdot\xi}\hat{f}(r\xi)\,d\sigma(\xi)\,dr\,d\sigma(x)\,d\sigma(y)\\
&= \int_0^\infty r^2\int_{S^2}\hat{f}(r\xi)\int_{S^2} e^{2\pi i r x\cdot\xi}\,d\sigma(x)\int_{S^2} e^{2\pi i r y\cdot\xi}\,d\sigma(y)\,d\sigma(\xi)\,dr\\
&= \int_0^\infty r^2\int_{S^2}\hat{f}(r\xi)\left(\frac{\sin 2\pi r}{r}\right)^2 d\sigma(\xi)\,dr\\
&= \int_{\mathbb{R}^3}\hat{f}(\xi)\left(\frac{\sin 2\pi|\xi|}{|\xi|}\right)^2 d\xi.
\end{aligned}$$

Now, by the Plancherel theorem, $\hat{f} \in L^2(\mathbb{R}^3)$ and $||\hat{f}||_2 = ||f||_2$. Moreover, $h(\xi) = \left(\frac{\sin 2\pi|\xi|}{|\xi|}\right)^2$ is in $L^2(\mathbb{R}^3)$ as well, since $h(\xi)^2$ is bounded near zero and decays like $|\xi|^{-4}$ near infinity. Therefore, Cauchy-Schwarz implies

$$|L(f)| \leqslant ||\hat{f}||_2\,||h||_2 = C||f||_2,$$

as required. □

Problem 3. Let $1 < p, q < \infty$ with $1/p + 1/q = 1$. Let $f \in L^p(\mathbb{R}^3)$ and $g \in L^q(\mathbb{R}^3)$. Show (a) that $f * g$ is continuous on $\mathbb{R}^3$ and (b) that $(f * g)(x) \to 0$ as $|x| \to \infty$.

Solution. (a) Fix $x \in \mathbb{R}^3$. We estimate

$$\begin{aligned}
|(f*g)(x) - (f*g)(x+h)| &= \left|\int_{\mathbb{R}^3}(f(x-y)g(y) - f(x+h-y)g(y))\,dy\right|\\
&\leqslant \int |g(y)||f(x+h-y) - f(x-y)|\,dy\\
&\leqslant ||g||_{L^q}\left(\int |f(x+h-y) - f(x-y)|^p\,dy\right)^{1/p}\\
&= ||g||_{L^q}\left(\int |f(y+h) - f(y)|^p\,dy\right)^{1/p}.
\end{aligned}$$

So it suffices to show that $\left(\int |f(y+h) - f(y)|^p\,dy\right)^{1/p} \to 0$ as $|h| \to 0$. This is just the L^p continuity of the translation operator, a proof of which is reproduced below.

For $f \in L^p$ define $\tau_h f(y) = f(y+h)$. We want to show that $||\tau_h f - f||_{L^p} \to 0$ as $|h| \to 0$. First suppose that $\phi \in C_c(\mathbb{R}^3)$. Let $S = \{x \in \mathbb{R}^3 : \mathrm{dist}(x, \mathrm{supp}(\phi)) \leqslant 1\}$ and let $M = \lambda_3(S) < \infty$. By uniform continuity of ϕ, let $|h| < 1$ be small enough so that $|\tau_h\phi(x) - \phi(x)| < \epsilon$ for all $x \in \mathbb{R}^3$. Then

$$||\tau_h\phi - \phi||_{L^p}^p \leqslant \epsilon^p M,$$

so the result is true for $C_c(\mathbb{R}^3)$ functions. For general $f \in L^p(\mathbb{R}^3)$, a standard density argument works: fix $\epsilon > 0$ and pick $\phi \in C_c(\mathbb{R}^3)$ with $||f - \phi||_{L^p} < \epsilon$. Then

$$||\tau_h f - f||_{L^p} \leqslant ||\tau_h f - \tau_h\phi||_{L^p} + ||\tau_h\phi - \phi||_{L^p} + ||\phi - f||_{L^p} < 2\epsilon + ||\tau_h\phi - \phi||_{L^p}.$$

Take $|h| \to 0$ and then $\epsilon \to 0$ and the result follows. $\square$

(b) Note that if f, g have compact support then $f * g$ also does. Pick sequences f_n, g_k with $f_n \to f$ in L^p, $g_k \to g$ in L^q, $||f_n||_{L^p} \leqslant ||f||_{L^p}$, $||g_k||_{L^p} \leqslant ||g||_{L^p}$, and each f_n, g_k has compact support (e.g. just cut off f and g at bigger and bigger balls). Fix $\epsilon > 0$ and pick n, k big enough so that $||f_n - f||_{L^p}, ||g_k - g||_{L^p} < \epsilon$. Then for any $x \in \mathbb{R}^3$ we have

$$\begin{aligned} |(f*g)(x)| &\leqslant |(f_n * g_k)(x)| + |((f - f_n) * g_k)(x)| + |(f * (g - g_k))(x)| \\ &\leqslant |(f_n * g_k)(x)| + ||(f - f_n) * g_k||_{L^\infty} + ||f * (g - g_k)||_{L^\infty} \\ &\leqslant |(f_n * g_k)(x)| + \epsilon\, ||g||_{L^q} + \epsilon\, ||f||_{L^p}. \end{aligned}$$

Take $|x| \to \infty$ and conclude $\lim_{|x|\to\infty}(f*g)(x) \leqslant \epsilon(||f||_{L^p} + ||g||_{L^q})$, then take $\epsilon \to 0$ to get the desired result. $\square$

Problem 4. Let $f \in C^\infty([0,\infty) \times [0,1])$ such that

$$\int_0^\infty \int_0^1 |\partial_t f(t,x)|^2 (1+t^2)\, dx\, dt < \infty.$$

Prove that there exists a function $g \in L^2([0,1])$ such that $f(t,\cdot)$ converges to $g(\cdot)$ in $L^2([0,1])$ as $t \to \infty$.

Solution. (There may be ways to make this proof more efficient, but it seems correct as far as I can tell.) For each t, $f(\cdot, t)$ is in $L^2([0,1])$, so by Parseval's theorem there exist complex numbers $a_n(t)$ such that

$$f(x,t) = \sum_{n\in\mathbb{Z}} a_n(t) e^{2\pi i n x}$$

in $L^2([0,1])$, where $\sum_n |a_n(t)|^2 = ||f(\cdot,t)||_2 < \infty$. By Parseval again it is enough to prove the existence of a sequence $\{b_n\}_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$ such that

$$\sum_{n\in\mathbb{Z}} |a_n(t) - b_n|^2 \to 0$$

as $t \to \infty$; the function $g(x) \sim \sum_n b_n e^{2\pi i n x}$ will then be the desired limit in $L^2([0,1])$. By completeness of $l^2(\mathbb{Z})$, this is the same as showing that $\{a_n(t)\}$ is Cauchy in $l^2(\mathbb{Z})$ as $t \to \infty$. In other words, given $\epsilon > 0$, we want to be able to find $T > 0$ so that $s, t > T$ implies

$$\sum_{n\in\mathbb{Z}} |a_n(t) - a_n(s)|^2 < \epsilon.$$

Assume for the moment that the coefficients $a_n(t)$ are continuously differentiable with respect to t and that

$$\partial_t f(x,t) = \sum_{n\in\mathbb{Z}} a_n'(t) e^{2\pi i n x}$$

in $L^2([0,1])$ for each t. Then by assumption, we have

$$\begin{aligned} \int_0^\infty \int_0^1 |\partial_t f(t,x)|^2 (1+t^2)\, dx\, dt &= \int_0^\infty \left(\sum_{n\in\mathbb{Z}} |a_n'(t)|^2 \right) (1+t^2)\, dt \\ &= \sum_{n\in\mathbb{Z}} \int_0^\infty |a_n'(t)|^2 (1+t^2)\, dt < \infty \qquad (1) \end{aligned}$$

(using the monotone convergence theorem to interchange the sum and integral). Since each $a_n(t)$ is C^1, we have

$$a_n(s) - a_n(t) = \int_t^s a_n'(\tau)\, d\tau.$$

Consequently, by Cauchy-Schwarz

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |a_n(s) - a_n(t)|^2 &= \sum_{n\in\mathbb{Z}} \left| \int_t^s a_n'(\tau)\, d\tau \right|^2 \\ &\leqslant \sum_{n\in\mathbb{Z}} \int_t^\infty \tau^2 |a_n'(\tau)|^2\, d\tau \int_t^\infty \frac{d\tau}{\tau^2} \quad \text{(assuming } s > t\text{)} \\ &\lesssim \sum_{n\in\mathbb{Z}} \int_t^\infty |a_n'(\tau)|^2 (1+\tau^2)\, d\tau.\end{aligned}$$

But by (1) above, this sum goes to 0 as $t \to \infty$. Hence, $\{a_n(t)\}_n$ is Cauchy in $l^2(\mathbb{Z})$ as $t \to \infty$, and so $f(\cdot, t) \to g(\cdot)$ in $L^2([0,1])$ as $t \to \infty$.

Now we just have to justify the continuous differentiability of the coefficients $a_n(t)$ and the fact that $\partial_t f(x,t)$ equals $\sum_n a_n'(t) e^{2\pi i n x}$ in $L^2([0,1])$. For any t, let $h > 0$; then by smoothness of f on $[0,\infty) \times [0,1]$,

$$\frac{f(x,t+h) - f(x,t)}{h} = \sum_{n\in\mathbb{Z}} \frac{a_n(t+h) - a_n(t)}{h} e^{2\pi i n x} \to \partial_t f(x,t)$$

as $h \to 0$, uniformly on $[0,1]$, and hence also in $L^2([0,1])$. But $\partial_t f(x,t)$ is also in $L^2([0,1])$, and hence has an L^2-Fourier series

$$\partial_t f(x,t) = \sum_{n\in\mathbb{Z}} \alpha_n(t) e^{2\pi i n x}.$$

Thus, by Parseval's theorem,

$$\sum_{n\in\mathbb{Z}} \left| \frac{a_n(t+h) - a_n(t)}{h} - \alpha_n(t) \right|^2 \to 0$$

as $h \to 0$, which implies $\frac{a_n(t+h)-a_n(t)}{h} \to \alpha_n(t)$ for each n. Thus, $a_n(t)$ is differentiable with derivative $a_n'(t) = \alpha(t)$, and

$$\partial_t f(x,t) = \sum_{n\in\mathbb{Z}} a_n'(t) e^{2\pi i n x}$$

in $L^2([0,1])$, as desired. The same argument applied to $\sum_n a_n'(t) e^{2\pi i n x}$ shows that the $a_n'(t)$ are themselves differentiable, and hence continuous; so the $a_n(t)$ are continuously differentiable, as required. $\square$

Problem 5. For $f \in L^1(\mathbb{R})$, recall the Hardy-Littlewood maximal function

$$Mf(x) \;:=\; \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |f(y)|\, dy.$$

Prove there is a constant A such that for any $\alpha > 0$,

$$\lambda\{x \in \mathbb{R} : Mf(x) > \alpha\} \;\leqslant\; \frac{A}{\alpha} \,||f||_{L^1}\,.$$

If you use a covering lemma, you should prove it.

Solution. Fix $\alpha > 0$ and let $E = \{x \in \mathbb{R} : Mf(x) > \alpha\}$. For each $x \in E$, by definition of Mf there is a radius r_x such that

$$\int_{x-r_x}^{x+r_x} |f| \;>\; 2\alpha r_x.$$

Note the above implies we must have $r_x < ||f||_{L^1}/(2\alpha)$ for each $x \in E$. Set $I_x = (x - r_x, x + r_x)$. Since the radii are uniformly bounded, we may apply the Vitali covering lemma to $\{I_x\}_{x \in E}$ to obtain a countable disjoint subcollection $I_j = (x_j - r_j, x_j + r_j)$ with $E \subseteq \bigcup_{j=1}^{\infty} 5I_j$. Thus we have

$$\lambda(E) \leqslant \sum_{j=1}^{\infty} \lambda(5I_j) = 5 \sum_{j=1}^{\infty} 2r_j \leqslant \frac{5}{\lambda} \sum_{j=1}^{\infty} \int_{x_j - r_j}^{x_j + r_j} |f| \leqslant \frac{5}{\lambda} ||f||_{L^1}$$

because the intervals I_j are pairwise disjoint. All that remains is to prove the Vitali covering lemma.

Let $\{I_\alpha\}$ be a collection of open balls with uniformly bounded radius. Let $R = \sup_\alpha \operatorname{rad}(I_\alpha)$. Let $\mathcal{F}_1$ be the collection of all balls I_α with radii in $(R/2, R]$. Let $\mathcal{B}_1$ be a maximal pairwise disjoint subcollection of $\mathcal{F}_1$ (a standard Zorn's lemma argument shows that this exists). Now let $\mathcal{F}_2$ be the subcollection of all balls I_α which are disjoint from every element of $\mathcal{B}_1$ and have radii in $(R/4, R/2]$, and let $\mathcal{B}_2$ be a maximal pairwise disjoint subcollection of $\mathcal{F}_2$ (same deal with Zorn's lemma). Inductively, we may construct $\mathcal{F}_n$ to be the collection of all balls I_α which do not intersect any ball in $\mathcal{B}_1 \cup \ldots \cup \mathcal{B}_{n-1}$ and have radii in $(R/2^n, R/2^{n-1}]$, and let $\mathcal{B}_n$ be a maximal disjoint subcollection of $\mathcal{F}_n$. Let $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$. It's clear that $\mathcal{B}$ is a pairwise disjoint (and therefore countable) subcollection of the I_α. Consider some $I_\alpha \notin \mathcal{B}$. We have $\operatorname{rad}(I\alpha) \in (R/2^n, R/2^{n-1}]$ for some n. By the maximality of $\mathcal{B}_n$, it must be the case that I_α intersects some $I_\beta \in \mathcal{B}_1 \cup \ldots \mathcal{B}_n$. So $\operatorname{rad}(I_\beta) > R/2^n \geqslant (1/2)\operatorname{rad}(I_\alpha)$. Thus $\mathcal{B}$ has the property that any $I_\alpha \notin \mathcal{B}$ intersects some $I_\beta \in \mathcal{B}$ with $\operatorname{rad}(I_\beta) > R/2^n \geqslant (1/2)\operatorname{rad}(I_\alpha)$. Thus a simple triangle inequality shows that $I_\alpha \subseteq 5I_\beta$, so $\bigcup_\alpha I_\alpha \subseteq \bigcup_{I \in \mathcal{B}} 5I$. $\square$

Problem 6. Let (X, d) be a compact metric space. Let μ_n be a sequence of positive Borel measures on X that converge in the weak-$*$ topology to a finite positive Borel measure μ, that is

$$\int_X f \, d\mu_n \to \int_X f \, d\mu \quad \text{for all } f \in C(X).$$

Show that

$$\mu(K) \geqslant \limsup_{n \to \infty} \mu_n(K) \quad \text{for all compact sets } K \subseteq X.$$

Solution. Fix K compact. First we show that the characteristic function χ_K is upper semicontinuous. We need to show

$$\chi_K(x_0) \geqslant \limsup_{x \to x_0} \chi_K(x)$$

for any $x_0 \in X$. If $x_0 \in K$, then the inequality obviously holds because $\chi_K(x_0)$ is equal to the maximum value χ_K can take. If $x_0 \notin K$, then since K^c is open there is a neighborhood around x_0 on which $\chi_K = 0$, so $\chi_K(x_0) = 0 = \lim_{x \to x_0} \chi_K(x)$. Thus χ_K is upper semicontinuous.

Now we prove the inequality

$$\int f \, d\mu \geqslant \limsup_{n \to \infty} f \, d\mu_n$$

for all upper semicontinuous $f : X \to \mathbb{R}$. This finishes the problem by taking $f = \chi_K$. It's equivalent to show

$$\int f \, d\mu \leqslant \liminf_{n \to \infty} f \, d\mu_n$$

whenever f is lower semicontinuous (by just taking the negative). Fix such an f. Since X is compact, f achieves a minimum on X (this is a property of lower semicontinuous functions). By an equivalent definition of lower semicontinuous, we have a sequence ϕ_k of continuous functions with $\phi_k \leqslant \phi_{k+1}$ and $\phi_k \to f$ pointwise. By replacing ϕ_k by $\max(\phi_k, \min(f))$ if necessary, we may assume that all of the ϕ_k are uniformly bounded from below. We have

$$\int_X \phi_k \, d\mu_n \leqslant \int_X f \, d\mu_n$$

for any k, n. Taking the liminf as $n \to \infty$, since ϕ_k is continuous we get

$$\int_X \phi_k \, d\mu \;\leqslant\; \liminf_{n\to\infty} \int_X f \, d\mu_n$$

for every k. Finally, since the right side is independent of k, apply the Monotone Convergence theorem to get the desired conclusion. $\square$

Problem 7. Compute $\int_0^\infty \frac{\cos(x)}{(1+x^2)^2}\, dx$.

Solution. Let $f(z) = \frac{e^{iz}}{(1+z^2)^2}$. Integrate f around a semicircle of radius R in the upper half plane. It's easy to show the contribution from the curved part of the contour vanishes as $R \to \infty$. The real part of the integral over the straight part is twice the desired integral because the original function is even. f has a double pole at $z = i$. Take the residue

$$\operatorname{Res}(f, i) \;=\; \lim_{z\to i} \frac{d}{dz}\left[(z-i)^2 f(z)\right] \;=\; \frac{-i}{2e}.$$

Set the two things equal to each other using the residue theorem and solve. The answer is $\pi/2e$. $\square$

Problem 8. Determine the number of solutions to

$$z - 2 - e^{-z} = 0$$

with z in the right half-plane $H = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Solution. Any such z satisfies $z = 2 + e^{-z}$, and therefore $|z| = |2 + e^{-z}| \leqslant 2 + |e^{-z}| < 3$, since $\operatorname{Re} z > 0$. Hence, we can restrict z to the half-disc $U = H \cap \{|z| < 3\}$. Consider the functions $f(z) = z - 2$ and $g(z) = -e^{-z}$ on ∂U. It is easy to see that $|g| < |f|$ on ∂U, since $|g| = e^{-x} < 1$ everywhere in H, whereas $|z - 2| > 1$ for all $x \in \partial U$ except at $z = 3$, at which point $|g(z)| = e^{-3} < 1$. Therefore, by Rouche's theorem, f and $f + g = z - 2 - e^{-z}$ have the same number of zeros in U; since f clearly has one zero in U, it follows that

$$z - 2 - e^{-z} = 0$$

has exactly one solution in H. $\square$

Problem 9. Suppose that f is a holomorphic function in the punctured open unit disc $\mathbb{D}^* := \mathbb{D}\backslash\{0\}$ such that

$$\int_{\mathbb{D}^*} |f(z)|^2 \, dA(z) \;<\; \infty$$

where integration is with respect to two dimensional Lebesgue measure. Show that f has a holomorphic extension to the unit disc $\mathbb{D}$.

Solution. Let $g(z) = zf(z)$. It's clear that g is also holomorphic on $\mathbb{D}^*$. By the mean value property, for $z \in \mathbb{D}^*$ fixed we have

$$\begin{aligned}
|g(z)| \;&=\; \frac{1}{\pi(1/2|z|)^2}\left|\int_{B(z,1/2|z|)} wf(w)\, dA(w)\right| \;\lesssim\; |z|^{-2}\left(\int_{B(z,1/2|z|)} |w|^2\, dA(w)\right)^{1/2}\left(\int_{B(z,1/2|z|)} |f(w)|^2\, dA(w)\right)^{1/2} \\
&\lesssim\; |z|^{-2}\left(\int_{B(0,3/2|z|)} |w|^2\, dA(w)\right)^{1/2} \;\lesssim\; |z|^{-2}\left(\int_0^{3/2|z|}\int_0^{2\pi} r^2 r\, d\theta\, dr\right)^{1/2} \lesssim |z|^{-2}\left(\int_0^{3/2|z|} r^3\, dr\right)^{1/2} \\
&\lesssim |z|^{-2}\left((\frac{3}{2}|z|)^4\right)^{1/2} \;\lesssim\; 1.
\end{aligned}$$

Thus g is bounded and holomorphic in the punctured disc $\mathbb{D}^*$, which means that the singularity at 0 must be removable. So $zf(z)$ has a removable singularity at 0, which implies that the singularity of f at 0 is either removable or a simple pole. But if f has a simple pole at zero, then there is a constant $C > 0$ and a neighborhood of 0 on which $|f(z)| \geqslant C|z|^{-1}$, which contradicts the fact that $\int_{\mathbb{D}*} |f(z)|^2 \, dA(z) < \infty$. So f has a removable singularity at 0 and therefore can be extended to a holomorphic function on $\mathbb{D}$. $\square$

Problem 10. Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and $f : \Omega \to \Omega$ be a holomorphic mapping. Suppose there are points $z_1 \neq z_2$ with $f(z_1) = z_1$ and $f(z_2) = z_2$. Show that f is the identity on Ω.

Solution. We need to assume f is conformal, otherwise it isn't true (as a counterexample take $\Omega = B(0,2)$ and $f(z) = z^2$, then 0 and 1 are both fixed points). By the Riemann mapping theorem, let $T : \Omega \to \mathbb{D}$ be a conformal map. Then $\phi = TfT^{-1} : \mathbb{D} \to \mathbb{D}$ is a conformal map with $\phi(\alpha_1) = \alpha_1$, $\phi(\alpha_2) = \alpha_2$ and $\alpha_1 \neq \alpha_2$ (take $\alpha_j = T(z_j)$). Let ψ be an automorphism of $\mathbb{D}$ that sends α_1 to 0. Then we have $\psi(\phi(\psi^{-1}(0))) = 0$, so the Schwartz lemma applies to $\psi\phi\psi^{-1}$. But note also that $\psi(\phi(\psi^{-1}(\psi(\alpha_2)))) = \psi(\alpha_2)$. So equality holds in the Schwartz lemma (actual equality, not just equality in absolute value), so $\psi\phi\psi^{-1}$ is the identity, which implies ϕ is the identity, which implies f is the identity. $\square$

Problem 11. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function with $f(z) \neq 0$ for all $z \in \mathbb{C}$. Define $U = \{z \in \mathbb{C} : |f(z)| < 1\}$. Show that all connected components of U are unbounded.

Solution. Since f is nonvanishing, $1/f$ is also entire. First note that U is clearly an open set because it's the preimage of $(0,1)$ under the continuous function $|f(z)|$. Suppose that Ω were a bounded connected component of U. Note that Ω is also open: let $z \in \Omega$ and let B be an open ball centered at z contained in U. If B were not contained in Ω, then there would be $w \in B$ where w belongs to a different connected component of U. But z and w can be joined by a path lying in U, so they must be in the same connected component. Thus Ω is a bounded connected open set, i.e. a region on which the maximum principle can be applied. First note that by continuity and by the fact that $\partial\Omega$ is disjoint from Ω, we must have $|f| = 1$ on $\partial\Omega$. Thus $|1/f| = 1$ on $\partial\Omega$ also. So by the maximum principle, we have $|1/f| \leqslant 1$ throughout Ω, implying $|f| \geqslant 1$ throughout Ω. But $|f| < 1$ in Ω by definition, which is a contradiction. $\square$

Problem 12. A holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is said to be of exponential type if there are constants $c_1, c_2 > 0$ such that
$$|f(z)| \leqslant c_1 e^{c_2|z|} \quad \text{for all } z \in \mathbb{C}.$$
Show that f is of exponential type if and only if f' is of exponential type.

Solution. First suppose f is of exponential type. For any z, the Cauchy estimates give
$$|f'(z)| \leqslant \frac{1}{R} \sup_{|w-z|=R} |f(w)| \leqslant \frac{1}{R} c_1 e^{c_2(|z|+R)}$$
for any $R > 0$. Pick $R = 1$, we get
$$|f'(z)| \leqslant c_1 e^{c_2(|z|+1)} = c_1 e^{c_2} e^{c_2|z|},$$
so f is of exponential type.

Now suppose f' is of exponential type. For any z we can write
$$f(z) = f(0) + \int_\gamma f(w) \, dw$$
where γ is a straight line from 0 to z. So we have
$$|f(z)| \leqslant |f(0)| + |z| \sup_{w \in \gamma} |f'(w)| \leqslant |f(0)| + |z| c_1 e^{c_2|z|} \leqslant (|f(0)| + c_1) e^{(c_2+1)|z|},$$
so f is of exponential type. $\square$

7 Spring 2012

Problem 1. $f_n \in L^3([0,1])$. True or false:

(a) If $f_n \to f$ almost everywhere then a subsequence converges to f in L^3.

(b) If $f_n \to f$ in L^3 then a subsequence converges almost everywhere.

(c) If $f_n \to f$ in measure then the sequence converges to f in L^3.

(d) If $f_n \to f$ in L^3 then the sequence converges to f in measure.

Solution.

(a) False. Let $f_n = n \cdot \chi_{[0,1/n]}$. Then $f_n \to 0$ almost everywhere but $\int_0^1 |f_n|^3 = \int_0^{1/n} n^3 = n^2$, so f_n doesn't converge to 0 in L^3.

(b) True. By part (d) we know that $f_n \to f$ in measure. So for each k, we have

$$\lim_{n\to\infty} \{x : |f_n(x) - f(x)| > 1/k\} = 0.$$

For each k, pick n_k large enough so that $\lambda\{x : |f_n(x) - f(x)| > 1/k\} < 2^{-k}$. Let $E_k = \{x : |f_n(x) - f(x)| > 1/k\}$. We claim that $f_{n_k} \to f$ almost everywhere. Note that since $\sum_{k=1}^{\infty} \lambda(E_k) < \infty$, the Borel-Cantelli lemma implies that the set of x that lie in infinitely many E_k has measure zero. Fix $\epsilon > 0$ and let x be one of the almost everywhere points lying in only finitely many E_k. Then, as long as k is big enough so that $1/k < \epsilon$ and $x \notin E_k$, we have $|f_{n_k}(x) - f(x)| \leqslant 1/k < \epsilon$. This shows that $f_{n_k}(x) \to f(x)$ for a.e. x. $\square$

(c) False. The same counterexample from part (a) works again.

(d) True. Fix $\alpha > 0$. Then we have

$$\int |f_n - f|^3 \;\geqslant\; \int_{\{x:|f_n(x)-f(x)|>\alpha\}} |f_n - f|^3 \;\geqslant\; \alpha^3 \cdot \lambda\{x : |f_n(x) - f(x)| > \alpha\}.$$

The left side goes to 0 as $n \to \infty$, so the right side does as well. $\square$

Problem 2. Let X and Y be topological spaces and $X \times Y$ the Cartesian product endowed with the product topology. $\mathcal{B}(X)$ denotes the Borel sets in X and similarly, $\mathcal{B}(Y)$ and $\mathcal{B}(X \times Y)$.

(a) Suppose $f : X \to Y$ is continuous. Prove that $E \in \mathcal{B}(Y)$ implies $f^{-1}(E) \in \mathcal{B}(X)$.

(b) Suppose $A \in \mathcal{B}(X)$ and $E \in \mathcal{B}(Y)$. Show that $A \times E \in \mathcal{B}(X \times Y)$.

Solution.

(a) Let $\mathcal{F} = \{E \subseteq Y : f^{-1}(E) \in \mathcal{B}(X)\}$. We want to show that $\mathcal{B}(Y) \subseteq \mathcal{F}$. It's enough to show that $\mathcal{F}$ is a σ-algebra containing all open sets of Y. It's clear that $\mathcal{F}$ contains all open sets in Y by the definition of continuous functions. Thus $\varnothing$ and Y are in $\mathcal{F}$ because they are open. Suppose $A \in \mathcal{F}$. Then we have $f^{-1}(A^c) = f^{-1}(A)^c \in \mathcal{B}(X)$, so $\mathcal{F}$ is closed under complementation. Finally, suppose $A_n \in \mathcal{F}$. Then we have $f^{-1}\left(\bigcup A_n\right) = \bigcup f^{-1}(A_n) \in \mathcal{B}(X)$, so $\mathcal{F}$ is closed under countable unions. Thus $\mathcal{F}$ is a σ-algebra, so we're done. $\square$

(b) Fix an open set $U \subseteq X$. We first show that $U \times E \in \mathcal{B}(X \times Y)$ for any $E \in \mathcal{B}(Y)$. Let $\mathcal{F}_U = \{E \subseteq Y : U \times E \in \mathcal{B}(X \times Y)\}$. To verify that claim, we just need to show $\mathcal{F}_U$ is a σ-algebra containing all open sets of Y. It's clear that $\mathcal{F}_U$ contains all open sets because the product of open sets is open. So $\mathcal{F}_U$ contains $\varnothing$ and Y. If $E \in \mathcal{F}_U$, then $U \times E^c = (U \times Y)\backslash(U \times E) \in \mathcal{B}(X \times Y)$, so $\mathcal{F}_U$ is closed under complementation. If $E_n \in \mathcal{F}_U$, then $U \times \bigcup E_n = \bigcup(U \times E_n) \in \mathcal{B}(X \times Y)$, so $\mathcal{F}_U$ is closed under countable unions, so it's a σ-algebra. This shows that $U \times E \in \mathcal{B}(X \times Y)$ for any open $U \subseteq X$ and any Borel $E \subseteq Y$.

Now fix a Borel set $E \subseteq Y$ and let $\mathcal{F}_E = \{A \subseteq X : A \times E \in \mathcal{B}(X \times Y)\}$. We want to show $\mathcal{F}_E$ contains all Borel sets in X, so it's enough to show $\mathcal{F}_E$ is a σ-algebra containing all open sets of X. We know it contains all open sets of X by the above work. The exact same argument as above shows that it's a σ-algebra. Thus we conclude that $A \times E \in \mathcal{B}(X \times Y)$ for any $A \in \mathcal{B}(X)$, $E \in \mathcal{B}(Y)$. $\square$

Alternate solution. (b) Let π_X (resp. π_Y) be the projection maps $X \times Y \to X$ (resp. Y). They are both continuous. Then by part (a),

$$A \times E \;=\; \pi_X^{-1}(A) \cap \pi_Y^{-1}(E) \;\in\; \mathcal{B}(X \times Y). \quad \square$$

Problem 3. Given $f : [0,1] \to \mathbb{R}$ belonging to L^1 and $n \in \mathbb{N}$, define

$$f_n(x) \;=\; n \int_{k/n}^{(k+1)/n} f(y)\, dy \quad \text{for } x \in [k/n,(k+1)/n) \text{ and } 0 \leqslant k \leqslant n-1.$$

Prove $f_n \to f$ in L^1.

Solution. First suppose f is the characteristic function of an interval $f = \chi_{[a,b]}$. Then note that for n large enough, f_n is constant and equal to f on each subinterval except for possibly the two subintervals containing a and b. On these two subintervals, we still have $0 \leqslant f_n \leqslant 1$. Thus we have

$$\int_0^1 |f_n - f| \;\leqslant\; 2 \cdot \frac{1}{n} \cdot \max|f_n - f| \;\leqslant\; \frac{2}{n},$$

which shows that $f_n \to f$ in L^1. Next note that the map $f \mapsto f_n$ is linear, so we also know that $f_n \to f$ in L^1 for any f which is a linear combination of characteristic functions of intervals. This class of functions is dense in L^1. So for a general $f \in L^1$, let g_k be a sequence of functions of the above form with $g_k \to f$ in L^1. Then for any n large enough we have

$$||f_n - f||_{L^1} \;\leqslant\; ||f - g_k||_{L^1} + ||g_k - (g_k)_n||_{L^1} + ||(g_k)_n - f_n||_{L^1}.$$

We estimate

$$\begin{aligned}
||(g_k)_n - f_n||_{L^1} \;&=\; \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} |g_k(x) - f(x)|\, dx \;=\; \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} n \left| \int_{k/n}^{(k+1)/n} (g_k(y) - f(y))\, dy \right| dx \\
&\leqslant\; \sum_{k=0}^{n-1} n \int_{k/n}^{(k+1)/n} |f(y) - g_k(y)| \int_{k/n}^{(k+1)/n} dx\, dy \quad \text{by Tonelli} \\
&=\; \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} |f(y) - g_k(y)|\, dy \;=\; ||f - g_k||_{L^1}.
\end{aligned}$$

Thus we have

$$||f_n - f||_{L^1} \;\leqslant\; 2\,||f - g_k||_{L^1} + ||g_k - (g_k)_n||_{L^1}.$$

This holds for any n, so taking $n \to \infty$ we get

$$\limsup_{n \to \infty} ||f_n - f||_{L^1} \;\leqslant\; 2\,||f - g_k||_{L^1}$$

since we already verified the desired property for each g_k. Now the above holds for any k, so we can take $k \to \infty$ and conclude $\lim_{n\to\infty} ||f_n - f||_{L^1} = 0$. $\square$

Problem 4. Let $S = \{f \in L^1(\mathbb{R}^3) : \int f(x)\, dx = 0\}$.

(a) Show that S is closed in the L^1 topology.

(b) Show that $S \cap L^2(\mathbb{R}^3)$ is a dense subset of $L^2(\mathbb{R}^3)$.

Solution.

(a) Let $f_n \in S$ and $f \in L^1$ with $f_n \to f$ in L^1. Then for each n we have

$$\left|\int f\right| \;=\; \left|\int f - \int f_n\right| \;\leqslant\; \int |f - f_n| \;\to\; 0,$$

so $\int f = 0$. $\square$

(b) We know that the set of L^2 functions with compact support is dense in L^2, so it suffices to show that for any $f \in L^2$ with compact support and any $\epsilon > 0$, there is some $g \in S \cap L^2$ with $||g - f||_{L^2} < \epsilon$. Fix $f \in L^2$ with compact support and $\epsilon > 0$. Say $\mathrm{supp}(f) \subseteq B(0, M)$ and let $I = \int f(x)\,dx$. We know that $I < \infty$ because L^2 functions with compact support are also L^1 (by Cauchy-Schwarz). We may assume $I > 0$ because if $I = 0$ then we're done, and if $I < 0$ then we can do the same argument with a negative sign on everything. The idea is to let $g = f$ on the support of f, and then let g be equal to a small negative value outside the support of f so that $\int g(x)\,dx = 0$.

Let $C > M$ be a solution to $4\pi/3(C^3 - M^3) = I^2/\epsilon$. Let $g(x) = f(x)$ for $|x| \leqslant M$, $g(x) = -\epsilon/I$ for $M < |x| \leqslant C$, and $g(x) = 0$ otherwise. It's clear that $g \in L^2$. We have

$$\int g(x)\,dx \;=\; \int_{|x|\leqslant M} f(x)\,dx + \int_{M<|x|\leqslant C} -\epsilon/I \;=\; I - \epsilon/I \cdot \lambda_3(M < |x| \leqslant C) \;=\; I - \epsilon/I \cdot \frac{4}{3}\pi(C^3 - M^3) \;=\; 0,$$

so $g \in S \cap L^2$. Also we have

$$||g - f||_{L^2} \;=\; \int_{M<|x|\leqslant C} \epsilon^2/I^2 \;=\; \epsilon^2/I^2 \cdot \lambda_3(M < |x| \leqslant C) \;=\; \epsilon. \quad \square$$

Problem 5. State and prove the Riesz representation theorem for linear functionals on a Hilbert space.

Solution. Statement: let H be a Hilbert space and let f be a bounded linear functional on H. Then there exists $z \in H$ such that $f(x) = \langle x, z\rangle$ for all $x \in H$.

Proof: Let $f \in H^*$. Since f is a continuous map into a 1-dimensional space, we know that $\ker(f)$ is a closed, co-dimension 1 subspace of H. Fix a nonzero $u \in \ker(f)^\perp$. Then we have the decomposition $H = \ker(f) \oplus \mathrm{span}(u)$. Let $\alpha = \overline{f(u)}/\,||u||^2$. Then we claim that $f(x) = \langle x, \alpha u\rangle$ for all $x \in H$. Since every $x \in H$ decomposes uniquely as the sum of something in $\ker(f)$ and something in $\mathrm{span}(u)$, we just need to show that $x \mapsto f(x)$ and $x \mapsto \langle x, \alpha u\rangle$ agree on $\ker(f)$ and $\mathrm{span}(u)$. For $y \in \ker(f)$, we clearly have $f(y) = 0$ and $\langle y, \alpha u\rangle = 0$ because u was chosen to be in $\ker(f)^\perp$. For $z \in \mathrm{span}(u)$, we have $z = cu$ for some c, so we have $f(z) = f(cu) = cf(u)$ and $\langle z, \alpha u\rangle = c\overline{\alpha}\,||u||^2 = cf(u)$ by choice of α. Thus $f(x) = \langle x, \alpha u\rangle$ for all $x \in H$. $\square$

Problem 6. Suppose $f \in L^2(\mathbb{R})$ and that the Fourier transform obeys $\widehat{f}(\xi) > 0$ for almost every ξ. Show that the set of finite linear combinations of translates of f is dense in the Hilbert space $L^2(\mathbb{R})$.

Solution. Let $M = \overline{\mathrm{span}\{x \mapsto f(x+a)\}_{a\in\mathbb{R}}}$ where the closure is with respect to the L^2 norm. Suppose for contradiction that $M \neq L^2$. Then there is some nonzero $\overline{g} \in M^\perp$. In particular we have $\int_{\mathbb{R}} f(x+a)g(x)\,dx = 0$ for all $a \in \mathbb{R}$. By Plancherel, this implies that

$$\int_{\mathbb{R}} \mathcal{F}(x \mapsto f(x+a)(\xi))\mathcal{F}(g)(\xi)\,d\xi \;=\; \int_{\mathbb{R}} e^{-2\pi i a\xi}\mathcal{F}(f)(\xi)\mathcal{F}(g)(\xi)\,d\xi \;=\; \mathcal{F}(\mathcal{F}(f)\mathcal{F}(g))(a) \;=\; 0$$

for all $a \in \mathbb{R}$, where $\mathcal{F}$ denotes the Fourier-Plancherel transform $L^2 \to L^2$. This formula is valid because since $f, g \in L^2$, $\mathcal{F}(f)\mathcal{F}(g) \in L^1$, and thus the Fourier-Plancherel transform agrees with the standard L^1 Fourier transform. But since $\mathcal{F}$ is a bijection this implies that $\mathcal{F}(f)(\xi)\mathcal{F}(g)(\xi) = 0$ for almost every ξ. And since $\mathcal{F}(f)(\xi) > 0$ almost everywhere, this implies $\mathcal{F}(g) = 0$ almost everywhere, so $g = 0$ almost everywhere, which is a contradiction. $\square$

Problem 7. Let $\{u_n(z)\}$ be a sequence of real-valued harmonic functions on $\mathbb{D}$ that obey

$$u_1(z) \;\geqslant\; u_2(z) \;\geqslant\; \cdots \;\geqslant 0 \quad \text{for all} z \in \mathbb{D}.$$

Prove that $z \mapsto \inf_n u_n(z)$ is a harmonic function on $\mathbb{D}$.

Solution. Let $u(z) = \inf_n u_n(z) = \lim_{n\to\infty} u_n(z)$ (the limit exists and equals the inf because the sequence is monotonically decreasing and bounded for each z). First we show that $u_n \to u$ uniformly on compact subsets of $\mathbb{D}$. Fix a compact subset $\overline{B(0,r)} \subseteq \mathbb{D}$. For any $n > m$, $u_m - u_n$ is a positive harmonic function on $\mathbb{D}$, so we can apply Harnack's inequality on the disc $B(0,(1+r)/2)$ to get, for any $|z| \leqslant r$,

$$|u_m(z) - u_n(z)| \;\leqslant\; \frac{(1+r)/2 + |z|}{(1+r)/2 - |z|}\,|u_m(0) - u_n(0)| \;\leqslant\; \frac{(1+r)/2 + r}{(1+r)/2 - r}\,|u_m(0) - u_n(0)| \;\to\; 0$$

as $n, m \to \infty$ uniformly in $|z| \leqslant r$ because $\{u_n(0)\}$ is a convergent sequence.

Since each u_n is continuous, the local uniform convergence implies that u is continuous. Also, for any $B(z_0, r) \subseteq \mathbb{D}$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta \;=\; \frac{1}{2\pi}\int_0^{2\pi} \lim_{n\to\infty} u_n(z_0 + re^{i\theta})\,d\theta \;=\; \lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi} u_n(z_0 + re^{i\theta})\,d\theta \;=\; \lim_{n\to\infty} u_n(0) \;=\; u(0)$$

where switching the limit and the integral is justified by uniform convergence on the compact set $\partial B(z_0, r)$. Thus u is continuous and satisfies the mean value property on every disc, so it's harmonic. $\square$

Problem 8. Let $\Omega = \{x + iy : x > 0, y > 0, xy < 1\}$. Give an example of an *unbounded* harmonic function on Ω that extends continuously to $\partial\Omega$ and vanishes there.

Solution. We want to conformally map Ω to a region where it will be easier to find such a function. Motivated by the fact that $(x + iy)^2 = (x^2 - y^2) + i(2xy)$, we see that the map $z \mapsto \pi z^2$ is a conformal map from Ω to the strip $S := \{z : 0 < \mathrm{Im}(z) < 2\pi\}$. Now note that $z \mapsto \mathrm{Im}(e^z)$ is an unbounded harmonic function in S which vanishes on the boundary of S: we have $\mathrm{Im}(\exp(x + 0i)) = \mathrm{Im}(\exp(x)) = 0$ and $\mathrm{Im}(\exp(x + 2\pi i)) = \mathrm{Im}(\exp(x)) = 0$, and $\mathrm{Im}(\exp(x + i\pi/2)) = \mathrm{Im}(i\exp(x)) = \exp(x)$, which is unbounded in S. Therefore the function $u(z) = \mathrm{Im}(\exp(\pi z^2))$ is a function that works. $\square$

Problem 9. Prove Jordan's lemma: If $f(z) : \mathbb{C} \to \mathbb{C}$ is meromorphic, $R > 0$, and $k > 0$, then

$$\left|\int_\Gamma f(z)e^{ikz}\,dz\right| \;\leqslant\; \frac{100}{k}\sup_{z\in\Gamma}|f(z)|$$

where Γ is the quarter circle $z = Re^{i\theta}$ with $0 \leqslant \theta \leqslant \pi/2$.

Solution. We have

$$\left|\int_\Gamma f(z)e^{ikz}\,dz\right| \;=\; \left|\int_0^{\pi/2} f(Re^{i\theta})e^{ikRe^{i\theta}}\,iRe^{i\theta}\,d\theta\right| \;\leqslant\; R\cdot\sup_{z\in\Gamma}|f(z)|\cdot\int_0^{\pi/2}\left|e^{ikR(\cos\theta + i\sin\theta)}\right|d\theta$$

$$=\; R\cdot\sup_{z\in\Gamma}|f(z)|\cdot\int_0^{\pi/2} e^{-kR\sin\theta}\,d\theta.$$

So we just need to show that $\int_0^{\pi/2} e^{-kR\sin\theta}\,d\theta \leqslant \frac{100}{kR}$. We break the integral in two:

$$\int_0^{\pi/2} e^{-kR\sin\theta}\,d\theta \;=\; \int_0^{\pi/4} e^{-kR\sin\theta}\,d\theta + \int_{\pi/4}^{\pi/2} e^{-kR\sin\theta}\,d\theta \;=:\; A + B.$$

Now we estimate

$$A = \int_0^{\pi/4} e^{-kR\sin\theta}\,d\theta = \int_0^{\sqrt{2}/2} e^{-u}\frac{du}{kR\cos\theta} \leqslant \frac{1}{kR\sqrt{2}/2}\int_0^{\sqrt{2}/2} e^{-u}\,du \leqslant \frac{\sqrt{2}}{kR}$$

$$B = \int_{\pi/4}^{\pi/2} e^{-kR\sin\theta}\,d\theta \leqslant \int_{\pi/4}^{\pi/2} e^{-kR\sqrt{2}/2}\,d\theta = \frac{\pi}{4}e^{-kR\sqrt{2}/2} \leqslant \frac{\pi\sqrt{2}}{4}\cdot\frac{1}{kR} \quad \text{because } e^{-x}\leqslant 1/x \text{ for } x>0.$$

Thus we conclude

$$\int_0^{\pi/2} e^{-kR\sin\theta}\,d\theta \leqslant \left(\sqrt{2}+\frac{\pi\sqrt{2}}{4}\right)\frac{1}{kR} \leqslant \frac{100}{kR}. \quad \square$$

Alternate solution. Same up to the bound

$$\left|\int_\Gamma f(z)\,dz\right| \leqslant R\cdot\sup_{z\in\Gamma}|f(z)|\cdot\int_0^{\pi/2} e^{-kR\sin(\theta)}\,d\theta.$$

Now note that on $[0,\pi/2]$, $\sin(\theta)\geqslant(2/\pi)\theta$, so we have

$$\left|\int_\Gamma f(z)\,dz\right| \leqslant R\cdot\sup_{z\in\Gamma}|f(z)|\cdot\int_0^{\pi/2} e^{-kR(2/\pi)\theta}\,d\theta = \frac{\pi/2}{k}\sup_{z\in\Gamma}|f(z)|\cdot\int_0^{\pi/2} e^{-\theta}\,d\theta \leqslant \frac{\pi/2}{k}\sup_{z\in\Gamma}|f(z)|\cdot\int_0^{\infty} e^{-\theta}\,d\theta \leqslant \frac{\pi/2}{k}\sup_{z\in\Gamma}|f(z)|,$$

and I think this is the optimal constant. $\square$

Problem 10. Let us define the Gamma function via

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$$

when the integral is absolutely convergent. Show that this function extends to a meromorphic function in the whole complex plane.

Solution. Note that for $\mathrm{Re}(z)>0$, we have

$$\int_0^\infty \left|t^{z-1}\right|e^{-t}\,dt = \int_0^\infty t^{\mathrm{Re}(z)-1}e^{-t}\,dt < \infty.$$

So the integral is absolutely convergent for all $\mathrm{Re}(z)>0$. First we show that it defines an analytic function for $\mathrm{Re}(z)>1$. We have

$$\frac{\Gamma(z+h)-\Gamma(z)}{h} = \int_0^\infty e^{-t}t^{z-1}\left(\frac{t^h-1}{h}\right).$$

We estimate

$$\begin{aligned}\left|e^{-t}t^{z-1}\left(\frac{t^h-1}{h}\right)\right| &= e^{-t}t^{\mathrm{Re}(z)-1}\left|\frac{e^{h\log t}-1}{h}\right| \leqslant e^{-t}t^{\mathrm{Re}(z)-1}\sum_{n=1}^\infty\frac{|h|^{n-1}|\log t|^n}{n!}\\ &\leqslant e^{-t}t^{\mathrm{Re}(z)-1}\sum_{n=1}^\infty\frac{|\log t|^n}{n!} \quad \text{for } |h|\leqslant 1\\ &\leqslant e^{-t}t^{\mathrm{Re}(z)-1}e^{|\log t|}.\end{aligned}$$

If $\mathrm{Re}(z)>1$, then $e^{-t}t^{\mathrm{Re}(z)-1}e^{|\log t|}$ is integrable on $[0,\infty)$, so by the Dominated Convergence theorem we see that the above difference quotient converges as $h\to 0$, so Γ is analytic. So far we have that Γ is analytic in $\mathrm{Re}(z)>1$. By integrating by parts we get, for any $\mathrm{Re}(z)>0$,

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t}\,dt = z\int_0^\infty t^{z-1}e^{-t}\,dt = z\Gamma(z).$$

So we can extend the definition of Γ by setting $\Gamma(z) := \frac{1}{z}\Gamma(z+1) = \frac{1}{z(z+1)}\Gamma(z+2)$ for all $-1 < \mathrm{Re}(z) \leqslant 0$ except for $z = 0$. This definition makes Γ analytic in $-1 < \mathrm{Re}(z) \leqslant 0$ except at 0 because for any nonzero point in that strip, we can take a neighborhood around that point on which $z \mapsto \frac{1}{z(z+1)}$ and $z \mapsto \Gamma(z+2)$ are both analytic. There is no problem even when taking neighborhoods around points with $\mathrm{Re}(z) = 0$ because in $0 < \mathrm{Re}(z) \leqslant 1$, the two definitions of Γ agree because of the functional equation.

We can extend this definition to all of $\mathbb{C}$. In general, for non-negative integers n, define Γ on the strip $-n-1 < \mathrm{Re}(z) \leqslant -n$ (except not at $z = -n$) by

$$\Gamma(z) = \frac{1}{z(z+1)\cdots(z+n+1)}\Gamma(z+n+2).$$

By the same reasoning, this definition makes Γ analytic everywhere except for at all of the non-positive integers. To show that Γ is meromorphic, we just need to show that it has poles at each non-positive integer. Fix a non-positive integer $-n$. In *any* neighborhood of $z = -n$, the representation

$$\Gamma(z) = \frac{1}{z(z+1)\cdots(z+n+1)}\Gamma(z+n+2).$$

is valid regardless of whether $\mathrm{Re}(z) \leqslant -n$ or $\mathrm{Re}(z) > -n$, because of the functional equation which is valid in the right half plane. Since $\Gamma(2) \neq 0$, it's clear that $\Gamma(z) \to \infty$ as $z \to -n$, and thus Γ has a pole at $-n$. $\square$

Problem 11. Let $P(z)$ be a polynomial. Show that there is an integer n and a second polynomial $Q(z)$ so that

$$P(z)Q(z) \;=\; z^n|P(z)|^2 \quad \text{whenever } |z| = 1.$$

Solution. Write $P(z) = (z-a_1)\cdots(z-a_m)$. Define $Q(z) = (1-\overline{a_1}z)\cdots(1-\overline{a_m}z)$. It's clear Q is a polynomial. On $|z| = 1$, we have

$$\begin{aligned} |P(z)|^2 &= P(z)\overline{P(z)} \;=\; (z-a_1)\cdots(z-a_m)(\overline{z}-\overline{a_1})\cdots(\overline{z}-\overline{a_m}) \\ &= (z-a_1)\cdots(z-a_m)(1/z-\overline{a_1})\cdots(1/z-\overline{a_m}) \\ &= (z-a_1)\cdots(z-a_m)(1/z)^m(1-\overline{a_1}z)\cdots(1-\overline{a_m}z) \;=\; \frac{1}{z^m}P(z)Q(z). \end{aligned}$$

So $P(z)Q(z) = z^m|P(z)|^2$ on $|z| = 1$. $\square$

Problem 12. Show that the only entire function $f(z)$ obeying both

$$|f'(z)| \;\leqslant\; e^{|z|} \quad \text{and} \quad f\left(\frac{n}{\sqrt{1+|n|}}\right) \;=\; 0 \quad \text{for all } n \in \mathbb{Z}$$

is the zero function.

Solution. Suppose f is not identically zero. Then since its zeros are discrete, it has countable many. Enumerate them $\{a_k\}$. By hypothesis f vanishes at every $n/\sqrt{1+|n|}$ for $n \in \mathbb{Z}$, so we know that $\sum_k |a_k|^{-2} = \infty$. This implies that the genus of f is at least 2 (proof below). By Hadamard's theorem, this also implies the order of f is at least 2. But by hypothesis, we have $f(0) = 0$, and so for any z we can write

$$|f(z)| \;=\; \left|\int_{\gamma_z} f'(w)\,dx\right| \;\leqslant\; |z| \sup_{w\in\gamma_z} |f'(w)| \;\leqslant\; |z|e^{|z|} \;\leqslant\; e^{2|z|}$$

where γ_z is a straight line from 0 to z. But this shows that the order of f is $\leqslant 1$, a contradiction.

Here is a proof that $\sum_k |a_k|^{-2} = \infty$ implies the genus of f is at least 2. It follows from the more general claim: If $\mathrm{genus}(f) \leqslant h$ and $\{a_k\}$ are the zeros of f, then $\sum_k |a_k|^{-(h+1)} < \infty$. If the genus is $\leqslant h$, then we know that the product

$$\prod_{k=1}^{\infty}\left(1-\frac{z}{a_k}\right)\exp\left(\frac{z}{a_k}+\frac{1}{2}\left(\frac{z}{a_k}\right)^2+\ldots+\frac{1}{h}\left(\frac{z}{a_k}\right)^h\right)$$

converges uniformly on compact sets. In particular, fix some z which is not a zero of f, then we know the series

$$\sum_{k=1}^{\infty} \log\left(1-\frac{z}{a_k}\right)+\frac{z}{a_k}+\frac{1}{2}\left(\frac{z}{a_k}\right)^2+\ldots+\frac{1}{h}\left(\frac{z}{a_k}\right)^h$$

convergs absolutely. For all $|a_k|>3|z|$, we have the estimate

$$\begin{aligned}
\left|\log\left(1-\frac{z}{a_k}\right)+\frac{z}{a_k}+\frac{1}{2}\left(\frac{z}{a_k}\right)^2+\ldots+\frac{1}{h}\left(\frac{z}{a_k}\right)^h\right| &= \left|\sum_{j=h+1}^{\infty}\frac{1}{j}\left(\frac{z}{a_k}\right)^j\right| \\
&= \frac{1}{h+1}\left|\frac{z}{a_k}\right|^{h+1}\left|\sum_{j=h+1}^{\infty}\frac{h+1}{j}\left(\frac{z}{a_k}\right)^{j-(h+1)}\right| \\
&\geqslant \frac{1}{h+1}\left|\frac{z}{a_k}\right|^{h+1}\left(1-\sum_{j=h+2}^{\infty}\frac{h+1}{j}\left|\frac{z}{a_k}\right|^{j-(h+1)}\right) \\
&\geqslant \frac{1}{h+1}\left|\frac{z}{a_k}\right|^{h+1}\left(1-\sum_{j=h+2}^{\infty}(1/3)^{j-(h+1)}\right) \\
&\geqslant \frac{|z|^{h+1}}{2(h+1)}|a_k|^{-(h+1)}.
\end{aligned}$$

Thus

$$\sum_{|a_k|>3|z|}|a_k|^{-(h+1)} \leqslant \frac{2(h+1)}{|z|^{h+1}}\sum_{|a_k|>3|z|}\left|\log\left(1-\frac{z}{a_k}\right)+\frac{z}{a_k}+\frac{1}{2}\left(\frac{z}{a_k}\right)^2+\ldots+\frac{1}{h}\left(\frac{z}{a_k}\right)^h\right| < \infty.$$

This establishes the desired claim because there are only finitely many a_k with $|a_k|\leqslant 3|z|$. $\square$

Alternate solution. By the same argument as in the other solution we have $|f(z)|\leqslant e^{2|z|}$. We want to use Jensen's formula. First multiply f by a power of z so that $f(0)\neq 0$. This preserves an inequality of the form $|f(z)|\leqslant e^{c|z|}$. For any R (assuming f has no zeros on $|z|=R$), Jensen's formula gives (enumerating the zeros of f as a_n)

$$\begin{aligned}
\log|f(0)| &= \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|\,d\theta+\sum_{|a_n|<R}\log\left|\frac{a_n}{R}\right| \\
&\lesssim \log e^{cR}+\sum_{|n|/\sqrt{1+|n|}<R}\log\left|\frac{n}{R\sqrt{1+|n|}}\right| \\
&\lesssim R+\sum_{n\leqslant R^2}\log\left|\frac{\sqrt{n}}{R}\right| \\
&\lesssim R-\sum_{n\leqslant R^2}\log R+\sum_{n\leqslant R^2}\log\sqrt{n} \\
&\lesssim R-R^2\log R+\frac{1}{2}\int_0^{R^2}\log x\,dx \\
&\lesssim R-R^2\log R+R^2\log R-\frac{1}{2}R^2
\end{aligned}$$

which goes to $-\infty$ as $R\to\infty$, a contradiction. $\square$

8 Fall 2012

Problem 1. Let $1 < p < \infty$ and let $f_n : \mathbb{R}^3 \to \mathbb{R}$ be a sequence of functions such that $\limsup ||f_n||_{L^p} < \infty$. Show that if f_n converges almost everywhere, then f_n converges weakly in L^p.

Solution. Let λ denote Lebesgue measure on $\mathbb{R}^3$. Say that $f_n \to f$ pointwise almost everywhere and also that $||f_n||_{L^p} \leqslant M$ for all n. To show that $f_n \to f$ weakly in L^p, we need to show that $\phi(f_n) \to \phi(f)$ for every bounded linear functional $\phi \in (L^p)^*$. By L^p-L^q duality, we know that every $\phi \in (L^p)^*$ is of the form $\phi(f) = \int fg\, d\lambda$ for some $g \in L^q$. So let g be any L^q function; it suffices to show that

$$\int f_n g \;\to\; \int fg.$$

Since $f_n \to f$ almost everywhere, we also know that $f_n g \to fg$ almost everywhere. By the Vitali Convergence Theorem, to show $\int f_n g\, d\lambda \;\to\; \int fg\, d\lambda$ it suffices to show that the sequence $\{f_n g\}$ is uniformly integrable and tight.

For uniform integrability, let $\epsilon > 0$. Since $|g|^q$ is integrable, let $\delta > 0$ be such that whenever $\lambda(A) < \delta$, we have $\int_A |g|^q\, d\lambda < \epsilon$. Then for any n and any $\lambda(A) < \delta$, we have by Hölder's inequality

$$\int_A |f_n g|\, d\lambda \;\leqslant\; \left(\int_A |f_n|^p\, d\lambda\right)^{1/p} \left(\int_A |g|^q\, d\lambda\right)^{1/q} \;<\; M\epsilon^{1/q},$$

which shows that $\{f_n g\}$ is a uniformly integrable family.

For tightness, let $\epsilon > 0$ and let E be a subset of $\mathbb{R}^3$ such that $\int_{E^c} |g|^q\, d\lambda < \epsilon$. Then for any n, we have by the same argument

$$\int_{E^c} |f_n g|\, d\lambda \;\leqslant\; \left(\int_{E^c} |f_n|^p\, d\lambda\right)^{1/p} \left(\int_{E^c} |g|^q\, d\lambda\right)^{1/q} \;<\; M\epsilon^{1/q},$$

so $\{f_n g\}$ is a tight family, so we are done. $\square$

Problem 2. Suppose $d\mu$ is a Borel probability measure on the unit circle in the complex plane such that

$$\lim_{n\to\infty} \int_{|z|=1} z^n\, d\mu(z) \;=\; 0.$$

For $f \in L^1(d\mu)$ show that

$$\lim_{n\to\infty} \int_{|z|=1} z^n f(z)\, d\mu(z) \;=\; 0.$$

Solution. By linearity, it is clear that the desired result holds for any trigonometric polynomial on the unit circle, i.e. any function of the form $P(z) = \sum_{n=-N}^{N} a_n z^n$. Since μ is a Borel measure and the unit circle is compact, we know that the set of continuous functions on S^1 is dense in $L^1(\mu)$ with respect to the norm $||\cdot||_{L^1(\mu)}$. We also know by the Stone-Weierstrass theorem that the set of trigonometric polynomials on S^1 is dense in the set of continuous functions on S^1 with respect to the norm $||\cdot||_{L^\infty(\mu)}$.

So let $f \in L^1(\mu)$ and fix $\epsilon > 0$. Let g be a continuous function on S^1 such that $||f - g||_{L^1(\mu)} < \epsilon$ and let P be a trigonometric polynomial such that $||g - P||_{L^\infty(\mu)}$. Since the result holds for trigonometric polynomials, we can pick n large enough so that

$$\left|\int_{|z|=1} z^n P(z)\, d\mu(z)\right| \;<\; \epsilon.$$

Then for such n, we have

$$\begin{aligned}\left|\int_{|z|=1} z^n f(z)\,d\mu(z)\right| \;&\leqslant\; \int_{|z|=1} |z^n(f(z)-g(z))|\;d\mu(z) + \int_{|z|=1} |z^n(g(z)-P(z))|\;d\mu(z) + \left|\int_{|z|=1} z^n P(z)\,d\mu(z)\right| \\ &\leqslant \int_{|z|=1} |f(z)-g(z)|\;d\mu(z) + \int_{|z|=1} |g(z)-P(z)|\;d\mu(z) + \epsilon \\ &\leqslant\; ||f-g||_{L^1(\mu)} + ||g-P||_{L^\infty(\mu)}\,\mu(S^1) + \epsilon \;<\; 3\epsilon,\end{aligned}$$

which shows that $\int_{|z|=1} z^n f(z)\,d\mu(z) \to 0$ as $n \to \infty$. $\square$

Problem 3. Let H be a Hilbert space and let E be a closed convex subset of H. Prove that there exists a unique element $x \in E$ such that

$$||x|| \;=\; \inf_{y\in E} ||y||\,.$$

Solution. First note that if $0 \in E$, then the statement is obviously true by taking $x = 0$, so assume $0 \notin E$. Let $\inf_{y\in E} ||y|| = \delta > 0$. First we prove that such an x must be unique. Suppose that $||x|| = ||x'|| = \delta$. Then since E is convex, we have $(1/2)x + (1/2)x' \in E$ and

$$\delta \;=\; \frac{1}{2}\,||x|| + \frac{1}{2}\,||x'|| \;=\; \left|\left|\frac{1}{2}x\right|\right| + \left|\left|\frac{1}{2}x'\right|\right| \;\geqslant\; \left|\left|\frac{1}{2}x + \frac{1}{2}x'\right|\right| \;\geqslant \delta.$$

But we know that equality in the triangle inequality occurs if and only if x and x' are scalar multiples of each other. Thus the above inequality yields the contradiction $\delta > \delta$ unless x and x' are scalar multiples of each other. So we can write $x = cx'$ where $|c| = 1$. Then since E is convex, $(1/2)(x + x') = \frac{c+1}{2}x' \in E$ also, so $\left|\left|\frac{c+1}{2}x'\right|\right| = |(c+1)/2|\,\delta \;\geqslant \delta$, which implies $c = 1$, so $x = x'$.

Now we show existence. Let $\{y_n\}$ be a sequence in E such that $||y_n|| \to \delta$ as $n \to \infty$. Then for any n and m, by the parallelogram law we can write

$$\left|\left|\frac{1}{2}y_n + \frac{1}{2}y_m\right|\right|^2 + \left|\left|\frac{1}{2}y_n - \frac{1}{2}y_m\right|\right|^2 \;=\; 2\left|\left|\frac{1}{2}y_n\right|\right|^2 + 2\left|\left|\frac{1}{2}y_m\right|\right|^2.$$

Since E is convex, $(1/2)y_n + (1/2)y_m \in E$, so we have

$$\frac{1}{4}\,||y_n - y_m||^2 \;=\; \frac{1}{2}\,||y_n||^2 + \frac{1}{2}\,||y_m||^2 - \left|\left|\frac{1}{2}y_n + \frac{1}{2}y_m\right|\right|^2 \;\leqslant\; \frac{1}{2}\,||y_n||^2 + \frac{1}{2}\,||y_m||^2 - \delta^2.$$

As $n, m \to \infty$, the right side of the above inequality tends to 0 by definition of the y_n, so we conclude that $||y_n - y_m||^2 \to 0$ as $n, m \to \infty$, so $\{y_n\}$ is a Cauchy sequence. Since H is complete, there is some $x \in H$ such that $y_n \to x$ as $n \to \infty$, and since E is closed, we must have $x \in E$. Finally, since the norm is a continuous function on H, we must have $||x|| = \lim_{n\to\infty} ||y_n|| = \delta$. $\square$

Problem 4. Fix $f \in C(\mathbb{T})$ where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Let s_n denote the nth partial sum of the Fourier series of f. Prove that

$$\lim_{n\to\infty} \frac{||s_n||_{L^\infty(\mathbb{T})}}{\log(n)} \;=\; 0.$$

Solution. Recall that we have $s_n(f)(x) = (f * D_n)(x)$, where D_n is the Dirichlet kernel

$$D_n(t) \;=\; \sum_{k=-n}^{n} e^{ikt} \;=\; \frac{\sin((n+1/2)t)}{\sin(t/2)}.$$

Therefore we immediately see that $||s_n(f)||_{L^\infty} \leqslant ||f||_{L^\infty}\,||D_n||_{L^1}$. We estimate

$$||D_n||_{L^1} \;\lesssim\; \int_{-\pi}^{\pi}\left|\frac{\sin((n+1/2)t)}{\sin(t/2)}\right| dt \;\lesssim\; \int_0^{\pi}\left|\frac{\sin((n+1/2)t)}{t}\right| dt$$

where the second inequality is valid because D_n is even and $\sin(t/2) \geqslant t/100$ on $[0,\pi]$. Continuing,

$$\begin{aligned} ||D_n||_{L^1} \;&\lesssim\; \int_0^{(n+1/2)\pi} \frac{|\sin(u)|}{u}\,du \;\lesssim\; \sum_{k=0}^{n}\int_{k\pi}^{(k+1)\pi}\frac{|\sin(u)|}{u}\,du \\ &\lesssim\; \sum_{k=0}^{n}\int_{k\pi}^{(k+1)\pi}\frac{|\sin(u)|}{(k+1)\pi}\,du \;\lesssim\; \sum_{k=0}^{n}\frac{1}{k+1} \;\lesssim\; \log(n). \end{aligned}$$

So we have established $||s_n(f)||_{L^\infty} \;\lesssim\; ||f||_{L^\infty}\log(n)$ for all $f \in C(\mathbb{T})$. Note that if P is a polynomial, then $s_n(P) \to P$ uniformly on $\mathbb{T}$ (this is proven by integrating by parts twice on the definition of the Fourier coefficients to get $|\widehat{P}(k)| \lesssim k^{-2}$, and then applying the Weierstrass M-test combined with the general fact that $s_n(P) \to P$ in L^2). In particular, $||s_n(P)||_{L^\infty}$ is bounded, so we clearly have $||s_n(P)||_{L^\infty}/\log(n) \to 0$. Fix $\epsilon > 0$ and any $f \in C(\mathbb{T})$. We can find a polynomial P with $||f-P||_{L^\infty} < \epsilon$. Then we have

$$\limsup_{n\to\infty}\frac{||s_n(f)||_{L^\infty}}{\log(n)} \;\leqslant\; \limsup_{n\to\infty}\frac{||s_n(f-P)||_{L^\infty}}{\log(n)} + \frac{||s_n(P)||_{L^\infty}}{\log(n)} \;\lesssim\; ||f-P||_{L^\infty} \;<\; \epsilon.$$

Take $\epsilon \to 0$ and we're done. $\square$

Problem 5. Let $f_n : \mathbb{R}^3 \to \mathbb{R}$ be a sequence of functions such that $\sup_n ||f_n||_{L^2} < \infty$. Show that if f_n converges almost everywhere to a function $f : \mathbb{R}^3 \to \mathbb{R}$, then

$$\int_{\mathbb{R}^3}\big||f_n|^2 - |f_n - f|^2 - |f|^2\big|\,dx \;\to\; 0.$$

Solution. Let M be such that $||f_n||_{L^2} \leqslant M$ for all n. Since $f_n \to f$ almost everywhere, we also have $|f_n|^2 \to |f|^2$ almost everywhere, so by Fatou's lemma,

$$\int |f|^2 \;=\; \int \liminf_{n\to\infty}|f_n|^2 \;\leqslant\; \liminf_{n\to\infty}\int |f_n|^2 \;\leqslant\; M^2,$$

which shows that $f \in L^2$ and $||f||_{L^2} \leqslant M$. Notice that we have the identity

$$\big||f_n|^2 - |f_n-f|^2 - |f|^2\big| \;=\; \big||f_n - f + f|^2 - |f_n - f|^2 - |f|^2\big| \;=\; 2|f_n - f||f|.$$

Fix $\epsilon > 0$. Since $|f|^2$ is integrable, there is a $\delta > 0$ such that $\lambda(E) < \delta$ implies $\int_E |f|^2 < \epsilon$. We can also pick an R which is big enough so that $\int_{|x|>R}|f|^2 < \epsilon$. Then on the set $|x| \leqslant R$, we can apply Egorov's theorem to get a set $E \subseteq \{|x| \leqslant R\}$ such that $f_n \to f$ uniformly on $\{|x| \leqslant R\}\backslash E$ and $\lambda(E) < \delta$. So we have the estimate

$$\int |f_n - f||f| \;=\; \int_{\{|x|\leqslant R\}\backslash E}|f_n-f||f| + \int_E |f_n - f||f| + \int_{\{|x|>R\}}|f_n - f||f| \;=:\; A + B + C.$$

Since $f_n \to f$ uniformly on $\{|x| \leqslant R\}\backslash E$, let n be big enough so that $\int_{\{|x|\leqslant R\}\backslash E}|f_n - f|^2 < \epsilon$. Now we estimate each of A, B, C separately using Cauchy-Schwarz. We have

$$\begin{aligned} A \;&\leqslant\; \left(\int_{\{|x|\leqslant R\}\backslash E}|f_n - f|^2\right)^{1/2}\left(\int_{\{|x|\leqslant R\}\backslash E}|f|^2\right)^{1/2} \;\leqslant\; M\sqrt{\epsilon} \\ B \;&\leqslant\; \left(\int_E |f_n - f|^2\right)^{1/2}\left(\int_E |f|^2\right)^{1/2} \;\leqslant\; \sqrt{2M^2}\sqrt{\epsilon} \\ C \;&\leqslant\; \left(\int_{\{|x|>R\}}|f_n - f|^2\right)^{1/2}\left(\int_{\{|x|>R\}}|f|^2\right)^{1/2} \;\leqslant\; \sqrt{2M^2}\sqrt{\epsilon}. \end{aligned}$$

This shows that $\int |f_n - f||f| \to 0$ as $n \to \infty$, which is enough to conclude the desired result. $\square$

Problem 6. Let $f \in L^1(\mathbb{R})$ and let Mf denote its maximal function, that is,

$$(Mf)(x) \;=\; \sup_{0<r<\infty} \frac{1}{2r}\int_{-r}^{r} |f(x-y)|\,dy.$$

By the Hardy-Littlewood maximal function theorem,

$$|\{x \in \mathbb{R} : (Mf)(x) > \lambda\}| \;\leqslant\; 3\lambda^{-1}\,||f||_{L^1} \quad \text{for all } \lambda > 0.$$

Using this show that

$$\limsup_{r\to 0} \frac{1}{2r}\int_{-r}^{r} |f(y) - f(x)|\,dy \;=\; 0 \quad \text{for almost every } x \in \mathbb{R}.$$

Solution. This is actually false as stated. As a counterexample, take $f = \chi_{[-1,1]}$. For any $x \notin [-1,1]$, we have $f(x) = 0$ but

$$\limsup_{r\to 0} \frac{1}{2r}\int_{-r}^{r} |f(y) - f(x)|\,dy \;=\; \limsup_{r\to 0} \frac{1}{2r}\int_{-r}^{r} |f(y)|\,dy \;=\; 1.$$

Presumably, what the question meant to say is to prove that

$$\limsup_{r\to 0} \frac{1}{2r}\int_{x-r}^{x+r} |f(y) - f(x)|\,dy \;=\; 0 \quad \text{for almost every } x \in \mathbb{R},$$

which is the Lebesgue differentiation theorem. Here is a proof of this:

Define

$$\begin{aligned} (T_r f)(x) \;&:=\; \frac{1}{2r}\int_{x-r}^{x+r} |f(y) - f(x)|\,dy \\ (Tf)(x) \;&:=\; \limsup_{r\to 0^+} (T_r f)(x). \end{aligned}$$

We want to prove that $Tf = 0$ almost everywhere. Fix some $\epsilon > 0$. Since the set of continuous functions with compact support is dense in $L^1(\mathbb{R})$, let g be a continuous function with compact support such that $||f - g||_{L^1} < \epsilon$. Define $h = f - g$ so that $f = g + h$. Note that for any $r > 0$ we have

$$T_r f \;=\; T_r(g+h) \;\leqslant\; T_r g + T_r h.$$

By the definition of continuity, it is clear that the desired result holds for continuous functions, so we have that Tg is identically zero, and thus we obtain $Tf \leqslant Th$.

To show that $Tf = 0$ almost everywhere, it suffices to show that $m\{x \in \mathbb{R} : (Tf)(x) > \delta\} = 0$ for any fixed $\delta > 0$, where m is Lebesgue measure on $\mathbb{R}$. So fix $\delta > 0$ and define $F := \{x \in \mathbb{R} : (Tf)(x) > \delta\}$ and $E := \{x \in \mathbb{R} : (Th)(x) > \delta\}$. Since $Tf \leqslant Th$, $F \subseteq E$, so we analyze the measure of E. Note that for any x and any $r > 0$, we have

$$(T_r h)(x) \;=\; \frac{1}{2r}\int_{x-r}^{x+r} |h(y) - h(x)|\,dy \;\leqslant\; \frac{1}{2r}\int_{x-r}^{x+r} |h(y)|\,dy + \frac{1}{2r}\int_{x-r}^{x+r} |h(x)|\,dy \;\leqslant\; (Mh)(x) + |h(x)|.$$

Therefore we have

$$E \;\subseteq\; \{x \in \mathbb{R} : (Mh)(x) > \delta/2\} \cup \{x \in \mathbb{R} : |h(x)| > \delta/2\},$$

so by the Hardy-Littlewood theorem, Chebyshev's inequality, and the definition of h,

$$m(E) \leqslant \frac{6}{\delta}\,||h||_{L^1} + \frac{2}{\delta}\,||h||_{L^1} < \frac{8}{\delta}\epsilon.$$

Thus we have $m(F) < (8/\delta)\epsilon$. Since the set F does not depend on ϵ, this holds for any $\epsilon > 0$ and thus we conclude $m(F) = 0$, which is enough to conclude that $Tf = 0$ almost everywhere. $\square$

Problem 7. Let f be a function holomorphic in $\mathbb{C}$ and suppose that $f(0) = 0$, $f(1) = 1$, and $f(\mathbb{D}) \subseteq \mathbb{D}$. Show that (a) $f'(1) \in \mathbb{R}$ and (b) $f'(1) \geqslant 1$.

Solution. (a) Suppose that $f'(1) \notin \mathbb{R}$. Then there exists $v \in \mathbb{C}$ with $\mathrm{Re}(v) < 0$ such that $\mathrm{Re}(f'(1)v) > 0$. The limit definition of the derivative, together with the fact that $f(1) = 1$ implies that

$$f'(1)v = \lim_{t\to 0^+} \frac{f(1+tv)-1}{t}.$$

For sufficiently small t, we have $1 + tv \in \mathbb{D}$. Since $f(\mathbb{D}) \subseteq \mathbb{D}$, But then $\mathrm{Re}\,\frac{f(1+tv)-1}{t} < 0$ small t. After passing to the limit, we have $\mathrm{Re}(f'(1)v) \leqslant 0$ which is a contradiction.

(b) Fix $t \in (0, 1)$. By the Schwarz lemma, $|f(1-t)| \leqslant 1-t$. Therefore

$$\frac{|f(1-t)-1|}{t} \geqslant \frac{1-|f(1-t)|}{t} \geqslant 1.$$

Taking the limit as $t \to 0^+$, we see that $|f'(1)| \geqslant 1$.

Problem 8. Let $f : \mathbb{C} \to \mathbb{C}$ be a nonconstant holomorphic function such that every zero of f has even multiplicity. Show that f has a holomorphic square root, i.e. there exists a holomorphic function $g : \mathbb{C} \to \mathbb{C}$ such that $f(z) = g(z)^2$ for all $z \in \mathbb{C}$.

Solution. If the set of zeros of f had a limit point, then f would have to be identically zero. But f is nonconstant by hypothesis, so the zeros of f are isolated. Since all of the multiplicities are even and the zeros are isolated, by Weierstrass's theorem there exists an entire function h such that h has the same zeros as f, but with each one half the multiplicity. Then h^2 is an entire function with exactly the same zeros as f with all the same multiplicities. Therefore the function f/h^2 is analytic at all points which are not zeros of f, and it has removable singularities at the zeros of f. So it can be extended to a function which is analytic everywhere, so we can assume without loss of generality that f/h^2 is a nonvanishing entire function. Since it is nonvanishing, it has a well-defined analytic logarithm, i.e. there is some entire function g such that $f/h^2 = \exp(g)$. Then $f = h^2\exp(g) = (h\exp(g/2))^2$, and $h\exp(g/2)$ is an entire function, so this is the desired result. $\square$

Problem 9. Suppose f is a holomorphic function in the unit disk $\mathbb{D}$ and $\{x_n\}$ is a sequence of real numbers satisfying $0 < x_{n+1} < x_n < 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = 0$. Show that if $f(x_{2n+1}) = f(x_{2n})$ for all n, then f is a constant function.

Solution. By translating by a constant, we may assume that $f(0) = 0$. Define $g(z) = f(z)\overline{f(\overline{z})}$. Since $\overline{f(\overline{z})}$ is also holomorphic, we see that g is also holomorphic and $g(z) \in \mathbb{R}$ whenever $z \in \mathbb{R}$. So we can consider the restriction of g to the positive real axis as a differential function on $\mathbb{R}$. Then since $g(x_{2n+1}) = g(x_{2n})$ for all n, by the mean value theorem there is a number $y_n \in (x_{2n+1}, x_{2n})$ such that $g'(y_n) = 0$. Since $x_n \to 0$, also $y_n \to 0$. Thus g' is zero on a set with a limit point, so g' is identically zero. Therefore g is a constant, and since $f(0) = 0$, we also have $g(0) = 0$, so g is identically zero. Therefore we have $f(z)\overline{f(\overline{z})} = 0$ for all $z \in \mathbb{D}$, which implies that f is identically zero because either $f(z)$ or $\overline{f(\overline{z})}$ is zero on a set with a limit point. $\square$

Problem 10. Let $\{f_n\}$ be a sequence of holomorphic functions on $\mathbb{D}$ satisfying $|f_n(z)| \leqslant 1$ for all z and all n Let $A \subseteq \mathbb{D}$ be the set of all $z \in \mathbb{D}$ for which the limit $\lim_{n\to\infty} f_n(z)$ exists. Show that if A has an accumulation point in $\mathbb{D}$, then there exists a holomorphic function f on $\mathbb{D}$ such that $f_n \to f$ locally uniformly on $\mathbb{D}$.

Solution. Since the sequence f_n is uniformly bounded, by Montel's theorem we know it is a normal family, so there is a subsequence f_{n_k} which converges locally uniformly on $\mathbb{D}$ to some function f. Since local uniform limits of holomorphic functions are holomorphic, we know that f is holomorphic. Now, to show that the whole sequence f_n converges locally uniformly to f, it suffices to prove that every subsequence has a further subsequence which converges locally uniformly to f. Since the whole sequence is uniformly bounded, clearly any subsequence is also uniformly bounded, so by applying Montel's theorem to the subsequence, we obtain a further subsequence which converges locally uniformly to some holomorphic function g on $\mathbb{D}$. But note that for every $z \in A$, since the limit of the whole sequence $\lim_{n\to\infty} f_n(z)$ exists, any subsequences which converge pointwise at z must have the same limit. This implies in particular, since local uniform convergence implies pointwise convergence, that $f(z) = g(z)$ for all $z \in A$. Since A has a limit point in $\mathbb{D}$ and f and g are both holomorphic, this implies that $f = g$ on $\mathbb{D}$. Thus we conclude that any subsequence of f_n has a further subsequence converging locally uniformly to f, which implies that f_n converges locally uniformly to f. $\square$

Problem 11. Find all holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ satisfying $f(z+1) = f(z)$ and $f(z+i) = e^{2\pi} f(z)$ for all $z \in \mathbb{C}$.

Solution. Note that $\exp(-2\pi i z)$ is one such function. Let $f : \mathbb{C} \to \mathbb{C}$ be any entire function satisfying $f(z+1) = f(z)$ and $f(z+i) = e^{2\pi} f(z)$ for all $z \in \mathbb{C}$. Define $g(z) = f(z)\exp(2\pi i z)$. Then g is also an entire function and it satisfies

$$\begin{aligned} g(z+1) &= f(z+1)\exp(2\pi i(z+1)) = f(z)\exp(2\pi i z)\exp(2\pi i) = g(z) \\ g(z+i) &= f(z+i)\exp(2\pi i(z+i)) = e^{2\pi} f(z)\exp(2\pi i z)\exp(-2\pi) = g(z). \end{aligned}$$

Thus g is a doubly periodic entire function, so it must be bounded and hence must be constant by Liouville's theorem. Thus we conclude that $f(z) = C\exp(-2\pi i z)$ for some $C \in \mathbb{C}$, and these are all of the functions f which satisfy the desired property. $\square$

Problem 12a. Let $M \in \mathbb{R}$, $\Omega \subseteq \mathbb{C}$ be a bounded open set, and $u : \Omega \to \mathbb{R}$ be a harmonic function. Show that if

$$\limsup_{z\to z_0} u(z) \leqslant M$$

for all $z_0 \in \partial\Omega$, then $u(z) \leqslant M$ for all $z \in \Omega$.

Solution. Fix $\epsilon > 0$. By the limsup condition, for each $z_0 \in \partial\Omega$, there is a radius $r(z_0)$ such that $|z - z_0| < r(z_0)$ implies that $u(z) \leqslant M + \epsilon$. Then the set

$$\bigcup_{z_0 \in \partial\Omega} B(z_0, r(z_0))$$

is an open cover of $\partial\Omega$, which is a compact set because Ω is bounded. Therefore $\partial\Omega$ is covered by only finitely many of these balls. Call them $B_1, \ldots, B_N$. Now the set

$$A = \Omega \backslash (\overline{B_1} \cup \ldots \cup \overline{B_N})$$

is an open set on which u is harmonic, extends continuously to the boundary, and satisfies $u(w) \leqslant M + \epsilon$ for all $w \in \partial A$. Thus by the maximum principle, we conclude that $u(z) \leqslant M + \epsilon$ for all $z \in A$. By construction of A, we also know that $u(z) \leqslant M + \epsilon$ for all $z \in \Omega\backslash A$, so we have $u(z) \leqslant M + \epsilon$ for all $z \in \Omega$. Since this argument holds for any $\epsilon > 0$ we conclude that $u(z) \leqslant M$ for all $z \in \Omega$. $\square$

Problem 12b. Show that if u is bounded from above and the above condition holds for all but finitely many $z_0 \in \partial\Omega$, then it still follows that $u(z) \leqslant M$ for all $z \in \Omega$.

Solution. Since Ω is bounded, let $d = \operatorname{diam}(\Omega) = \sup_{z,w\in\Omega} |z - w| < \infty$. Let $p_1, \ldots, p_N$ be the points in $\partial\Omega$ for which the limsup condition above does not hold. Define the function

$$v(z) := -\log\left|\frac{z - p_1}{d}\right| - \ldots - \log\left|\frac{z - p_N}{d}\right|.$$

Note that v is a nonnegative harmonic function in Ω because the function

$$z \mapsto \left(\frac{z-p_1}{d}\right)\cdots\left(\frac{z-p_N}{d}\right)$$

is a nonvanishing analytic function in Ω.

Fix $\epsilon > 0$ and define $f(z) = u(z) - \epsilon v(z)$. For any $z_0 \in \partial\Omega\backslash\{p_1, \ldots, p_N\}$, the limsup condition holds, and so as in the previous problem we have a radius $r(z_0)$ such that $|z - z_0| < r(z_0)$ implies $u(z) \leqslant M + \epsilon$, and since $v \geqslant 0$ we also have $f(z) \leqslant M + \epsilon$ for all such z. However, for any p_j, since u is bounded above and $v(z) \to \infty$ as $z \to p_j$, there is also a radius $r(j)$ such that $|z - p_j| < r(j)$ implies $f(z) \leqslant M + \epsilon$. Now we proceed as in the previous problem. Since $\partial\Omega$ is compact, it can be covered by finitely many of the balls $B(z_0, r(z_0))$ and $B(p_j, r(j))$. So we obtain a smaller set $A \subseteq \Omega$ on which f is harmonic, extends continuously to the boundary, and satisfies $f(w) \leqslant M + \epsilon$ on the boundary of A. So by the maximum principle and by construction of A we have $f(z) \leqslant M + \epsilon$ for all $z \in \Omega$, i.e. $u(z) \leqslant M + \epsilon + \epsilon v(z)$ for all $z \in \Omega$. And this argument holds for any $\epsilon > 0$, so we conclude that $u(z) \leqslant M$ for all $z \in \Omega$. $\square$

9 Spring 2013

Problem 1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is bounded, Lebesgue measurable, and

$$\lim_{h\to 0} \int_0^1 \frac{|f(x+h) - f(x)|}{h} \, dx \;=\; 0.$$

Show that f is a.e. constant on $[0,1]$.

Solution. Let $F(x) = \int_0^x f(t)\, dt$. By the Lebesgue differentiation theorem, there is a set E of measure zero such that

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \;=\; f(x)$$

for all $x \notin E$. Then for any $a, b \notin E$, pick h small enough so that without loss of generality we have $a, a+h < b, b+h$, then we have

$$\begin{aligned} |f(a) - f(b)| \;&=\; \lim_{h\to 0} \left| \frac{F(a+h) - F(a)}{h} - \frac{F(b+h) - F(b)}{h} \right| \;=\; \lim_{h\to 0} \frac{1}{h} \left| \int_a^b f(t)\, dt - \int_{a+h}^{b+h} f(t)\, dt \right| \\ &\leqslant\; \lim_{h\to 0} \frac{1}{h} \int_{a+h}^{b+h} |f(t+h) - f(t)|\, dt \;\leqslant\; \lim_{h\to 0} \frac{1}{h} \int_0^1 |f(t+h) - f(t)|\, dt \;=\; 0, \end{aligned}$$

so f is constant a.e. $\square$

Problem 2. Consider the Hilbert space $\ell^2(\mathbb{Z})$. Show that the Borel σ-algebra $\mathcal{N}$ on $\ell^2(\mathbb{Z})$ associated to the norm topology agrees with the Borel σ-algebra $\mathcal{W}$ on $\ell^2(\mathbb{Z})$ associated to the weak topology.

Solution. Note: I'm pretty sure this argument still works if $\ell^2(\mathbb{Z})$ is replaced by any separable Hilbert space.

It's known that the weak topology is coarser than the norm topology, so we automatically have $\mathcal{W} \subseteq \mathcal{N}$. We just need to show that any norm-open set in $\ell^2(\mathbb{Z})$ is in $\mathcal{W}$. Since $\ell^2(\mathbb{Z})$ with the norm topology is separable, any norm-open set is a countable union of open balls, so it suffices to show that every norm-open ball is in $\mathcal{W}$. Fix $B(x,r) = \{y \in \ell^2(\mathbb{Z}) : ||y - x||_{\ell^2}^2 < r^2\}$. We can view this as a preimage $f^{-1}([0, r^2))$ where $f : \ell^2(\mathbb{Z}) \to \mathbb{R}$ is given by

$$f(y) \;:=\; ||y - x||^2 \;=\; ||y||^2 + ||x||^2 - 2\operatorname{Re}\langle y, x\rangle \;=\; \sum_{n=1}^{\infty} |\langle y, e_n\rangle|^2 + ||x||^2 - 2\operatorname{Re}\langle y, x\rangle$$

where $\{e_n\}$ is an orthonormal basis for $\ell^2(\mathbb{Z})$ and we have used Parseval's theorem. We claim that this function is $\mathcal{W}$-measurable. This is because by definition of the weak topology, the function $y \mapsto \langle y, z\rangle$ is weak-continuous for any $z \in \ell^2(\mathbb{Z})$ and therefore $\mathcal{W}$-measurable. So the first term in f is a countable sum of non-negative measurable functions, which is measurable (combination of the facts that g measurable implies $|g|^2$ measurable, sum of measurable functions is measurable, and pointwise limit of measurable functions is measurable). The second term in f is a constant, which is measurable, and the third term in f is the real part of a measurable function, again measurable. So f is a $\mathcal{W}$-measurable function, and therefore $B(x,r) = f^{-1}([0, r^2)) \in \mathcal{W}$. $\square$

Problem 3. Given $f : \mathbb{R}^2 \to \mathbb{R}$ continuous, we define

$$[A_r f](x,y) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x + r\cos(\theta), y + r\sin(\theta)) d\theta$$

and

$$[Mf](x,y) := \sup_{0<r<1} [A_r f](x,y).$$

By a theorem of Borgain, there is an absolute constant C so that

$$||Mf||_{L^3(\mathbb{R}^2)} \leqslant C\, ||f||_{L^3(\mathbb{R}^2)}$$

for all $f \in C_c(\mathbb{R}^2)$. Use this to show the following: If $K \subset \mathbb{R}^2$ is compact, then $[A_r\chi_K](x,y) \to 1$ as $r \to 0$ at almost every point (x,y) in K (with respect to Lebesgue measure).

Solution. We would like to mimic the proof of the Lebesgue differentiation theorem. This doesn't work directly since we are only given Borgain's result for continuous functions, so we start by expanding this result slightly. In what follows C will always denote an absolute constant which may change from line to line.

Claim. Let S be a bounded open subset of $\mathbb{R}^2$ with $\lambda(S) < \infty$. Then for $t > 0$ we have

$$\lambda(\{(x,y) \in \mathbb{R}^2 : [M\chi_S](x,y) > t\}) \leqslant C\frac{\lambda(S)^3}{t^3}.$$

Proof. First note that the restriction of χ_S to a circle is Borel measurable with respect to the uniform measure on the circle, since the restriction of an open set to a subset of $\mathbb{R}^2$ is open in the subspace topology. So $[M\chi_S]$ is defined.

Note that χ_S is the characteristic function of an open set and is therefore lower semi-continuous. Thust we may find an increasing sequence of functions $f_k \in C_c(\mathbb{R}^2)$ converging monotonically to χ_S. By replacing f_k with $\max(f_k, 0)$, we may assume that each f_k is non-negative. From the weak-type L^3 estimate which follows from Borgain's result, we have

$$\lambda(\{(x,y) : [Mf_k](x,y) > t\}) \leqslant Ct^{-3}\, ||f_k||_3^3 \leqslant Ct^{-3}\, ||\chi_S||_3^3 = Ct^{-3}\lambda(S).$$

If $[M\chi_S](x,y) > t$, then there exists $r \in (0,1)$ such that $[A_r\chi_S](x,y) > t$, and by monotone convergence, we have $[A_r f_k](x,y) > t$ for sufficiently large k. Since Mf_k is an increasing sequence of functions, we can write

$$\{(x,y) : [M\chi_S](x,y) > t\} = \bigcup_{k=1}^{\infty}\{(x,y) : [Mf_k](x,y) > t\}.$$

Then applying continuity from below along with the earlier weak-type estimate gives

$$\lambda(\{(x,y) : [M\chi_S](x,y) > t\}) \leqslant Ct^{-3}\, ||f||_3^3\,,$$

which proves the claim.

To prove the main result, we define

$$S_n = \{(x,y) \in K : \limsup_{r\to 0} |A_r\chi_K(x,y) - 1| > \frac{1}{n}\}.$$

Next we fix $\epsilon > 0$ and approximate K by a bounded open set $U \supseteq K$ where $\lambda(U\backslash K) < \epsilon$. Note that the stated theorem is true if we replaced K with U. For fixed $r \in (0,1)$ and $(x,y) \in K$ we have

$$\begin{aligned} |A_r\chi_K(x,y) - 1| &\leqslant |A_r\chi_K(x,y) - A_r\chi_U(x,y)| + |A_r\chi_U(x,y) - 1]| \\ &= [A_r\chi_{U\backslash K}](x,y) + |A_r\chi_U(x,y) - 1]| \\ &\leqslant [M\chi_{U\backslash K}](x,y) + |A_r\chi_U(x,y) - 1]|\,. \end{aligned}$$

As $r \to 0$ the last term tends to 0, so if (x,y) lies in S_n then $[M\chi_{U\backslash K}](x,y) > 1/n$. Note that $U\backslash K$ is open, so the claim applies and gives

$$\lambda^*(S_n) \leqslant C(1/n)^{-3}\lambda(U\backslash K)^3 \leqslant Cn^3\epsilon^3.$$

But ϵ was arbitrary, so $\lambda^*(S_n) = \lambda(S_n) = 0$. Finally we have $\lambda(\bigcup_{n=1}^{\infty} S_n) = 0$, so

$$\limsup_{r\to 0} |A_r\chi_K(x,y) - 1| = 0$$

for a.e. (x, y) in K, and the main result follows.

Problem 4. Let K be a non-empty compact subset of $\mathbb{R}^3$. For any Borel probability measure μ on K, define the *Newtonian energy* $I(\mu) \in (0, +\infty]$ by

$$I(\mu) \;:=\; \int_K \int_K \frac{1}{|x-y|}\, d\mu(x)\, d\mu(y)$$

and let R_K be the infimum of $I(\mu)$ over all Borel probability measures μ on K. Show that there exists a Borel probability measure μ such that $I(\mu) = R_K$.

Solution. Let M be the set of all Borel probability measures on K. By the Riesz representation theorem, M is a subset of the unit ball in the dual space $C(K)^*$. Let μ_n be a sequence in M with $I(\mu_n) \to R_K$. By the Banach-Alaoglu theorem, the unit ball in $C(K)^*$ is weak-$*$ compact, and since $C(K)$ is separable, it is also sequentially compact. So by passing to a subsequence if necessary, we have a measure μ in the unit ball of $C(K)^*$ with $\mu_n \to \mu$ in weak-$*$. By applying weak-$*$ convergence to the constant function 1, we see that μ is also a probability measure on K.

Now we claim that $I(\mu) = R_K$. We first need to show that $\mu_n \otimes \mu_n \to \mu \otimes \mu$ in weak-$*$, i.e. that

$$\iint f(x,y)\, d\mu_n(x)\, d\mu_n(y) \;\to\; \iint f(x,y)\, d\mu(x)\, d\mu(y)$$

for all $f \in C(K \times K)$. This is clear for all functions of the form $(x, y) \mapsto g(x)h(y)$ with $g, h \in C(K)$ by the weak-$*$ convergence of μ_n to μ. Let $\mathcal{F}$ be the span of all functions of the above form. Then it's easy to check that $\mathcal{F}$ is dense in $C(K \times K)$ by the Stone-Weierstrass theorem. Thus the desired result holds for all of $C(K \times K)$. This establishes that $\mu_n \otimes \mu_n \to \mu \otimes \mu$ in weak-$*$.

We want to conclude that

$$I(\mu) \;=\; \lim_{n\to\infty} I(\mu_n) \;=\; R_K.$$

We would be done by the weak-$*$ convergence of $\mu_n \otimes \mu_n$ to $\mu \otimes \mu$, except $(x, y) \mapsto \frac{1}{|x-y|}$ isn't continuous on $K \times K$. However, it is lower semicontinuous, so by the portmanteau theorem, we have

$$\liminf_{n\to\infty} I(\mu_n) \;\geqslant\; I(\mu).$$

But $\liminf_{n\to\infty} I(\mu_n) = R_K$ and R_K is the inf of all values of $I(\mu)$, so also $R_K \leqslant I(\mu)$ and thus $I(\mu) = R_K$, so I achieves its minimum. $\square$

Problem 5. Define a Hilbert space

$$H \;:=\; \left\{ u : \mathbb{D} \to \mathbb{R} :\; u \text{ is harmonic and } \int_{\mathbb{D}} |u(x,y)|^2\, dx\, dy < \infty \right\}$$

with inner product $\langle f, g\rangle = \int_{\mathbb{D}} fg\, dx\, dy$.

(a) Show that $f \mapsto f_x(0,0)$ is a bounded linear functional on H.

(b) Compute the norm of this linear functional.

Solution (bad). We show that the norm is $2/\sqrt{\pi}$. Since u is harmonic, u_x also is. So we apply the mean value property on a disc of radius $r \in (0,1)$ to get

$$|u_x(0)| \;=\; \frac{1}{\pi r^2}\left|\int_{B(0,r)} u_x\, dA\right| \;=\; \frac{1}{\pi r^2}\left|\int_{\partial B(0,r)} u\, dy\right|$$

by Green's theorem. So

$$\begin{aligned}
|u_x(0)| &= \frac{1}{\pi r^2}\left|\int_0^{2\pi} u(r\cos\theta, r\sin\theta) r\cos(\theta)\,d\theta\right| \\
|u_x(0)|^2 &\leqslant \frac{1}{\pi^2 r^2}\left(\int_0^{2\pi} u(r\cos\theta, r\sin\theta)^2\,d\theta\right)\left(\int_0^{2\pi}\cos^2\theta\right) \quad \text{by Cauchy-Schwarz} \\
\pi r^2 |u_x(0)|^2 &\leqslant \int_0^{2\pi} u(r\cos\theta, r\sin\theta)^2\,d\theta.
\end{aligned}$$

Multiplying both sides by r and integrating over $r \in [0,1]$ we get

$$\frac{\pi}{4}|u_x(0)|^2 \leqslant \int_{\mathbb{D}} u^2\,dA,$$

so $|u_x(0)| \leqslant \frac{2}{\sqrt{\pi}}||u||_H$. Finally, it's easy to check that $u(x,y) = x$ achieves this bound, so $2/\sqrt{\pi}$ is the operator norm. $\square$

Alternate solution (way better). Since $\mathbb{D}$ is simply connected, u is the real part of an analytic function $f = u + iv$ on $\mathbb{D}$. Write $f(z) = \sum_{n=0}^{\infty} a_n z^n$. We know this power series converges uniformly on compact subsets of $\mathbb{D}$. We have

$$u(re^{i\theta}) = \sum_{n=0}^{\infty}\mathrm{Re}(a_n r^n e^{in\theta}) = \sum_{n=0}^{\infty} r^n(\mathrm{Re}(a_n)\cos(n\theta) - \mathrm{Im}(a_n)\sin(n\theta)).$$

We also know that $u_x = \mathrm{Re}(f')$, so we have $u_x(0) = \mathrm{Re}(a_1)$. We have

$$\begin{aligned}
\int_{\mathbb{D}} u^2\,dA &= \int_0^1\int_0^{2\pi}\left(\sum_{n=0}^{\infty} r^n(\mathrm{Re}(a_n)\cos(n\theta) - \mathrm{Im}(a_n)\sin(n\theta))\right)^2 r\,d\theta\,dr \\
&= \int_0^1 r\int_0^{2\pi}\sum_{n,k=0}^{\infty} r^n r^k(\mathrm{Re}(a_n)\cos(n\theta) - \mathrm{Im}(a_n)\sin(n\theta))(\mathrm{Re}(a_k)\cos(k\theta) - \mathrm{Im}(a_k)\sin(k\theta))\,d\theta\,dr.
\end{aligned}$$

Using the orthonormality properties of sin and cos and the fact that the power series converges uniformly on compact sets, this is equal to

$$\begin{aligned}
&= \int_0^1\sum_{n=0}^{\infty} r^{2n+1}\int_0^{2\pi}\left(\mathrm{Re}(a_n)^2\cos^2(n\theta) + \mathrm{Im}(a_n)^2\sin^2(n\theta)\right)\,d\theta\,dr \\
&= \int_0^1\sum_{n=0}^{\infty} r^{2n+1}\pi(\mathrm{Re}(a_n)^2 + \mathrm{Im}(a_n)^2)\,dr \\
&\geqslant \int_0^1 r^3\pi\,\mathrm{Re}(a_1)^2 = \frac{\pi}{4}\mathrm{Re}(a_1)^2.
\end{aligned}$$

Thus we see that

$$\mathrm{Re}(a_1)^2 \leqslant \frac{4}{\pi}\int_{\mathbb{D}} u^2\,dA,$$

so

$$u_x(0) = \mathrm{Re}(a_1) \leqslant \frac{2}{\sqrt{\pi}}||u||_H.$$

This shows that the operator norm is at most $2/\sqrt{\pi}$. And by inspecting the above proof, we see that equality holds if $\mathrm{Re}(a_n) = \mathrm{Im}(a_n) = 0$ for $n \neq 1$ and $\mathrm{Im}(a_1) = 0$. This is achieved when $f(z) = z$, i.e. $u(x,y) = x$, so the operator norm is exactly $2/\sqrt{\pi}$. Alternatively one could compute directly that $u(x,y) = x$ achieves this bound. $\square$

Problem 6. Let
$$X \ := \ \left\{ \xi \mapsto \int_{\mathbb{R}} e^{i\xi x} f(x)\, dx : f \in L^1(\mathbb{R}) \right\}.$$
Show that (a) X is a subset of $C_0(\mathbb{R})$, (b) X is a *dense* subset of $C_0(\mathbb{R})$, and (c) $X \neq C_0(\mathbb{R})$.

Solution. Note that $\xi \mapsto \int_{\mathbb{R}} e^{i\xi x} f(x)$ is the function $\widehat{f}(-\xi)$. For the sake of a having a convenient notation, we will prove each of these results for the Fourier transform. Obviously (a)-(c) will follow.

(a) Continuity follows immediately from the dominated convergence theorem, since $\left|e^{-i\xi x} f(x)\right| \leqslant |f(x)|$, which is integrable by hypothesis.

By directly calculating the integral, it is easy to see that $\widehat{s}$ lies in $C_0(\mathbb{R})$ when s is a sum of characteristic functions of open intervals. The set of such functions is dense in $L^1(\mathbb{R})$, so given $f \in L^1$ choose s with $||f - s||_1 < \epsilon$. Then $\left|\left|\widehat{f} - \widehat{s}\right|\right|_\infty \leqslant ||f - s||_1 < \epsilon$, and so
$$\lim_{|\xi| \to \infty} \widehat{f}(\xi) \leqslant \lim_{|\xi| \to \infty} s(\xi) + \epsilon = \epsilon.$$
But ϵ was arbitrary, so the limit is 0.
Remark. One could also solve this problem by invoking the density of C_c^∞ (or even C_c^1) in $L^1(\mathbb{R})$ and then applying integration by parts.

(b) We claim that $C_c^\infty(\mathbb{R})$ is dense in $C_c(\mathbb{R})$. To see this, fix $f \in C_c(\mathbb{R})$ and choose M large enough so that $|f(x)| < \epsilon$ when $|x| > M$. Let g be a smooth function such that $|f(x) - g(x)| < \epsilon$ for $x \in [-(M+1), M+1]$. Also let $\beta : \mathbb{R} \to [0,1]$ be a smooth bump function with $\operatorname{supp}(\beta) \subseteq [-(M+1), M+1]$ and which takes the value 1 on $[-M, M]$. Then βg is smooth, and we have $||f - \beta g||_\infty < 2\epsilon$.

So $C_c^\infty(\mathbb{R})$ is dense in $C_c(\mathbb{R})$, and in particular the space of Schwartz functions is dense in $C_c(\mathbb{R})$. The Fourier transform is a bijection on the space of Schwartz functions, so X contains all Schwartz functions which gives a dense subset.

(c) Recall that the Fourier transform $\mathcal{F}$ is an injective bounded linear map from $L^1(\mathbb{R})$ to $C_0(\mathbb{R})$. If the Fourier transform was surjective onto $C_0(\mathbb{R})$ then by the open mapping theorem $\mathcal{F}^{-1} : C_0(\mathbb{R}) \to L^1(\mathbb{R})$ would be bounded.

Let $h = \chi_{[-1,1]}$ and let $h_i \in C_c^\infty(\mathbb{R})$ be a uniformly bounded sequence of functions which converges to h in L^2 (for instance, bump functions would suffice). Also let $g_i = \mathcal{F}^{-1}(h_i)$. Note that the g_i's are Schwartz functions and therefore lie in L^1. (Alternatively, this must be true by the hypothesis of surjectivity.) Now h lies in L^2 and is therefore the Fourier-Plancherel transform of a function g. Since the Fourier-Plancherel transform is an L^2 isometry, we have that $g_i \to g$ in L^2. By passing to a subsequence if necessary, we may assume that $g_i \to g$ pointwise almost everywhere.

On the other hand g is not in L^1, otherwise its Fourier transform would be continuous. Thus by Fatou's lemma, $\lim_{i \to \infty} ||g_i||_1 = \infty$. However this contradicts the boundedness of $\mathcal{F}^{-1}$, since we assumed that the h_i's were uniformly bounded.

Remark. It turns out that $g(x) = \frac{\sin(x)}{x}$. However this wasn't important to us. In fact we could have taken h to be any bounded L^2 function which doesn't agree a.e. with a continuous function.

Problem 7. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $\log|f|$ is absolutely integrable with respect to planar Lebesgue measure. Show that f is constant.

Solution. Suppose that f is not constant. By Liouville there exists $z_0 \in \mathbb{C}$ such that $\log|f(z_0)| > 1$. Recall that $\log|f|$ is subharmonic. By the mean value property we have
$$\int_{\mathbb{R}^2} \log|f(z)| d\lambda = \int_{r=0}^\infty r \int_0^{2\pi} \log|f(z_0 + re^{i\theta})|\, d\theta\, dr \geqslant \int_{r=0}^\infty 2\pi r\, dr = \infty. \quad \square$$

Problem 8a. Let A and B be positive definite $n \times n$ real symmetric matrices with the property

$$\left|\left|BA^{-1}x\right|\right| \leqslant ||x||$$

for all $x \in \mathbb{R}^n$, where $||x||$ denotes the usual Euclidean norm. Show that for each pair $x, y \in \mathbb{R}^n$,

$$z \mapsto \left\langle y, B^z A^{-z} x \right\rangle$$

admits an analytic continuation from $0 < z < 1$ to the whole complex plane.

Solution. Since A and B are symmetric and positive definite, we can write $A = S_A \Lambda_A S_A^{-1}$ and $B = S_B \Lambda_B S_B^{-1}$ where Λ_A and Λ_B are diagonal matrices with positive diagonal entries. Then for $z \in (0,1)$, $A^{-z} = S_A \Lambda_A^z S_A^{-1}$ and $B^z = S_B \Lambda_B^z S_B^{-1}$, where Λ_A^z is simply the matrix gotten by raising each diagonal entry to the power z. The given function is seen to be a polynomial in the zth powers of the eigenvalues of B and the inverses of the eigenvalues of B, and therefore extends to a holomorphic function on $\mathbb{C}$. (Note that $\lambda^z = e^{\log(\lambda) z}$, which is holomorphic.)

Problem 8b. Show that $\left|\left|B^\theta A^{-\theta} x\right|\right| \leqslant ||x||$ for all $0 \leqslant \theta \leqslant 1$.

Solution. For $x, y \in \mathbb{R}^n$, let $f_{x,y}(z)$ be holomorphic function from part (a).

When $\mathrm{Re}(z) = 0$ we note that the eigenvalues of B^z and A^{-z} have norm 1. These matrices are symmetric, so they each have operator norm 1, which implies that

$$|f_{x,y}(z)| = \left|\left\langle y, B^z A^{-z} x\right\rangle\right| \leqslant ||y|| \left|\left|B^z A^{-z} x\right|\right| \leqslant ||y|| \, ||x|| \, .$$

When $\mathrm{Re}(z) = 1$, write $z = 1 + bi$. Then

$$\left|\left|B^z A^{-z}\right|\right|_{\mathrm{op}} = \left|\left|B^{iz} B A^{-1} A^{-iz}\right|\right|_{\mathrm{op}} \leqslant \left|\left|B^{iz}\right|\right|_{\mathrm{op}} \left|\left|BA^{-1}\right|\right|_{\mathrm{op}} \left|\left|A^{-iz}\right|\right|_{\mathrm{op}} \leqslant 1,$$

and so

$$|f_{x,y}(z)| \leqslant ||y|| \left|\left|B^z A^{-z} x\right|\right| \leqslant ||y|| \, ||x|| \, .$$

Also note that $f_{x,y}$ is bounded on the strip $S = \{z : \mathrm{Re}(z) \in [0,1]\}$, since each function λ^z is bounded on the strip (recall the solution to part (a)). By the Hadamard three lines theorem, we conclude that $f_{x,y}$ is bounded by $||x|| \, ||y||$ everywhere in S. (Alternatively one can mimic the proof of this theorem by applying the Phragmen-Lindelof method.)

Finally for $\theta \in [0,1]$ we have

$$\left|\left|B^\theta A^{-\theta} x\right|\right| = \sup_{||y||=1} |f_{x,y}(\theta)| \leqslant ||x|| \, .$$

Problem 9. Let $P(z)$ be a non-constant polynomial, all of whose zeros lie in a half plane $\{z \in \mathbb{C} : \mathrm{Re}(z) < \sigma\}$. Show that all zeros of $P'(z)$ also lie in the same half plane.

Solution. Write $P(z) = (z - a_1) \cdots (z - a_n)$. Then we have

$$\frac{P'(z)}{P(z)} = \frac{1}{z - a_1} + \ldots + \frac{1}{z - a_n}.$$

Suppose that $P'(z) = 0$. If $P(z) = 0$ also, then z is obviously in the same half plane, so assume otherwise. Then in particular we have

$$\begin{aligned} 0 &= \mathrm{Re}\left(\frac{1}{z - a_1}\right) + \ldots + \mathrm{Re}\left(\frac{1}{z - a_n}\right) \\ &= \frac{\mathrm{Re}(z) - \mathrm{Re}(a_1)}{|z - a_1|^2} + \ldots + \frac{\mathrm{Re}(z) - \mathrm{Re}(a_n)}{|z - a_n|^2}. \end{aligned}$$

So

$$\mathrm{Re}(z)\sum_{j=1}^{n}\frac{1}{|z-a_j|^2} \;=\; \sum_{j=1}^{n}\frac{\mathrm{Re}(a_j)}{|z-a_j|^2} \;<\; \sigma\sum_{j=1}^{n}\frac{1}{|z-a_j|^2},$$

so $\mathrm{Re}(z) < \sigma$. $\square$

Problem 10. Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Without using either of the Picard theorems, show that there exist arbitrarily large complex numbers z for which $f(z)$ is a positive real.

Solution. Fix a closed ball B_r centered at 0 of radius r so that $f(z) \in \mathbb{C}\backslash\mathbb{R}_{\geqslant 0}$ for $|z| > r$. By compactness, $|f(z)|$ attains a maximum value R on B_r. Then $f(z) - R$ is a holomorphic function which avoids the poitive real axis.

Let $\phi : \mathbb{C}\backslash\mathbb{R}_{\geqslant 0} \to \mathbb{D}$ be a conformal equivalence of the complex plane with the positive real axis removed, and the open unit disc. Such a map exists by the Riemann mapping theorem. For the sake of being concrete we may take

$$\phi(z) = \frac{\sqrt{z}-i}{\sqrt{z}+i}$$

where $\sqrt{e^{i\theta}} = e^{i\theta/2}$ for $\theta \in [0, 2\pi)$.

The map $z \mapsto \phi(f(z) - R)$ is holomorphic and bounded, and therefore constant by Liouville. So for some constant C, we have $f(z) = \phi^{-1}(C) + R$. We conclude that f is constant.

Problem 11. Let $f(z) = -\pi z\cot(\pi z)$ be a meromorphic function on $\mathbb{C}$.

(a) Locate all poles of f and determine their residues.

(b) Show that for each $n \geqslant 1$ the coefficient of z^{2n} in the Taylor expansion of $f(z)$ about $z = 0$ coincides with

$$a_n \;=\; \sum_{k=1}^{\infty}\frac{2}{k^{2n}}.$$

Solution. (a) We have

$$-\pi z\cot(\pi z) \;=\; \frac{-\pi z\cos(\pi z)}{\sin(\pi z)}.$$

From this representation it is clear that f has simple poles at every nonzero integer. (because $\sin(\pi z)$ has a simple pole at every integer). So to calculate the residue at $z = n$ we have

$$\mathrm{Res}(f, z=n) \;=\; \lim_{z\to n}\frac{-\pi z(z-n)\cos(\pi z)}{\sin(\pi z)} \;=\; \lim_{z\to n} -z\cdot\cos(\pi z)\cdot\frac{\pi(z-n)}{\sin(\pi(z-n))} \;=\; (-1)^{n+1}n.$$

(b) Here we use the other standard representation

$$\pi\cot(\pi z) \;=\; \sum_{k=-\infty}^{\infty}\frac{1}{z-k} \;=\; \frac{1}{z} + \sum_{k=1}^{\infty}\frac{2z}{z^2-k^2},$$

so we have

$$f(z) \;=\; -1 - \sum_{k=1}^{\infty}\frac{2z^2}{z^2-k^2}.$$

Write $f(z) = g(z^2)$ where $g(z) = -1 - \sum_{k=1}^{\infty}\frac{2z}{z-k^2}$. Note that g is holomorphic except at the points where it equals ∞ because the series defining it converges uniformly on compact sets. So the coefficient of z^{2n} in the power series for f is the same as the coefficient of z^n in the power series for g. It now suffices to show that $g^{(n)}(0) = n! \cdot \sum_{k=1}^{\infty} \frac{2}{k^{2n}}$. Write $g(z) = -1 - 2zh(z)$, where $h(z) = \sum_{k=1}^{\infty} \frac{1}{z-k^2}$. Again, h is holomorphic except for at the points where it blows up. Therefore we have

$$g^{(n)}(0) \;=\; -2\sum_{j=0}^{n}\binom{n}{j}(z \mapsto z)^{(j)}(0)h^{(n-j)}(0) \;=\; -2h^{(n-1)}(0).$$

Since the series defining h converges uniformly on compact sets, it can be differentiated term-by-term, so it's easy to see by induction that

$$h^{(n)}(z) \;=\; \sum_{k=1}^{\infty}\frac{(-1)^n n!}{(z-k^2)^{n+1}}.$$

Therefore

$$g^{(n)}(0) \;=\; -2h^{(n-1)}(0) \;=\; n!\sum_{k=1}^{\infty}\frac{2}{k^{2n}}. \quad \square$$

Problem 12. Let $f : \mathbb{H} \to \mathbb{H}$ be a holomorphic function obeying

$$\lim_{y\to\infty} yf(iy) \;=\; i \quad \text{and} \quad |f(z)| \;\leqslant\; \frac{1}{\operatorname{Im}(z)} \quad \text{for all } z \in \mathbb{H}.$$

(a) For $\epsilon > 0$, write $g_\epsilon(x) := \frac{1}{\pi}\operatorname{Im} f(x + i\epsilon)$. Show that

$$f(z + i\epsilon) \;=\; \int_{\mathbb{R}} \frac{g_\epsilon(x)}{x - z}\, dx.$$

(b) Show that there exists a Borel probability measure μ on $\mathbb{R}$ such that

$$f(z) \;=\; \int_{\mathbb{R}} \frac{d\mu(x)}{x - z}\, dx.$$

Solution. (a) We have the Schwarz integral formula for the upper-half plane:

$$f(z) = \frac{1}{\pi i}\int_{\mathbb{R}} \frac{\operatorname{Re} f(x)}{x - z}\,\mathrm{d}x + C$$

for $z \in \mathbb{H}$ and some real constant C. Replacing f with $-if$, we see

$$-if(z) = \frac{1}{\pi i}\int_{\mathbb{R}} \frac{\operatorname{Im} f(x)}{x - z}\,\mathrm{d}x + C \Longrightarrow f(z + i\varepsilon) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\operatorname{Im} f(x + i\varepsilon)}{x - z}\,\mathrm{d}x + iC.$$

We will show in (b) that $C = 0$.

(b) The map $h \mapsto \int_{\mathbb{R}} hg_\epsilon\, dx$ defines a linear function on $C_0(\mathbb{R})$, i.e., the closure of compactly supported continuous functions on $\mathbb{R}$. Thus, we may view $\{g_\epsilon\}_\epsilon \subseteq C_0(\mathbb{R})^*$, the space of Radon measures. We wish to apply Banach-Alaoglu, so we need to show that $||g_\epsilon||_{C_0(\mathbb{R})^*}$ is uniformly bounded.

Write $z = x + iy$. For $\epsilon > 0$, consider the contour γ, which is an upper semicircle centered at $x + i\epsilon$ of radius R. Notice that if $0 < \epsilon < y < R$, then z is in the interior of γ and $\overline{z} + 2i\epsilon$ is outside. Then by the Cauchy integral formula,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_\gamma \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - \overline{z} - 2i\varepsilon}\right) f(\zeta)\mathrm{d}\zeta \\
&= \frac{1}{\pi}\int_\gamma \frac{y - \varepsilon}{(\zeta - z)(\zeta - \overline{z} - 2i\varepsilon)} f(\zeta)\mathrm{d}\zeta \\
&= \frac{1}{\pi}\int_{-R}^{R} \frac{y - \varepsilon}{t^2 + (y - \varepsilon)^2} f(x + i\varepsilon + t)\mathrm{d}t + \frac{i}{\pi}\int_0^{\pi} \frac{y - \varepsilon}{R^2 e^{2i\theta} + (y - \varepsilon)^2} f\left(x + i\varepsilon + Re^{i\theta}\right) Re^{i\theta}\mathrm{d}\theta.
\end{aligned}$$

In the second integral, $\left|f\left(x+i\varepsilon+Re^{i\theta}\right)\right| \leqslant \frac{1}{\varepsilon+R\sin\theta}$, and so the integrand tends to 0 as $R\to\infty$. Thus, by dominated convergence, we may send $R\to\infty$ to get

$$f(z)=\frac{1}{\pi}\int_{\mathbb{R}}\frac{y-\varepsilon}{(t-x)^2+(y-\varepsilon)^2}f(t+i\varepsilon)\mathrm{d}t$$

via a translation change of variables. Taking imaginary parts and multiplying both sides by y, we see

$$\begin{aligned}&\int_{\mathbb{R}}\frac{y-\varepsilon}{(t-x)^2+(y-\varepsilon)^2}g_\varepsilon(t)\mathrm{d}t=\operatorname{Im}f(z)\leqslant\frac{1}{\operatorname{Im}z}\\ \Longrightarrow&\int_{\mathbb{R}}\frac{y(y-\varepsilon)}{(t-x)^2+(y-\varepsilon)^2}g_\varepsilon(t)\mathrm{d}t\leqslant 1.\end{aligned}$$

Sending $y\to\infty$ and applying dominated convergence, we see $\int_{\mathbb{R}}g_\varepsilon(t)\mathrm{d}t\leqslant 1$. Thus, $\{g_\epsilon\}_\epsilon$ is uniformly bounded, so by Banach-Alaoglu and taking $\epsilon=\frac{1}{n}$, we get a subsequence g_{n_k} which converges weakly to a Radon measure μ. In particular, $\frac{1}{x-z}\in C_0(\mathbb{R})$, so by weak convergence,

$$\int_{\mathbb{R}}\frac{g_{n_k}(x)}{x-z}\,\mathrm{d}x\xrightarrow{k\to\infty}\int_{\mathbb{R}}\frac{\mathrm{d}\mu(x)}{x-z},$$

which gives the representation

$$f(z)=\int_{\mathbb{R}}\frac{\mathrm{d}\mu(x)}{x-z}+iC.$$

Notice that if $z=iy$, then $\left|\frac{y}{x-iy}\right|=\frac{y}{\sqrt{x^2+y^2}}$ is bounded. Since μ is a finite measure, we see that $\frac{1}{x-z}\in L^1(\mathbb{R},\mu)$ is dominated by 1, so by dominated convergence,

$$\lim_{n\to\infty}\int_{\mathbb{R}}\frac{y}{x-iy}\,\mathrm{d}\mu(x)=-\frac{1}{i}\mu(\mathbb{R})=i\mu(\mathbb{R}).$$

If $C\neq 0$, then $yf(iy)$ would be unbounded, which implies that $C=0$. Thus,

$$i=\lim_{n\to\infty}yf(iy)=i\mu(\mathbb{R})\Longrightarrow\mu(\mathbb{R})=1,$$

so μ is a probability measure. □

10 Fall 2013

Problem 1. Let U and V be open and connected sets in the complex plane $\mathbb{C}$, and $f:U\to\mathbb{C}$ be a holomorphic function with $f(U)\subseteq V$. Suppose that f is a proper map from U into V, i.e., $f^{-1}(K)\subseteq U$ is compact, whenver $K\subseteq V$ is compact. Then f is surjective.

Solution. We use a connectedness argument. First note that f can't be constant on U, otherwise f isn't proper. Then by the open mapping theorem, $f(U)$ is open.

We claim that $V\backslash f(U)$ is also open. Fix $v\in V\backslash f(U)$, and let $B_1\subseteq B_2\subseteq\ldots\subseteq V$ be a seqence of nested closed balls around v such that $\bigcap_{i\in\mathbb{N}}B_i=v$. We have

$$\varnothing=f^{-1}(\{v\})=f^{-1}\left(\bigcap_{i\in\mathbb{N}}B_i\right)=\bigcap_{i\in\mathbb{N}}f^{-1}(B_i).$$

By properness, each $f^{-1}(B_i)$ is compact. In general, a nested sequence of nonempty compact sets has nontrivial intersection[1] It follows that one of the sets $f^{-1}(B_i)$ must be empty. The interior of B_i is an open neighborhood of v lying in $V\backslash f(U)$. But $v\in V\backslash f(U)$ was arbitrary, so $V\backslash f(U)$ is open.

[1]To see this, consider a sequence consisting of a point from each set.

Since $f(U)$ is nonempty, and V is connected we must have $V = f(U)$. $\square$

Problem 2. Show that there is no function f that is holomorphic near $0 \in \mathbb{C}$ and satisfies

$$f(1/n^2) \;=\; \frac{n^2-1}{n^5}$$

for all large $n \in \mathbb{N}$.

Solution. Since f is holomorphic near 0, there is an $r > 0$ such that f has a power series expansion

$$f(z) \;=\; \sum_{j=0}^{\infty} a_j z^j$$

valid in $B(0, r)$. If f is identically zero then it obviously does not satisfy the condition, so assume it isn't. Then let k be the smallest j for which $a_j \neq 0$, so we can write

$$f(z) \;=\; x^k \sum_{j=k}^{\infty} a_j z^{j-k}.$$

When n is big enough so that $1/n^2 < r$, we have

$$f(1/n^2) \;=\; \frac{1}{n^{2k}} \sum_{j=1}^{\infty} \frac{a_j}{n^{2(j-k)}}.$$

We have the inequalities

$$\begin{aligned} |f(1/n^2)| \;&\leqslant\; \frac{1}{n^{2k}}\left(|a_k| + \frac{1}{n^2}\left|\sum_{j=k+1}^{\infty} \frac{a_j}{n^{2(j-k-1)}}\right|\right) \;\leqslant\; \frac{(3/2)|a_k|}{n^{2k}} \\ |f(1/n^2)| \;&\geqslant\; \frac{1}{n^{2k}}\left(|a_k| - \frac{1}{n^2}\left|\sum_{j=k+1}^{\infty} \frac{a_j}{n^{2(j-k-1)}}\right|\right) \;\geqslant\; \frac{(1/2)|a_k|}{n^{2k}} \end{aligned}$$

for sufficiently large n. Thus if the condition $f(1/n^2) = (n^2-1)/n^5$ is satisfied, we would have

$$\frac{(1/2)|a_k|}{n^{2k}} \;\leqslant\; \frac{n^2-1}{n^5} \;\leqslant\; \frac{(3/2)|a_k|}{n^{2k}}$$

for all sufficiently large n. But since $(n^2-1)/n^5$ is asymptotic to n^{-3} as $n \to \infty$, it can't be $\Theta(n^{-2k})$ for any integer k, and so there is no integer k for which this is true. So f can't satisfy the condition. $\square$

Alternate Solution. By setting $x = 1/n$, we have $f(x^2) = x^3 - x^5$ for all x of the form $1/n$ where $n \in \mathbb{N}$ is large enough. We also have $f(0) = 0$ by continuity. Thus $f(x^2)$ is a holomorphic function on a neighborhood of 0 which agrees with $x^3 - x^5$ on a set with a limit point. So $f(x^2) = x^3 - x^5$ everywhere on a neighborhood of 0. Then for $|z|$ small enough we must have

$$z^3 - z^5 = f(z^2) = f((-z)^2) = (-z)^3 - (-z)^5,$$

which is false for $z \neq 0$.

Problem 3. Does there exist a holomorphic function $f : \mathbb{D} \to \mathbb{C}$ such that

$$\lim_{n\to\infty} |f(z_n)| \;=\; +\infty$$

for all sequences $\{z_n\}$ in $\mathbb{D}$ with $\lim_{n\to\infty} |z_n| = 1$?

Solution. There does not exist such a function. Roughly, we would like to apply the minimum principle on the disk. Unfortunately f may take on the value 0 so this doesn't work directly. We can rectify the situation as follows.

By hypothesis, f cannot have a sequence of zeros approaching the boundary of $\mathbb{D}$. Moreover the zeros of f cannot have a limit point in the interior of $\mathbb{D}$, otherwise f would be identically 0. Moreover each zero of f occurs with finite multiplicity. So by compactness, f has only finitely many zeros $\alpha_1, \ldots a\alpha_n$ in $\mathbb{D}$ counting multiplicity. Let $p(z) = (z - \alpha_1) \ldots (z - \alpha_n)$. Then $p(z)/f(z)$ has removable singularities at the zeros of f, and hence may be regarded as an analytic function on $\mathbb{D}$. By hypothesis, $p(z)/f(z)$ extends continuously to take the value 0 on the boundary of $\mathbb{D}$. But then by the maximum principle, $p(z)/f(z)$ is identically 0, which is a contradiction. $\square$

Problem 4. Let u be a non-negative continuous function on $\overline{\mathbb{D}}\backslash\{0\}$ that is subharmonic on $\mathbb{D}\backslash\{0\}$. Suppose that $u|_{\partial\mathbb{D}} = 0$ and

$$\lim_{r \to 0^+} \frac{1}{r^2 \log(1/r)} \int_{\{z \in \mathbb{C} : 0 < |z| < r\}} u(z)\, d\lambda(z) \;=\; 0,$$

where integration is with respect to Lebesgue measure λ on $\mathbb{C}$. Show that then $u \equiv 0$.

Solution. First we want to show that $u(z) = o(\log|1/z|)$ as $|z| \to 0$. Fix $\epsilon < 0$. By the hypothesis, let $|z|$ be small enough so that

$$\int_{\{z \in \mathbb{C} : 0 < |w| < 3|z|/2\}} u(w)\, d\lambda(w) \;<\; \epsilon |z|^2 \log|1/z|.$$

Then by the mean value property for subharmonic functions we have

$$u(z) \;\leqslant\; \frac{1}{\pi(|z|/2)^2} \int_{\{w \in \mathbb{C} : |w - z| < (1/2)|z|\}} u(w)\, d\lambda(w) \;\leqslant\; \frac{4\pi}{|z|^2} \int_{\{w \in \mathbb{C} : 0 < |w| < 3|z|/2\}} u(w)\, d\lambda(w) \;<\; \frac{4\pi\epsilon |z|^2 \log|1/z|}{|z|^2},$$

which shows that $u(z) = o(\log|1/z|)$ as $|z| \to 0$.

Now let $\alpha > 0$ and note that the function $f(z) := \alpha \log|1/z|$ is harmonic on $\mathbb{D}\backslash\{0\}$. Thus we know that $u - f$ does not have a maximum value inside $\mathbb{D}\backslash\{0\}$. Notice that since $u(z) = o(\log|1/z|)$ as $|z| \to 0$, $u(z) - f(z) \to -\infty$ as $|z| \to 0$. Thus there exists an $r > 0$ such that $u(z) - f(z) \leqslant 0$ for $|z| \leqslant r$. Now on the compact set $S := \{z \in \mathbb{C} : r \leqslant |z| \leqslant 1\}$, $u - f$ is continuous so it achieves a maximum. But the maximum must be achieved on the boundary of f because $u - f$ doesn't have any maxima inside $\mathbb{D}\backslash\{0\}$. Since $u - f = 0$ on $\partial\mathbb{D}$ and $u - f \leqslant 0$ on $|z| = r$ by choice of r, this implies that $u - f \leqslant 0$ in all of $\mathbb{D}\backslash\{0\}$. So $u(z) - \alpha \log|1/z| \leqslant 0$ for all $z \in \mathbb{D}\backslash\{0\}$, and since α is arbitrary this implies $u(z) \leqslant 0$ for all $z \in \mathbb{D}\backslash\{0\}$, which since $u \geqslant 0$ by hypothesis gives that u is identically zero. $\square$

Problem 5. Let $\{f_n\}$ be a sequence of holomorphic functions on $\mathbb{D}$ and suppose that

$$\int_{\mathbb{D}} |f_n(z)|\, d\lambda(z) \;\leqslant 1$$

for all $n \in \mathbb{N}$. Show that then there exists a subsequence $\{f_{n_k}\}$ that converges uniformly on all compact subsets of $\mathbb{D}$.

Solution. We would like to show that the functions f_n form a normal family. Since each f_n is holomorphic, this is equivalent to verifying that the f_n's are uniformly bounded on the closed ball $B_r = B(0, r)$ for each $r \in (0, 1)$. (Note that each compact subset of $\mathbb{D}$ is contained in some such ball.) Fix $z_0 \in B_r$ and let $U = B(z_0, 1 - |z_0|)$. Applying the mean value property we have

$$1 \geqslant \int_U |f_n(z)| d\lambda(z) \geqslant \left| \int_U f_n(z) d\lambda \right| \geqslant \pi(1 - |z_0|)^2 |f(z_0)| \geqslant \pi(1 - r)^2 |f(z_0)|.$$

Therefore $|f(z_0)| \leqslant \frac{1}{\pi(1-r)^2}$ for all $z_0 \in B_r$, and so f is uniformly bounded on compact sets. $\square$

Problem 6. Let $U \subseteq \mathbb{C}$ be a bounded open set with $0 \in U$, and $f : U \to \mathbb{C}$ be holomorphic with $f(U) \subseteq U$ and $f(0) = 0$. Show that $|f'(0)| \leqslant 1$. Hint: Consider the iterates $f^n = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}$ of f.

Solution. First we prove by induction that $(f^n)'(0) = (f'(0))^n$. The case $n = 1$ is obviously true. Supposing $(f^{n-1})'(0) = (f'(0))^{n-1}$, since $f(0) = 0$ we have

$$(f^n)'(0) \;=\; (f^{n-1} \circ f)'(0) \;=\; (f^{n-1})'(f(0))f'(0) \;=\; (f'(0))^n,$$

so the induction is finished. Note that since U is a bounded set and $f(U) \subseteq U$, also $f^n(U) \subseteq U$ for all n and there is an M such that $|f^n(z)| \leqslant M$ for all $z \in U$ and all n. Since U is open, let $R > 0$ be such that $\overline{B(0,R)} \subseteq U$. Then applying the Cauchy estimate to f^n, we get

$$|f'(0)|^n \;=\; |(f^n)'(0)| \;\leqslant\; \frac{1}{R} \sup_{|z|=R} |f^n(z)| \;\leqslant\; \frac{M}{R}$$

for all n. If $|f'(0)| > 1$ this would be impossible because $|f'(0)|^n$ would tend to infinity as $n \to \infty$, so $|f'(0)| \leqslant 1$. $\square$

Problem 7. Show that there is a dense set of functions $f \in L^2([0,1])$ such that $x \mapsto x^{-1/2} f(x) \in L^1([0,1])$ and $\int_0^1 x^{-1/2} f(x)\,dx = 0$.

Solution. Let $S := \{f \in L^2([0,1]) : x \mapsto x^{-1/2} f(x) \in L^1([0,1]) \text{ and } \int_0^1 x^{-1/2} f(x)\,dx = 0\}$. Since the set of continuous functions with compact support properly contained in $[0,1]$ is dense in $L^2([0,1])$, it suffices to show that S is dense in that set. Let g be a function which is continuous on $[\delta, 1]$ and identically zero on $[0, \delta]$ for some fixed $\delta > 0$. Fix $\epsilon > 0$. Define

$$I \;:=\; \int_\delta^1 x^{-1/2} g(x)\,dx \;=<\; \infty$$

because $x^{-1/2}$ is bounded on $[\delta, 1]$. Now define the function f_ϵ by

$$f_\epsilon(x) \;:=\; \begin{cases} g(x) & x \in [\delta, 1] \\ \frac{-I\epsilon}{\delta^\epsilon} x^{-1/2+\epsilon} & x \in (0, \delta) \\ 0 & x = 0 \end{cases}.$$

We calculate

$$\int_0^1 x^{-1/2} f_\epsilon(x)\,dx \;=\; \frac{-I\epsilon}{\delta^\epsilon} \int_0^\delta x^{-1+\epsilon}\,dx + I \;=\; 0$$

and

$$\begin{aligned} ||f_\epsilon - g||_2^2 \;&=\; \int_0^\delta |f\epsilon(x) - g(x)|^2\,dx \;\leqslant\; 4\int_0^\delta |f_\epsilon(x)|^2\,dx \\ &<\; \frac{4I^2\epsilon^2}{\delta^{2\epsilon}} \int_0^\delta x^{-1+2\epsilon}\,dx \;=\; \frac{4I^2\epsilon^2}{\delta^{2\epsilon}} \cdot \frac{\delta^{2\epsilon}}{2\epsilon} \;=\; 2I^2\epsilon, \end{aligned}$$

which can be made as small as desired. So S is dense in $L^2([0,1])$. $\square$

Problem 8(a). Compute

$$\lim_{k\to\infty} \int_0^k x^n \left(1 - \frac{x}{k}\right)^k\,dx$$

where $n \in \mathbb{N}$.

Solution. Define the functions $f_k(x) := x^n(1 - x/k)^k \cdot \chi_{[0,k]}$. For each $x \in [0, \infty)$, as soon as $k \geqslant x$

we have $f_k(x) = x^n(1 - x/k)^k$, so we see that $f_k(x) \to x^n e^{-x}$ pointwise on $[0, \infty)$. Also note that for each k, $f_k(x) \geqslant 0$ for all $x \in [0, \infty)$ because $(1 - x/k) \geqslant 0$ for $x \in [0, k]$ and $f_k(x) = 0$ for $x > k$. We want to show that $f_k(x) \leqslant f_{k+1}(x)$ for all x so that we can use the Monotone Convergence Theorem. By the AM-GM inequality, we have

$$\left(1 \cdot \left(1 - \frac{x}{k}\right)^k\right)^{1/(k+1)} \leqslant \frac{1 + k\left(1 - \frac{x}{k}\right)}{k+1} = \frac{1 + k - x}{k+1} = 1 - \frac{x}{k+1},$$

so $(1 - x/k)^k \leqslant (1 - x/(k+1))^{k+1}$. This establishes that $f_k \leqslant f_{k+1}$. Since $x^n e^{-x}$ is integrable on $[0, \infty)$, the Monotone Convergence Theorem gives

$$\lim_{k \to \infty} \int_0^k x^n \left(1 - \frac{x}{k}\right)^k dx = \int_0^\infty f_k(x)\, dx = \int_0^\infty x^n e^{-x}\, dx = n! \quad \square$$

Problem 8(b). Compute

$$\lim_{k \to \infty} \int_0^\infty \left(1 + \frac{x}{k}\right)^{-k} \cos(x/k)\, dx.$$

Solution. For each $k \geqslant 2$ define $f_k(x) := (1+(x/k))^{-k} \cos(x/k)$. For a fixed $x \in [0, \infty)$, we have $\cos(x/k) \to 1$ as $k \to \infty$ and $(1 + (x/k))^{-k} \to e^{-x}$ as $k \to \infty$. Thus $f_k(x)$ converges pointwise to e^{-x} on $[0, \infty)$. Using the same AM-GM inequality argument as in the problem above, we see

$$\left(1 \cdot \left(1 + \frac{x}{k}\right)^k\right)^{1/(k+1)} \leqslant \frac{1 + k\left(1 + \frac{x}{k}\right)}{k+1} = \frac{k + 1 + x}{k+1} = 1 + \frac{x}{k+1},$$

which establishes $(1 + x/k)^k \leqslant (1 + x/(k+1))^{k+1}$. Thus $f_k(x) \geqslant f_{k+1}(x)$ for all $x \in [0, \infty)$. So we have the estimate

$$|f_k(x)| \leqslant \left(1 + \frac{x}{k}\right)^{-k} \leqslant \frac{1}{(1 + x/2)^2}$$

which is integrable on $[0, \infty)$, for all $k \geqslant 2$. Thus by the Dominated Convergence Theorem we have

$$\lim_{k \to \infty} \int_0^\infty \left(1 + \frac{x}{k}\right)^{-k} \cos(x/k)\, dx = \int_0^\infty e^{-x} = 1. \quad \square$$

Note. Alternate way of showing that Dominated Convergence applies: we just need to show that $0 \leqslant (1 - x/k)^k \leqslant e^{-x}$ for all k and all $x \in [0, k]$. Equivalent, we want $k \log(1 - x/k) \leqslant -x$. Expanding $t \mapsto \log(1 - t)$ in a power series around $t = 0$ gives this.

Problem 9. Let X be a Banach space, Y be a normed linear space, and $B : X \times Y \to \mathbb{R}$ be a bilinear function. Suppose that for each $x \in X$ there exists a constant $C_x \geqslant 0$ such that $|B(x, y)| \leqslant C_x ||y||$ for all $y \in Y$, and for each $y \in Y$ there exists $C_y \geqslant 0$ such that $|B(x, y)| \leqslant C_y ||x||$ for all $x \in X$.

Show that then there exists a constant $C \geqslant 0$ such that $|B(x, y)| \leqslant C||x||||y||$ for all $x \in X$ and all $y \in Y$.

Solution. For each $y \in Y$, define the function $T_y : X \to \mathbb{R}$ by $T_y(x) = B(x, y)$. Since B is bilinear, T_y is a linear functional on X. By hypothesis, for each y we have $|T_y(x)| = |B(x, y)| \leqslant C_y ||x||$, so T_y is actually a bounded linear functional. Let $\mathcal{F} = \{T_y : ||y|| = 1\}$. This is a family of bounded linear functionals on X, and for each $x \in X$ we have by the other hypothesis

$$\sup_{||y||=1} |T_y(x)| = \sup_{||y||=1} |B(x, y)| \leqslant C_x < \infty.$$

Thus since X is a Banach space, we can apply the uniform boundedness principle to conclude that $\sup_{||y||=1} ||T_y|| < \infty$. This means that there is a $C \geqslant 0$ such that $||T_y|| \leqslant C$ for any $||y|| = 1$, which means that $|T_y(x)| =$

$|B(x,y)| \leqslant C||x||$ for any $x \in X$ and any $||y|| = 1$. Then by linearity in the second variable we get that $|B(x,y)| \leqslant C||x||||y||$ for any $x \in X$, $y \in Y$. $\square$

Problem 10a. Let $f \in L^2(\mathbb{R})$ and define $h(x) = \int_{\mathbb{R}} f(x-y)f(y)\,dy$ for $x \in \mathbb{R}$. Show that then there exists a function $g \in L^1(\mathbb{R})$ such that

$$h(\xi) \;=\; \int_{\mathbb{R}} e^{-i\xi x} g(x)\,dx$$

for $\xi \in \mathbb{R}$, i.e. h is the Fourier transform of a function in $L^1(\mathbb{R})$.

Solution. We are motivated by the fact that if g were such a function, then we would have $\mathcal{F}(g) = f * f = \mathcal{F}(\mathcal{F}^{-1}(f)) * \mathcal{F}(\mathcal{F}^{-1}(f)) = \mathcal{F}(\mathcal{F}^{-1}(f)^2)$, so $g = \mathcal{F}^{-1}(f)^2$.

Let $\mathcal{F}$ denote the Fourier-Plancherel transform. Recall it is an isometric isomorphism $L^2 \to L^2$. Given $f \in L^2$, define $g := \mathcal{F}^{-1}(f)^2$. It's clear that $g \in L^1$. Let $\widehat{\cdot}$ denote the regular Fourier transform $L^1 \to L^\infty$. Recall that $\widehat{\cdot}$ and $\mathcal{F}(\cdot)$ agree on $L^1 \cap L^2$. We verify

$$\widehat{g} \;=\; \mathcal{F}^{-1}\widehat{(f)\mathcal{F}^{-1}}(f) \;=\; \mathcal{F}(\mathcal{F}^{-1}(f)) * \mathcal{F}(\mathcal{F}^{-1}(f)) \;=\; f * f.$$

In the previous line we used the identity $\widehat{ab} = \mathcal{F}(a) * \mathcal{F}(b)$ for $a, b \in L^2$. Here is a proof of it (not sure if this would be required on the qual or not):
We know the identity holds for Schwartz functions (this follows from basic properties of the Fourier transform and a lot of Fubini's theorem). Let a_n, b_n be Schwartz functions with $a_n \to a$ and $b_n \to b$ in L^2. We know that $\widehat{a_n b_n} = \mathcal{F}(a_n) * \mathcal{F}(b_n)$ for each n, so it suffices to show that $\widehat{a_n b_n} \to \widehat{ab}$ and $\mathcal{F}(a_n) * \mathcal{F}(b_n) \to \mathcal{F}(a) * \mathcal{F}(b)$ in L^∞. We have

$$\begin{aligned}
\left|\left|\widehat{a_n b_n} - \widehat{ab}\right|\right|_{L^\infty} &= \left|\left|\widehat{a_n b_n - ab}\right|\right|_{L^\infty} \;\leqslant\; ||a_n b_n - ab||_{L^1} \;\leqslant\; ||(a_n - a)b||_{L^1} + ||(b_n - b)a||_{L^1} \\
&\leqslant ||a_n - a||_{L^2}\,||b||_{L^2} + ||b_n - b||_{L^2}\,||a||_{L^2} \;\to\; 0 \\
||\mathcal{F}(a_n) * \mathcal{F}(b_n) - \mathcal{F}(a) * \mathcal{F}(b)||_{L^\infty} &\leqslant ||\mathcal{F}(a_n - a) * \mathcal{F}(b)||_{L^\infty} + ||\mathcal{F}(b_n - b) * \mathcal{F}(a)||_{L^\infty} \\
&\leqslant ||\mathcal{F}(a_n - a)||_{L^2}\,||\mathcal{F}(b)||_{L^2} + ||\mathcal{F}(b_n - b)||_{L^2}\,||\mathcal{F}(a)||_{L^2} \\
&= ||a_n - a||_{L^2}\,||b||_{L^2} + ||b_n - b||_{L^2}\,||a||_{L^2} \;\to\; 0. \quad \square
\end{aligned}$$

Problem 10b. Conversely, show that if $g \in L^1(\mathbb{R})$, then there is a function $f \in L^2(\mathbb{R})$ such that the Fourier transform of g is given by $x \mapsto h(x) := \int_{\mathbb{R}} f(x-y)f(y)\,dy$.

Solution. Using a similar motivating argument as in part (a), we see that we want to set $f = \mathcal{F}^{-1}(\sqrt{\breve{g}})$ (recall that $\breve{g}(x) := g(-x)$ and that for Schwartz functions, $\mathcal{F}^2(s) = \breve{s}$). This is a little annoying because $\sqrt{\breve{g}}$ isn't even necessarily defined. But in general, for measurable functions $h : \mathbb{R} \to \mathbb{C}$, we can define $\sqrt{h(x)}$ to be the square root defined by removing the positive real axis if $h(x)$ is not a positive real, and define it to be the positive real square root if $h(x)$ is a positive real. The representation

$$\sqrt{h} \;=\; sqrt_1(h \cdot \chi_{\{x : h(x) \notin \mathbb{R}^+\}}) + sqrt_2(h \cdot \chi_{\{x : h(x) \in \mathbb{R}^+\}})$$

where $sqrt_1$ is the branch cut square root and $sqrt_2$ is the positive real square root immediately shows that the square root defined this way is measurable, and it's clear that $\sqrt{h} \in L^2$ if and only if $h \in L^1$. So the definition $f := \mathcal{F}^{-1}(\sqrt{\breve{g}}) \in L^2$ makes sense. Again, we just verify

$$f * f \;=\; \mathcal{F}^{-1}(\sqrt{\breve{g}}) * \mathcal{F}^{-1}(\sqrt{\breve{g}}) \;=\; \mathcal{F}(\breve{\breve{g}}) \;=\; \mathcal{F}(g).$$

Here we have used the identity $\mathcal{F}^{-1}(a) * \mathcal{F}^{-1}(b) = \mathcal{F}(\breve{ab})$ for $a, b \in L^2$. This is proven using a similar argument as for the corresponding identity in part (a), recalling that $\mathcal{F}^{-1} = \mathcal{F}^3$ for Schwartz functions. $\square$

Problem 11. Consider the space $C([0,1])$ of real-valued continuous functions on the unit interval $[0,1]$. We denote by $||f||_\infty := \sup_{x\in[0,1]} |f(x)|$ the supremum norm and by $||f||_2 := \left(\int_0^1 |f(x)|^2\right)^{1/2}$ the L^2-norm of a function $f \in C([0,1])$.

Let S be a subspace of $C([0,1])$. Show that if there exists a constant $K \geqslant 0$ such that $||f||_\infty \leqslant K\,||f||_2$ for all $f \in S$, then S is finite-dimensional.

Solution. Let $\overline{S}$ denote the closure of S with respect to the L^2 norm. It obviously suffices to show that $\overline{S}$ is finite-dimensional. First we show that $\overline{S}$ is still contained in $C([0,1])$. Suppose $f \in \overline{S}$, then there is a sequence $f_n \in S$ with $||f_n - f||_2 \to 0$ as $n \to \infty$. For any n, m, we have $||f_n - f_m||_\infty \leqslant K\,||f_n - f_m||_2$, and since $\{f_n\}$ converges in L^2, it is also Cauchy in L^2, so by the above inequality it is also a Cauchy sequence in $C([0,1])$. Since $C([0,1])$ is complete, there is some $g \in C([0,1])$ with $||f_n - g||_\infty \to 0$ as $n \to \infty$. Note that since $||h||_2 \leqslant ||h||_\infty$ for any $h \in C([0,1])$, we have

$$||g - f||_2 \;\leqslant\; ||g - f_n||_2 + ||f_n - f||_2 \;\leqslant\; ||g - f_n||_\infty + ||f_n - f||_2 \to 0$$

as $n \to \infty$. Thus $||g - f||_2 = 0$, so $f = g$ in L^2, hence f is continuous. Thus $\overline{S} \subseteq C([0,1])$.

For each $x \in [0,1]$, define the map between normed vector spaces $\phi_x : (\overline{S}, ||\cdot||_2) \to \mathbb{R}$ by $f \mapsto f(x)$. This is clearly a linear functional on the space $\overline{S}$. For any $f \in \overline{S}$, we have

$$|\phi_x(f)| \;=\; |f(x)| \;\leqslant\; ||f||_\infty \;\leqslant\; K\,||f||_2\,,$$

so in fact ϕ_x is a bounded linear functional on $\overline{S}$. Since $\overline{S}$ is a closed subspace of the Hilbert space $L^2([0,1])$, it is also a Hilbert space, and thus by the Riesz representation theorem for each x there exists some $g_x \in \overline{S}$ such that $f(x) = \phi_x(f) = \langle f, g_x\rangle$ for all $f \in \overline{S}$. Note also that for each x

$$||g_x||_2^2 \;=\; |\langle g_x, g_x\rangle| \;=\; |g_x(x)| \;\leqslant\; ||g_x||_\infty \;\leqslant\; K\,||g_x||_2\,,$$

so $||g_x||_2 \leqslant K$.

Now let $\{f_1, \ldots, f_N\}$ be any linearly independent set in $\overline{S}$. By applying the Gram-Schmidt process if necessary we may assume that it is an orthonormal set. Then by Bessel's inequality, we have for each x that

$$\sum_{j=1}^N |f_j(x)|^2 \;=\; \sum_{j=1}^N |\langle f_j, g_x\rangle|^2 \;\leqslant\; ||g_x||_2^2 \;\leqslant\; K^2.$$

Then integrating both sides from 0 to 1 we get

$$K^2 \;\geqslant\; \sum_{j=1}^N \int_0^1 |f_j(x)|^2\,dx \;=\; \sum_{j=1}^N ||f_j||_2^2 \;=\; N.$$

This shows that a linearly independent set in $\overline{S}$ can have at most K^2 elements and thus $\dim(\overline{S}) \leqslant K^2 < \infty$. □

Problem 12(a). Let $f : [0,1] \to \mathbb{R}$ be a continuous function that is absolutely continuous on each interval $[\epsilon, 1]$ with $0 < \epsilon \leqslant 1$. Show that f is not necessarily absolutely continuous on $[0,1]$.

Solution. Let $f(x) = x\sin(1/x)$ for $x > 0$ and $f(0) = 0$. For any $x > 0$, f is differentiable and

$$f'(x) \;=\; \sin(1/x) - \frac{cos(1/x)}{x}.$$

So for a fixed $\epsilon > 0$ and any $x \in [\epsilon, 1]$, we have

$$|f'(x)| \;\leqslant\; |sin(1/x)| + \left|\frac{\cos(1/x)}{x}\right| \;\leqslant\; 1 + \frac{1}{\epsilon}.$$

Thus f' is bounded on $[\epsilon, 1]$, so f is Lipschitz and thus f is absolutely continuous on $[\epsilon, 1]$.

Let $x_n = 1/2\pi n$ and $y_n = 1/(\pi + 2\pi n)$. Note that we have

$$\begin{aligned} |x_n - y_n| &= \left|\frac{\pi}{4\pi^2 n^2 + 2\pi^2 n}\right| < \frac{1}{n^2} \\ |f(x_n) - f(y_n)| &= |x_n + y_n| = \left|\frac{\pi + 4\pi n}{4\pi^2 n^2 + 2\pi^2 n}\right|. \end{aligned}$$

In particular, $\sum_{n=1}^{\infty} |x_n - y_n| < \infty$ and $\sum_{n=1}^{\infty} |f(x_n) - f(y_n)| = \infty$. Suppose that f were absolutely continuous on $[0, 1]$. Then pick $\epsilon = 1$ and let δ be such that for any N, M, $\sum_{n=N}^{M} |x_n - y_n| < \delta$ implies $\sum_{n=N}^{M} |f(x_n) - f(y_n)| < 1$. But by the convergence and divergence of the above series, we can pick an N such that $\sum_{n=N}^{\infty} |x_n - y_n| < \delta$ and then we can pick an M such that $\sum_{n=N}^{M} |f(x_n) - f(y_n)| > 1$, which is a contradiction. Thus f is not absolutely continuous on $[0, 1]$. □

Problem 12(b). Show that if f is of bounded variation on $[0, 1]$, then f is absolutely continuous on $[0, 1]$.

Solution. Let $TV_{[a,b]}$ denote the total variation of f on the interval $[a, b]$. Since f is continuous and of bounded variation on $[0, 1]$, we can show that $TV_{[0,x]}$ is a continuous function of x. Fix $\epsilon > 0$. Since f is of bounded variation, pick a partition $\{0 = t_0 < t_1 < \cdots < t_n = 1\}$ such that

$$\sum_{j=1}^{n} |f(t_j) - f(t_{j-1})| > TV_{[0,1]} - \epsilon.$$

Since f is continuous, we can pick an $h \in (0, t_1)$ such that $|f(h) - f(0)| < \epsilon$. By adding h into the original partition, the variation can only increase. Furthermore, $\{h, t_1, \ldots, t_n\}$ is a partition of $[h, 1]$, so we get

$$\epsilon + TV_{[h,1]} > |f(h) - f(0)| + |f(t_1) - f(h)| + \sum_{j=2}^{n} |f(t_j) - f(t_{j-1})| > TV_{[0,1]} - \epsilon,$$

which implies $TV_{[0,h]} = TV_{[0,1]} - TV_{[h,1]} < 2\epsilon$. Since $TV[0, x]$ is an increasing function, this shows that it is continuous at 0.

Now we want to show that f is absolutely continuous on $[0, 1]$. Fix $\epsilon > 0$ and let $h > 0$ be such that $TV_{[0,h]} < \epsilon$. By hypothesis, f is absolutely continuous on $[h, 1]$, so let $\delta > 0$ be as in the definition of absolute continuity on $[h, 1]$. Let $a_1 < b_1 \leqslant a_2 < \cdots \leqslant a_n < b_n$ be such that $\sum_{k=1}^{n} b_k - a_k < \delta$. By dividing one of the intervals into two subintervals, the variation can only increase, so without loss of generality we may assume that $h \notin (a_k, b_k)$ for any k. Let ℓ be the index such that $b_\ell \leqslant h \leqslant a_{\ell+1}$. Since $\{a_1, b_1, \ldots, a_\ell, b_\ell\}$ is a partition of $[0, h]$, by the choice of h we have

$$\sum_{j=1}^{\ell} |f(b_j) - f(a_j)| \leqslant TV_{[0,h]} < \epsilon.$$

By absolute continuity on $[h, 1]$, we have

$$\sum_{j=\ell+1}^{n} |f(b_j) - f(a_j)| < \epsilon$$

and hence

$$\sum_{j=1}^{n} |f(b_j) - f(a_j)| < 2\epsilon,$$

which establishes that f is absolutely continuous on $[0, 1]$. □

11 Spring 2014

Problem 1. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. For each $t \in \mathbb{R}$ let e_t be the characteristic function of the interval (t, ∞). Prove that if $f, g : X \to \mathbb{R}$ are $\mathcal{A}$-measurable, then $||f - g||_{L^1(X)} = \int_{\mathbb{R}} ||e_t \circ f - e_t \circ g||_{L^1(X)} \, dt$.

Solution. We have

$$\begin{aligned}\int_{\mathbb{R}} ||e_t \circ f - e_t \circ g||_{L^1} \, dt &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |e_t \circ f(x) - e_t \circ g(x)| dx \right) dt \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |e_t \circ f(x) - e_t \circ g(x)| dt \right) dx,\end{aligned}$$

where we are justified in switching the order of integration by Tonelli's theorem since μ is σ-finite. Now observe that $|e_t \circ f(x) - e_t \circ g(x)|$ is equal to 1 if either $f(x) < t \leqslant g(x)$ or $g(x) < t \leqslant f(x)$ and 0 otherwise. Thus the inner integral evaluates to $|f(x) - g(x)|$, which gives the desired result. $\square$

Problem 2. Let $f \in L^1(\mathbb{R}, dx)$ and $\beta \in (0, 1)$. Prove that

$$\int_{\mathbb{R}} \frac{|f(x)|}{|x - a|^\beta} dx < \infty$$

for (Lebesgue) a.e. $a \in \mathbb{R}$.

Solution. Write $F(a) = \int_{\mathbb{R}} \frac{|f(x)|}{|x-a|^\beta} dx$. We would be done if we could show that $\int_{\mathbb{R}} F(a) da < \infty$. Unfortunately this isn't true. However it is enough to show that $\int_{\mathbb{R}} u(a) F(a) da < \infty$ for some strictly positive u.

We take $u(a) = \min(a^{-2}, 1)$, with the convention that $u(0) = 1$. By Tonelli's theorem, we write

$$\begin{aligned}\int_{\mathbb{R}} u(a) F(a) da &= \int_{\mathbb{R}} u(a) \left(\int_{\mathbb{R}} \frac{|f(x)|}{|x - a|^\beta} dx \right) da \\ &= \int_{\mathbb{R}} |f(x)| \left(\int_{\mathbb{R}} \frac{u(a)}{|a - x|^\beta} da \right) dx.\end{aligned}$$

Let I be the interval $[x - 1, x + 1]$. We bound the inner integral as follows:

$$\begin{aligned}\int_{\mathbb{R}} \frac{u(a)}{|a - x|^\beta} da &= \int_I \frac{u(a)}{|a - x|^\beta} da + \int_{\mathbb{R} \setminus I} \frac{u(a)}{|a - x|^\beta} da \\ &\leqslant \int_I \frac{1}{|a - x|^\beta} da + \int_{\mathbb{R} \setminus I} u(a) da \\ &\leqslant \int_I \frac{1}{|a - x|^\beta} da + \int_{\mathbb{R}} u(a) da \\ &= \int_{[-1,1]} \frac{1}{|a|^\beta} da + \int_{\mathbb{R}} u(a) da,\end{aligned}$$

where we applied a linear change of variables in the last step. But $\beta \in (0, 1)$ so the first integral is finite, and it's clear the second integral integral is finite. So there is a constant C, independent of x such that $\int_{\mathbb{R}} \frac{u(a)}{|a-x|^\beta} < C$. Returning to the original integral, we have

$$\int_{\mathbb{R}} u(a) F(a) da \leqslant \int_{\mathbb{R}} C |f(x)| dx = C \, ||f||_{L^1},$$

which is finite by hypothesis. It follows that $F(a) < \infty$ for a.e. $a \in \mathbb{R}$. $\square$

Problem 3.1. Let $[a,b]$ be a finite interval and let $f : [a,b] \to \mathbb{R}$ be a bounded Borel measurable function. Prove that $\{x \in [a,b] : f \text{ is continuous at } x\}$ is Borel measurable.

Solution. Let

$$E_n \;:=\; \{x \in [a,b] : \text{there exists a } \delta > 0 \text{ such that } |f(a) - f(b)| < 1/n \text{ for any } a, b \in (x-\delta, x+\delta)\}.$$

Note that f is continuous at x if and only if $x \in \bigcap_{n=1}^{\infty} E_n$. So to show the set of continuities of f is Borel it suffices to show that each E_n is an open set. Let $x \in E_n$ and let δ be as in the definition of E_n. We show that $(x - \delta/2, x + \delta/2) \subseteq E_n$. Indeed, if $|y - x| < \delta/2$, then for any $a, b \in (y - \delta/2, y + \delta/2)$ we have $|a - x|, |b - x| < \delta$, so $|f(a) - f(b)| < 1/n$. Thus $y \in E_n$ with the choice $\delta/2$, so E_n is open. $\square$

Problem 3.2. Prove that f is Riemann integrable if and only if it is continuous almost everywhere.

Solution. Let $\overline{I}$ be the upper Riemann integral of f and $\underline{I}$ be the lower Riemann integral of f. We know that we can find a sequence of nested partitions $P_1 \subseteq P_2 \subseteq \ldots$ of $[a,b]$ such that the mesh size of P_n tends to 0 as $n \to \infty$ and $\lim_{n\to\infty} U(f, P_n) = \overline{I}$ and $\lim_{n\to\infty} L(f, P_n) = \underline{I}$. Denote by $E_{k,n}$ the kth subinterval of the partition P_n and let $m_{k,n}$ and $M_{k,n}$ be the infimum and supremum respectively of f on $E_{k,n}$. Define the functions U_n and L_n by

$$\begin{aligned} L_n \;&:=\; \sum_k m_{k,n} \chi_{E_{k,n}} \\ U_n \;&:=\; \sum_k M_{k,n} \chi_{E_{k,n}}. \end{aligned}$$

By construction we have $\int_a^b U_n = U(f, P_n)$ and $\int_a^b L_n = L(f, P_n)$. Also, since the partitions are nested, we have

$$L_1 \leqslant L_2 \leqslant \ldots \leqslant f \leqslant \ldots \leqslant U_2 \leqslant U_1.$$

Since $\{U_n\}$ and $\{L_n\}$ are both monotone, they converge pointwise to functions U and L respectively such that $L \leqslant f \leqslant U$. By applying the Dominated Convergence Theorem to both L_n and U_n with U_1 as the dominating function, we see that $\int_a^b L = \underline{I}$ and $\int_a^b U = \overline{I}$. Now we have that f is Riemann intergrable if and only if $\overline{I} = \underline{I}$, which happens if and only if $\int_a^b L = \int_a^b U$, which since $L \leqslant U$ happens if and only if $L = U$ almost everywhere, and since $L \leqslant f \leqslant U$ this happens if and only if $L(x) = f(x) = U(x)$ almost everywhere. Note that the set of x which appear as a partition point of some P_n is at most countable, and thus has measure zero and can be ignored. For other x, the statement that $L(x) = f(x) = U(x)$ is exactly the statement that f is continuous at x (because the mesh size of the partition tends to 0). Thus we conclude that f is Riemann integrable if and only if f is continuous almost everywhere. $\square$

Problem 4a. Consider a sequence $\{a_n\} \subseteq [0,1]$. For $f \in C([0,1])$, let us denote

$$\phi(f) \;=\; \sum_{n=1}^{\infty} 2^{-n} f(a_n).$$

Prove that there is no $g \in L^1([0,1])$ such that $\phi(f) = \int f(x) g(x)\, dx$ is true for all $f \in C([0,1])$.

Solution. Suppose there was such a g. Let f_k be the function which is zero outside $[a_1 - 1/k, a_1 + 1/k]$, equal to 1 at a_1, and linear in between (the graph is a triangle of height 1 and width $2/k$ centered at a_1). Then for each k we have $\phi(f_k) \geqslant 1/2$. But we also have $f_k \to 0$ pointwise almost everywhere and $|f_k| \leqslant 1$, so by the dominated convergence theorem, $\int_0^1 f_k g \to 0$, which is a contradiction. $\square$

Problem 4b. Each $g \in L^1([0,1])$ defines a continuous functional T_g on $L^\infty([0,1])$ by

$$T_g(f) \;=\; \int f(x) g(x)\, dx.$$

Prove that there are continuous functionals on $L^\infty([0,1])$ that are not of this form.

Solution. Suppose not, i.e. that every element of $(L^\infty)^*$ is of the form T_g for some $g \in L^1$. Then the map $g \mapsto T_g$ is a normed vector space isomorphism $L^1 \to (L^\infty)^*$. Indeed, it is surjective by assumption, injective because $T_g = 0$ implies $\int_0^1 fg = 0$ for all $f \in C([0,1])$, which implies $g = 0$, and bounded because

$$||T_g||_{op} \;=\; \sup_{||f||_{L^\infty}=1} \left|\int_0^1 fg\right| \;\leqslant\; \left|\int_0^1 g\right| \;\leqslant\; ||g||_{L^1}\,.$$

Thus by the open mapping theorem, it's inverse is also bounded and therefore it's an isomorphism. Thus $L^1 \simeq (L^\infty)^*$. Since L^1 is separable, this implies $(L^\infty)^*$ is separable, which implies L^∞ is separable. But this is a contradiction: $\{\chi_{[0,r]}\}_{0<r<1}$ is an uncountable discrete set in L^∞. $\square$

Alternate Solution (using part a). Note that ϕ is a bounded linear functional on the space $C([0,1])$, so by Hahn-Banach ϕ extends to a bounded linear functional $\widetilde{\phi}$ on $L^\infty([0,1])$. If $\widetilde{\phi}$ was of the form T_g then its restriction ϕ would also be of this form, which contradicts part (a).

Problem 5a. Prove that $\ell^1(\mathbb{N})$ and $\ell^2(\mathbb{N})$ are separable Banach spaces but $\ell^\infty(\mathbb{N})$ is not.

Solution. Let X be either $\ell^1(\mathbb{N})$ or $\ell^2(\mathbb{N})$ (the proof that follows works for both). Define the set

$$S_n \;:=\; \{f \in X : f(k) \in \mathbb{Q} + i\mathbb{Q} \text{ for all } k \text{ and } f(k) = 0 \text{ for } k > n\}$$

and let $S = \bigcup_{n=1}^\infty S_n$. Note that each S_n can be identified with $(\mathbb{Q} + i\mathbb{Q})^n$, which is countable, so S is countable as well. We now show that S is dense in X. Let $f \in X$ and fix $\epsilon > 0$. Let e be either 1 or 2 depending on if X is $\ell^1(\mathbb{N})$ or $\ell^2(\mathbb{N})$. Since $\sum_{k=1}^\infty |f(k)|^e < \infty$, there is an N such that $\sum_{k=N+1}^\infty |f(k)|^e < \epsilon$. For each $k \leqslant N$, since $\mathbb{Q} + i\mathbb{Q}$ is dense in $\mathbb{C}$, pick $q_k \in \mathbb{Q} + i\mathbb{Q}$ such that $|q_k - f(k)| < (\epsilon/N)^{1/e}$. Now define g by $g(k) = q_k$ for $k \leqslant N$ and $g(k) = 0$ for $k > N$. Then we see that $g \in S_N \subseteq S$ and

$$||f-g||_X \;=\; \sum_{k=1}^\infty |f(k)-g(k)|^e \;=\; \sum_{k=1}^N |f(k)-q_k|^e + \sum_{k=N+1}^\infty |f(k)|^e \;<\; \epsilon + \epsilon \;=\; 2\epsilon.$$

Thus S is dense in X, so X is separable.

For $\ell^\infty(\mathbb{N})$, for any subset $A \subseteq \mathbb{N}$, define $f_A \in \ell^\infty(\mathbb{N})$ by $f_A(k) = 1$ if $k \in A$ and 0 otherwise. Note that for any two subsets A and B, if $A \neq B$ then $||f_A - f_B||_{\ell^\infty} = 1$. But since there are uncountably many subsets of $\mathbb{N}$, the collection $\{f_A\}_{A \subseteq \mathbb{N}}$ is an uncountable discrete subset of $\ell^\infty(\mathbb{N})$, which means $\ell^\infty(\mathbb{N})$ can't be separable. $\square$

Problem 5b. Prove that there exists no bounded linear surjective map $T : \ell^2(\mathbb{N}) \to \ell^1(\mathbb{N})$.

Solution. If such a map existed then it would induce a bounded injective map $T^* : l^\infty(\mathbb{N}) \to l^2(\mathbb{N})$ between the dual spaces. Taking duals again, we obtain a surjective bounded linear map $T^{**} : l^2(\mathbb{N}) \to (l^\infty(\mathbb{N}))^*$. But the image of a separable space under a bounded linear map is separable, so $(l^\infty(\mathbb{N}))^*$ must be separable. But then $l^\infty(\mathbb{N})$ is separable, which is a contradiction.

Problem 6a. Given a Hilbert space $\mathcal{H}$, let $\{a_n\}$ be a sequence with $||a_n|| = 1$ for all n. Recall that the closed convex hull of $\{a_n\}$ is the closure of the set of all convex combinations of elements in $\{a_n\}$. Show that if $\{a_n\}$ spans $\mathcal{H}$ linearly, then $\mathcal{H}$ is finite dimensional.

Solution. Suppose $\{a_n\}$ linearly spans $\mathcal{H}$ and suppose that $\mathcal{H}$ is infinite-dimensional. By inductively removing any elements a_n which are in the span of $\{a_1, \ldots, a_{n-1}\}$, we may assume that $\{a_n\}$ is a linearly independent set in $\mathcal{H}$. Define $S_N := \text{span}(a_1, \ldots, a_N)$. We know that S_N is a finite-dimensional subspace of $\mathcal{H}$ and is therefore closed. We also know that S_N does not contain any open sets because if S_N contained the open ball $B(x,r)$, then since S is a subspace it would also contain the set $B(x,r)-x = B(0,r)$, and then it would also have to contain the set $n \cdot B(0,r) = B(0,nr)$ for all integers n, implying that S_N would be equal to all of $\mathcal{H}$. But since $\mathcal{H}$ is infinite dimensional this is not the case. Hence S_N has empty interior and since S_N is closed, S_N is nowhere dense. By the assumption that $\{a_n\}$ spans $\mathcal{H}$, we see that $\mathcal{H} = \bigcup_{N=1}^{\infty} S_N$. But this is a countable union of nowhere dense sets, and since Hilbert spaces are complete, this contradicts the Baire category theorem. Thus $\mathcal{H}$ must be finite dimensional. $\square$

Problem 6b. Show that if $\langle a_n, \xi \rangle \to 0$ for all $\xi \in \mathcal{H}$, then 0 is in the closed convex hull of $\{a_n\}$.

Solution. Fix $\epsilon > 0$. It suffices to show the existence of a convex combination of the a_n with norm less than ϵ. Set $a_{N_1} = a_1$. Since $\langle a_n, a_{N_1} \rangle \to 0$ as $n \to \infty$, pick a_{N_2} so that $|\langle a_{N_2}, a_{N_1} \rangle| < \epsilon$. Now since $\langle a_n, a_{N_1} \rangle$ and $\langle a_n, a_{N_2} \rangle$ both tend to 0 as $n \to \infty$, we can pick a_{N_3} so that $|\langle a_{N_3}, a_{N_1} \rangle|, |\langle a_{N_3}, a_{N_2} \rangle| < \epsilon$. Continuing this construction inductively we get a subsequence a_{N_k} with the property that every pairwise inner product in the subsequence has absolute value less than ϵ. Now let r be big enough so that $1/r < \epsilon$ and consider the convex combination $(1/r)a_{N_1} + \ldots + (1/r)a_{N_r}$. We have

$$\begin{aligned} \left\| \frac{1}{r} a_{N_1} + \ldots + \frac{1}{r} a_{N_r} \right\|^2 &= \frac{1}{r^2} \langle a_{N_1} + \ldots a_{N_r}, a_{N_1} + \ldots a_{N_r} \rangle \\ &= \frac{1}{r^2} \left(\sum_{j=1}^{r} \|a_{N_j}\|^2 + \sum_{i \neq j} \langle a_{N_i}, a_{N_j} \rangle \right) < \frac{1}{r^2} \left(r + r^2 \epsilon \right) < \frac{3}{2} \epsilon. \quad \square \end{aligned}$$

Problem 7. Characterize all entire functions f with $|f(z)| > 0$ for z large and

$$\limsup_{z \to \infty} \frac{|\log |f(z)||}{|z|} < \infty.$$

Solution. The condition that $|f(z)| > 0$ for $|z|$ large implies that all of the zeros of f lie in some bounded set, and since the zeros have to be discrete, f has only finitely many zeros. Let $p(z)$ be the polynomial with the same zeros as f, counting multiplicity. Then $f(z)/p(z)$ is a nonvanishing entire function, so we can write $f(z)/p(z) = e^{h(z)}$ for some entire function h. So we have the representation $f(z) = p(z)e^{h(z)}$ where p is a polynomial and h is entire. We have

$$\limsup_{z \to \infty} \frac{|\log |f(z)||}{|z|} = \limsup_{z \to \infty} \frac{|\log |p(z)||}{|z|} + \frac{|\log |\operatorname{Re}(h(z))||}{|z|} = \limsup_{z \to \infty} \frac{|\log |\operatorname{Re}(h(z))||}{|z|} < \infty.$$

Thus we have $|\operatorname{Re}(h(z))| \leqslant C|z|$ for some constant C and all z. We claim this implies that h is a degree 1 polynomial. It would be obvious if the bound had $|h(z)|$ instead of $|\operatorname{Re}(h(z))|$, but it doesn't, so we have to do more work. Write $h = u + iv$ and also write

$$h(z) = h(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}.$$

Then we have $u(re^{i\theta}) = \sum_{n=0}^{\infty} r^n (\operatorname{Re}(a_n)\cos(n\theta) - \operatorname{Im}(a_n)\sin(n\theta))$. Using various orthonormality properties and the fact the the power series converges uniformly on compact sets, one can compute

$$\int_0^{2\pi} u(re^{i\theta}) e^{-ik\theta} \, d\theta = \pi r^k a_k$$

for each fixed k. Thus

$$|a_k| r^k \leqslant \frac{1}{\pi} \int_0^{2\pi} |u(re^{i\theta})| \, d\theta.$$

Combining this with the mean value property for u, we have

$$|a_k|r^k + 2u(0) \;\leqslant\; \frac{1}{\pi}\int_0^{2\pi} (|u(re^{i\theta})| + u(re^{i\theta}))\,d\theta \;\leqslant\; \frac{1}{\pi}\cdot 2\pi \cdot 2Cr \;=\; 4Cr$$

by the estimate on $|\operatorname{Re}(h)|$ from above. Thus we have $|a_k| \leqslant 4Cr^{1-k} - 2u(0)r^{-k}$. This holds for any r, so we can take $r \to \infty$ to conclude that $a_k = 0$ for any $k > 1$. This implies that h is a degree 1 polynomial.

So we conclude that if f satisfies the given conditions, then $f(z) = p(z)e^{az+b}$ for some polynomial p and $a, b \in \mathbb{C}$. It's clear that every function of this form satisfies the conditions, so this is a complete characterization. $\square$

Problem 8. Construct a non-constant entire function $f(z)$ such that the zeros of f are simple and coincide with the set of all (positive) natural numbers.

Solution. Use the canonical product representation. Let

$$f(z) \;=\; \prod_{n=1}^{\infty}\left(1 - \frac{z}{n}\right)e^{z/n}.$$

This clearly has the right zeros. We just need to show f is entire. It's enough to show that the product converges uniformly and absolutely on compact sets. Equivalently, we need to show that

$$\sum_{n=1}^{\infty} |\log(1 - z/n) + z/n|$$

converges uniformly on compact sets. Examining the power series expansion of $\log(1-x)$ around 0, we see that there exists $\delta > 0$ such that $|x| < \delta$ implies $|\log(1-x) + x| \leqslant |x|^2$. Fix a compact set $\overline{B(0,R)}$. Pick n big enough so that $R/n < \delta$ and also so that $n > R$. Then for any $|z| \leqslant R$, we have $|z|/n < \delta$, so

$$|\log(1 - z/n) + z/n| \;\leqslant\; \frac{|z|^2}{n^2} \;\leqslant\; \frac{R^2}{n^2}.$$

Thus the series in question is eventually majorized by the convergent series $\sum_{n=1}^{\infty} R^2/n^2$ for all $|z| \leqslant R$, which shows that it converges uniformly and absolutely on $B(0,R)$. $\square$

Problem 9. Prove Hurwitz' Theorem: Let $\Omega \subseteq \mathbb{C}$ be a connected open set and $f_n, f : \Omega \to \mathbb{C}$ holomorphic functions. Assume that $f_n(z)$ converges uniformly to $f(z)$ on compact subsets of Ω. Prove that if $f_n(z) \neq 0$ for all $z \in \Omega$ and all n, then either f is identically zero or $f(z) \neq 0$ for all $z \in \Omega$.

Solution. Since $f_n \to f$ uniformly on compact sets, we also know that $f_n' \to f'$ uniformly on compact sets. Suppose that f is not identically zero. Then the zeros of f are isolated. Fix any $z_0 \in \Omega$. Choose an $r > 0$ small enough so that f has no zeros in $B(z_0, r)$ except for possibly at z_0 and $|f(z)| \geqslant \delta > 0$ for $|z - z_0| = r$. Because $\partial B(z_0, r)$ is compact and each f_n is nonvanishing, each f_n is bounded away from 0 on $\partial B(z_0, r)$, and since f is also bounded away from zero on it, we have $1/f_n \to 1/f$ uniformly on $\partial B(z_0, r)$. Therefore by the argument principle, we have

$$0 \;=\; \lim_{n\to\infty}(\#\text{ zeros of } f_n \text{ inside } B(z_0,r)) \;=\; \lim_{n\to\infty}\int_{\partial B(z_0,r)} \frac{f_n'(z)}{f_n(z)}\,dz \;=\; \int_{\partial B(z_0,r)} \frac{f'(z)}{f(z)}\,dz \;=\; (\#\text{ zeros of } f \text{ inside } B(z_0,r))\,.$$

Therefore $f(z_0) \neq 0$, and since this argument can be applied at any point z_0, we conclude that f is nonvanishing in Ω. $\square$

Problem 10. Let $\alpha \in [0,1]\backslash\mathbb{Q}$ and let $\{a_n\} \in \ell^1(\mathbb{N})$ with $a_n \neq 0$ for all n. Show that

$$f(z) \;=\; \sum_{n\geqslant 1} \frac{a_n}{z - e^{i\alpha n}}$$

converges and defines a function that is analytic in $\mathbb{D}$ which does not admit an analytic continuation to any domain larger than $\mathbb{D}$.

Solution. Each of the summands is analytic in $\mathbb{D}$, so to show that f is analytic in $\mathbb{D}$ it suffices to show that the sum converges uniformly on compact sets. Note that it is enough to show that sum converges uniformly on $\mathbb{D}_r = \{z : |z| < r\}$ For $z \in D_r$ we have

$$\left|\sum_{n=k}^{\infty} \frac{a_n}{z - e^{i\alpha n}}\right| \leqslant \sum_{n=k}^{\infty} \frac{|a_n|}{|z - e^{i\alpha n}|} < \frac{1}{1-r} \sum_{n=k}^{\infty} |a_n|\,,$$

which converges to 0 as $k \to \infty$. Thus the sequence of partial sums for f is uniformly Cauchy on $\mathbb{D}_r$. This establishes that the, sum converges everywhere in $\mathbb{D}$, and defines an analytic function in $\mathbb{D}$.

Let Ω be any region containing $\mathbb{D}$. Then Ω contains an open arc of the unit circle. Since α is irrational, the points $\{e^{i\alpha n}\}$ are dense in the unit circle, so there is some $e^{i\alpha k} \in \Omega$. The intuition is that this is a contradiction because f will blow up near $e^{i\alpha k}$, but it's hard to show this directly. Instead let $g(z) = (z - e^{i\alpha k})f(z)$. Since f is analytic in Ω by assumption, $g(e^{i\alpha k}) = 0$. Consider for $0 < r < 1$

$$g(re^{i\alpha k}) \;=\; a_k + \sum_{n \neq k} \frac{a_n(r-1)e^{i\alpha k}}{re^{i\alpha k} - e^{i\alpha n}}$$

where changing the order of summation is allowed because the series converges absolutely on each circle $|z| = r$ for $r < 1$. Now note that we have

$$\left|\frac{a_n(r-1)e^{i\alpha k}}{re^{i\alpha k} - e^{i\alpha n}}\right| \;\leqslant\; |a_n|\frac{1-r}{1-r} \;\leqslant\; |a_n|$$

for all $r < 1$, so by the Dominated Convergence theorem we have

$$g(e^{i\alpha k}) \;=\; \lim_{r\to 1^-} g(re^{i\alpha k}) \;=\; a_k + \sum_{n\neq k} \lim_{r \to 1^-} \frac{a_n(r-1)e^{i\alpha k}}{re^{i\alpha k} - e^{i\alpha n}} \;=\; a_k \;\neq\; 0,$$

which is a contradiction. $\square$

Problem 11. For each $p \in (-1, 1)$, compute the improper Riemann integral

$$\int_0^{\infty} \frac{x^p}{x^2+1} dx.$$

Solution. Define $\log(z)$ to be the branch with the negative imaginary axis removed, i.e. $\operatorname{Im}(\log(re^{i\theta})) = \theta \in (-\pi/2, 3\pi/2)$. Then define

$$f(z) \;:=\; \frac{z^p}{z^2+1} \;=\; \frac{\exp(p \log z)}{z^2+1}.$$

Integrate f over the contour which consists of a half circle in the upper half plane from $-R$ to R, then along the negative real axis from $-R$ to $-\epsilon$, then a half circle in the upper half plane from $-\epsilon$ to ϵ, then along the positive real axis from ϵ to R. The contributions from the two half circles go to 0 as $\epsilon \to 0$, $R \to \infty$ and you are left with

$$(1 + \exp(p\pi i)) \int_0^{\infty} \frac{x^p}{x^2+1} dx \;=\; 2\pi i \cdot \operatorname{Res}_{z=i} f(z) \;=\; \pi \cdot \exp(p\pi i/2)$$

(I left out the computation of the residue). After rearranging you get that the answer is $\frac{\pi}{2\cos(p\pi/2)}$. $\square$

Problem 12. Compute the number of zeros, including multiplicity, of $f(z) = z^6 + iz^4 + 1$ in the upper half plane.

Solution. Since the polynomial is even, z is a root of multiplicity m if and only if $-z$ is a root of multiplicity m. Therefore the roots in the open upper half plane are in bijection with the roots in the open lower half plane. If $r \neq 0$ is real, then $\operatorname{Im}(f(r)) = r^4$ which is nonzero. Since $f(0) \neq 0$ we see that f has no real roots. Since z has 6 total roots (counting multiplicity), exactly 3 of them must lie in the upper half plane. $\square$

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Problem 1. Show that

$$A := \{f \in L^3(\mathbb{R}) : \int_{\mathbb{R}} |f(x)|^2 \, dx < \infty\}$$

is a Borel subset of $L^3(\mathbb{R})$.

Solution. Define the functional ϕ_n on $L^3(\mathbb{R})$ by

$$\phi_n(f) = \int_{-n}^{n} |f|^2.$$

Note that we have

$$A = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \{f \in L^3(\mathbb{R}) : \phi_n(f) \leqslant m\}.$$

So to show A is Borel it suffices to prove that ϕ_n is a continuous function from $L^3(\mathbb{R}) \to \mathbb{R}$. For $f, g \in L^3$, we have

$$\begin{aligned}
|\phi_n(f) - \phi_n(g)| &\leqslant \int_{-n}^{n} \left|f^2 - g^2\right| \leqslant \int_{-n}^{n} |f - g| \left(|f| + |g|\right) \\
&\leqslant \int_{-n}^{n} |f - g| \, |f| + \int_{-n}^{n} |f - g| \, |g| \\
&\leqslant \left(\int_{-n}^{n} |f - g|^3\right)^{1/3} \left(\int_{-n}^{n} |f|^3\right)^{1/3} \left(\int_{-n}^{n} 1^3\right)^{1/3} + \left(\int_{-n}^{n} |f - g|^3\right)^{1/3} \left(\int_{-n}^{n} |g|^3\right)^{1/3} \left(\int_{-n}^{n} 1^3\right)^{1/3} \\
&\leqslant (2n)^{1/3} \, ||f - g||_{L^3} \left(||f||_{L^3} + ||g||_{L^3}\right).
\end{aligned}$$

Fix $\epsilon > 0$. If $||f - g||_{L^3} < \epsilon \cdot (3(2n)^{1/3} \, ||f||_{L^3})^{-1}$ and $||f - g||_{L^3} < ||f||_{L^3}$, then

$$|\phi_n(f) - \phi_n(g)| < (2n)^{1/3}(3 \, ||f||_{L^3}) \, ||f - g||_{L^3} < \epsilon.$$

Thus $\phi_n(f)$ is continuous at f for every $f \in L^3(\mathbb{R})$, so we're done. $\square$

Problem 2. Construct an $f \in L^1(\mathbb{R})$ so that $f(x + y)$ does *not* converge almost everywhere to $f(x)$ as $y \to 0$. Prove that your f has this property.

Solution. Let K be a fat Cantor set contained in $[0, 1]$. Recall that K is closed, has positive measure, and that each point in K is a boundary point. Take $f = \chi_K$. Since K is closed, f is measurable, and since K has finite measure, f lies in L^1. But for each $x \in K$ every neighborhood U of x contains a point u which lies outside K and hence has $f(u) = 0$. Therefore for each $x \in K$, $f(x + y)$ does not converge to $f(x)$ as $y \to 0$. This is enough, since K has positive measure. $\square$

Problem 3. Let (f_n) be a bounded sequence in $L^2(\mathbb{R})$ and suppose that $f_n \to 0$ Lebesgue almost everywhere. Show that $f_n \to 0$ in the weak topology on $L^2(\mathbb{R})$.

Solution. To show that $f_n \to 0$ in the weak topology on $L^2(\mathbb{R})$, we need to show that $\phi(f_n) \to 0$ for every bounded linear functional ϕ on $L^2(\mathbb{R})$. Since $L^2(\mathbb{R})$ is a Hilbert space, by the Riesz representation theorem we know that every bounded linear functional ϕ is of the form $\phi(f) = \int f(x)g(x) \, dx$ for some $g \in L^2(\mathbb{R})$. So it suffices to show that for any $g \in L^2(\mathbb{R})$, we have $\int f_n(x)g(x) \, dx \to 0$ as $n \to \infty$. Since $f_n \to 0$ pointwise almost everywhere, we also have that $f_n g \to 0$ pointwise almost everywhere. By the Vitali Convergence Theorem, to conclude that $\int f_n g \to 0$, it suffices to show that the sequence $\{f_n g\}$ is both uniformly integrable and tight.
As a reminder, uniformly integrable means that for every $\epsilon > 0$ there exists a $\delta > 0$ such that for any n, $m(A) < \delta$ implies $\int_A |f_n g| < \epsilon$. Tight means that for any $\epsilon > 0$, there exists a subset $E \subseteq \mathbb{R}$ such that for

any n, $\int_{E^c} |f_n g| < \epsilon$.

We know that $\{f_n\}$ is a bounded sequence in $L^2(\mathbb{R})$, so let $||f_n||_{L^2} \leqslant M$ for all n. First we show uniform integrability. Fix $\epsilon > 0$. Since $|g|^2$ is integrable, there is a δ so that $m(A) < \delta$ implies $\int_A |g|^2 < \epsilon/M$. Now for any n, we have by Cauchy-Schwarz that if $m(A) < \delta$,

$$\int_A |f_n g| \;\leqslant\; \left(\int_A |f_n|^2\right)^{1/2} \left(\int_A |g|^2\right)^{1/2} \;\leqslant\; ||f_n||_{L^2} \frac{\epsilon}{M} \;\leqslant\; \epsilon,$$

so the family $\{f_n g\}$ is uniformly integrable.

For tightness, fix $\epsilon > 0$. Since $|g|^2$ is integrable, there is a set E such that $\int_{E^c} |g|^2 < \epsilon/M$. Then for any n, by the same Cauchy-Schwarz argument we have

$$\int_A |f_n g| \;\leqslant\; \epsilon.$$

Thus $\{f_n g\}$ is tight, so we conclude that $\int f_n g \to 0$ as $n \to \infty$. $\square$

Problem 4. Given $f \in L^2([0,\pi])$, we say that $f \in \mathcal{G}$ if f admits a representation of the form

$$f(x) \;=\; \sum_{n=0}^{\infty} c_n \cos(nx) \quad \text{with} \quad \sum_{n=0}^{\infty} (1+n^2)|c_n|^2 \;<\; \infty.$$

Show that if $f \in \mathcal{G}$ and $g \in \mathcal{G}$ then $fg \in \mathcal{G}$.

Solution. The motivation for this is that the c_n are basically the Fourier coefficients of f, so the condition for membership in $\mathcal{G}$ translates as $(1+n^2)^{1/2}\widehat{f}(n) \in \ell^2$. So $\mathcal{G}$ is basically a "Fourier series version" of the Sobolev space H^1.

First we want to make a technical modification so that we can work directly with the regular Fourier coefficients (it makes stuff easier later). It's clear that $L^2([0,\pi])$ is in bijection with the space $L^2_e :=$ the subspace of $L^2([-\pi,\pi])$ consisting of even functions. So we identify each $f \in \mathcal{G}$ with its even extension to $[-\pi,\pi]$. For $f \in \mathcal{G}$, the given condition implies that

$$\sum_{n=0}^{\infty} |c_n| \;=\; \sum_{n=0}^{\infty} |c_n|(1+n^2)^{1/2}(1+n^2)^{-1/2} \;\leqslant\; \left(\sum_{n=0}^{\infty} |c_n|^2(1+n^2)\right)^{1/2} \left(\sum_{n=0}^{\infty} (1+n^2)^{-1}\right)^{1/2} \;<\; \infty.$$

Thus by the Weierstrass M-test, we know that the given series representation for f converges absolutely and uniformly on $[-\pi,\pi]$. Recall that $\{\cos(nx)\}_{n=0}^{\infty}$ is an orthonormal basis for the Hilbert space L^2_e. For a fixed n, we calculate in two different ways the inner product

$$\begin{aligned}
\langle f, \cos(nx)\rangle \;&=\; \left\langle f, \frac{1}{2}(e^{inx} + e^{-inx})\right\rangle \;=\; \frac{1}{2}(\widehat{f}(n) + \widehat{f}(-n)) \;=\; \widehat{f}(n) \quad \text{because } f \text{ is even} \\
\langle f, \cos(nx)\rangle \;&=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{m=1}^{\infty} c_m \cos(mx)\cos(nx)\,dx \\
&=\; \sum_{m=1}^{\infty} c_m \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx \;=\; \begin{cases} \frac{1}{2}c_n & n \neq 0 \\ c_0 & n = 0 \end{cases}
\end{aligned}$$

where switching the order is justified because of the uniform convergence. Thus we conclude that for $f \in \mathcal{G}$, the coefficients c_n are exactly equal to $2\widehat{f}(n)$ for $n \neq 0$ and $\widehat{f}(0)$ for $n = 0$. So the problem is equivalent to showing that for $f, g \in \mathcal{G}$, we have $(1+n^2)^{1/2}\widehat{fg}(n) \in \ell^2$.

Let $f, g \in \mathcal{G}$. The same argument from above that showed the uniform convergence of the series representations also shows that the representations f or $g(x) = \sum_{n=-\infty}^{\infty} \widehat{f \text{ or } g}(n) e^{inx}$ converge uniformly, so we can compute the Fourier coefficients

$$\begin{aligned} \widehat{fg}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) g(x) e^{-inx}\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{ikx} \sum_{\ell=-\infty}^{\infty} \widehat{g}(\ell) e^{i\ell x} e^{-inx}\, dx \\ &= \sum_{k,\ell=-\infty}^{\infty} \widehat{f}(k) \widehat{g}(\ell) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k+\ell-n)x}\, dx = \sum_{k=-\infty}^{\infty} \widehat{f}(k) \widehat{g}(n-k) = (\widehat{f} * \widehat{g})(n). \end{aligned}$$

Also note the elementary estimate

$$\begin{aligned} (1+n^2)^{1/2} &= (1+(n-k+k)^2)^{1/2} = (1+(n-k)^2+k^2+2(n-k)k)^{1/2} \leqslant (1+2(n-k)^2+2k^2)^{1/2} \\ &\leqslant (2+2(n-k)^2+2+2k^2)^{1/2} \lesssim (1+(n-k)^2)^{1/2} + (1+k^2)^{1/2}, \end{aligned}$$

valid for any $k \in \mathbb{R}$. So we estimate

$$\begin{aligned} (1+n^2)^{1/2} \widehat{fg}(n) &\lesssim \sum_{k=-\infty}^{\infty} (1+k^2)^{1/2} \widehat{f}(k) \widehat{g}(n-k) + \sum_{k=-\infty}^{\infty} (1+(n-k)^2)^{1/2} \widehat{g}(n-k) \widehat{f}(k) \\ &= ((1+k^2)^{1/2} \widehat{f}(k) * \widehat{g})(n) + ((1+k^2)^{1/2} \widehat{g}(k) * \widehat{f})(n). \end{aligned}$$

Thus we have

$$\begin{aligned} \left\| (1+n^2)^{1/2} \widehat{fg}(n) \right\|_{\ell^2} &\lesssim \left\| (1+k^2)^{1/2} \widehat{f}(k) * \widehat{g} \right\|_{\ell^2} + \left\| (1+k^2)^{1/2} \widehat{g}(k) * \widehat{f} \right\|_{\ell^2} \\ &\leqslant \left\| (1+k^2)^{1/2} \widehat{f}(k) \right\|_{\ell^2} ||\widehat{g}||_{\ell^1} + \left\| (1+k^2)^{1/2} \widehat{g}(k) \right\|_{\ell^2} \left\| \widehat{f} \right\|_{\ell^1} \quad \text{by Young's convolution inequality} \\ &< \infty \end{aligned}$$

because we showed at the very beginning that $f \in \mathcal{G}$ implies $\widehat{f} \in \ell^1$. Thus $(1+n^2)^{1/2} \widehat{fg}(n) \in \ell^2$ so we're done. $\square$

Problem 5. Let $\phi : [0,1] \to [0,1]$ be continuous and let $d\mu$ be a Borel probability measure on $[0,1]$. Suppose $\mu(\phi^{-1}(E)) = 0$ for every Borel set $E \subseteq [0,1]$ with $\mu(E) = 0$. Show that there is a Borel measurable function $w : [0,1] \to [0,\infty)$ so that

$$\int f \circ \phi(x)\, d\mu(x) = \int f(y) w(y) d\mu(y)$$

for all continuous $f : [0,1] \to \mathbb{R}$.

Solution. Since ϕ is continuous, it is Borel measurable. The condition that $\mu(\phi^{-1}(E)) = 0$ whenever $\mu(E) = 0$ says that the measure $\phi_* \mu$ is absolutely continuous with respect to μ. Both μ and $\phi_* \mu$ are finite measures on $[0,1]$, so by the Radon-Nikodym theorem there is a Borel measurable function w such that

$$(\phi_* \mu)(A) = \int_A w(x)\, d\mu(x)$$

for all Borel sets A. Since $\phi_* \mu$ is a positive measure, we know that w is a nonnegative function. Also, if f is any continuous function on $[0,1]$, then it is also integrable on $[0,1]$, so by a well-known property of the Radon-Nikodym derivative,

$$\int_0^1 f(\phi(x))\, d\mu(x) = \int_0^1 f(x)\, d(\phi_* \mu)(x) = \int_0^1 f(x) w(x)\, d\mu(x). \quad \square$$

Problem 6. Let X be a Banach space and let X^* be its dual space. Suppose X^* is separable; show that X is separable (you should assume the Axiom of Choice).

Solution. Let $\{f_n\}_{n=1}^{\infty}$ be a countable dense subset of X^*. By definition of operator norm, for each n pick $x_n \in X$ with $||x_n|| = 1$ such that $|f_n(x_n)| > (1/2)\,||f_n||$. Let $M = \text{span}\{x_n\}$. We first want to show that M is dense in X, i.e. $\overline{M} = X$. Suppose that $y \notin \overline{M}$. Then by the Hahn-Banach theorem, there is a linear functional $f \in X^*$ such that $f = 0$ on $\overline{M}$ and $f(y) \neq 0$. By the separability of X^*, there is a subsequence $\{f_{n_k}\}$ that converges to f in the operator norm topology. We have

$$||f_{n_k} - f|| \;\geqslant\; |f_{n_k}(x_{n_k}) - f(x_{n_k})| \;=\; |f_{n_k}(x_{n_k})| \;>\; \frac{1}{2}\,||f_{n_k}||\,,$$

and since $||f_{n_k} - f|| \to 0$ as $k \to \infty$, this implies that $||f_{n_k}|| \to 0$ as $k \to \infty$ as well, which implies that $f_{n_k} \to 0$. But $f_{n_k} \to f$, and f is not identically zero, so this is a contradiction. Thus $\overline{M} = X$, so M is dense in X.

Now to show X is separable, it suffices to find a countable set which is dense in M. Let S be the subset of M which consists only of linear combinations with coefficients in $\mathbb{Q} + i\mathbb{Q}$. S is a countable set because it can be put in bijection with $\bigcup_{n=1}^{\infty}(\mathbb{Q} + i\mathbb{Q})^n$, which is countable. Since $\mathbb{Q} + i\mathbb{Q}$ is dense in $\mathbb{C}$, it follows that S is dense in M, so S is dense in X and hence X is separable. $\square$

Problem 7. Find an explicit conformal mapping from the upper half plane slit along the vertical segment

$$\{z \in \mathbb{C} : \text{Im}(z) > 0\}\backslash(0, 0 + ih], \quad h > 0$$

to the unit disk.

Solution. Start with $\Omega_1 = \{z \in \mathbb{C} : \text{Im}(z) > 0\}\backslash(0, 0 + ih]$. Let $f_1(z) = i(h/z)$. This is a conformal map $\Omega \to \Omega_2 := \{z : \text{Re}(z) > 0\}\backslash[1, \infty)$. Let $f_2(z) = z^2$. This is a conformal map $\Omega_1 \to \Omega_2 := \mathbb{C}\backslash[1, \infty)\backslash(-\infty, 0]$. Let $f_3(z) = 1/z - 1$. This is a conformal map $\Omega_2 \to \Omega_3 := \mathbb{C}\backslash(-\infty, 0]$. Let $f_4(z)$ be the branch of $\sqrt{z}$ that you get by removing the negative real axis. Then this is a conformal map $\Omega_3 \to \mathbb{H}$. Finally let $f_5(z) = (z - i)/(z + i)$; this is a conformal map $\mathbb{H} \to \mathbb{D}$. Thus $f := f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$ is a conformal map $\Omega \to \mathbb{D}$. $\square$

Problem 8. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Show that

$$|f(z)| \;\leqslant\; Ce^{a|z|}$$

for some constants C and a if and only if we have

$$|f^{(n)}(0)| \;\leqslant\; M^{n+1}$$

for some constant M.

Solution. First suppose that $|f(z)| \leqslant Ce^{a|z|}$ for all $z \in \mathbb{C}$. Then by applying the Cauchy estimates to a disk of radius R centered at 0, we get

$$|f^{(n)}(0)| \;\leqslant\; \frac{n!}{R^n}Ce^{aR}.$$

Since f is entire, the above inequality is valid for any $R > 0$, so we choose $R = n/a$ to get

$$|f^{(n)}(0)| \;\leqslant\; \frac{n!a^n}{n^n}Ce^n \;\leqslant\; C \cdot (ea)^n \;\leqslant\; M^{n+1}$$

for some constant M.

Conversely, suppose that $|f^{(n)}(0)| \leqslant M^{n+1}$ for all n. Then, since f is entire, we can write f as a power series

$$f(z) \;=\; \sum_{n=0}^{\infty} a_n z^n$$

and it is valid for all $z \in \mathbb{C}$. We know that the power series coefficients are given by

$$a_n = \frac{f^{(n)}(0)}{n!},$$

so we have

$$|f(z)| \leqslant \sum_{n=0}^{\infty} |a_n||z|^n \leqslant \sum_{n=0}^{\infty} \frac{M^{n+1}}{n!}|z|^n = Me^{M|z|}$$

for all $z \in \mathbb{C}$. $\square$

Problem 9. Let $\Omega \subseteq \mathbb{C}$ be open and connected. Suppose (f_n) is a sequence of injective holomorphic functions defined on Ω such that $f_n \to f$ locally uniformly in Ω. Show that if f is not constant, then f is also injective in Ω.

Solution. Since $f_n \to f$ locally uniformly, we know that f is also holomorphic. We first prove the following variation of Hurwitz's theorem: If each f_n has at most one zero in Ω, then either f is identically zero or f has at most one zero in Ω.

Suppose that f is not identically zero. Then the zeros of f are isolated. Suppose that $f(z_0) = 0$. Pick $r > 0$ small enough so that f has no other zeros in $\overline{B(z_0, r)}$. Since f is nonzero on $\partial B(z_0, r)$, which is compact, we have $|f(z)| \geqslant \delta > 0$ for $|z - z_0| = r$. This shows that $1/f_n \to 1/f$ uniformly on $\partial B(z_0, r)$. We also know that $f_n' \to f'$ uniformly on compact sets. Thus we conclude that

$$\lim_{n\to\infty} \int_{\partial B(z_0,r)} \frac{f_n'(z)}{f_n(z)}\, dz = \int_{\partial B(z_0,r)} \frac{f'(z)}{f(z)}\, dz.$$

By the argument principle, the right side of this equation is equal to the number of zeros of f inside $B(z_0, r)$, which is one. Similarly, the left side is equal to the number of zeros of f_n inside $B(z_0, r)$. Thus the above equation implies that for sufficiently large n, f_n has exactly one zero inside $B(z_0, r)$. So we have shown that given a zero of f and a sufficiently small ball around that zero, then n can be made sufficiently large so that f_n has zero inside that ball. Thus, if f had two zeros, we could put two disjoint balls around them, then the previous statement would imply that f_n would eventually have to have two zeros, which is a contradiction. Thus we conclude that f has only one zero.

Now, for any $w \in \mathbb{C}$, we have that $f_n - w$ converges locally uniformly to $f - w$. Since each f_n is injective, $f_n - w$ has at most one zero in Ω. Thus $f - w$ is either identically zero or has at most one zero. Since this is true for every $w \in \mathbb{C}$, it implies that f is either constant or injective. $\square$

Problem 10. Let us introduce a vector space $\mathcal{B}$ as follows.

$$\mathcal{B} = \left\{ u : \mathbb{C} \to \mathbb{C} : u \text{ is holomorphic and } \iint_{\mathbb{C}} |u(x+iy)|^2 e^{-(x^2+y^2)}\, dx\, dy < \infty \right\}.$$

Show that $\mathcal{B}$ becomes a *complete* vector space when equipped with the norm

$$||u||^2 = \iint_{\mathbb{C}} |u(x+iy)|^2 e^{-(x^2+y^2)}\, dx\, dy.$$

Solution. Define a measure μ on $\mathbb{C}$ by $d\mu = e^{-(x^2+y^2)} dx\, dy$, i.e.

$$\mu(A) := \int_A e^{-(x^2+y^2)} dx\, dy.$$

Note that μ is a finite measure on $\mathbb{C}$, and $L^2(\mu)$ is a complete vector space. Thus $\mathcal{B}$ is simply the subspace of $L^2(\mu)$ consisting of holomorphic functions, so to show that $\mathcal{B}$ is complete it suffices to show that $\mathcal{B}$ is closed with respect to the $L^2(\mu)$ norm.

Let $\{f_n\}$ be a sequence in $\mathcal{B}$ converging to $f \in L^2(\mu)$. We need to show that f is holomorphic. To do that, it suffices to show that $f_n \to f$ uniformly on compact subsets of $\mathbb{C}$. Let $K \subseteq \mathbb{C}$ be compact. Then $K' := \{z \in \mathbb{C} : \operatorname{dist}(z, K) \leqslant 1\}$ is also compact, so in particular, we have $e^{-(x^2+y^2)} \geqslant c > 0$ on K' and $\lambda(K') < \infty$ where λ denotes Lebesgue measure on $\mathbb{C}$. For any $z \in K$, we use the mean value property of holomorphic functions to write

$$f_n(z) - f_m(z) \;=\; \frac{1}{\pi}\int_{B(z,1)} (f_n(w) - f_m(w))\, d\lambda(w),$$

thus we have by Cauchy-Schwarz

$$\begin{aligned}
|f_n(z) - f_m(z)| &\leqslant \frac{1}{\pi}\int_{B(z,1)} |f_n(w) - f_m(w)|\, d\lambda(w) \\
&\leqslant \frac{1}{\pi}\lambda(B(z,1))^{1/2}\left(\int_{B(z,1)} |f_n(w) - f_m(w)|^2\, d\lambda(w)\right)^{1/2} \\
&\leqslant \frac{1}{\pi}\lambda(K')^{1/2}\left(\frac{1}{c}\int_{B(z,1)} |f_n(w) - f_m(w)|^2 c\, d\lambda(w)\right)^{1/2} \\
&\leqslant M_K\left(\int_{B(z,1)} |f_n(w) - f_m(w)|^2 e^{-(x^2+y^2)} d\lambda(w)\right) \\
&\leqslant M_K\, ||f_n - f_m||_{L^2(\mu)}\,.
\end{aligned}$$

Since $\{f_n\}$ converges in the $L^2(\mu)$ norm, the above inequality implies that $||f_n - f_m||_{L^\infty(K)} \to 0$ as $n, m \to \infty$, meaning that $\{f_n\}$ is uniformly Cauchy on K. Since L^∞ is complete, this means that f_n converges uniformly on K to some function g. In particular, f_n converges pointwise to g on K. But we know that f_n converges to f in $L^2(\mu)$, and thus (by passing to a subsequence if necessary) we also know that f_n converges to f pointwise. Thus we must have $f = g$, so we conclude that f_n converges uniformly to f on K. This holds for any compact set $K \subseteq \mathbb{C}$ and thus we know that f must be holomorphic, so $\mathcal{B}$ is a closed subspace of $L^2(\mu)$ and therefore complete. $\square$

Problem 11. Let $\Omega \subseteq \mathbb{C}$ be open, bounded, and simply connected. Let u be harmonic in Ω and assume that $u \geqslant 0$. Show the following: for each compact set $K \subseteq \Omega$, there exists a constant $C_K > 0$ such that

$$\sup_{x \in K} u(x) \;\leqslant\; C_K \inf_{x \in K} u(x).$$

Solution. Since Ω is open, simply connected and not all of $\mathbb{C}$, by the Riemann mapping theorem there is a conformal map $\phi : \mathbb{D} \to \Omega$. Then the function $v(z) = u(\phi(z))$ is a harmonic function on $\mathbb{D}$. Let K be any compact subset of Ω. Then $\phi^{-1}(K)$ is a compact subset of $\mathbb{D}$, so there is some $r \in (0, 1)$ such that $\phi^{-1}(K) \subseteq B(0, r) \subseteq \overline{B(0, r)} \subseteq \mathbb{D}$. Since u is nonnegative, so is v, and thus by Harnack's inequality, for any $z \in \phi^{-1}(K)$ we have

$$\frac{1-r}{1+r}v(0) \;\leqslant\; \frac{1-|z|}{1+|z|}v(0) \;\leqslant\; v(z) \;\leqslant\; \frac{1+|z|}{1-|z|}v(0) \;\leqslant\; \frac{1+r}{1-r}v(0).$$

The left inequality shows that $\inf_{z \in \phi^{-1}(K)} v(z) \geqslant \frac{1-r}{1+r}v(0)$, which implies $v(0) \leqslant \frac{1+r}{1-r}\inf_{z \in \phi^{-1}(K)} v(z)$. Then by putting this into the right inequality we get

$$v(z) \;\leqslant\; \left(\frac{1+r}{1-r}\right)^2 \inf_{z \in \phi^{-1}(K)} v(z)$$

for any $z \in \phi^{-1}(K)$, so $\sup_{z\in\phi^{-1}(K)} v(z) \leqslant \left(\frac{1+r}{1-r}\right)^2 \inf_{z\in\phi^{-1}(K)} v(z)$. The constant $\left(\frac{1+r}{1-r}\right)^2$ depends only on the set K, so we conclude

$$\sup_{z\in\phi^{-1}(K)} u(\phi(z)) \;\leqslant\; C_K \inf_{z\in\phi^{-1}(K)} u(\phi(z)),$$

and since ϕ is a bijection this is the same as saying $\sup_{w\in K} u(w) \;\leqslant\; C_K \inf_{w\in K} u(w)$. $\square$

Problem 12. Let $\Omega = \{z \in \mathbb{C} : |z| > 1\}$. Suppose $u : \overline{\Omega} \to \mathbb{R}$ is bounded and continuous on $\overline{\Omega}$ and subharmonic on Ω. Prove the following: if $u(z) \leqslant 0$ for all $|z| = 1$ then $u(z) \leqslant 0$ for all $z \in \Omega$.

Solution. Let $v(z) = u(1/z)$. Then v is subharmonic on $A := \mathbb{D}\backslash\{0\}$ and bounded and continuous on $\overline{A}\backslash\{0\}$ because $z \mapsto 1/z$ is a conformal map from $A \to \Omega$. Fix $\epsilon > 0$ and let $f(z) = v(z) - \epsilon \log|1/z|$. Since $\log|z|$ is harmonic on A, we know that f does not have a local maximum in A. Also, since u is bounded, v also is, and thus $f(z) \to -\infty$ as $|z| \to 0$. So there exists an $r > 0$ such that $f(z) \leqslant 0$ for $|z| \leqslant r$. Now f is continuous on the compact set $\{z \in \mathbb{C} : r \leqslant |z| \leqslant 1\}$, so it achieves a maximum somewhere. But since $f(z) \leqslant 0$ for all $|z| = r$ and all $|z| = 1$, if that maximum were positive then it would have to be achieved on the interior of A, which contradicts the maximum principle. Thus the maximum is at most zero, so $f(z) \leqslant 0$ for all $r \leqslant |z| \leqslant 1$, and by choice of r this implies that $f \leqslant 0$ on A. Thus we have $v(z) \leqslant \epsilon \log|1/z|$ for all $z \in A$. Since ϵ is arbitrary, this means $v(z) = u(1/z) \leqslant 0$ for all $z \in A$, which means that $u(w) \leqslant 0$ for all $w \in \Omega$. $\square$

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Problem 1. Let $f \in L^1(\mathbb{R})$. Show that

$$\lim_{n\to\infty} \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{(k+1)/n} f(x)\,dx \right| \;=\; \int |f(x)|\,dx.$$

Solution. Let V be the set of functions which are finite linear combinations of characteristic functions of closed intervals. First we show that the result holds for elements of V. Let $g \in V$ and write

$$g \;=\; \sum_{j=1}^{M} \alpha_j \cdot \chi_{[a_j,b_j]}.$$

Let n be sufficiently large so that for each $-n^2 \leqslant k \leqslant n^2$, the interval $[k/n,(k+1)/n]$ does not intersect more than one of the intervals $[a_j,b_j]$. Then in particular, on each subinterval $[k/n,(k+1)/n]$, f is either non-negative or non-positive, depending on the sign of α_j. Thus we have, for such sufficiently large n,

$$\sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{(k+1)/n} f(x)\,dx \right| \;=\; \sum_{k=-n^2}^{n^2} \int_{k/n}^{(k+1)/n} |f(x)|\;dx \;=\; \int_{-n}^{n} |f(x)|\;dx,$$

so

$$\lim_{n\to\infty} \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{(k+1)/n} f(x)\,dx \right| \;=\; \int |f(x)|\,dx.$$

Thus the result holds for functions in V.

We know that V is dense in $L^1(\mathbb{R})$. Let $f \in L^1(\mathbb{R})$ and fix $\epsilon > 0$. We need to show that when n is sufficiently large, we have

$$\left| \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{(k+1)/n} f(x)\,dx \right| - \int |f(x)|\,dx \right| \;<\; \epsilon.$$

Let g be an element of V such that $||f-g||_{L^1} < \epsilon/3$. We have the estimate

$$\begin{aligned}
\left| \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{(k+1)/n} f(x)\,dx \right| - \int |f(x)|\,dx \right| \;&\leqslant\; \left| \int |f(x)|\,dx - \int |g(x)|\,dx \right| + \left| \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{(k+1)/n} g(x)\,dx \right| - \int |g(x)|\,dx \right| \\
&\quad + \left| \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{(k+1)/n} f(x)\,dx \right| - \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{(k+1)/n} g(x)\,dx \right| \right| \\
&=: \; I + II + III.
\end{aligned}$$

By choice of g, we have $I < \epsilon/3$. Since we have already proved the result for elements of V, let n be large enough so that $II < \epsilon/3$. Finally, by taking absolute values inside multiple times we have

$$\begin{aligned}
III \;&\leqslant\; \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{(k+1)/n} f(x)\,dx - \int_{k/n}^{(k+1)/n} g(x)\,dx \right| \;\leqslant\; \sum_{k=-n^2}^{n^2} \int_{k/n}^{(k+1)/n} |f(x)-g(x)|\;dx \;=\; \int_{-n}^{n} |f(x)-g(x)|\,dx \\
&\leqslant\; ||f-g||_{L^1} \;<\; \epsilon/3.
\end{aligned}$$

Thus we conclude that

$$\left| \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{(k+1)/n} f(x)\,dx \right| - \int |f(x)|\,dx \right| \;<\; \epsilon$$

for all sufficiently large n and thus the result holds for all $f \in L^1(\mathbb{R})$. $\square$

Problem 2. Let $f \in L^2_{loc}(\mathbb{R}^n)$, $g \in L^3_{loc}(\mathbb{R}^n)$. Assume that for all real $r \geqslant 1$, we have

$$\int_{r \leqslant |x| \leqslant 2r} |f(x)|^2 \, dx \ \leqslant \ r^a, \quad \int_{r \leqslant |x| \leqslant 2r} |g(x)|^3 \, dx \ \leqslant \ r^b.$$

Here a and b are such that $3a + 2b + n < 0$. Show that $fg \in L^1(\mathbb{R}^n)$.

Solution. Let $E_0 = \{x \in \mathbb{R}^n : |x| \leqslant 1\}$ and for $k \geqslant 1$ let $E_k = \{x \in \mathbb{R}^n : 2^{k-1} \leqslant |x| \leqslant 2^k\}$. Since each E_k is compact for $k \geqslant 0$, $|f|^2$ and $|g|^3$ are integrable on each E_k, which also implies by compactness that $|f|$ and $|g|$ are integrable on each E_k. To show that $fg \in L^1(\mathbb{R}^n)$ it suffices to show that

$$\sum_{k=1}^{\infty} \int_{E_k} |f(x)g(x)| \, dx \ < \ \infty.$$

For each $k \geqslant 1$, by Hölder's inequality using $1/6 + 1/2 + 1/3 = 1$, we have

$$\begin{aligned}\int_{E_k} |f(x)g(x)| \, dx \ &\leqslant \ \left(\int_{E_k} 1^6 \, dx\right)^{1/6} \left(\int_{E_k} |f(x)|^2 \, dx\right)^{1/2} \left(\int_{E_k} |g(x)|^3 \, dx\right)^{1/3} \\ &\leqslant \ (\lambda_n(E_k))^{1/6}((2^{k-1})^a)^{1/2}((2^{k-1})^b)^{1/3}.\end{aligned}$$

Since $E_k \subseteq [-2^k, 2^k]$, we have $\lambda_n(E_k) \leqslant (2^{k+1})^n$. Thus we have

$$\int_{E_k} |f(x)g(x)| \, dx \ \leqslant \ (2^{k+1})^{n/6}(2^{k-1})^{a/2}(2^{k-1})^{b/3} \ = \ 4^{n/6} \cdot (2^{k-1})^{n/6+a/2+b/3}.$$

By hypothesis, $n/6 + a/2 + b/3 < 0$, so let $-\delta \in (n/6 + a/2 + b/3, 0)$. Then we have

$$\sum_{k=1}^{\infty} \int_{E_k} |f(x)g(x)| \, dx \ \leqslant \ 4^{n/6} \sum_{k=1}^{\infty} (2^{k-1})^{-\delta} \ = \ 4^{n/6} \sum_{k=1}^{\infty} \left(\frac{1}{2^\delta}\right)^{k-1} \ < \ \infty$$

because $2^\delta > 1$. Thus $fg \in L^1(\mathbb{R}^n)$. $\square$

Problem 3a. Let $f \in L^1_{loc}(\mathbb{R}^n)$ and let

$$Mf(x) \ = \ \sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y)| \, dy$$

be the Hardy-Littlewood maximal function. Show that

$$m(\{x : Mf(x) > s\}) \ \leqslant \ \frac{C_n}{s} \int_{|f(x)|>s/2} |f(x)| \, dx, \quad s > 0,$$

where the constant C_n depends on n only. The Hardy-Littlewood maximal theorem may be used.

Solution. Suppose that $B \subseteq \mathbb{R}^n$ is a ball and that $\frac{1}{m(B)} \int_B |f(y)| dy > s$. Then we have

$$\begin{aligned} s \cdot m(B) &< \int_{B \cap \{x : |f(x)| \leqslant s/2\}} |f(y)| dy + \int_{B \cap \{x : |f(x)| > s/2\}} |f(y)| dy \\ &\leqslant \frac{s}{2} \cdot m(B) + \int_{B \cap \{x : |f(x)| > s/2\}} |f(y)| dy. \end{aligned}$$

Define $\tilde{f}(x)$ to be $f(x)$ if $|f(x)| > s/2$ and 0 otherwise. It follows from the work above that

$$\int_B |\tilde{f}(y)| dy > \frac{s}{2}.$$

Thus if $Mf(x) > s$, then $M\tilde{f}(x) > s/2$. Applying the Hardy-Littlewood maximal inequality to $\tilde{f}$ gives

$$\begin{aligned} m(\{x : Mf(x) > s\}) &\leqslant m(\{x : M\tilde{f}(x) > s/2\}) \\ &\leqslant \frac{C_n}{s} \int |\tilde{f}(y)| dy \\ &= \frac{C_n}{s} \int_{|f(x)|>s/2} |f(y)| dy, \end{aligned}$$

for some constant C_n. $\square$

Problem 3b. Prove that if $\phi \in C^1(\mathbb{R}), \phi(0) = 0$, and $\phi' > 0$, then

$$\int \phi(Mf(x))\, dx \;\leqslant\; C_n \int |f(x)| \left(\int_{0<t<2|f(x)|} \frac{\phi'(t)}{t}\, dt \right) dx.$$

Solution. Using part (a), we estimate the integral on the right by

$$\begin{aligned} C_n \int |f(x)| \left(\int_{0<t<2|f(x)|} \frac{\phi'(t)}{t}\, dt \right) dx \;&=\; C_n \iint_{\{(x,t):0<t<2|f(x)|\}} |f(x)| \frac{\phi'(t)}{t}\, dx\, dt \quad \text{by Tonelli because } \phi' > 0 \\ &=\; C_n \int_0^\infty \frac{\phi'(t)}{t} \int_{|f(x)|>t/2} |f(x)|\, dx\, dt \\ &\geqslant\; \int_0^\infty \frac{\phi'(t)}{t} t \cdot m\{x : Mf(x) > t\}\, dt \;=\; \int_0^\infty \phi'(t) \cdot m\{x : Mf(x) > t\}\, dt \\ &=\; \int_0^\infty \phi'(t) \int_{Mf(x)>t} dx\, dt \\ &=\; \iint_{\{(x,t):0<t<Mf(x)\}} \phi'(t)\, dx\, dt \quad \text{again by Tonelli because } \phi' > 0 \\ &=\; \int_{x\in\mathbb{R}} \int_0^{Mf(x)} \phi'(t)\, dt\, dx \;=\; \int (\phi(Mf(x)) - \phi(0))\, dx \;=\; \int \phi(Mf(x))\, dx. \quad \square \end{aligned}$$

Problem 4. Let $f \in L^1_{loc}(\mathbb{R})$ be 2π-periodic. Show that the linear combinations of the translates $f(x-a),\ a \in \mathbb{R}$, are dense in $L^1((0, 2\pi))$ if and only if each Fourier coefficient of f is $\neq 0$.

Solution. For a function $u \in L^1([0, 2\pi])$, denote by $\hat{u}(n)$ the nth Fourier coefficient of u. First suppose that $\hat{f}(n) = 0$ for some n. Then note that for any linear combination of translates of f, $h(x) = \alpha_1 f(x - a_1) + \ldots + \alpha_m f(x - a_m)$, we have $\hat{h}(n) = \alpha_1 e^{-ina_1} \hat{f}(n) + \ldots + \alpha_m e^{-ina_m} \hat{f}(n) = 0$. But then the span of the linear translates of f can't possibly be dense in L^1, because if we let $g(x) = e^{inx}$, then $\hat{g}(n) = 1$, and since the map $u \mapsto \hat{u}$ is a continuous mapping $L^1 \to \ell^\infty$, there can't be a sequence of linear combinations of translates of f converging to g in L^1.

Conversely, suppose that $\hat{f}(n) \neq 0$ for every n. Let M be the closure (with respect to the L^1 norm) of $\text{span}\{f(x - a) : a \in \mathbb{R}\}$ and suppose that $M \neq L^1$. Then by the Hahn-Banach theorem, there is a nonzero bounded linear functional $\phi \in (L^1)^*$ which is zero on M. Since $(L^1)^* \simeq L^\infty$, we get that there exists a nonzero $g \in L^\infty$ such that

$$\int_0^{2\pi} g(x) f(x - a)\, dx \;=\; 0$$

for every $a \in \mathbb{R}$. If we consider the above integral as a function of a, call it $h(a)$, then h is identically zero, so in particular it is 2π-periodic, so we can look at its Fourier coefficients. A standard computation shows that $\hat{h}(n) = \hat{g}(n)\overline{\hat{f}(n)}$ for all n, and since h is identically zero, $\hat{h}(n) = 0$ for all n. Since $\hat{f}(n) \neq 0$ for all n, this implies that $\hat{g}(n) = 0$ for all n, but this contradicts the fact that g is nonzero, so we're done. $\square$

Problem 5. Let $u \in L^2(\mathbb{R})$ and let us set

$$U(x,\xi) \;=\; \int e^{-(x+i\xi-y)^2/2}u(y)\,dy, \quad x,\xi \in \mathbb{R}.$$

Show that $U(x,\xi)$ is well-defined on $\mathbb{R}^2$ and that there exists a constant $C > 0$ such that for all $u \in L^2(\mathbb{R})$, we have

$$\iint |U(x,\xi)|^2\, e^{-\xi^2}\,dx\,d\xi \;=\; C\int |u(y)|^2\,dy.$$

Determine C explicitly.

Solution. To show that $U(x,\xi)$ is well-defined, note that by Cauchy-Schwarz

$$\int \left|e^{-(x+i\xi-y)^2/2}u(y)\right|\,dy \;\leqslant\; \left(\int e^{-(x+i\xi-y)^2}\,dy\right)^{1/2}\left(\int |u(y)|^2\,dy\right)^{1/2} \;<\; \infty.$$

Now we expand

$$U(x,\xi) \;=\; e^{-x^2/2}e^{\xi^2/2}e^{-ix\xi}\int e^{xy-y^2/2}u(y)e^{i\xi y}\,dy.$$

For a fixed x, let

$$f_x(y) \;=\; u(y)e^{xy-y^2/2}.$$

Then we see that

$$\hat{f}_x(\xi) \;=\; \int e^{xy-y^2/2}u(y)e^{-2\pi i\xi y}\,dy,$$

so

$$U(x,\xi) \;=\; e^{-x^2/2}e^{\xi^2/2}e^{-ix\xi}\hat{f}_x(-\xi/(2\pi)).$$

Therefore, by Plancherel and Tonelli since everything is non-negative, we have

$$\begin{aligned}
\iint |U(x\xi)|^2 e^{-\xi^2}\,dx\,d\xi \;&=\; \iint e^{-x^2}|\hat{f}_x(-\xi/(2\pi))|^2\,dx\,d\xi \;=\; 2\pi\int e^{-x^2}\int \left|\hat{f}_x(\xi)\right|^2\,d\xi\,dx \\
&=\; 2\pi\int e^{-x^2}\int |f_x(y)|^2\,dy\,dx \;=\; 2\pi\iint e^{-x^2+2xy-y^2}|u(y)|^2\,dy\,dx \\
&=\; 2\pi\int |u(y)|^2\left(\int e^{-(x-y)^2}\,dx\right)\,dy \;=\; 2\pi^{3/2}\int |u(y)|^2\,dy. \quad \square
\end{aligned}$$

Problem 6. When B_1 and B_2 are Banach spaces, we say a linear operator $T : B_1 \to B_2$ is compact if for any bounded sequence (x_n) in B_1, the sequence (Tx_n) has a convergent subsequence. Show that if T is compact then $\mathrm{Im}(T)$ has a dense countable subset.

Solution. Since T is a compact operator, we know that for any bounded set $A \subseteq B_1$, $T(A)$ is a relatively compact subset of B_2. Let $A_n = \{x \in B_1 : ||x||_{B_1} \leqslant n\}$. Then we can write $B_1 = \bigcup_{n=1}^{\infty} A_n$, so we have $\mathrm{Im}(T) = \bigcup_{n=1}^{\infty} T(A_n)$. Since each A_n is a bounded set, each $T(A_n)$ is relatively compact. This means that $\overline{T(A_n)}$ is compact. Since compact sets are separable (this follows from the totally bounded definition of compactness), it follows that $\overline{T(A_n)}$ has a countable dense subset. We need to upgrade this to a countable dense subset of $T(A_n)$. Let E be a countable dense subset of $\overline{T(A_n)}$. Start with $\widetilde{E} := E \cap T(A_n)$. For any $x \in E \backslash T(A_n)$, there is a sequence $\{x_k\} \in T(A_n)$ converging to x. Add the sequence $\{x_k\}$ to $\widetilde{E}$. Repeating this process for every $x \in E\backslash T(A_n)$, we see that $\widetilde{E}$ is at most a countable union of countable sequences and is thus countable, and it's clear that it is dense in $T(A_n)$. Thus $T(A_n)$ also has a countable dense subset for each n. Thus by taking the (countable) union of these dense subsets, we see that $\mathrm{Im}(T) = \bigcup_{n=1}^{\infty} T(A_n)$ has a countable dense subset. $\square$

Problem 7. Suppose $f_n : \mathbb{D} \to \mathbb{H}$ is a sequence of holomorphic functions and $f_n(0) \to 0$ as $n \to \infty$. Show that $f_n(z) \to 0$ uniformly on compact subsets of $\mathbb{D}$.

Solution. Any compact subset of $\mathbb{D}$ is contained in $\overline{B(0,r)}$ for some $0 < r < 1$, so it suffices to show that $f_n \to 0$ uniformly on $\overline{B(0,r)}$ for each $0 < r < 1$. Fix such an r. Note that since each f_n takes values only in $\mathbb{H}$, we can define a single-valued analytic branch of $g_n(z) := \sqrt{f_n(z)}$ on $\mathbb{D}$. Each g_n is a holomorphic function from $\mathbb{D}$ to $\Omega := \{z \in \mathbb{C} : \mathrm{Re}(z), \mathrm{Im}(z) > 1\}$ and it is still true that $g_n(0) \to 0$ as $n \to \infty$. Let $u_n = \mathrm{Re}(g_n)$ and $v_n = \mathrm{Im}(g_n)$. We also have $u_n(0), v_n(0) \to 0$ as $n \to \infty$. Since g_n is holomorphic and takes values in Ω, u_n and v_n are both positive harmonic functions on $\mathbb{D}$. Thus for any $z \in \overline{B(0,r)}$, we can apply Harnack's inequality to get

$$|u_n(z)| \;\leqslant\; \frac{1+|z|}{1-|z|}|u_n(0)| \;\leqslant\; \frac{1+r}{1-r}|u_n(0)|,$$

which shows that $u_n \to 0$ uniformly on $\overline{B(0,r)}$. The same argument holds for v_n. Thus since $\mathrm{Re}(g_n)$ and $\mathrm{Im}(g_n)$ both converge uniformly to 0 on $\overline{B(0,r)}$, g_n also does. Finally, since $|f_n(z)| = |g_n(z)|^2$, this also shows that $f_n \to 0$ uniformly on $\overline{B(0,r)}$, so we are done. $\square$

Alternate solution. Let $g_n = \frac{f_n - i}{f_n + i}$. The relation $f_n = \frac{(-i)(g_n+1)}{g_n - 1}$ shows that it suffices to show that the g_n converge locally uniformly to -1. Note the g_n are holomorphic maps $\mathbb{D} \to \mathbb{D}$. Let ψ_n^{-1} be an automorphism of $\mathbb{D}$ which takes $g_n(0)$ to 0 and let $h_n = \psi_n^{-1} \circ g_n$. Then h_n is holomorphic with $h_n(0) = 0$. Write $g_n = \psi_n \circ h_n$. We want to show that g_n converges locally uniformly to -1. Fix a compact set $K := \overline{B(0,r)} \subseteq \mathbb{D}$. By the Schwarz lemma, $h_n(K) \subseteq K$. So to show $g_n \to -1$ uniformly on K, it's enough to show $\psi_n \to -1$ uniformly on K. This is just a calculation: for any $|z| \leqslant r$, we have

$$|\psi_n(z) - g_n(0)| \;=\; \left|\frac{z + g_n(0)}{1 + \overline{g_n(0)}z} - g_n(0)\right| \;=\; \frac{|z|}{\left|1 + \overline{g_n(0)}z\right|}(1 - |g_n(0)|^2) \;\leqslant\; \frac{2r}{1-r}(1 - |g_n(0)|^2)$$

for sufficiently large n (where "sufficiently large" here only depends on the convergence of $g_n(0)$ to -1, so this is uniform in $|z| \leqslant r$). Since $g_n(0) \to -1$ by hypothesis (because $f_n(0) \to 0$), this shows $\psi_n \to -1$ uniformly on K, so we're done. $\square$

Problem 8. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and suppose

$$\sup_{x \in \mathbb{R}}\{|f(x)|^2 + |f(ix)|^2\} < \infty \text{ and } |f(z)| \leqslant e^{|z|} \text{ for all } z \in \mathbb{C}.$$

Deduce that f is constant.

Solution. By Liouville's theorem, to show f is constant it is enough to show that f is bounded. The first given condition implies that there is some $M < \infty$ such that $|f(z)| \leqslant M$ for all z with either $\mathrm{Re}(z) = 0$ or $\mathrm{Im}(z) = 0$. First we show that f is bounded in the first quadrant $A := \{z : \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0\}$.

We use the Phragmen-Lindelöf method. Fix $\epsilon > 0$, and define

$$g(z) \;=\; f(z) \cdot \exp(-\epsilon(e^{-i\pi/4}z)^{3/2})$$

where $w \mapsto w^{3/2}$ is defined by removing the branch cut along the negative real axis, so that $(re^{i\theta})^{3/2} = r^{3/2}e^{i3\theta/2}$. We wish to show that $|g(z)| \to 0$ as $|z| \to \infty$ in A. Writing $z = re^{i\theta}$, we have

$$\begin{aligned} |g(z)| \;&=\; |f(z)| \exp(\mathrm{Re}(-\epsilon(e^{-i\pi/4}z)^{3/2})) \;\leqslant\; \exp(r)\exp(-\epsilon r^{3/2}\,\mathrm{Re}(e^{-i3\pi/8}e^{i3\theta/2})) \\ &\leqslant \exp(r)\exp(-\epsilon r^{3/2}\cos(3\theta/2 - 3\pi/8)). \end{aligned}$$

On A, since $\theta \in (0, \pi/2)$, we have $3\theta/2 - 3\pi/8 \in (-3\pi/8, 3\pi/8)$, and thus $\cos(3\theta/2 - 3\pi/8) > \cos(3\pi/8) =: \delta > 0$. So we have

$$|g(z)| \;\leqslant\; \exp(r - \epsilon\delta r^{3/2})$$

and this tends to 0 as $|z| = r \to \infty$.

So pick R big enough so that $|g(z)| \leqslant M$ for all $z \in A$ with $|z| \geqslant R$. Now $A \cap B(0,R)$ is a bounded domain such that $|g(z)| \leqslant M$ everywhere on the boundary. Thus, since g is holomorphic, it follows from the maximum principle that $|g| \leqslant M$ everywhere in $A \cap B(0,R)$. Thus by choice of R, $|g| \leqslant M$ on all of A. This means that for any $z \in A$,

$$|f(z)| \;\leqslant\; M \cdot \left|\exp(\epsilon(e^{-i\pi/4}z)^{3/2})\right|.$$

Since ϵ is arbitrary, we can take $\epsilon \to 0$ and thus conclude that $|f(z)| \leqslant M$ for all $z \in A$.

Since M is a bound for $|f(z)|$ on the entirety of the real and imaginary axes, we can repeat this argument in each of the other three quadrants and hence obtain that $|f(z)| \leqslant M$ for all $z \in \mathbb{C}$, implying that f is a bounded entire function and thus f must be constant. $\square$

Problem 9. Let $\Omega = \{z \in \mathbb{C} : |z| > 1 \text{ and } \mathrm{Re}(z) > -2\}$. Suppose $u : \overline{\Omega} \to \mathbb{R}$ is bounded, continuous, and harmonic on Ω and also that $u(z) = 1$ when $|z| = 1$ and that $u(z) = 0$ when $\mathrm{Re}(z) = -2$. Determine $u(2)$.

Solution. Note that Ω is a region on which the Dirichlet problem can be solved, so the function u is uniquely determined by its boundary values. We want to conformally map Ω to an annulus, on which we can determine u easily. Note that the map $z \mapsto 1/z$ is a conformal map from Ω to $\Omega' = \mathbb{D}\backslash\{z \in \mathbb{C} : |z+1/4| \leqslant 1/4\}$. We now want to conformally map Ω' to the annulus $\{z \in \mathbb{C} : r < |z| < 1\}$. It suffices to find a conformal map which fixes the unit circle and maps 0 to r and $-1/2$ to $-r$. We know that the map

$$\phi : z \to \frac{z-\alpha}{1-\overline{\alpha}z}$$

fixes the unit circle, so we just need to pick an α such that $\phi(0) = r$ and $\phi(-1/2) = -r$. Solving the system of equations, we find that $-\alpha = r = 2-\sqrt{3}$ is the right choice.

So we know that $z \mapsto \phi(1/z)$ is a conformal map from Ω to the annulus $A = \{z \in \mathbb{C} : r < |z| < 1\}$, with the line $\mathrm{Re}(z) = -2$ mapping onto the inner circle $|z| = r$ and the unit circle mapping to itself. So we find a harmonic function v on A with $v(z) = 0$ for $|z| = r$ and $v(z) = 1$ for $|z| = 1$. The function

$$v(z) \;=\; \frac{\log|z/r|}{\log(1/r)}$$

accomplishes this. Thus the original function u is given by

$$u(z) \;=\; v(\phi(1/z)) \;=\; \frac{1}{\log(1/r)}\log\left|\frac{1+rz}{rz+r^2}\right|.$$

So $u(2) = \frac{1}{\log(1/r)}\log\left|\frac{1+2r}{2r+r^2}\right|$. $\square$

Problem 10. Determine

$$\int_{-\infty}^{\infty} \frac{dy}{(1+y^2)(1+(x-y)^2)}$$

for all $x \in \mathbb{R}$.

Solution. For a fixed $x \in \mathbb{R}$, integrate the function

$$f(z) \;=\; \frac{1}{(1+z^2)(1+(x-z)^2)}$$

around a half circle in the upper half plane from R to $-R$ and then along the real axis from $-R$ to R. After computing the residues and taking the limit (the contribution from the half circle goes to 0) you get that the answer is $\frac{2\pi}{x^2+4}$. $\square$

Problem 11. Let $\Omega = \mathbb{D}\backslash\{0\}$. Prove that for every bounded harmonic function $u : \Omega \to \mathbb{R}$ there is a harmonic function $v : \Omega \to \mathbb{R}$ obeying

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Solution. Let ${}^*du = -u_y\,dx + u_x\,dy$ be the conjugate differential of u. We know that for any $0 < r < 1$, the function u satisfies

$$\int_{|z|=r} u(re^{i\theta})\,d\theta = \alpha\log(r) + \beta$$

for some constants α and β, and α is given by the quantity

$$\int_{|z|=r} {}^*du,$$

which is constant with respect to r. Since u is bounded on Ω, write $|u| \leqslant M$, then we have

$$\left|\int_{|z|=r} u(re^{i\theta})\,d\theta\right| \leqslant \int_{|z|=r} |u(re^{i\theta})|\,d\theta \leqslant 2\pi rM,$$

which tends to 0 as $r \to 0^+$. This implies that we must have $\alpha = 0$. Thus in particular $\int_{|z|=1/2} {}^*du = 0$. Since the circle $|z| = 1/2$ forms a homology basis for Ω, this implies that $\int_\gamma {}^*du = 0$ for any curve $\gamma \subseteq \Omega$, so ${}^*du$ is an exact differential on Ω. This implies that there is a function v on Ω satisfying $dv = {}^*du$, i.e. $v_x = -u_y$ and $v_y = u_x$. The only thing left to verify is that v is harmonic. Note that we can define $f = u+iv$ on Ω and since f satisfies the Cauchy-Riemann equations, it is holomorphic on Ω, and therefore its real and imaginary parts are harmonic, so v is harmonic on Ω. $\square$

Alternate solution. It is a standard fact that a harmonic function on a simply connected domain has a harmonic conjugate. So to show the existence of v it suffices to show that u can be extended to be harmonic on all of $\mathbb{D}$. We know that u is continuous on the circle $|z| = 1/2$, so let h be the function which is harmonic in $|z| < 1/2$ and solves the Dirichlet problem with boundary values $u(w)$ for $|w| = 1/2$. If we show that $u = h$ everywhere where they are both defined, then this shows that u can be extended to be harmonic at 0. Let $f = u - h$. Then f is a function which is harmonic in $|z| < 1/2$ and is equal to 0 everywhere on $|z| = 1/2$. Also, since u and h are both bounded, f is bounded. We now proceed with the standard ϵ argument. Fix $\epsilon > 0$ and consider the function $z \mapsto f(z) + \epsilon\log|2z|$. This function is harmonic in $|z| < 1/2$ and is equal to 0 on the boundary $|z| = 1/2$. Furthermore, since f is bounded, this function tends to $-\infty$ as $z \to 0$. Therefore, we may pick $0 < r > 1/2$ such that $f(z) + \epsilon\log|2z| \leqslant 0$ for $|z| \leqslant r$. Now since $f(z) + \epsilon\log|2z|$ is harmonic on $r < |z| < 1/2$ and vanishes on the boundary, by the maximum principle we conclude that $f(z) \leqslant -\epsilon\log|2z|$ for all $r < |z| < 1/2$, and by choice of r we also have that $f(z) \leqslant -\epsilon\log|2z|$ for all $z \in \Omega$. Now taking $\epsilon \to 0$ we conclude that $f(z) \leqslant 0$ for all $z \in \Omega$, so $u(z) \leqslant h(z)$ in Ω. Now we can repeat the entire argument again with $\tilde{f} := h - u$ in place of f, and conclude that $h(z) \leqslant u(z)$ in Ω, so $h = u$ and we are done. $\square$

Problem 12. Find all entire functions $f : \mathbb{C} \to \mathbb{C}$ that obey

$$f'(z)^2 + f(z)^2 = 1.$$

Prove your list is exhaustive.

Solution. By taking the derivative of the above equation, we see that a necessary condition is

$$2f'(z)f''(z) + 2f(z)f'(z) = 2f'(z)(f''(z) + f(z)) = 0$$

for all $z \in \mathbb{C}$. This means we have $\{z \in \mathbb{C} : f'(z) = 0\} \cup \{z \in \mathbb{C} : f''(z) + f(z) = 0\} = \mathbb{C}$, so at least one of those sets must have a limit point, and since f is holomorphic, both f' and $f'' + f$ also are, and thus we either have $f' = 0$ or $f'' + f = 0$ on all of $\mathbb{C}$.

If $f' = 0$, then f is a constant, and the only constants which satisfy the original equation are $f(z) = \pm 1$. Now focus on the case $f'' + f = 0$. We show that the most general function that satisfies this is given by $f(z) = a\cos(z) + b\sin(z)$. We can write f as a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and since $f''(z) = -f(z)$ and power series can be differentiated term by term, we conclude that $a_n = -(n+2)(n+1)a_{n+2}$ for each n. This shows that a solution f is uniquely determined by its first two coefficients a_0 and a_1, which means the set of solutions is a 2-dimensional subspace of the vector space of entire functions. Since we know that $\cos(z)$ and $\sin(z)$ are two linearly independent solutions, it follows that $f(z) = a\cos(z) + b\sin(z)$ is the most general solution. Plugging this into the original condition, we get

$$(-a\sin(z) + b\cos(z))^2 + (a\cos(z) + b\sin(z))^2 \;=\; a^2 + b^2 \;=\; 1.$$

Thus we conclude that all of the solutions of the original equation are $f(z) = \pm 1$ or $f(z) = a\cos(z) + b\sin(z)$ where $a^2 + b^2 = 1$. $\square$

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Problem 1. Let g_n be a sequence of measurable functions on $\mathbb{R}^d$, such that $|g_n(x)| \leqslant 1$ for all x, and assume that $g_n \to 0$ almost everywhere. Let $f \in L^1(\mathbb{R}^d)$. Show that the sequence

$$f * g_n(x) = \int f(x-y)g_n(y) \to 0$$

uniformly on each compact subset of $\mathbb{R}^d$, as $n \to \infty$.

Solution. Fix $r > 0$ and let B_r denote the closed ball of radius r centered at the origin. We will show that $f * g_n$ converges uniformly on B_r.

For an arbitrary $a > 0$, we and $x \in B_r$ have

$$\begin{aligned}
|f * g_n(x)| &\leqslant \int |f(x-y)g_n(y)|dy \\
&= \int_{B_a} |f(x-y)| \cdot |g_n(y)|dy + \int_{\mathbb{R}\backslash B_a} |f(x-y)| \cdot |g_n(y)|dy \\
&\leqslant \int_{B_a} |f(x-y)| \cdot |g_n(y)|dy + \int_{\mathbb{R}\backslash B_a} |f(x-y)|dy
\end{aligned}$$

We analyze each of these last two integrals separately.

For the second integral, we recall that $x \in B_r$, so we have

$$\int_{\mathbb{R}\backslash B_a} |f(x-y)| \leqslant \int_{\mathbb{R}\backslash B_{a-r}} |f(y)|dy,$$

after a linear change of variables. Then for fixed $\epsilon > 0$ we may choose an $a = a(\epsilon)$ so that this integral is bounded by ϵ^2.

For the first integral, recall that the integral of an L^1 function over a set of small measure is small. So by Egarov we may find a measurable set $E \subseteq B_a$ so that $f_n \to f$ uniformly on $B_a \backslash E$, and $\int_E f(x-y)dy < \epsilon'$. Then for large enough n we have

$$\begin{aligned}
\int_{B_a} |f(x-y)| \cdot |g_n(y)|dy &= \int_E |f(x-y)| \cdot |g_n(y)|dy + \int_{B_a\backslash E} |f(x-y)| \cdot |g_n(y)|dy \\
&\leqslant \int_E |f(x-y)|dy + \epsilon' \int_{B_a\backslash E} |g_n(y)|dy \\
&\leqslant \epsilon'(1 + \lambda_d(B_a)).
\end{aligned}$$

Combining the two pieces, we have

$$|f * g_n(x)| \leqslant \epsilon' \cdot (1 + \lambda_d(B_{a(\epsilon)})) + \epsilon.$$

By choosing $\epsilon' = \epsilon/(1 + \lambda_d(B_{a(\epsilon)}))$, we see that $|f * g_n(x)| < 2\epsilon$ for large enough n. Since this bound is independent of x, we conclude that $f * g_n \to 0$ uniformly on B_r. $\square$

Remark. One can also solve this problem by first solving it when f has compact support and then applying an approximation argument. This is equivalent, but perhaps conceptually simpler since some of the details get abstracted into the compact support case.

Problem 2. Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$, and let $a \in \mathbb{R}$ be such that $a > 1 - 1/p$. Show that the series

$$\sum_{n=1}^{\infty} \int_n^{n+n^{-a}} |f(x+y)|\, dy$$

[2] This follows by "continuity from below" for general measures.

converges for almost all $x \in \mathbb{R}$.

Solution. Let q be the conjugate exponent so that $1/p + 1/q = 1$. Define

$$g(x) \;=\; \sum_{n=1}^{\infty} \int_n^{n+n^{-a}} |f(x+y)|\, dy.$$

With a change of variables we can write

$$g(x) \;=\; \sum_{n=1}^{\infty} n^{-a} \int_0^1 |f(x+n+n^{-a}z)|\, dz.$$

Applying Hölder's inequality for sums we have

$$|g(x)| \;\leqslant\; \left(\sum_{n=1}^{\infty} n^{-aq}\right)^{1/q} \left(\sum_{n=1}^{\infty} \left(\int_0^1 |f(x+n+n^{-a}z)|\, dz\right)^p\right)^{1/p}.$$

Since $aq > 1$ by hypothesis, the first term on the right side is just a constant C, and applying Hölder to the integral in the second term we get

$$|g(x)| \;\leqslant\; C\left(\sum_{n=1}^{\infty}\left(\left(\int_0^1 1^q\right)^{1/q}\left(\int_0^1 |f(x+n+n^{-a}z|^p\, dz\right)^{1/p}\right)^p\right)^{1/p} \;=\; C\left(\sum_{n=1}^{\infty}\int_0^1 |f(x+n+n^{-a}z)|^p\, dz\right)^{1/p}.$$

To show g is finite almost everywhere it is sufficient to show that $\int_N^{N+1} |g(x)|^p\, dx < \infty$ for each $N \in \mathbb{Z}$. We have

$$\begin{aligned}
\int_N^{N+1} |g(x)|^p\, dx \;&\leqslant\; C^p \int_N^{N+1} \sum_{n=1}^{\infty} \int_0^1 |f(x+n+n^{-a}z)|^p\, dz\, dx \\
&=\; C^p \int_0^1 \sum_{n=1}^{\infty} \int_N^{N+1} |f(x+n+n^{-a}z)|^p\, dx\, dz
\end{aligned}$$

by two applications of the Monotone Convergence Theorem and one application of Tonelli's Theorem. Changing variables again we get

$$\begin{aligned}
\int_N^{N+1} |g(x)|^p\, dx \;&\leqslant\; C^p \int_0^1 \sum_{n=1}^{\infty} \int_{N+n+n^{-a}z}^{N+1+n+n^{-a}z} |f(u)|^p\, du\, dz \\
&\leqslant\; C^p \int_0^1 \sum_{n=1}^{\infty} \int_{N+n}^{N+n+2} |f(u)|^p\, du\, dz \\
&\leqslant\; C^p \int_0^1 2\,||f||_{L^p}^p\; dz \;=\; 2C^p\,||f||_{L^p}^p \;<\; \infty.
\end{aligned}$$

Thus $\int_N^{N+1} |g|^p$ is finite for any integer N, so we conclude that $g(x)$ is finite almost everywhere. $\square$

Problem 3. Let $f \in L^1_{loc}(\mathbb{R}^d)$ be such that for some $0 < p < 1$, we have

$$\left|\int f(x)g(x)\, dx\right| \;\leqslant\; \left(\int |g(x)|^p\right)^{1/p},$$

for all $g \in C_0(\mathbb{R}^d)$ (continuous functions with compact support). Show that $f(x) = 0$ a.e.

Solution.

We would like to apply the condition of the problem when g is a characteristic function. Unfortunately characteristic functions aren't continuous, but we're able to recover the same information via a suitable approximation.

Lemma. Let K be a compact set. Then $|\int_K f(x)dx| \leqslant \lambda_d(K)^{1/p}$.

Proof: Fix $\epsilon > 0$ and let U be an open set with compact closure containing K such that $\int_{U\setminus K} |f(x)|dx < \epsilon$. (This is possible by continuity from above together with the fact that the integral of f over a set of small measure is small.) By replacing U with a set of smaller measure if necessary, we may suppose in addition that $\lambda_d(U\setminus K) < \epsilon$. Let g_K be a continuous function $\mathbb{R}^d \to [0,1]$ which takes the value 1 on K and 0 outside of U (such a function exists by Urysohn). We have

$$\begin{aligned}\left|\int f(x)g_K(x)dx - \int_K f(x)dx\right| &= \left|\int f(x)(g_K(x) - \chi_K(x))dx\right| \\ &\leqslant \int_{U\setminus K} |f(x)| \\ &< \epsilon.\end{aligned}$$

Then we have

$$\begin{aligned}\left|\int_K f(x)dx\right| &\leqslant \epsilon + \left|\int f(x)g_K(x)dx\right| \\ &\leqslant \epsilon + \left(\int |g_K(x)|^p\right)^{1/p} \\ &\leqslant \epsilon + \lambda_d(U)^{1/p} \\ &\leqslant \epsilon + (\lambda_d(K) + \epsilon)^{1/p}.\end{aligned}$$

But ϵ was arbitrary, so we the lemma follows by taking the limit as $\epsilon \to 0^+$.

Now fix a cube $C \subseteq \mathbb{R}^d$ of side length s. For any positive integer N we may dissect C into N^d cubes $\{C_i\}_{i\in[N^d]}$ of side lengths s/N. By the lemma,

$$\int_{C_i} f(x)dx \leqslant \lambda_d(C_i)^{1/p} = \left(\frac{s}{N}\right)^{d/p}.$$

Summing over all C_i we find that

$$\int_C f(x)dx \leqslant N^d \cdot \left(\frac{s}{N}\right)^{d/p} = s^{d/p} \cdot N^{d(1-1/p)}.$$

But $1 - \frac{1}{p} < 0$, so the right-hand side tends to 0 as $N \to \infty$. Thus we conclude that $\int_C f(x)dx$ for all cubes C.

Every open set is a union of countably many cubes with disjoint interiors. Therefore $\int_U f(x)dx = 0$ for any open set U. Then by continuity from above, $\int_M f(x)$ must be zero for any measurable set M, from which it follows that f is 0 a.e. $\square$

Alternate solution. Same idea as the first solution but the technical details are different.

Fix a large closed ball $S = \overline{B(0,R)}$, it's enough to show $f = 0$ a.e. on S. Suppose not. Then
Claim: There exists a $\delta > 0$ and a set $E \subseteq S$ with $\lambda(E) > 0$ with the property that for any subset $F \subseteq E$, $\left|\int_F f(x)\,dx\right| > \delta\lambda(F)$.
Assume the claim for now. A corollary of the claim is that there exist sets E of arbitrarily small positive measure satisfying the inequality in the claim. Fix such a set E with measure small enough to satisfy $\delta\lambda(E) > \lambda(E)^{1/p}$ (possible because $1/p > 1$).

Fix $\epsilon > 0$ (assume w.l.o.g that $\epsilon < \lambda(E)/10$). Since f is integrable on S, let $\alpha > 0$ be small enough so that $\lambda(A) < 2\alpha$ and $A \subseteq S$ implies $\int_A |f| < \epsilon$. We may also pick $\alpha < \epsilon$. Take a compact set K and an open set U with $K \subseteq E \subseteq U \subseteq S$ and $\lambda(E \backslash K), \lambda(U \backslash E) < \epsilon$. Let g be a continuous function with $0 \leqslant g \leqslant 1$, $g = 1$ on K, and $g = 0$ outside U. Then g also has compact support. We have the estimates

$$\begin{aligned}
\left(\int |g(x)|^p \, dx \right)^{1/p} &= \left(\int_K |g(x)|^p + \int_{U \backslash K} |g(x)|^p \, dx \right)^{1/p} \leqslant (\lambda(K) + 2\epsilon)^{1/p} \leqslant (\lambda(E) + 2\epsilon)^{1/p} \\
\left| \int f(x) g(x) \, dx \right| &= \left| \int_K f(x) g(x) \, dx + \int_{U \backslash K} f(x) g(x) \, dx \right| \geqslant \left| \int_K f(x) \, dx \right| - \int_{U \backslash K} |f(x)| \, dx \\
&\geqslant \delta \lambda(K) - \epsilon \geqslant \delta(\lambda(E) + \epsilon) - \epsilon = \delta \lambda(E) - (\delta + 1)\epsilon.
\end{aligned}$$

By the hypothesis of the problem, this implies

$$\delta \lambda(E) - (\delta + 1)\epsilon \leqslant (\lambda(E) + 2\epsilon)^{1/p}.$$

Since ϵ was arbitrary, taking $\epsilon \to 0$ gives $\delta \lambda(E) \leqslant \lambda(E)^{1/p}$, a contradiction by the choice of E at the beginning.

We need to prove the claim. Suppose f is not a.e. 0. Then by continuity from below, there is some $\delta > 0$ such that $\lambda\{x \in S : |f(x)| > 2\delta\} > 0$. For any k, we have the decomposition

$\{x \in S : |f(x)| > 2\delta\} =$
$\{x \in S : |f(x)| > 2\delta, \arg(f) \in [-2\pi/k, 2\pi/k)\} \cup \ldots \cup \{x \in S : |f(x)| > 2\delta, \arg(f) \in [-2\pi(k-3)/k, 2\pi(k-1)/k), \}$

so one of those sets has positive measure. By multiplying f by a rotation, without loss of generality we can assume

$$\lambda(E) := \lambda\{x \in S : |f(x)| > 2\delta, \arg(f) \in [-2\pi/k, 2\pi/k)\} > 0.$$

Let k be big enough so that $|f(x)| > 2\delta$ and $\arg(f) \in [-2\pi/k, 2\pi/k)$ implies $\mathrm{Re}(f) > \delta$. Then for any subset $F \subseteq E$, we have

$$\left| \int_F f \right| \geqslant \left| \int_F \mathrm{Re}(f) \right| > \delta \lambda(F).$$

This proves the claim, so we're done. $\square$

Problem 4a. Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space and assume that (e_n) is an orthonormal system in $\mathcal{H}$. Let (f_n) be another orthonormal system which is complete, i.e. the closure of the span of (f_n) is all of $\mathcal{H}$. Show that if $\sum_{n=1}^{\infty} ||f_n - e_n||^2 < 1$ then the orthonormal system (e_n) is also complete.

Solution. Let v be a vector which is orthogonal to each of the e_i. It suffices to show that $v = 0$. Since (f_i) is an orthonormal system, we can write $v = \sum_{n=1}^{\infty} \langle v, f_n \rangle f_n$. Using this expression as motivation, we define $w = \sum_{n=1}^{\infty} \langle v, f_n \rangle e_n$. Note that v and w are orthogonal, while the original condition suggests that they should be close in some suitable sense. More precisely, by applying Cauchy-Schwarz we have

$$\begin{aligned}
||v - w||^2 = \left|\left| \sum_{n=1}^{\infty} \langle v, f_n \rangle (f_n - e_n) \right|\right|^2 &\leqslant \left(\sum_{n=1}^{\infty} |\langle v_n, f_n \rangle| \, ||f_i - e_i|| \right)^2 \\
&\leqslant \left(\sum_{n=1}^{\infty} |\langle v, f_i \rangle|^2 \right) \cdot \left(\sum_{n=1}^{\infty} ||f_n - e_n||^2 \right) \leqslant ||v||^2 .
\end{aligned}$$

On the other hand, v and w are orthogonal, so $||v - w||^2 = ||v||^2 + ||w||^2$. Thus $||w||^2 = 0$, and by our original definition of w we must have $\langle v, f_n \rangle = 0$ for all n. Since (f_n) is a complete system, this means that $v = 0$ as desired.

Problem 4b. Assume we only have $\sum_{n=1}^{\infty} ||f_n - e_n||^2 < \infty$. Prove that it is still true that (e_n) is complete.

Solution. Let $E_N = \overline{\mathrm{span}(e_N, e_{N+1}, \ldots)}$ and $F_N = \overline{\mathrm{span}(f_N, f_{N+1}, \ldots)}$. The condition that $\sum_{n=1}^{\infty} ||f_n - e_n||^2 <$

∞ tells us that for big n, e_n and f_n are very close together, so the subspaces E_N and F_N should also be "close together" when N is big enough. For a closed subspace $M \subseteq \mathcal{H}$, let $\pi_M : \mathcal{H} \to M$ be the orthogonal projection onto M. We show that $||\pi_{E_N} - \pi_{F_N}||_{op} \to 0$ as $N \to \infty$ (this is one way of saying the subspaces are close to each other). For any $x \in \mathcal{H}$ we have

$$\begin{aligned}
||(\pi_{E_N} - \pi_{F_N})(x)|| &= \left\| \sum_{n=N+1}^{\infty} \langle x, e_n \rangle e_n - \langle x, f_n \rangle f_n \right\| = \left\| \sum_{n=N+1}^{\infty} \langle x, e_n \rangle (e_n - f_n) + \sum_{n=N+1}^{\infty} \langle x, e_n - f_n \rangle f_n \right\| \\
&\leqslant \sum_{n=N+1}^{\infty} |\langle x, e_n \rangle| \, ||e_n - f_n|| + \left(\left\| \sum_{n=N+1}^{\infty} \langle x, e_n - f_n \rangle f_n \right\|^2 \right)^{1/2} \\
&\leqslant \left(\sum_{n=N+1}^{\infty} |\langle x, e_n \rangle|^2 \right)^{1/2} \left(\sum_{n=N+1}^{\infty} ||e_n - f_n||^2 \right)^{1/2} + \left(\sum_{n=N+1}^{\infty} \left| \langle x, e_n - f_n \rangle^2 \right| \right)^{1/2} \\
&\leqslant ||x|| \left(\sum_{n=N+1}^{\infty} ||e_n - f_n||^2 \right)^{1/2} + \left(\sum_{n=N+1}^{\infty} ||x||^2 \, ||e_n - f_n||^2 \right)^{1/2} \\
&\leqslant ||x|| \cdot 2 \left(\sum_{n=N+1}^{\infty} ||e_n - f_n||^2 \right)^{1/2}
\end{aligned}$$

where we have used Cauchy-Schwarz for sums, the Pythagorean theorem, and Cauchy-Schwarz in $\mathcal{H}$. This shows that $||\pi_{E_N} - \pi_{F_N}||_{op}^2 \leqslant 4 \sum_{n=N+1}^{\infty} ||e_n - f_n||^2$, which goes to 0 as $N \to \infty$ by hypothesis.

We know that $\mathcal{H} = E_N \oplus E_N^{\perp}$ for any N because E_N is closed. So to show that $\overline{\operatorname{span}(\{e_n\})} = \mathcal{H}$, it's enough to find an N such that $\{e_1, \ldots, e_N\}$ spans $E_N^{\perp}$. Since the e_n are orthonormal, we at least know that $\operatorname{span}(e_1, \ldots, e_N) \subseteq E_N^{\perp}$ for each N. The e_j are also independent, so it suffices to find an N such that $\dim(E_N^{\perp}) \leqslant N$. By the assumption that $\{f_n\}$ is a complete system, we also know that $\operatorname{span}(f_1, \ldots, f_N) = F_N^{\perp}$, so $\dim(F_N^{\perp}) = N$. Finally, since $\pi_{S^{\perp}} = id - \pi_S$ for any closed subspace S, we have $\left\| \pi_{E_N^{\perp}} - \pi_{F_N^{\perp}} \right\|_{op} = ||\pi_{E_N} - \pi_{F_N}||_{op} \to 0$ as $N \to \infty$. Pick N to be large enough so that $\left\| \pi_{E_N^{\perp}} - \pi_{F_N^{\perp}} \right\|_{op} \leqslant 1/2$. Now the desired result follows from the following lemma.

Claim: Let S and T be two closed subspaces of $\mathcal{H}$ with $||\pi_S - \pi_T||_{op} \leqslant 1/2$ and $\dim(T) = N < \infty$. Then $\dim(S) \leqslant N$.
Proof: Let $x_1, \ldots, x_{N+1}$ be any $N+1$ vectors in S. Then $\pi_T(x_1), \ldots, \pi_T(x_{N+1})$ are $N+1$ vectors in an N-dimensional space, so we have

$$0 = \alpha_1 \pi_T(x_1) + \ldots + \alpha_{N+1} \pi_T(x_{N+1}) = \pi_T(\alpha_1 x_1 + \ldots + \alpha_{N+1} x_{N+1})$$

But also since each $x_j \in S$, we have $\pi_S(\alpha_1 x_1 + \ldots + \alpha_{N+1} x_{N+1}) = \alpha_1 x_1 + \ldots + \alpha_{N+1} x_{N+1}$, so

$$\begin{aligned}
||\alpha_1 x_1 + \ldots + \alpha_{N+1} x_{N+1}|| &= ||\pi_S(\alpha_1 x_1 + \ldots + \alpha_{N+1} x_{N+1}) - \pi_T(\alpha_1 x_1 + \ldots + \alpha_{N+1} x_{N+1})|| \\
&\leqslant \frac{1}{2} ||\alpha_1 x_1 + \ldots + \alpha_{N+1} x_{N+1}|| ,
\end{aligned}$$

which implies $\alpha_1 x_1 + \ldots + \alpha_{N+1} x_{N+1} = 0$, so the x_j are a dependent set. So any set of $N+1$ vectors in S is dependent, so $\dim(S) \leqslant N$. $\square$

Problem 5. A function $f \in C([0,1])$ is called Hölder continuous of order $\delta > 0$ if there is a constant C such that $|f(x) - f(y)| \leqslant C|x - y|^{\delta}$ for all $x, y \in [0,1]$. Show that the Hölder continuous functions form a meager set in $C([0,1])$.

Solution. Define Λ^{δ} to be the set of all Hölder continuous functions of order δ on $[0,1]$ and let Λ be the set of all Hölder continuous functions of any order on $[0,1]$. First note that $\delta > \eta$ implies that $\Lambda^{\delta} \subseteq \Lambda^{\eta}$, so we can write

$$\Lambda = \bigcup_{n=1}^{\infty} \Lambda^{1/n}.$$

Since a countable union of meager sets is meager, it suffices to show that Λ^δ is a meager subset of $C([0,1])$ for any fixed δ. We can write

$$\Lambda^\delta \;=\; \bigcup_{m=1}^{\infty} \{f \in \Lambda^\delta : ||f||_{\Lambda^\delta} \leqslant m\} \;=:\; \bigcup_{m=1}^{\infty} E_m$$

where the norm $||f||_{\Lambda^\delta}$ is defined by

$$|f(0)| + \sup_{x,y\in[0,1]} \frac{|f(x)-f(y)|}{|x-y|^\delta}$$

(this is one of the standard norms on the space of Hölder continuous functions). So it suffices to show that each E_m is closed and nowhere dense with respect to the L^∞ norm.

To show E_m is closed, suppose that $f_n \in E_m$ and f_n converges uniformly to $f \in C([0,1])$. Fix $\epsilon > 0$, and for any $x, y \in [0,1]$, let n be big enough so that $|f - f_n| < \epsilon|x-y|^\delta \leqslant \epsilon$ on $[0,1]$. Then we have

$$\begin{aligned} |f(0)| + \frac{|f(x)-f(y)|}{|x-y|^\delta} \;&\leqslant\; |f(0)-f_n(0)| + |f_n(0)| + \frac{|f(x)-f_n(x)|}{|x-y|^\delta} + \frac{|f_n(x)-f_n(y)|}{|x-y|^\delta} + \frac{|f_n(y)-f(y)|}{|x-y|^\delta} \\ &\leqslant\; ||f_n||_{\Lambda^\delta} + 3\epsilon \;\leqslant\; M + 3\epsilon, \end{aligned}$$

and since the left side does not depend on ϵ, we conclude that

$$|f(0)| + \frac{|f(x)-f(y)|}{|x-y|^\delta} \;\leqslant\; m$$

for all x, y, so $||f||_{\Lambda^\delta} \leqslant m$. Therefore E_m is closed.

For nowhere dense, let $f \in E_m$ and fix $\epsilon > 0$. We just need to show the existence of some $h \notin E_m$ with $||h-f||_{L^\infty} \leqslant \epsilon$. Fix any $g \notin \Lambda^\delta$ (for example, $g(x) = x^{\delta/2}$ works) and by scaling, we may assume $||g||_{L^\infty} = 1$. Then let $h \;=\; f + \epsilon g$. Then we clearly have $||h-f||_{L^\infty} = \epsilon$. Since $g \notin \Lambda^\delta$, we can find points x_n, y_n such that

$$\frac{|g(x_n)-g(y_n)|}{|x_n-y_n|^\delta} \;\geqslant\; \frac{n}{\epsilon}.$$

Then we have

$$\begin{aligned} \frac{|h(x_n)-h(y_n)|}{|x_n-y_n|^\delta} \;&=\; \frac{|f(x_n)+\epsilon g(x_n)-f(y_n)-\epsilon g(y_n)|}{|x_n-y_n|^\delta} \\ &\geqslant\; \epsilon\frac{|g(x_n)-g(y_n)|}{|x_n-y_n|^\delta} - \frac{|f(x_n)-f(y_n)|}{|x_n-y_n|^\delta} \;\geqslant\; n-m, \end{aligned}$$

which goes to ∞ as $n \to \infty$, so $h \notin \Lambda^\delta$. Therefore E_m is closed and nowhere dense, so we're done. □

Problem 6. Let $u \in L^2(\mathbb{R}^d)$ and say that $u \in H^{1/2}(\mathbb{R}^d)$ (a Sobolev space) if

$$\left(1 + |\xi|^{1/2}\right)\hat{u}(\xi) \in L^2(\mathbb{R}^d).$$

Here $\hat{u}$ is the Fourier transform of u. Show that $u \in H^{1/2}(\mathbb{R}^d)$ if and only if

$$\iint \frac{|u(x+y)-u(x)|^2}{|y|^{d+1}}\,dx\,dy \;<\; \infty.$$

Solution. Since $u \in L^2(\mathbb{R}^d)$, we know immediately that $\hat{u} \in L^2(\mathbb{R}^d)$ also, so we just need to show that $\left(1 + |\xi|^{1/2}\right) \hat{u}(\xi) \in L^2(\mathbb{R}^d)$ if and only if the above double integral is finite. It suffices to prove that

$$\int |\xi| \, |\hat{u}(\xi)|^2 \, d\xi \;\lesssim\; \iint \frac{|u(x+y) - u(x)|^2}{|y|^{d+1}} \, dx \, dy \;\lesssim\; \int |\xi| \, |\hat{u}(\xi)|^2 \, d\xi,$$

where throughout this problem $\lesssim$ denotes an implied constant which depends only on d. First note that by Plancherel, we have

$$\iint \frac{|u(x+y) - u(x)|^2}{|y|^{d+1}} \, dx \, dy \;=\; \int \frac{1}{|y|^{d+1}} \int \left|1 - e^{2\pi i y \cdot \xi}\right|^2 |\hat{u}(\xi)|^2 \, d\xi \, dy \;=\; \int |\hat{u}(\xi)|^2 \int \frac{\left|1 - e^{2\pi i y \cdot \xi}\right|^2}{|y|^{d+1}} \, dy \, d\xi,$$

so it now suffices just to prove the estimates

$$|\xi| \;\lesssim\; \int \frac{\left|1 - e^{2\pi i y \cdot \xi}\right|^2}{|y|^{d+1}} \, dy \;\lesssim\; |\xi|.$$

For the upper bound, we have the estimate

$$\begin{aligned}
\int \frac{\left|1 - e^{2\pi i y \cdot \xi}\right|^2}{|y|^{d+1}} \, dy \;&=\; \int_{|y| \leqslant 1/(2|\xi|)} \frac{\left|1 - e^{2\pi i y \cdot \xi}\right|^2}{|y|^{d+1}} \, dy + \int_{|y| > 1/(2|\xi|)} \frac{\left|1 - e^{2\pi i y \cdot \xi}\right|^2}{|y|^{d+1}} \, dy \\
&\leqslant\; \int_{|y| \leqslant 1/(2|\xi|)} \frac{|4\pi y \cdot \xi|^2}{|y|^{d+1}} \, dy + \int_{|y| > 1/(2|\xi|)} \frac{4}{|y|^{d+1} \, dy} \quad \text{because } |1 - e^z| \leqslant 2|z| \text{ for } |z| \leqslant 1/2 \\
&\lesssim\; |\xi|^2 \int_{|y| \leqslant 1/(2|\xi|)} \frac{|y|^2}{|y|^{d+1}} \, dy + \int_{|y| > 1/(2|\xi|)} \frac{1}{|y|^{d+1}} \, dy \\
&\lesssim\; |\xi| + |\xi| \;\lesssim\; |\xi|.
\end{aligned}$$

Now we do the lower bound. For ξ fixed, define $E = \{y \in \mathbb{R}^d : |y \cdot \xi| \geqslant (1/2)|y||\xi|\}$. We estimate

$$\begin{aligned}
\int \frac{\left|1 - e^{2\pi i y \cdot \xi}\right|^2}{|y|^{d+1}} \, dy \;&\geqslant\; \int_{|y| \leqslant 1/(3|\xi|), y \in E} \frac{\left|1 - e^{2\pi i y \cdot \xi}\right|^2}{|y|^{d+1}} \, dy \\
&\geqslant\; \int_{|y| \leqslant 1/(3|\xi|), y \in E} \frac{|\pi y \cdot \xi|^2}{|y|^{d+1}} \, dy \quad \text{because } |e^z - 1| \geqslant (1/2)|z| \text{ for } |z| \leqslant 1/3 \\
&\gtrsim\; \int_{|y| \leqslant 1/(3|\xi|), y \in E} \frac{(1/2)|y|^2|\xi|^2}{|y|^{d+1}} \, dy \;\gtrsim\; |\xi|^2 \int_{|y| \leqslant 1/(3|\xi|), y \in E} \frac{1}{|y|^{d-1}} \, dy.
\end{aligned}$$

Now note that membership in E is determined only by the direction of y and is independent of the magnitude of y. So since the above integrand is a function only of $|y|$, and E takes up a "positive proportion" of all of $\mathbb{R}^d$ (this can be made precise), it follows that the above integral is

$$\gtrsim\; |\xi|^2 \int_{|y| \leqslant 1/(3|\xi|)} \frac{1}{|y|^{d-1}} \, dy \;\gtrsim\; |\xi|,$$

which concludes the proof of the lower bound, so we are done. $\square$

Problem 7. Assume that $f(z)$ is analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. If $f(z) = f(1/z)$ when $|z| = 1$, prove that $f(z)$ is constant.

Solution. Define the function g by

$$g(z) \;:=\; \begin{cases} f(z) & |z| \leqslant 1 \\ f(1/z) & |z| \geqslant 1 \end{cases}.$$

Because of the condition that $f(z) = f(1/z)$ for $|z| = 1$, we see that g is continuous on all of $\mathbb{C}$. We now mimic the proof of the Schwarz reflection principle to show that g is analytic on all of $\mathbb{C}$. By Morera's theorem, it is enough to show that

$$\int_{\partial R} g(z)\,dz \;=\; 0$$

for any rectangle R. It is clear from the definition that g is analytic inside $\mathbb{D}$ and so we don't need to consider rectangles R that are contained in $\mathbb{D}$. Also, since $z \mapsto 1/z$ is a conformal map from $\mathbb{C}\backslash\mathbb{D}$ into $\mathbb{D}\backslash\{0\}$, we also see that g is analytic on the exterior of $\mathbb{D}$, so we also don't need to consider rectangles that are contained in the exterior of $\mathbb{D}$. Thus we only need to consider rectangles which intersect the unit circle. For such a rectangle, split the contour along the arc of the unit circle into a band of width δ (this is hard to explain without a picture). Since g is analytic on both the inside and the outside of $\mathbb{D}$, the integral over this split contour is necessarily 0. Then, since g is continuous everywhere, as we let $\delta \to 0$, the integral over the split contour approaches the integral over the original rectangle, and so we conclude that $\int_{\partial R} g(z)\,dz = 0$ for all rectangles R and thus g is analytic on all of $\mathbb{C}$.

Now note that since f is continuous on $\overline{\mathbb{D}}$, which is compact, f must be bounded, and thus g must also be bounded. But g is entire, so g must be a constant, which means f must also be a constant. $\square$

Problem 8. Assume that $f(z)$ is an entire function that is 2π-periodic in the sense that $f(z+2\pi) = f(z)$, and

$$|f(x+iy)| \;\leqslant\; Ce^{\alpha|y|}$$

for some $C > 0$, where $0 < \alpha < 1$. Prove that f is constant.

Solution. Since $f(z)$ is 2π periodic, we can express f as the pullback of a holomorphic function on the cylinder. More formally, we can write

$$f(z) = g(e^{iz})$$

where we define g on $\mathbb{C}\backslash\{0\}$ by $g(z) = f(\frac{1}{i}\log(z))$. Since f is 2π-periodic, the branch of log is irrelevant, and g is well-defined.

The given bound implies that $|g(e^y \cdot e^{ix})| \leqslant Ce^{\alpha|y|}$. Thus we have

$$|g(z)| \leqslant C\exp(\alpha|\log|z||).$$

As $|z| \to 0$, we have $|g(z)| \leqslant Cz^{-\alpha}$, but $\alpha < 1$, so g has a removable singularity at 0, and we can extend g to an analytic function on $\mathbb{C}$. Similarly as $|z| \to 0$, we have $|g(z)| \leqslant Cz^{\alpha}$, and so g must be constant. This immediately implies that f is constant.

Problem 9. Let (f_j) be a sequence of entire functions such that, writing $z = x+iy$, we have

$$\iint_{\mathbb{C}} |f_j(z)|^2 e^{-|z|^2}\,dx\,dy \;\leqslant\; C, \quad j = 1, 2, \ldots$$

for some constant $C > 0$. Show that there exists a subsequence (f_{j_k}) and an entire function f such that we have

$$\iint_{\mathbb{C}} |f_{j_k}(z) - f(z)|^2 e^{-2|z|^2}\,dx\,dy \to 0, \quad k \to \infty.$$

Solution. By the mean value property and Cauchy-Schwarz, for any $z \in \mathbb{C}$ with $|z| \geqslant 2$ and any j we can write

$$|f_j(z)| \;\lesssim\; \int_{B(z,1)} |f_j(w)|\,dx\,dy \;\lesssim\; \left(\int_{B(z,1)} |f_j(w)|^2\,dx\,dy\right)^{1/2} \;\leqslant\; e^{\frac{1}{2}(|z|+1)^2}\left(\int_{B(z,1)} |f_j(w)|^2\,e^{-|w|^2}\,dx\,dy\right)^{1/2} \;\leqslant\; Ce^{\frac{1}{2}(|z|+1)^2}.$$

In particular, this implies that the sequence $\{f_j\}$ is uniformly bounded on every compact subset of $\mathbb{C}$, so it is a normal family. Thus it has a subsequence $\{f_{j_k}\}$ which converges uniformly on every compact subset of $\mathbb{C}$. Since each f_j is entire, we also know that the limit function f is entire and also satisfies the estimate

$$|f(z)| \lesssim e^{\frac{1}{2}(|z|+1)^2}.$$

for $|z| \geqslant 2$.

To show the desired conclusion, fix $\epsilon > 0$. Let R be big enough so that

$$\int_{|z|>R} e^{-|z|^2+|z|+1}\,dx\,dy \;<\; \epsilon.$$

Since $f_{j_k} \to f$ uniformly on every compact subset of $\mathbb{C}$, we may choose k to be big enough so that

$$\int_{|z|\leqslant R} |f_{j_k}(z) - f(z)|^2 e^{-2|z|^2}\,dx\,dy \;<\; \epsilon.$$

Thus we have the estimate

$$\begin{aligned}
\int_{\mathbb{C}} |f_{j_k}(z) - f(z)|^2 e^{-2|z|^2}\,dx\,dy \;&=\; \int_{|z|\leqslant R} |f_{j_k}(z) - f(z)|^2 e^{-2|z|^2}\,dx\,dy + \int_{|z|>R} |f_{j_k}(z) - f(z)|^2 e^{-2|z|^2}\,dx\,dy \\
&<\; \epsilon + \int_{|z|>R} (C' \cdot 2e^{\frac{1}{2}(|z|+1)^2})^2 e^{-2|z|^2}\,dx\,dy \\
&\leqslant\; \epsilon + C'' \int_{|z|>R} e^{-|z|^2+|z|+1}\,dx\,dy \;<\; (1+C'')\epsilon,
\end{aligned}$$

which establishes the desired conclusion. $\square$

Problem 10. Use the Residue Theorem to prove that

$$\int_0^\infty e^{\cos x} \sin(\sin x)\frac{dx}{x} \;=\; \frac{\pi}{2}(e-1)$$

Use a large semicircle as part of the contour.

Solution. For real x, the integrand can be written as $\frac{1}{x}\operatorname{Im}(e^{e^{ix}})$. We can rewrite our integral as

$$\int_0^\infty \operatorname{Im}(e^{e^{ix}})\frac{dx}{x} = \operatorname{Im}\int_{-\infty}^\infty e^{e^{ix}}\frac{dx}{x},$$

where the equality holds provided the second integral exists (which it will).

Set $f(z) = \frac{1}{z}e^{e^{iz}}$ and let Γ_R denote a large semicircular contour of radius R with endpoints at $-R$ and R. Also let γ_r denote a small *clockwise* contour of radius r with endpoints at $-r$ and r.

Note that f is holomorphic everywhere except $z = 0$, where it has a simple pole with residue e. Thus by (a variant of) the residue theorem for "indented contours", we have

$$\lim_{r\to 0}\int_{\gamma_r} f(z)dz = -\frac{1}{2}\cdot 2\pi i \cdot e = -i\pi e.$$

On the outer contour we have

$$\int_{\Gamma_R} f(z)dz = i\int_0^\pi e^{e^{iR\exp(i\theta)}}\,d\theta.$$

Note that for $\theta \in [0, \pi]$,

$$\left|e^{iR\exp(i\theta)}\right| = e^{-R\sin(\theta)} \leqslant 1.$$

Thus by the bound $|e^z| \leqslant e^{|z|}$, our integrand is dominated by e. Also as $R \to \infty$, the same bound shows that the integrand tends pointwise to $e^0 = 1$ (except at $\theta = 0$ and $\theta = \pi$), so by dominated convergence,

$$\int_{\Gamma_R} f(z)dz \to i\pi \text{ as } R \to \infty.$$

By Cauchy's applying Cauchy's theorem to a contour joining the two semicircles, we have

$$0 = 2\int_{-r}^{R} f(z)dz + \int_{\gamma_r} f(z)dz + \int_{\Gamma_R} f(z)dz,$$

and taking the limit as $r \to 0$ and $R \to \infty$ gives

$$\int_0^\infty f(x)dx = i\frac{\pi}{2}(e-1).$$

Finally, the imaginary part of this is the desired value.

Problem 11. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$ and let u be subharmonic in Ω, continuous in $\overline{\Omega}$, such that

$$u(x,y) \;\leqslant\; |x+iy|\,,$$

for large $(x,y) \in \Omega$. Assume that

$$u(x,0) \;\leqslant\; ax, \quad u(0,y) \;\leqslant\; by, \quad x,y \geqslant 0,$$

for some $a,b > 0$. Show that

$$u(x,y) \;\leqslant\; ax+by, \quad (x,y) \in \Omega.$$

Solution. We use the Phregman-Linedlöf method. Fix $\epsilon > 0$ and, writing $(x,y) = re^{i\theta}$, define

$$\phi(x,y) \;=\; ax+by+\epsilon r^{3/2}\cos\left(\frac{-3\pi}{8}+\frac{3\theta}{2}\right).$$

Note that $\epsilon r^{3/2}\cos\left(\frac{-3\pi}{8}+\frac{3\theta}{2}\right)$ is the real part of the function $f(z) = -\epsilon(e^{-i\pi/4}z)^{3/2}$, which is single-valued and analytic in Ω, so ϕ is harmonic in Ω (because $ax+by$ is clearly harmonic). Thus, since u is subharmonic in Ω, we know that $v := u - \phi$ does not have any local maximum in Ω.

We want to show that $v(x,y) \to -\infty$ as $r \to \infty$ in Ω. Note that since for $(x,y) \in \Omega$ we have $\theta \in (0, \pi/2)$, we have $-3\pi/8 + 3\theta/2 \in (-3\pi/8, 3\pi/8)$ and thus $\cos(-3\pi/8 + 3\theta/2) > \cos(3\pi/8) =: \delta > 0$. So as $r \to \infty$, by the hypothesis that $u(x,y) < r$ for r sufficiently large, we have

$$v(x,y) \;=\; u(x,y) - ax - by - \epsilon r^{3/2}\cos\left(\frac{-3\pi}{8}+\frac{3\theta}{2}\right) \;\leqslant\; r - \epsilon\delta r^{3/2} \to -\infty$$

as $r \to \infty$. Thus we can pick an R large enough so that $v(x,y) \leqslant 0$ for all $r \geqslant R$. We also know from the other hypotheses that on the x-axis,

$$v(x,0) = u(x,y) - ax - \epsilon r^{3/2}\cos\left(\frac{-3\pi}{8}+\frac{3\theta}{2}\right) \;\leqslant\; 0$$

and similarly on the y-axis $v(0,y) \leqslant 0$. Thus we can now apply the maximum principle to v on the bounded region $\{(x,y) \in \Omega : r \leqslant R\}$, and since $v \leqslant 0$ on the boundary, we conclude that $v \leqslant 0$ throughout the entire region, and thus by choice of R, $v(x,y) \leqslant 0$ for all $(x,y) \in \Omega$. This means that

$$u(x,y) \;\leqslant\; ax+by+\epsilon r^{3/2}\cos\left(\frac{-3\pi}{8}+\frac{3\theta}{2}\right)$$

for each $(x, y) \in \Omega$, and since ϵ is arbitrary, we conclude that $u(x, y) \leqslant ax + by$ for all $(x, y) \in \Omega$. $\square$

Problem 12. Find a function $u(x, y)$ harmonic in the region between the circles $|z| = 2$ and $|z - 1| = 1$ which equals 1 on the outer circle and 0 on the inner circle (except at the point where the circles are tangent to each other).

Solution. Let $\Omega = \{z \in \mathbb{C} : |z| < 2, |z - 1| > 1\}$ be the original region. We want to conformally map Ω to a region on which such a function can easily be found and then pull it back. The map $z \mapsto 1/(z - 2)$ sends Ω to the strip $\{z \in \mathbb{C} : -1/2 < \mathrm{Re}(z) < -1/4\}$, with the circle $|z| = 2$ going to the line $\mathrm{Re}(z) = -1/4$ and the circle $|z - 1| = 1$ going to the circle $\mathrm{Re}(z) = -1/2$. So we are looking for a harmonic function v which satisfies $v(z) = 0$ when $\mathrm{Re}(z) = -1/2$ and $v(z) = 1$ when $\mathrm{Re}(z) = -1/4$. The function $v(z) = \mathrm{Re}(4z + 2)$ clearly satisfies this and is harmonic because it is the real part of an analytic function. Therefore the function

$$u(z) \;=\; v\left(\frac{1}{z-2}\right) \;=\; \mathrm{Re}\left(\frac{4}{z-2} + 2\right) \;=\; \mathrm{Re}\left(\frac{2z}{z-2}\right)$$

is a harmonic function on Ω with the desired properties. $\square$

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Problem 1a. Let

$$K_t(x) \;=\; (4\pi t)^{-3/2} e^{-|x|^2/4t}, \quad x \in \mathbb{R}^2,\ t > 0,$$

where $|x|$ is the Euclidean norm of $\mathbb{R}^3$. Show that the linear map

$$f \;\mapsto\; t^{1/2}(K_t * f), \quad L^3(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$$

is bounded uniformly in $t > 0$.

Solution. Throughout this problem, we use the symbol $\lesssim$ to denote an implied constant which does not depend on f, x or t. For any $x \in \mathbb{R}^3$, we calculate

$$\left|t^{1/2}(K_t * f)(x)\right| \;\lesssim\; t^{-1}\int_{\mathbb{R}^3} \exp\left(\frac{-1}{4t}|x-y|^2\right)|f(y)|\,dy \;\leqslant\; t^{-1}\left(\int_{\mathbb{R}^3}|f(y)|^3\,dy\right)^{1/3}\left(\int_{\mathbb{R}^3}\exp\left(\frac{-3}{8t}|x-y|^2\,dy\right)\right)^{2/3}$$

by Hölder's inequality. Making the change of variables $z = \frac{\sqrt{3}}{\sqrt{8}}(x-y)$ in the last integral, we get

$$\begin{aligned}
\left|t^{1/2}(K_t * f)(x)\right| \;&\lesssim\; t^{-1}\,||f||_{L^3}\left(\int_{\mathbb{R}^3}\exp\left(-\left|\frac{z}{\sqrt{t}}\right|^2\right)\,dz\right)^{2/3}\\
&=\; t^{-1}\,||f||_{L^3}\left(\left(\int_{\mathbb{R}}\exp(-(u/\sqrt{t})^2)\,du\right)^3\right)^{2/3} \quad \text{by Tonelli's theorem}\\
&\lesssim\; t^{-1}\,||f||_{L^3}\,(\sqrt{\pi t})^2 \;\lesssim ||f||_{L^3}\,.
\end{aligned}$$

Thus $\left|\left|t^{1/2}(K_t * f)\right|\right|_{L^\infty} \lesssim ||f||_{L^3}$, so we see that $f \mapsto t^{1/2}(K_t * f)$ is a bounded linear operator whose operator norm is bounded uniformly in $t > 0$. $\square$

Problem 1b. Prove that $t^{1/2}\,||K_t * f||_{L^\infty} \to 0$ as $t \to 0$, for $f \in L^3(\mathbb{R}^3)$.

Solution. We know that $C_c(\mathbb{R}^3)$, the set of continuous functions with compact support, is dense in $L^3(\mathbb{R})$. If $g \in C_c(\mathbb{R}^3)$, then we have

$$|(K_t * g)(x)| \;\leqslant\; \int_{\mathbb{R}^3}|K_t(x-y)g(y)|\,dy \;\leqslant\; ||g||_{L^\infty}\int_{\mathbb{R}^3}|K_t(x-y)|\,dy \;\lesssim\; ||g||_{L^\infty}$$

where again the implied constant here does not depend on t. Thus we have $t^{1/2}\,||K_t * g||_{L^\infty} \to 0$ as $t \to 0$ for all $g \in C_c(\mathbb{R}^3)$.

Now let f be any function in $L^3(\mathbb{R}^3)$. Let the linear operator $\phi_t : L^3(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ be defined by

$$\phi_t(f) \;=\; t^{1/2}(K_t * f).$$

Recall that in part (a) we showed that there is a constant C, independent of t, such that $||\phi_t(f)||_{L^\infty} \leqslant C\,||f||_{L^3}$ for all $f \in L^3$. Fix $\epsilon > 0$. By density, we can pick $g \in C_c(\mathbb{R}^3)$ such that $||f-g||_{L^3} < \epsilon/2C$. Since we have proved the result for functions in $C_c(\mathbb{R}^3)$, we can now pick a $\delta > 0$ such that for all $t < \delta$, $||\phi_t(g)||_{L^\infty} < \epsilon/2$. Then we conclude that for any $t < \delta$ we have

$$t^{1/2}\,||K_t * f||_{L^\infty} \;=\; ||\phi_t(f)||_{L^\infty} \;\leqslant\; ||\phi_t(g)||_{L^\infty} + ||\phi_t(f-g)||_{L^\infty} \;<\; \frac{\epsilon}{2} + C\,||f-g||_{L^3} \;<\; \epsilon.$$

This shows that $\lim_{t\to 0} t^{1/2}\,||K_t * f||_{L^\infty} = 0$ for any $f \in L^3(\mathbb{R}^3)$. $\square$

Problem 2. Let $f \in L^1(\mathbb{R})$. Show that the series

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}f(x-\sqrt{n})$$

converges absolutely for almost all $x \in \mathbb{R}$.

Solution. Let

$$g(x) \;=\; \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left| f(x - \sqrt{n}) \right|.$$

We show that $\int_M^{M+1} g(x)\, dx$ is finite for every integer M, which is enough to conclude that $g(x) < \infty$ for almost every $x \in [M, M+1]$, which in turn implies that $g(x)$ is finite almost everywhere, which is exactly what we need to prove.

For a fixed integer M, we have

$$\int_M^{M+1} g(x)\, dx \;=\; \int_M^{M+1} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left| f(x - \sqrt{n}) \right| \;=\; \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \int_M^{M+1} \left| f(x - \sqrt{n}) \right| dx$$

by the Monotone Convergence Theorem, and after changing variables we get

$$\int_M^{M+1} g(x)\, dx \;=\; \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \int_{M-\sqrt{n}}^{M+1-\sqrt{n}} |f(y)|\, dy.$$

For each integer k, there are $2k+1$ integers n such that $k < \sqrt{n} \leqslant k+1$. For each of these integers n, we have $[M - \sqrt{n}, M+1-\sqrt{n}] \subseteq [M-k-1, M+1-k]$. Thus the above sum is bounded by

$$\sum_{k=1}^{\infty} (2k+1) \cdot \frac{1}{k} \int_{M-k-1}^{M+1-k} |f(y)|\, dy \;\leqslant\; 3 \sum_{k=1}^{\infty} \left[\int_{M-k-1}^{M-k} |f(y)|\, dy + \int_{M-k}^{M+1-k} |f(y)|\, dy \right] \;\leqslant\; 6\, ||f||_{L^1}.$$

Thus we conclude that $\int_M^{M+1} g(x) < \infty$, so $g(x)$ is finite almost everywhere. $\square$

Problem 3. Let $f \in L^1_{loc}(\mathbb{R})$ be real-valued and assume that for each integer $n > 0$, we have

$$f\left(x + \frac{1}{n}\right) \;\geqslant\; f(x),$$

for almost all $x \in \mathbb{R}$. Show that for each real number $a \geqslant 0$ we have

$$f(x+a) \;\geqslant\; f(x)$$

for almost all $x \in \mathbb{R}$.

Solution. Let E be the (measure zero) set of $x \in \mathbb{R}^n$ that do not have the property of the hypothesis. Define $F = \bigcup_{p \in \mathbb{Q}} (E + p)$. This is a countable union of measure zero sets so it also has measure zero. If $a = 0$, the result is obvious, so let $a > 0$ be fixed. By the Lebesgue differentiation theorem, we know that

$$f(x+a) - f(x) \;=\; \lim_{r \to 0^+} \frac{1}{2r} \int_{x-r}^{x+r} (f(y+a) - f(y))\, dy.$$

for all x outside of some measure zero set G. We show that $f(x+a) - f(x) \geqslant 0$ for all x outside of G. It is enough to show that for any interval $[b, c]$,

$$\int_b^c f(y+a)\, dy \;\geqslant\; \int_b^c f(y)\, dy,$$

or equivalently

$$\int_{b+a}^{c+a} f(y)\, dy \;\geqslant\; \int_b^c f(y)\, dy.$$

We can write a in binary as

$$a \;=\; m + \sum_{j=1}^{\infty} \frac{\epsilon_j}{2^j} \;=\; \sum_{j=1}^{\infty} \frac{1}{k_j}$$

where $\{k_j\}$ is some sequence of integers (not necessarily distinct, because there could by many 1s at the beginning). Let $a_N = \sum_{j=1}^{N} 1/k_j$. For any $y \notin F$ and any N, we know that $y + a_N \notin E$ by construction of F. Therefore we have $f(y + a_N) = f(y + a_{N-1} + 1/k_N) \geqslant f(y + a_{N-1})$. By induction and the fact that $y + a_N \notin E$ for each N, we see that $f(y + a_N) \geqslant f(y)$ for all N. Therefore, since F has measure zero, this means

$$\int_{b+a_N}^{c+a_N} f(y)\,dy \;=\; \int_b^c f(y + a_N)\,dy \;\geqslant\; \int_b^c f(y)\,dy.$$

Defining $f_N(y) = f(y)\chi_{[b+a_N,c+a_N]}(y)$, we see that

$$\int_{\mathbb{R}} f_N(y)\,dy \;\geqslant\; \int_b^c f(y)\,dy.$$

Since $f_N \to f\chi_{[b+a,c+a]}$ pointwise as $N \to \infty$ and $|f_N| \leqslant |f|\chi_{[b,c+a]}$ for all N, and $|f|\chi_{[b,c+a]}$ is integrable, by the Dominated Convergence Theorem we conclude that

$$\int_{b+a}^{c+a} f(y)\,dy \;=\; \int_{\mathbb{R}} f\chi_{[b+a,c+a]} \;\geqslant\; \int_b^c f(y)\,dy.$$

Thus we conclude that $f(x + a) - f(x) \geqslant 0$ for all x for which the Lebesgue differentiation theorem applies to the function $x \mapsto f(x + a) - f(x)$, which is almost all $x \in \mathbb{R}$. $\square$

Problem 4. Let V_1 be a finite-dimensional subspace of the Banach space V. Show that there exists a continuous projection $P : V \to V_1$, i.e., a continuous linear map $P : V \to V_1$ such that $P^2 = P$ and the range of P is equal to V_1.

Solution. Let $\{e_1, \ldots, e_n\}$ be a basis for V_1. Without loss of generality we may assume that $||e_j|| = 1$ for each j. For a fixed j, we know that $\text{span}\{e_i\}_{i\neq j}$ is a closed subspace of V. Thus by the Hahn-Banach theorem, there is a linear functional $f_j \in V^*$ such that $f_j(e_j) = ||e_j|| = 1$ and $f_j(x) = 0$ for all $x \in \text{span}\{e_i\}_{i\neq j}$. Now define the map $P : V \to V_1$ by

$$P(x) \;:=\; \sum_{j=1}^{n} f_j(x)e_j.$$

It is clear that $\text{Im}(P) \subseteq V_1$ by construction, and since each f_j is linear, P is also linear. We see that P is continuous because

$$||Px - Py|| \;=\; \left\|\sum_{j=1}^{n} f_j(x-y)e_j\right\| \;\leqslant\; \sum_{j=1}^{n} |f_j(x-y)|\,||e_j|| \;\leqslant\; \left(\sum_{j=1}^{n} ||f_j||\right) ||x-y||\,.$$

Finally, for any $v \in V_1$, we write $v = v_1e_1 + \ldots + v_ne_n$ and note that

$$Pv \;=\; \sum_{j=1}^{n} f_j(v_1e_1 + \ldots + v_ne_n)e_j \;=\; \sum_{j=1}^{n} v_je_j \;=\; v.$$

This implies both that $P^2 = P$ and that $V_1 \subseteq \text{Im}(P)$, so $\text{Im}(P) = V_1$. Thus P is the desired map. $\square$

Problem 5. For $f \in C_0^\infty(\mathbb{R}^2)$ define $u(x,t)$ by

$$u(x,t) \;=\; \int_{\mathbb{R}^2} e^{ix\cdot\xi}\frac{\sin(t|\xi|)}{|\xi|} f(\xi)\,d\xi, \quad x \in \mathbb{R}^2, \quad t > 0.$$

Show that $\lim_{t\to\infty} ||u(\cdot,t)||_{L^2} = \infty$ for a set of f that is dense in $L^2(\mathbb{R})$.

Solution. We claim the desired result holds for all f in the set

$$S \;:=\; \{f \in L^2 : \lim_{x\to 0} |f(x)| = \infty\}.$$

Define

$$g_t(\xi) \;=\; \frac{\sin(t|\xi|)}{|\xi|}\overline{f(\xi)},$$

then we see that

$$u(x,t) \;=\; \int_{\mathbb{R}^2} e^{ix\cdot\xi}\overline{g_t(\xi)}\,d\xi \;=\; \overline{\hat{g}_t(x)}.$$

Therefore by Plancherel we have

$$\begin{aligned}
||u(\cdot,t)||_{L^2}^2 \;&=\; ||\hat{g}_t||_{L^2}^2 \;=\; ||g_t||_{L^2}^2 \;=\; \int \left(\frac{\sin(t|\xi|)}{|\xi|}\right)^2 |f(\xi)|^2\,d\xi \;\geqslant\; \int_{B(0,\pi/(2t))} \left(\frac{\sin(t|\xi|)}{|\xi|}\right)^2 |f(\xi)|^2\,d\xi \\
&\gtrsim \int_{B(0,\pi/(2t))} \left(\frac{t|\xi|}{|\xi|}\right)^2 |f(\xi)|^2\,d\xi \;=\; t^2 \int_{B(0,\pi/(2t))} |f(\xi)|^2\,d\xi \\
&\geqslant\; t^2\cdot\lambda_2(B(0,\pi/(2t)))\cdot \min_{|\xi|=\pi/(2t)} |f(\xi)|^2 \;\gtrsim\; \min_{|\xi|=\pi/(2t)} |f(\xi)|^2,
\end{aligned}$$

which goes to ∞ as $t\to\infty$ for $f\in S$.

Now we need to show S is dense in L^2. Fix $f\in L^2$, $\epsilon>0$. Let $g(x) = |x|^{-1/2}\cdot\chi_{B(0,1)}(x) \in L^2(\mathbb{R}^2)$. Pick a continuous function ϕ with $||f-\phi||_{L^2} < \epsilon$ and let $h = \phi + \epsilon g$. It's clear that $h\in S$ and we have

$$||f-h||_{L^2} \;\leqslant\; ||f-\phi||_{L^2} + ||\epsilon g||_{L^2} \;\leqslant\; \epsilon(1+||g||_{L^2}).$$

So S is dense in L^2. $\square$

Problem 6. Suppose that $\{\phi_n\}$ is an orthonormal system of continuous functions in $L^2([0,1])$ and let S be the closure of the span of $\{\phi_n\}$. If $\sup_{f\in S\setminus\{0\}} \frac{||f||_{L^\infty}}{||f||_{L^2}}$ is finite, prove that S is finite dimensional.

Solution. We consider S as a subspace of $L^2([0,1])$ equipped with the L^2 norm on $[0,1]$. The sup condition on S tells us that there exists a constant M such that for any $f\in S$, $||f||_{L^\infty} \leqslant M\,||f||_{L^2}$. For a fixed $x\in[0,1]$, note that the map $f\mapsto f(x)$ is a linear functional on S and that

$$|f(x)| \;\leqslant\; ||f||_{L^\infty} \;\leqslant\; M\,||f||_{L^2}\,,$$

which shows that this is in fact a bounded linear functional on S. Since S is a closed subspace of the Hilbert space $L^2([0,1])$, S is also a Hilbert space by itself, and thus by the Riesz representation theorem we know that there exists a function $g_x\in S$ such that $f(x) = \langle f, g_x\rangle$ for all $f\in S$. Moreover, notice that

$$||g_x||_{L^2}^2 \;=\; \langle g_x, g_x\rangle \;=\; |g_x(x)| \;\leqslant\; ||g_x||_{L^\infty} \;\leqslant\; M\,||g_x||_{L^2}\,,$$

which implies that $||g_x||_{L^2} \leqslant M$ for each $x\in[0,1]$.

Now let $\{f_1,\ldots,f_N\}$ be any orthonormal set in S. By Bessel's inequality, for each $x\in[0,1]$ we have

$$M^2 \;\geqslant\; ||g_x||_{L^2}^2 \;\geqslant\; \sum_{n=1}^N |\langle f_n, g_x\rangle|^2 \;=\; \sum_{n=1}^N |f_n(x)|^2.$$

Integrating both sides from 0 to 1 we obtain

$$M^2 \;\geqslant\; \sum_{n=1}^N \int_0^1 |f_n(x)|^2\,dx \;=\; \sum_{n=1}^N ||f_n||_{L^2}^2 \;=\; N.$$

This shows that any orthonormal set in S can contain no more than M^2 elements, which implies that $\dim(S) \leqslant M^2$. $\square$

Problem 7. Determine

$$\int_0^\infty \frac{x^{a-1}}{x+z}\,dx$$

for $0 < a < 1$ and $\mathrm{Re}(z) > 0$.

Solution. Pick the branch of log with the positive real axis cut out and integrate

$$f(w) \;:=\; \frac{w^{a-1}}{w+z} \;=\; \frac{\exp((a-1)\log(w))}{w+z}$$

along a "Pac-Man" contour with a circle of radius ϵ around 0, a large semicircle of radius R, and an angle of α away from the positive real axis. The integrals over the circles go to 0 in the limit and the two integrals along the straight paths combine in the limit as $\alpha \to 0$ to give

$$(1-\exp(2\pi i a))\int_0^\infty \frac{t^{a-1}}{t+z}\,dt.$$

Then calculate the residue at $w = -z$, it's equal to $(-z)^{a-1}$ (this is well-defined because since $\mathrm{Re}(z) > 0$, $-z$ does not lie on the positive real axis). So we conclude that the answer is

$$\int_0^\infty \frac{t^{a-1}}{t+z}\,dt \;=\; \frac{2\pi i(-z)^{a-1}}{1-\exp(2\pi i a)}. \quad \square$$

Problem 8. Let $f_n : \mathbb{H} \to \mathbb{H}$ be a sequence of holomorphic functions. Show that unless $|f_n| \to \infty$ uniformly on compact subsets of $\mathbb{H}$, there exists a subsequence converging uniformly on compact subsets of $\mathbb{H}$.

Solution. By Marty's Theorem, we know that the family $\{f_n\}$ is either a normal family or tends uniformly to ∞ on every compact set if and only if the spherical derivatives

$$\rho_n(z) \;=\; \frac{|f_n'(z)|}{1+|f_n(z)|^2}$$

are uniformly bounded on every compact set. So suppose that f_n does not tend uniformly to ∞ on every compact set. Then if we show that $\{f_n\}$ is a normal family, it implies that $\{f_n\}$ has a subsequence that converges uniformly on all compact sets. So it suffices to show that the quantites $\rho_n(z)$ above are uniformly bounded on compact sets.

Define

$$g_n(z) \;=\; \frac{f_n(z)-i}{f_n(z)+i}.$$

Then each g_n is a holomorphic function $\mathbb{H} \to \mathbb{D}$. In particular, the family $\{g_n\}$ is uniformly bounded on all of $\mathbb{H}$, so $\{g_n\}$ is a normal family. Thus we know that the quantities

$$\frac{|g_n'(z)|}{1+|g_n(z)|^2}$$

are uniformly bounded on compact subsets of $\mathbb{H}$. Now we have the calculation

$$\frac{|g_n'(z)|}{1+|g_n(z)|^2} \;=\; \frac{4\frac{|f_n'(z)|^2}{|f_n(z)+i|^2}}{1+\frac{|f_n(z)-i|^2}{|f_n(z)+i|^2}} \;=\; \frac{4|f_n'(z)|^2}{|f_n(z)+i|^2+|f_n(z)-i|^2} \;=\; 2\cdot\frac{|f_n'(z)|}{1+|f_n(z)|^2} \;=\; 2\rho_n(z).$$

This shows that $\rho_n(z)$ must also be uniformly bounded on compact subsets of $\mathbb{H}$ and thus $\{f_n\}$ is a normal family, so we are done. $\square$

Alternate solution. Without using Marty's theorem (it's not such a standard result).

Let g_n be defined as in the first solution, so that $g_n : \mathbb{H} \to \mathbb{D}$ is holomorphic. Fix a compact set $K \subseteq \mathbb{H}$. The g_n are uniformly bounded, so there is a subsequence g_{n_k} converging uniformly to another function g on K. Let $v_k = g_{n_k}$. First suppose that $g \neq 1$ anywhere on K. Then, since $g(K)$ is compact (g is continuous as a local uniform limit of continuous functions), $|g(z) - 1|$ is bounded away from 0 for $z \in K$. Therefore, letting

$$f \;=\; \frac{-i(g+1)}{(g-1)},$$

we have for any $z \in K$

$$|f_{n_k}(z) - f(z)| \;=\; \left|\frac{v_k(z)+1}{v_k(z)-1} - \frac{g(z)+1}{g(z)-1}\right| \;=\; 2\left|\frac{v_k(z)-g(z)}{(v_k(z)-1)(g(z)-1)}\right| \;\lesssim\; 2\,|v_k(z) - g(z)|\,,$$

which shows that $f_{n_k} \to f$ uniformly on K. This is the "subsequence converging uniformly on compact subsets of $\mathbb{H}$" part of the problem.

On the other hand, now assume that $g(z_0) = 1$ for some $z_0 \in K$. We want to show that in fact g is identically 1 and $v_k \to 1$ uniformly on K. Fix a conformal map $T : \mathbb{D} \to \mathbb{H}$ with $T(0) = z_0$ and let $h_k = v_k \circ T$. Let

$$\psi_k(z) \;=\; \frac{z + h_k(0)}{1 + \overline{h_k(0)}}$$

be an automorphism of $\mathbb{D}$ taking 0 to $h_k(0)$. Let $u_k = \psi_k^{-1} \circ h_k$ so that we have $h_k = \psi_k \circ u_k$ where $u_k : \mathbb{D} \to \mathbb{D}$ is holomorphic and satisfies $u_k(0) = 0$. Since T is conformal, to show $v_k \to 1$ locally uniformly it is enough to show $h_k \to 1$ locally uniformly. It's enough to show $h_k \to 1$ uniformly on the closed ball $\overline{B(0,r)}$ for $0 < r < 1$. By the Schwarz lemma, we have $u_n(\overline{B(0,r)}) \subseteq \overline{B(0,r)}$, so to show $h_k \to 1$ uniformly on $\overline{B(0,r)}$ it's enough to show $\psi_k \to 1$ uniformly on $\overline{B(0,r)}$. This is true because for any $z \in \overline{B(0,r)}$ we have

$$|\psi_k(z) - h_k(0)| \;=\; \frac{|z|}{|1 + \overline{h_k(0)}z|}(1 - |h_n(0)|^2) \;\leqslant\; \frac{2r}{1-r}(1 - |h_n(0)|^2)$$

which tends to 0 uniformly for $z \in \overline{B(0,r)}$. So we have shown $h_k \to 1$ locally uniformly on $\mathbb{D}$, which shows $v_k \to 1$ locally uniformly. It then follows that

$$f_{n_k} \;=\; \frac{(-i)(v_k+1)}{v_k - 1}$$

tends locally uniformly to ∞.

So far we've only shown that a subsequence of the f_n tends locally uniformly to ∞. But the argument above can be applied to any subsequence of the f_n to conclude that any subsequence of the f_n has a further subsequence converging locally uniformly to ∞, which implies that $f_n \to \infty$ locally uniformly. $\square$

Problem 9. Let $f : \mathbb{C} \to \mathbb{C}$ be entire and assume that $|f(z)| = 1$ when $|z| = 1$. Show that $f(z) = Cz^m$ for some integer $m > 0$ and $C \in \mathbb{C}$ with $|C| = 1$.

Solution. We know that f is not identically zero, so the zeros of f are isolated and thus f has only finitely many zeros inside $\mathbb{D}$. Denote them by $a_1, \ldots, a_n$, where each root is listed as many times as its multiplicity. Define

$$B(z) \;:=\; \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j}z}.$$

Notice that B is a function which is analytic in $\mathbb{D}$, has exactly the same zeros as f in $\mathbb{D}$, and satisfies $|B(z)| = 1$ for all $|z| = 1$. Thus f/B and B/f are two nonvanishing analytic functions in $\mathbb{D}$ which have modulus 1 on $\partial\mathbb{D}$. By the maximum modulus principle, we conclude that $|B/f| \leqslant 1$ and $|f/B| \leqslant 1$ throughout

$\mathbb{D}$, which implies that $|f/B| = 1$ throughout $\mathbb{D}$, which by the open mapping theorem implies that f/B must be equal to a constant C with $|C| = 1$ on all of $\mathbb{D}$.

So we can write

$$f(z) \;=\; CB(z) \;=\; C\prod_{j=1}^{n}\frac{z-a_j}{1-\overline{a_j}z}$$

for all $z \in \mathbb{D}$. Since f is entire, by the uniqueness of analytic continuations we know that B must also be entire. But notice that if any a_j is nonzero, then B has a pole at $\overline{a_j}$, which would be a contradiction. So we must have all $a_j = 0$ and thus $B(z) = z^m$ for some integer m. Since we know $f(z) = CB(z) = Cz^m$ for all $z \in \mathbb{D}$, since both sides are entire functions this implies that $f(z) = Cz^m$ for all $z \in \mathbb{C}$. $\square$

Alternate solution. This solution is basically just a worse version of the first one, but it uses the reflection principle so it's cool.

The fact that $|f| = 1$ on the unit circle essentially allows us to use the reflection principle. But we need to get rid of the roots at 0 first. More concretely:

Let m be the order of vanishing of f at 0 and let $g(z) = z^{-m}f(z)$. Then g is entire, $g(0) \neq 0$, and we still have $|g(z)| = 1$ for all $|z| = 1$. We can write this as $1 \;=\; g(z)\overline{g(z)} \;=\; g(z)\overline{g(1/\overline{z})}$ for $|z| = 1$. The function $z \mapsto \frac{1}{\overline{g(1/\overline{z})}}$ is analytic in a neighborhood of the unit circle (because $g(1/\overline{z})$ does not vanish on the unit circle) and agrees with g on the unit circle. Therefore since the unit circle has a limit point, by uniqueness of analytic continuation we have

$$g(z) = \frac{1}{\overline{g(1/\overline{z})}} \quad \text{for all } z \neq 0.$$

Taking $z \to \infty$, we see that $\lim_{z\to\infty} g(z) \;=\; 1/\overline{g(0)} < \infty$ because g does not vanish at 0. So g is bounded, but it's not necessarily entire because zeros of g inside $\mathbb{D}$ reflect to poles outside of $\mathbb{D}$. Let $a_1, \ldots, a_m$ be the zeros of g inside $\mathbb{D}$, counted with multiplicity. Then

$$z \mapsto g(z)\frac{(z-1/\overline{a_1})\cdots(z-1/\overline{a_n})}{(z-a_1)\cdots(z-a_n)}$$

is bounded and entire, so it must be a constant. Therefore we conclude

$$f(z) = Cz^m\frac{(z-a_1)\cdots(z-a_n)}{(z-1/\overline{a_1})\cdots(z-1/\overline{a_n})},$$

but since f is entire, it can't have any of those poles, so it also can't have any of the corresponding zeros, so $f(z) = Cz^m$. $\square$

Problem 10. Does there exist a function $f(z)$ holomorphic in the disk $|z| < 1$ such that $\lim_{|z|\to 1}|f(z)| = \infty$? Either find one or prove that none exist.

Solution. No such function exists. Suppose f had that property. Then in particular f is not identically zero, so f has only finitely many zeros $r_1, \ldots, r_n \in \mathbb{D}$ (where roots are listed as many times as their multiplicity). Let $g(z) = f(z)/(z-r_1)\cdots(z-r_n)$. Then g is a function which is holomorphic and nonvanishing in $\mathbb{D}$, and since $(z-r_1)\cdots(z-r_n)$ does not tend to ∞ as $|z| \to 1$, we still have that $|g(z)| \to \infty$ as $|z| \to 1$. Since g is nonvanishing, $1/g$ is also holomorphic in $\mathbb{D}$ and $|1/g(z)| \to 0$ as $|z| \to 1$. But applying the maximum principle to $1/g$, we see that $|1/g|$ can't have any local maximum inside $\mathbb{D}$, and since it extends continuously to be identically zero on $\partial\mathbb{D}$, this implies that $1/g$ must be identically zero on all of $\mathbb{D}$, which is a contradiction because g is a holomorphic function on $\mathbb{D}$. Thus no such function f can exist. $\square$

Problem 11. Assume that $f(z)$ is holomorphic on $|z| < 2$. Show that

$$\max_{|z|=1}\left|f(z) - \frac{1}{z}\right| \;\geqslant\; 1.$$

Solution. Let M be the max in question, and let γ be the counterclockwise contour around the unit circle. By the ML inequality

$$\left|\int_\gamma f(z) - \frac{1}{z}\ dz\right| \leqslant 2\pi M.$$

On the other hand,

$$\int_\gamma f(z) - \frac{1}{z}\ dz = 0 - 2\pi i = -2\pi i.$$

Therefore $2\pi \leqslant 2\pi M$, hence the result. $\square$

Alternate solution. I think these two solutions are essentially equivalent but this one feels less like a trick.

Suppose instead that $|f(z) - 1/z| < 1$ for all $|z| = 1$. Let C be the unit circle. The idea is that the image of C under $1/z$ has winding number -1 around the origin, and if $f(z)$ is always less than 1 away from $1/z$, then f should also wind C around the origin -1 times, which is bad.

By assumption we have $|zf(z) - 1| < |z| = 1$ for all $z \in C$. So the image of C under $zf(z)$ is contained in $B(1,1)$, which implies it has winding number 0 around the origin. Therefore by the argument principle, $zf(z)$ has no zeros inside $\mathbb{D}$, which is impossible if f is analytic. Alternatively, one can apply Rouche's theorem to the inequality $|zf(z)-1| < |z| = 1$ to conclude that $zf(z)$ has the same number of zeros in $\mathbb{D}$ as the constant function 1, which is zero (the first argument given here is essentially just a proof of Rouche's theorem). $\square$

Problem 12a. Find a real-valued harmonic function v defined on the disk $|z| < 1$ such that $v(z) > 0$ and $\lim_{z\to 1} v(z) = \infty$.

Solution. Define $v(z) = \log\left|\frac{z+1}{z-1} - 1\right|$. It is clear that $v(z) \to \infty$ as $z \to 1$. To see that v is harmonic in $\mathbb{D}$, note that the map $z \mapsto \frac{z+1}{z-1} - 1$ is nonvanishing on $\mathbb{D}$, so $z \mapsto \log\left(\frac{z+1}{z-1} - 1\right)$ is a well-defined analytic function on $\mathbb{D}$, and $v(z) = \mathrm{Re}\left(\log\left(\frac{z+1}{z-1} - 1\right)\right)$, so v is harmonic in $\mathbb{D}$. To show that $v(z) > 0$ on $\mathbb{D}$, note that $z \mapsto \frac{z-1}{z+1} - 1$ is a conformal map from $\mathbb{D}$ to $\{z \in \mathbb{C} : \mathrm{Im}(z) < -1\}$, so $\left|\frac{z-1}{z+1} - 1\right| > 1$ for all $z \in \mathbb{D}$ and thus $v(z) > 0$. $\square$

"Alternate" Solution Simply define $v(z) = -\log\left|\frac{z-1}{2}\right|$. On the disc, $\frac{z-1}{2}$ is nonzero and holomorphic, so $v(z)$ is harmonic. It is also non-negative since $\frac{z-1}{2} < 1$ for $|z| < 1$. The blowup near 1 is clear.

Problem 12b. Let u be a real-valued harmonic function in the disk $|z| < 1$ such that $u(z) \leqslant M < \infty$ and $\lim_{r\to 1} u(re^{i\theta}) \leqslant 0$ for almost all θ. Show that $u(z) \leqslant 0$.

Solution. For any $0 < r < 1$, u is harmonic on the closed disk $|z| \leqslant r$. So for any $0 < s < 1$, we can use the Poisson integral formula to write

$$u(rse^{i\theta}) \;=\; \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - (rs)^2}{|re^{i\phi} - rse^{i\theta}|^2} u(re^{i\phi})\, d\phi. \tag{2}$$

For a fixed s and θ, define

$$g_r(\phi) \;=\; \frac{r^2 - (rs)^2}{|re^{i\phi} - rse^{i\theta}|^2} u(re^{i\phi}).$$

We see that g_r is bounded on $[0, 2\pi]$ because $u \leqslant M$ on all of $\mathbb{D}$ by hypothesis and $|re^{i\phi} - rse^{i\theta}|^2$ is bounded away from 0 because $s < 1$. So say that $|g_r(\phi)| \leqslant A$ for all $\phi \in [0, 2\pi]$. Therefore we can apply Fatou's lemma to the functions $A - g_r(\phi)$ to get

$$\int_0^{2\pi} \liminf_{r\to 1}(A - g_r(\phi))\, d\phi \;\leqslant\; \liminf_{r\to 1} \int_0^{2\pi} (A - g_r(\phi))\, d\phi,$$

which implies that

$$\int_0^{2\pi} \limsup_{r\to 1} g_r(\phi)\, d\phi \;\geqslant\; \limsup_{r\to 1} \int_0^{2\pi} g_r(\phi)\, d\phi.$$

So taking the $\limsup$ as $r \to 1$ on both sides of equation (1) yields, since u is continuous on $\mathbb{D}$,

$$u(se^{i\theta}) \;=\; \limsup_{r\to 1} u(rse^{i\theta}) \;=\; \limsup_{r\to 1} \int_0^{2\pi} g_r(\phi)\, d\phi \;\leqslant\; \int_0^{2\pi} \limsup_{r\to 1} g_r(\phi)\, d\phi \;=\; \int_0^{2\pi} \frac{1-s^2}{|e^{i\phi} - se^{i\theta}|^2} \limsup_{r\to 1} u(re^{i\phi})\, d\phi.$$

By hypothesis, the integral on the far right is an integral of a function which is $\leqslant 0$ almost everywhere, so we have $u(se^{i\theta}) \leqslant 0$. This argument holds for any $0 < s < 1$ and any $\theta \in [0, 2\pi]$, so we conclude that $u \leqslant 0$ on $\mathbb{D}$. $\square$

16 Fall 2016

Problem 1. We consider the space $L^1(\mu)$ of integrable functions on a measure space $(X, \mathcal{M}, \mu)$. For $f \in L^1(\mu)$ let

$$||g||_1 = \int |g(x)| d\mu$$

be the corresponding L^1-norm. Suppose that f and f_n for $n \in \mathbb{N}$ are functions in $L^1(\mu)$ such that

(i) $f_n(x) \to f(x)$ for μ-almost every $x \in X$ and

(ii) $||f_n||_1 \to ||f||_1$.

Show that then $||f_n - f||_1 \to 0$.

Solution. Note that the function $|f| + |f_n| - |f - f_n|$ is nonnegative for all n (this just follows from the triangle inequality). Then we apply Fatou's lemma to get

$$\int \liminf_{n\to\infty}(|f| + |f_n| - |f - f_n|)\, d\mu \;\leqslant\; \liminf_{n\to\infty} \int (|f| + |f_n| - |f - f_n|)\, d\mu.$$

Since $f_n \to f$ pointwise almost everywhere, the left side of the above inequality reduces to

$$2 \int |f|\, d\mu.$$

Since $||f_n||_{L^1} \to ||f||_{L^1}$ as $n \to \infty$, the right side reduces to

$$2 \int |f|\, d\mu - \limsup_{n\to\infty} \int |f - f_n|\, d\mu.$$

Together these imply that

$$\limsup_{n\to\infty} \int |f - f_n|\, d\mu \;\leqslant\; 0,$$

which implies that $||f - f_n||_{L^1} \to 0$ as $n \to \infty$. $\square$

Problem 2. Let μ be a finite positive Borel measure on $\mathbb{R}$ that is singular to the Lebesgue measure. Show that

$$\lim_{r\to 0^+} \frac{\mu([x - r, x + r])}{2r} = +\infty$$

for μ-almost every $x \in \mathbb{R}$.

Solution. Let λ be Lebesgue measure on $\mathbb{R}$. It suffices to show that

$$\lim_{r\to 0^+} \frac{\lambda([x - r, x + r])}{\mu([x - r, x + r])} = 0$$

for μ-almost every $x \in \mathbb{R}$. Since λ and μ are singular, write $\mathbb{R} = A \cup A^c$ where $\lambda(A) = 0$ and $\mu(A^c) = 0$. It suffices to just look at $x \in A$ because $\mu(A^c) = 0$. Define

$$E_k \;=\; \left\{ x \in A : \limsup_{r\to 0^+} \frac{\lambda([x - r, x + r])}{\mu([x - r, x + r])} > \frac{1}{k} \right\}.$$

To prove the desired result it suffices to show that $\mu(E_k) = 0$ for each fixed k. Fix $\epsilon > 0$. By the regularity of Lebesgue measure, let V be an open set with $E_k \subseteq V$ and $\lambda(V) < \epsilon$. By definition of E_k, for each $x \in E_k$ there is an open interval $I(x) = (x - r(x), r + r(x))$ such that

$$\frac{\lambda(I(x))}{\mu(I(x))} \;\geqslant\; \frac{\lambda([x - r(x), x + r(x)])}{\mu([x - r(x), x + r(x)])} \;>\; \frac{1}{k},$$

and $r(x)$ may be chosen small enough so that $I(x) \subseteq V$ for each x. Then $\bigcup_{x \in E_k} (1/5)I(x)$ is a covering of E_k by open intervals, so by the Vitali covering lemma, we can pick a countable subcollection $\{(1/5)I(x_n)\}$ which is pairwise disjoint and satisfies

$$E_k \;\subseteq\; \bigcup_{x \in E_k} (1/5)I(x) \;\subseteq\; \bigcup_{n=1}^{\infty} I(x_n).$$

Therefore we have the estimate

$$\mu(E_k) \;\leqslant\; \sum_{n=1}^{\infty} \mu(I(x_n)) \;\leqslant\; k \sum_{n=1}^{\infty} \lambda(I(x_n)) \;=\; k\lambda\left(\bigcup_{n=1}^{\infty} I(x_n)\right) \;\leqslant\; k\lambda(V) \;<\; k\epsilon.$$

Since $\mu(E_k)$ is independent of ϵ, we may take $\epsilon \to 0$ and conclude that $\mu(E_k) = 0$, so we are done. $\square$

Problem 3a. If X is a compact metric space, we denote by $\mathcal{P}(X)$ the set of all positive Borel measures μ on X with $\mu(X) = 1$. Let $\phi : X \to [0, \infty]$ be lower semicontinuous function on X. Show that if μ and μ_n are in $\mathcal{P}(X)$ and $\mu_n \to \mu$ with respect to the weak-star topology on $\mathcal{P}(X)$, then

$$\int \phi \, d\mu \;\leqslant\; \liminf_{n\to\infty} \int \phi \, d\mu_n.$$

Solution. Since ϕ is lower semicontinuous, we can write it as a monotonically increasing limit of continuous functions, and since $\phi \geqslant 0$ we may also take these continuous functions to be nonnegative. So say that $0 \leqslant f_k \nearrow \phi$ as $k \to \infty$. Then, by definition of weak-$*$ convergence of measures and applying the Monotone Convergence Theorem twice, we have

$$\int \phi \, d\mu \;=\; \lim_{k\to\infty} \int f_k \, d\mu \;=\; \lim_{k\to\infty} \lim_{n\to\infty} \int f_k \, d\mu_n \;\leqslant\; \liminf_{n\to\infty} \lim_{k\to\infty} \int f_k \, d\mu_n \;=\; \liminf_{n\to\infty} \int \phi \, d\mu_n.$$

The interchange of the limits with the inequality is justified by the following statement:

Let $\{a_{n,k}\}_{n,k=1}^{\infty}$ be nonnegative numbers such that $\lim_{n\to\infty} a_{n,k}$ and $\lim_{k\to\infty} a_{n,k}$ both exist for each fixed k and n respectively, $\lim_{k\to\infty} \lim_{n\to\infty} a_{n,k}$ exists, and for each fixed n, $a_{n,k}$ is increasing in k. Then $\lim_{k\to\infty} \lim_{n\to\infty} a_{n,k} \leqslant \liminf_{n\to\infty} \lim_{k\to\infty} a_{n,k}$.

Proof: Define

$$b_n \;:=\; \lim_{k\to\infty} a_{n,k} \qquad c_k \;:=\; \lim_{n\to\infty} a_{n,k} \qquad L \;:=\; \lim_{k\to\infty} c_k.$$

Fix $\epsilon > 0$. Let K be big enough so that $c_K > L - \epsilon$. By the increasing condition, we have $b_n \geqslant a_{n,K}$ for each n. Therefore

$$\liminf_{n\to\infty} b_n \;\geqslant\; \liminf_{n\to\infty} a_{n,K} \;=\; c_K \;>\; L - \epsilon.$$

Since $\liminf_{n\to\infty} b_n$ does not depend on ϵ, we conclude that $\liminf_{n\to\infty} b_n \geqslant L$. $\square$

Problem 3b. Let $K \subseteq \mathbb{R}^d$ be a compact set. For $\mu \in \mathcal{P}(K)$, define

$$E(\mu) \;=\; \int_K \int_K \frac{1}{|x-y|} \, d\mu(x) \, d\mu(y).$$

Show that the function $E : \mathcal{P}(K) \to [0, \infty]$ attains its minimum on $\mathcal{P}(K)$ (which could possibly be infinity).

Solution. See Spring 2013 # 4

Problem 4. Let $L^1 = L^1([0,1])$ be the space of integrable functions and $L^2 = L^2([0,1])$ be the space of square-integrable functions on $[0, 1]$. Then $L^2 \subset L^1$. Show that L^2 is a *meager subset* of L^1., i.e., L^2 can be written as a countable union of sets in L^1 that are closed and have empty interior in L^1.

Solution. Write

$$L^2 \;=\; \bigcup_{N=1}^{\infty} \underbrace{\left\{ f \in L^1 : \int_0^1 |f|^2 \leqslant N \right\}}_{} \;=:\; E_N.$$

To show that L^2 is a meager subset of L^1, it suffices to show that each E_N is closed and nowhere dense with respect to the L^1 norm. To show E_N is closed, let f_k be a sequence in E_N and suppose that $f_k \to f$ in the L^1 norm. This implies that a subsequence converges to f almost everywhere, so by relabeling if necessary we may just assume that $f_k \to f$ almost everywhere, so also $|f_k|^2 \to |f|^2$ almost everywhere. Therefore by Fatou's lemma we have

$$\int_0^1 |f|^2 \;=\; \int_0^1 \liminf_{k\to\infty} |f_k|^2 \;=\; \liminf_{k\to\infty} \int_0^1 |f_k|^2 \;\leqslant\; N,$$

so $f \in E_N$. Thus E_N is closed.

To show E_N is nowhere dense, fix $f \in E_N$ and $\epsilon > 0$. It suffices to find a function g such that $g \notin E_N$ and $||g - f||_{L^1} < \epsilon$. Define $g(x) = f(x) + \epsilon x^{-1/2}$. It is clear that $g \notin E_N$ because if g were in L^2, then $x^{-1/2}$ would also be, which is a contradiction. It is also clear that

$$||g - f||_{L^1} \;=\; \epsilon \int_0^1 x^{-1/2}\, dx \;=\; 2\epsilon,$$

so we are done. $\square$

Problem 5. Let $X = C([0, 1])$ be the Banach space of real valued continuous functions on $[0, 1]$ equipped with the sup norm. Let $\mathcal{A}$ be the Borel σ-algebra on X. Show that $\mathcal{A}$ is the smallest σ-algebra on X that contains all sets of the form

$$S(t, B) \;=\; \{f \in X : f(t) \in B\}$$

for $t \in [0, 1]$ and B a Borel subset of $\mathbb{R}$.

Solution. First we show that each set of the form $S(t, B)$ is actually a Borel set in X. Note that for each t, the evaluation map $\phi_t : X \to \mathbb{R}$ given by $f \mapsto f(t)$ is a bounded linear functional on X because $|f(t)| \leqslant ||f||_X$. Therefore ϕ_t is a continuous function $X \to \mathbb{R}$, and since $S(t, B) = \phi_t^{-1}(B)$ where B is a Borel set in $\mathbb{R}$, we see that $S(t, B)$ must be a Borel set in X.

Let $\mathcal{F}$ denote the σ-algebra generated by the sets of the form $S(t, B)$. To show that $\mathcal{F} = \mathcal{A}$, it suffices to show that every closed neighborhood in X is in $\mathcal{F}$. So fix $g \in X$ and $\epsilon > 0$. We need to show that $E := \{f \in X : ||f - g||_X \leqslant \epsilon\}$ is an element of $\mathcal{F}$. For any $q \in \mathbb{Q} \cap [0, 1]$, define $B_q \;:=\; [g(q) - \epsilon, g(q) + \epsilon]$. It is clear that B_q is a Borel subset of $\mathbb{R}$. Now we claim that

$$E \;=\; \bigcap_{q \in \mathbb{Q} \cap [0,1]} S(q, B_q).$$

Proving this is enough to conclude that E is an element of $\mathcal{F}$, so this will finish the problem.

If $f \in E$, then $||f - g||_X \leqslant \epsilon$, so in particular $|f(q) - g(q)| \leqslant \epsilon$ for every $q \in \mathbb{Q} \cap [0, 1]$, which implies that $f(q) \in B_q$ for every q, so f is an element of the set on the right side of the above equation. Conversely, let f be an element of the right side and suppose that $f \notin E$. Then we have $|f(x) - g(x)| > \epsilon$ for some $x \in [0, 1]$, and since f and g are both continuous, we can find a rational number q near x such that $|f(q) - g(q)| > \epsilon$, which contradicts the assumption that $f \in S(q, B_q)$. Therefore we conclude that $E \;=\; \bigcap_{q \in \mathbb{Q} \cap [0,1]} S(q, B_q) \in \mathcal{F}$, so we are done. $\square$

Problem 6a. Consider the Banach space ℓ^1 consisting of all sequences $u = \{x_i\}$ in $\mathbb{R}$ with

$$||u||_{\ell^1} \;=\; \sum_{i=1}^{\infty} |x_i| \;<\; \infty$$

and the Banach space ℓ^∞ consisting of all sequences $v = \{y_i\}$ in $\mathbb{R}$ with

$$||v||_{\ell^\infty} \;=\; \sup_{i\in\mathbb{N}} |y_i| \;<\; \infty.$$

There is a well-defined dual pairing between ℓ^1 and ℓ^∞ given by

$$\langle u, v\rangle \;=\; \sum_{i=1}^{\infty} x_i y_i$$

for $u = \{x_i\} \in \ell^1$ and $v = \{y_i\} \in \ell^\infty$. With this dual pairing, $\ell^\infty = (\ell^1)^*$ is the dual space of ℓ^1.

Show that there exists no sequence $\{u_n\}$ in ℓ^1 such that $||u_n||_{\ell^1} \geqslant 1$ for all n and $\langle u_n, v\rangle \to 0$ for each $v \in \ell^\infty$.

Solution. Let $\{u_n\}$ be a sequence in ℓ^1 satisfying $||u_n||_{\ell^1} \geqslant 1$ for all n. We can assume by scaling that $||u_n||_{\ell^1} = 1$ for each n because scaling the sequences down can only decrease $\langle u_n, v\rangle$ for any $v \in \ell^\infty$. Suppose that $\langle u_n, v\rangle \to 0$ as $n \to \infty$ for all $v \in \ell^\infty$. We will get a contradiction by constructing a sequence $v \in \ell^\infty$ such that $\langle u_n, v\rangle$ is bounded away from zero infinitely often.

First note that by letting v be the sequence which has a 1 in the jth spot and 0 everywhere else, we know that $(u_n)_j \to 0$ as $n \to \infty$ for each fixed j. Also note that since $||u_n||_{\ell^1} = 1$ for each n, necessarily $||u_n||_{\ell^\infty} \leqslant 1$ for all n. Now, for any fixed $\epsilon \in (0, 1/2)$, we can do the following construction:

Pick J_1 to be large enough so that

$$\sum_{j\in[1,J_1]} |(u_1)_j| \;>\; 1-\epsilon.$$

Now, since we know that $(u_n)_j$ tends to zero in each slot individually, pick N_1 to be large enough so that

$$\max(|(u_{N_1})_1|, \ldots, |(u_{N_1})_{J_1}|) \;<\; \frac{\epsilon}{2J_1}.$$

Then we see that

$$\sum_{j\in[1,J_1]} |(u_{N_1})_j| \;<\; \epsilon/2,$$

so we may pick J_2 such that

$$\sum_{j\in[J_1+1,J_2]} |(u_{N_1})_j| \;>\; 1-\epsilon.$$

Now pick N_2 to be large enough so that

$$\max(|(u_{N_2})_1|, \ldots, |(u_{N_2})_{J_2}|) \;<\; \frac{\epsilon}{2J_2}.$$

We may repeat this process indefinitely, and so we obtain a sequence $\{N_k\}$ and a sequence $\{J_k\}$ such that for each k

$$\sum_{j\in[J_k+1,J_{k+1}]} |(u_{N_k})_j| \;>\; 1-\epsilon.$$

Now, letting $s(x)$ denote the function which is 1 if $x \geqslant 0$ and -1 if $x < 0$, define the sequence $v \in \ell^\infty$ by

$$(v)_j \;=\; s((u_{N_k})_j) \quad \text{when } j \in [J_k+1, J_{k+1}].$$

Note that each $(v)_j$ is an entry of some u_n, so we have $||v||_{\ell^\infty} \leqslant 1$. By construction, for each k we have

$$\sum_{j \in [J_k+1, J_{k+1}]} (u_{N_k})_j (v)_j \;=\; \sum_{j \in [J_k+1, J_{k+1}]} |(u_{N_k})_j| \;>\; 1 - \epsilon,$$

so

$$\langle u_{N_k}, v \rangle \;=\; \sum_{j \in [J_k+1, J_{k+1}]} (u_{N_k})_j (v)_j + \sum_{j \notin [J_k+1, J_{k+1}]} (u_{N_k})_j (v)_j \;\geqslant\; 1 - \epsilon - ||v||_{\ell^\infty} \sum_{j \notin [J_k+1, J_{k+1}]} |(u_{N_k})_j| \;>\; 1 - 2\epsilon.$$

Therefore, picking (for example) $\epsilon = 1/3$, we see that $\langle u_{N_k}, v \rangle$ is bounded away from zero for every k, which is our contradiction. $\square$ (Note: I would really prefer a nicer, non-constructive solution)

Problem 6b. Show that every weakly convergent sequence $\{u_n\}$ in ℓ^1 converges in the norm topology of ℓ^1.

Solution. Suppose that $u_n \to u$ weakly in ℓ^1. This means that $\phi(u_n) \to \phi(u)$ for every bounded linear functional $\phi \in (\ell^1)^*$, and by the given dual pairing this means that $\langle u_n, v \rangle \to \langle u, v \rangle$ for every $v \in \ell^\infty$, i.e. $\langle u_n - u, v \rangle \to 0$ for every $v \in \ell^\infty$. Suppose that u_n did not converge to u in the norm topology on ℓ^1. Then there is a subsequence u_{n_k} and a $\delta > 0$ such that $||u_{n_k} - u||_{\ell^1} \geqslant \delta$ for all k. Replacing $u_{n_k} - u$ with $(1/\delta)(u_{n_k} - u)$ if necessary, we may assume that $||u_{n_k} - u||_{\ell^1} \geqslant 1$ for all k. But we still must have $\langle u_{n_k} - u, v \rangle \to 0$ for every $v \in \ell^\infty$, which contradicts part (a). Therefore we must have $u_n \to u$ in the norm topology on ℓ^1. $\square$

Problem 7a. Let $\mathcal{H}$ be the space of holomorphic functions f on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f(z)|^2 \, dA(z) \;<\; \infty.$$

Here integration is with respect to Lebesgue measure A on $\mathbb{D}$. The vector space $\mathcal{H}$ is a Hilbert space if equipped with the inner product

$$\langle f, g \rangle \;=\; \int_{\mathbb{D}} f(z) \overline{g(z)} \, dA(z)$$

for $f, g \in \mathcal{H}$. Fix $z_0 \in \mathbb{D}$ and define $L_{z_0}(f) = f(z_0)$ for $f \in \mathcal{H}$.

Show that $L_{z_0} : \mathcal{H} \to \mathbb{C}$ is a bounded linear functional on $\mathcal{H}$.

Solution. It's obvious that L_{z_0} is a linear functional. For z_0 fixed, let $\delta > 0$ be small enough so that $B(z_0, \delta) \subseteq \mathbb{D}$. Then for any $f \in \mathcal{H}$, we have by the mean value formula

$$\begin{aligned}
|L_{z_0}(f)| \;&=\; |f(z_0)| \;=\; \left| \frac{1}{\pi\delta^2} \int_{B(z_0,\delta)} f(z) \, dA(z) \right| \;\leqslant\; \frac{1}{\pi\delta^2} \int_{B(z_0,\delta)} |f(z)| \, dA(z) \;\leqslant\; \frac{1}{\pi\delta^2} \int_{\mathbb{D}} |f(z)| \, dA(z) \\
&\leqslant\; \frac{1}{\pi\delta^2} \left(\int_{\mathbb{D}} 1^2 \, dA(z) \right)^{1/2} \left(\int_{\mathbb{D}} |f(z)|^2 \, dA(z) \right)^{1/2} \quad \text{by Cauchy-Schwarz} \\
&\leqslant\; \frac{1}{\sqrt{\pi}\delta^2} \, ||f||_{\mathcal{H}} \,,
\end{aligned}$$

so L_{z_0} is a bounded linear functional. $\square$

Problem 7b. Find an explicit function $g_{z_0} \in \mathcal{H}$ such that

$$L_{z_0}(f) \;=\; f(z_0) \;=\; \langle f, g_{z_0} \rangle$$

for all $f \in \mathcal{H}$.

Solution. Note that such a g_{z_0} exists for each $z_0 \in \mathbb{D}$ by the Riesz representation theorem. First we claim that the set

$$\left\{ e_n(z) := \sqrt{\frac{n+1}{\pi}} z^n \right\}$$

is an orthonormal basis for $\mathcal{H}$. It's easy to compute directly using polar coordinates that it's an orthonormal set. To show it's a basis, it's enough to show that $\langle f, e_n \rangle = 0$ for all n implies $f = 0$. We compute

$$\langle f, e_n \rangle \; = \; C(n) \int_{\mathbb{D}} f(z) \overline{z^n} \, dA(z) \; = \; C(n) \int_0^1 \int_0^{2\pi} f(re^{i\theta}) r^{n+1} e^{-in\theta} \, d\theta \, dr.$$

The Cauchy integral formula gives

$$f^{(n)}(0) \; = \; C(n) \int_0^{2\pi} \frac{f(re^{i\theta})}{r^{n+1} e^{i(n+1)\theta}} re^{i\theta} \, d\theta.$$

Combining these two we can observe that

$$\langle f, e_n \rangle \; = \; C(n) \int_0^1 r^{2n+1} f^{(n)}(0) \, dr \; = \; C(n) f^{(n)}(0).$$

($C(n)$ is a constant in terms of n that is different from line to line). This implies that $\langle f, e_n \rangle = 0$ implies $f^{(n)}(0) = 0$. Therefore because holomorphic functions have power series expansions, $\langle f, e_n \rangle = 0$ for all n implies $f = 0$. This shows that the e_n form an orthonormal basis for $\mathcal{H}$.

Now we determine g_{z_0}. For $z \in \mathbb{D}$ we have

$$\begin{aligned} g_{z_0}(z) \; &= \; \langle g_{z_0}, g_z \rangle \; = \; \sum_{n=0}^{\infty} \langle g_{z_0}, e_n \rangle \overline{\langle g_z, e_n \rangle} \quad \text{by Parseval} \\ &= \; \sum_{n=0}^{\infty} \overline{\langle e_n, g_{z_0} \rangle} \langle e_n, g_z \rangle \; = \; \sum_{n=0}^{\infty} \overline{e_n(z_0)} e_n(z) \\ &= \; \sum_{n=0}^{\infty} \frac{n+1}{\pi} (\overline{z_0} z)^n \; = \; \frac{1}{\pi(1 - \overline{z_0} z)^2}. \quad \square \end{aligned}$$

Problem 8a. Let f be a continuous complex-valued function on $\overline{\mathbb{D}}$ which is holomorphic on $\mathbb{D}$ and $f(0) \neq 0$. Show that if $0 < r < 1$ and $\inf_{|z|=r} |f(z)| > 0$, then

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left| f(re^{i\theta}) \right| \, d\theta \; \geqslant \; \log |f(0)| \, .$$

Solution. Let r be such that $\inf_{|z|=r} |f(z)| > 0$. Since f is not identically zero, it has only finitely many zeros inside the disc $|z| < r$. Denote them by $a_1, \ldots, a_n$. Define the function

$$g(z) \; = \; \left(\frac{r(z - a_1)}{r^2 - \overline{a_1} z} \right) \cdots \left(\frac{r(z - a_n)}{r^2 - \overline{a_n} z} \right).$$

We know that $|g(z)| = 1$ for all $|z| = r$ and g has the same zeros as f and no poles in $|z| \leqslant r$. Therefore the function f/g is a nonvanishing holomorphic function on $|z| < r$ with $|f(z)/g(z)| = |f(z)|$ for $|z| = r$. Since it is nonvanishing we know that it has a holomorphic single-valued logarithm, so $\log |f(z)/g(z)| = \mathrm{Re}(\log(f(z)/g(z)))$ is harmonic in $|z| < r$. Therefore we can apply the mean value property to $\log |f/g|$ to obtain

$$\log \left| \frac{f(0)}{g(0)} \right| \; = \; \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{f(re^{i\theta})}{g(re^{i\theta})} \right| \, d\theta \; = \; \frac{1}{2\pi} \int_0^{2\pi} \log \left| f(re^{i\theta}) \right| \, d\theta.$$

We compute

$$\log\left|\frac{f(0)}{g(0)}\right| \;=\; \log|f(0)| - \sum_{j=1}^{n}\log\left|\frac{a_j}{r}\right|.$$

Since each $|a_j| < r$, we have $\log|a_j/r| < 0$ and therefore

$$\log|f(0)| \;\leqslant\; \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(re^{i\theta})\right|\,d\theta. \quad \square$$

Problem 8b. Show that $\left|\{\theta \in [0,2\pi] : f(e^{i\theta}) = 0\}\right| = 0$, where $|E|$ denotes the Lebesgue measure of E.

Solution. Let $E = \{\theta \in [0,2\pi] : f(e^{i\theta}) = 0\}$. Suppose that $|E| > 0$. Since $\overline{\mathbb{D}}$ is compact, we know that f is uniformly continuous on $\overline{\mathbb{D}}$. Fix $\epsilon > 0$. Then we know that there is some $r_\epsilon > 0$ such that $|f(r_\epsilon e^{i\theta})| < \epsilon$ for every $\theta \in E$. We can also say $|f| \leqslant M$ on $\overline{\mathbb{D}}$. Now we have the following estimate:

$$\int_0^{2\pi}\log\left|f(r_\epsilon e^{i\theta})\right|\,d\theta \;=\; \int_E \log\left|f(r_\epsilon e^{i\theta})\right|\,d\theta + \int_{E^c}\log\left|f(r_\epsilon e^{i\theta})\right|\,d\theta \;\leqslant\; |E|\log(\epsilon) + 2\pi\log(M).$$

But since $f(0) \neq 0$, we can pick $\epsilon > 0$ small enough so that the right side above is smaller than $2\pi\log|f(0)|$, but part (a) says that we must have $\int_0^{2\pi}\log\left|f(re^{i\theta})\right|\,d\theta \geqslant 2\pi\log|f(0)|$ for any $r > 0$, so this is a contradiction. $\square$

Alternate Solution. Since f is continuous on the compact set $\overline{\mathbb{D}}$, we can say $|f| \leqslant M$. Thus $\log|f|$ takes values in $[-\infty, M]$. Let $g_r(\theta) = M - \log|f(re^{i\theta})|$. Then each g_r for $0 < r < 1$ takes values in $[0,\infty]$, so we can apply Fatou's lemma:

$$\begin{aligned}
\int_0^{2\pi}\liminf_{r\to 1} g_r(\theta)\,d\theta \;&\leqslant\; \liminf_{r\to 1}\int_0^{2\pi} g_r(\theta)\,d\theta \\
2\pi M - \int_0^{2\pi}\limsup_{r\to 1}\log|f(re^{i\theta})|\,d\theta \;&\leqslant\; 2\pi M - \limsup_{r\to 1}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta \\
\int_0^{2\pi}\log|f(e^{i\theta})|\,d\theta \;&\geqslant\; \limsup_{r\to 1}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta \;\geqslant\; 2\pi\log|f(0)| \;>\; -\infty
\end{aligned}$$

by part (a). But if E had positive measure, then the integral on the left side would be $-\infty$, a contradiction. $\square$

Problem 9a. Let μ be a positive Borel measure on $[0,1]$ with $\mu([0,1]) = 1$. Show that the function f defined as

$$f(z) = \int_{[0,1]} e^{izt}d\mu(t)$$

for $z \in \mathbb{C}$ is holomorphic on $\mathbb{C}$.

Solution. For $h_k \in \mathbb{C}$ with $|h_k| \to 0$ we have

$$\frac{1}{h}(f(z+h_k) - f(z)) = \int_{[0,1]} e^{izt}\cdot\frac{e^{ih_kt}-1}{h_k}d\mu(t)$$

Notice that

$$\lim_{k\to\infty}\frac{e^{ih_kt}-1}{h_k} = \left(\frac{d}{dz}e^{itz}\right)(0) = it.$$

Thus for fixed z, the magnitude of the integrand is bounded by $2\sup_{t\in[0,1]}|e^{izt}| < \infty$ for k large enough. By dominated convergence, we have

$$f'(z) = \int_{[0,1]} ite^{izt}d\mu(t).$$

Note that all functions in question are continuous, and hence Borel measurable, so applying dominated convergence was justified. $\square$

Alternate solution. We are motivated by the fact that if f is holomorphic it should have $f'(z) = \int_0^1 ite^{izt}\, d\mu(t)$. We estimate, for a fixed z,

$$\begin{aligned}\left|\frac{1}{h}(f(z+h)-f(z)) - \int_0^1 ite^{izt}\, d\mu(t)\right| &= \left|\frac{1}{h}\int_0^1 (e^{i(z+h)t} - e^{izt} - ihte^{izt})\, d\mu(t)\right| \\ &\leqslant \frac{1}{|h|}\int_0^1 \left|e^{izt}\right| \left|e^{iht} - (1+iht)\right|\, d\mu(t).\end{aligned}$$

We can pick $|h|$ to be small enough so that $\left|e^{iht} - (1+iht)\right| \leqslant C\,|iht|^2 = Ct^2|h|^2$ for some absolute constant C. Then we have

$$\left|\frac{1}{h}(f(z+h)-f(z)) - \int_0^1 ite^{izt}\, d\mu(t)\right| \leqslant \frac{1}{|h|}C|h|^2\int_0^1 (e^{|z|})^t t^2\, d\mu(t) \leqslant Ce^{|z|}|h|\int_0^1 d\mu(t) = Ce^{|z|}|h|,$$

which tends to 0 as $|h| \to 0$, so we conclude that $f'(z) = \int_0^1 ite^{izt}\, d\mu(t)$. $\square$

Problem 9b. Suppose that there exists $n \in \mathbb{N}$ such that

$$\limsup_{|z|\to\infty} |f(z)|/|z|^n < \infty$$

Show that then μ is equal to the Dirac measure δ_0 at 0.

Solution. By the given condition, we have for large $|z|$ that $|f(z)| < C|z|^n$ for some constant C. Since f is polynomially bounded and holomorphic, f must in fact be a polynomial.

For z real,

$$|f(z)| \leqslant \int_{[0,1]} |e^{izt}| d\mu(t) \leqslant 1.$$

But a polynomial which is bounded on the real line must be constant. Since $f(0) = 1$, we have $f(z) = 1$ for all z.

For real z, we must therefore have equality in the rightmost inequality above. This occurs only if e^{izt} is real, outside a subset of $[0,1]$ with measure 0. However e^{izt} is real only for t an integer multiple of $\pi k/z$. It follows that the set of multiples M_z of $\pi k/z$ has μ-measure 1 for all z. But M_z and $M_{\sqrt{2}z}$ intersect only at 0, so we must have $\mu(\{0\}) = 1$. (Is there a nicer way to finish off the problem?)

Alternate solution. Using the same argument from above, we know that f is a polynomial of degree n and the derivatives of f are given by $f^{(j)}(z) = \int_0^1 (it)^j e^{izt}\, d\mu(t)$. Since it's a polynomial of degree n, the $(n+1)$st derivative is identically zero, so

$$\int_0^1 t^{n+1}e^{izt}\, d\mu(t) = 0$$

for all $z \in \mathbb{C}$. If μ is not a point mass at 0, then $\mu(0,1] > 0$, so by continuity, $\mu[\delta,1] > 0$ for some $\delta > 0$. Then taking $z = -i$ we have

$$0 = \int_0^1 t^{n+1}e^t\, d\mu(t) \geqslant \int_\delta^1 t^{n+1}e^t\, d\mu(t) \geqslant \delta^{n+1}e^\delta\mu[\delta,1] > 0,$$

a contradiction. $\square$

Problem 10 a. Consider the quadratic polynomial $f(z) = z^2 - 1$ on $\mathbb{C}$. We are interested in the iterates f^n of f for $n \in \mathbb{N}$. Find an explicit constant $M > 0$ such that the following dichotomy holds for each point $z \in \mathbb{C}$: either (i) $|f^n(z)| \to \infty$ as $n \to \infty$ or (ii) $|f^n(z)| \leqslant M$ for all $n \in \mathbb{N}_0$.

Solution. We take $M = 2$. For $|z| \geqslant 2$, we have

$$\begin{aligned}|f(z)| &= |z^2 - 1| \\ &= \left|z - \frac{1}{z}\right| \cdot |z| \\ &\geqslant \left(|z| - \frac{1}{|z|}\right) \cdot |z| \\ &\geqslant \frac{3}{2}|z|.\end{aligned}$$

Thus if $|z| \geqslant 2$, we have $f^n(z) > 2 \cdot (3/2)^n$. So if $|f^k(z)|$ is greater than 2 for some k, then $|f^n(z)| \to \infty$ as $k \to \infty$. In particular if (i) does not hold, then (ii) must hold. It is clear that (i) and (ii) cannot hold simultaneously. $\square$

Problem 10b. Let U be the set of all $z \in \mathbb{C}$ for which the first alternative (i) holds and K be the set of all $z \in \mathbb{C}$ for which the second alternative (ii) holds. Show that U is an open set and K is a compact set without "holes", i.e., $\mathbb{C}\backslash K$ has no bounded connected components.

Solution. For $k \in \mathbb{N}$, let U_k be the set of all $z \in \mathbb{C}$ where $|f^k(z)| > M$. Then U_k is the preimage of an open set, and hence open. By part (a) we have that U is the union of the sets U_k, so U is open.

It is immediate that K is closed, since K is the complement of U. Any element z in K must satisfy $|z| \leqslant M$, so K is compact.

Suppose that S was a bounded connected component of U. By part (a) we have that $f^k(x) < M$ for all $x \in K$, and hence for all $x \in \partial S$. But then the maximum principle implies that $f^k(x)$ is bounded by M for all x in S. Thus (i) is not satisfied, and so $x \notin U$, which is a contradiction. $\square$

Problem 11 a. Suppose $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function such that the function $z \mapsto g(z) = f(z)f(1/z)$ is bounded on $\mathbb{C}\backslash\{0\}$. Show that if $f(0) \neq 0$, then f is constant.

Solution. Let $|g(z)|$ be bounded by M. Since $f(0) \neq 0$, there is a constant $m > 0$ such that $|f(z)| > m$ on a δ-neighborhood of 0. For $|z| < \delta$, we then have

$$M \geqslant f(z)f(1/z) \geqslant mf(1/z).$$

So $f(1/z) \leqslant M/m$ for $|z| < \delta$, and hence $f(z)$ is bounded for $|z| > 1/\delta$. It follows that f is bounded and therefore constant. $\square$

Problem 11 b. Show that if $f(0) = 0$, then there exists $n \in \mathbb{N}$ and $a \in \mathbb{C}$ such that $f(z) = az^n$ for all $z \in \mathbb{C}$.

Solution. Let n be the order of f's zero at 0. Then we can write $f(z) = z^n h(z)$ where h is holomorphic and $h(0) \neq 0$. Note that $h(z)h(1/z) = f(z)f(1/z) = g(z)$ for $z \neq 0$. By part (a) $h(z) = a$ identically for some constant a, and then we have $f(z) = az^n$. $\square$

Problem 12a. Let $U \subseteq \mathbb{C}$ be an open set and $K \subseteq U$ be a compact subset of U. Prove that there exists a bounded open set V with $K \subseteq V \subseteq \overline{V} \subseteq U$ such that ∂V consists of finitely many closed line segments.

Solution. Since K is compact and U^c is closed, we have $\operatorname{dist}(K, U^c) = \delta > 0$. Tile the complex plane with squares of side length $\delta/100$. Let $\mathcal{Q}$ be the family of all squares Q such that $\operatorname{dist}(Q, K) \leqslant \delta/10$. This is a finite family because K is compact and therefore bounded. Then let V be the interior of $\bigcup_{Q \in \mathcal{Q}} Q$. This is clearly a bounded open set such that $K \subseteq V \subseteq \overline{V} \subseteq U$, and ∂V just consists of finitely many edges of squares. $\square$

Problem 12b. Let f be a holomorphic function on U. Show that there exists a sequence $\{R_n\}$ of rational functions such that $R_n \to f$ uniformly on K and none of the functions R_n has a pole in K.

Solution. Let the set V be as in the previous part. For any $z \in K$, by the Cauchy integral formula we can write

$$f(z) \;=\; \frac{1}{2\pi i}\int_{\partial V}\frac{f(w)}{w-z}\,dw \;=\; \frac{1}{2\pi i}\sum_{j=1}^{N}\int_{\gamma_j}\frac{f(w)}{w-z}\,dw$$

where each γ_j is a straight line and they all have the same length. We parametrize each of these integrals and write

$$f(z) \;=\; \frac{1}{2\pi i}\sum_{j=1}^{N}\int_0^1\frac{f(\gamma_j(t))\gamma_j'(t)}{\gamma_j(t)-z}\,dt$$

and we know that $|\gamma_j'(t)| = c$ for some constant c and all j.

We want to show that the above integral can be approximated uniformly in $z \in K$ by its Riemann sums. Fix $\epsilon > 0$. By construction of the set V, we know that $|\gamma_j(t) - z|$ is bounded away from zero uniformly for $z \in K$ and $t \in [0,1]$, and therefore, since everything involved is continuous, we know that there is a $\delta > 0$ such that $|t_1 - t_2| < \delta$ implies

$$\left|\frac{f(\gamma_j(t_1))\gamma_j'(t_1)}{\gamma_j(t_1)-z} - \frac{f(\gamma_j(t_2))\gamma_j'(t_2)}{\gamma_j(t_2)-z}\right| \;<\; \epsilon$$

for every $z \in K$. So for each j, let $\{0 = t_{j,0} < t_{j,1} < \ldots < t_{j,M(j)} = 1\}$ be a partition of $[0,1]$ with mesh size less than δ. Then we have, for any $z \in K$,

$$\left|f(z) - \sum_{j=1}^{N}\sum_{i=1}^{M(j)}\frac{f(\gamma_j(t_{j,i}))\gamma_j'(t_{j,i})}{\gamma_j(t_{j,i})-z}(t_{j,i}-t_{j,i-1})\right| \;=\; \left|\sum_{j=1}^{N}\frac{1}{2\pi i}\int_0^1\frac{f(\gamma_j(t))\gamma_j'(t)}{\gamma_j(t)-z}\,dt - \sum_{i=1}^{M(j)}\frac{f(\gamma_j(t_{j,i}))\gamma_j'(t_{j,i})}{\gamma_j(t_{j,i})-z}(t_{j,i}-t_{j,i-1})\right|$$

$$= \left|\sum_{j=1}^{N}\sum_{i=1}^{M(j)}\int_{t_{j,i-1}}^{t_{j,i}}\left(\frac{f(\gamma_j(t))\gamma_j'(t)}{\gamma_j(t)-z} - \frac{f(\gamma_j(t_{j,i}))\gamma_j'(t_{j,i})}{\gamma_j(t_{j,i})-z}\right)dt\right|$$

$$\leqslant \sum_{j=1}^{N}\sum_{i=1}^{M(j)}\int_{t_{j,i-1}}^{t_{j,i}}\left|\frac{f(\gamma_j(t))\gamma_j'(t)}{\gamma_j(t)-z} - \frac{f(\gamma_j(t_{j,i}))\gamma_j'(t_{j,i})}{\gamma_j(t_{j,i})-z}\right|dt \;<\; \sum_{j=1}^{N}\sum_{i=1}^{M(j)}\epsilon(t_{j,i}-t_{j,i-1}) \;<\; N\epsilon.$$

Finally, notice that the big double sum in the first term is exactly a rational function in z which only has poles on the lines γ_j, which are all outside of K, so this gives us the desired result. $\square$

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Problem 1. Let $K \subseteq \mathbb{R}$ be a compact set of positive measure and let $f \in L^\infty(\mathbb{R})$. Show that the function

$$F(x) \;=\; \frac{1}{|K|}\int_K f(x+t)\,dt$$

is *uniformly* continuous on $\mathbb{R}$. Here $|K|$ denotes the Lebesgue measure of K.

Solution. We calculate

$$\begin{aligned}
|F(x)-F(y)| \;&=\; \frac{1}{|K|}\left|\int_K f(x+t)\,dt - \int_K f(y+t)\,dt\right| \;=\; \frac{1}{|K|}\left|\int_{K-x} f(t)\,dt - \int_{K-y} f(t)\,dt\right| \\
&\leqslant\; \frac{1}{|K|}\int_{(K-x)\Delta(K-y)} |f(t)|\,dt \;\leqslant\; \frac{||f||_{L^\infty}}{|K|}\lambda((K-x)\Delta(K-y)) \;=\; \frac{||f||_{L^\infty}}{|K|}\lambda((K-(x-y))\Delta K)
\end{aligned}$$

where Δ denotes the symmetric difference of two sets and λ is Lebesgue measure.

Fix $\epsilon > 0$. Let $h = x-y$; we want to estimate the measure of $(K-h)\Delta K$. Since K is compact, there is a set V which is a finite union of disjoint open intervals such that $K \subseteq V$ and $\lambda(V\backslash K) < \epsilon$. Say $V = I_1 \cup \ldots \cup I_n$. We have

$$\begin{aligned}
(K-h)\Delta K \;&=\; ((K-h)\backslash K) \cup (K\backslash(K-h)) \\
&\subseteq\; ((V-h)\backslash V) \cup (V\backslash K) \cup (V\backslash(V-h)) \cup ((V-h)\backslash(K-h)) \\
&=\; ((V-h)\Delta V) \cup (V\backslash K) \cup ((V-h)\backslash(K-h)).
\end{aligned}$$

Since V is a finite union of disjoint open intervals, it is clear that

$$\lambda((V-h)\Delta V) \;\leqslant\; 2n|h|.$$

Therefore we have $\lambda((K-h)\Delta K) \leqslant 2\epsilon + 2n|h|$. So for any $x, y \in \mathbb{R}$ satisfying $|x-y| < \frac{\epsilon}{2n+2}$, we have

$$|F(x)-F(y)| \;<\; \frac{||f||_{L^\infty}}{|K|}\lambda((K-(x-y))\Delta K) \;<\; \frac{||f||_{L^\infty}}{|K|}\epsilon.$$

Since n is a parameter depending only on ϵ and the set K, this shows that F is uniformly continuous on $\mathbb{R}$. $\square$

Problem 2. Let $f_n : [0,1] \to [0,\infty)$ be a sequence of functions, each of which is non-decreasing on the interval $[0,1]$. Suppose the sequence is uniformly bounded in $L^2([0,1])$. Show that there exists a subsequence that converges in $L^1([0,1])$.

Solution. Let M be a uniform upper bound for $||f_n||_{L^2}$. Since each f_n is nondecreasing, we get the bound $0 \leqslant f_n(t) \leqslant \frac{M}{\sqrt{1-t}}$ for $t \in [0,1]$. In particular note that for fixed t, $f_n(t)$ is restricted to a compact set. Therefore the standard diagonalization argument allows us to construct a subsequence f_{n_k} which converges on $[0,1] \cap \mathbb{Q}$.

We claim that f_{n_k} converges pointwise a.e. as $k \to \infty$. For a rational q, let a_q be the limit of the sequence $f_{n_k}(q)$. Note that $a_q \leqslant a_{q'}$ for $q < q'$, since each f_{n_k} is nondecreasing. For $r \in \mathbb{R}$ let $L_r = \sup_{q<r} a_q$ and $U_r = \inf_{q'>r} a_{q'}$. Observe that the intervals (L_r, U_r) are all disjoint, so at most countably many of them are nonempty. The interval is empty exactly when $L_r = U_r$, so this equality holds for almost every r. But when $L_r = U_r$, the sequence $f_{n_k}(r)$ converges to this value. This establishes pointwise a.e. convergence.

Let f be a function on $[0,1]$ such that $f_{n_k} \to f$ pointwise a.e. We have $|f_{n_k}(t) - f(t)| \leqslant \frac{M}{\sqrt{1-t}}$ for almost every t. Since $\frac{M}{\sqrt{1-t}}$ lies in $L^1([0,1])$, Dominated Convergence implies that $f_n \to f$ in L^1.

Note that there are no issues of measurability to worry about; an increasing function is continuous a.e. (in fact everywhere except possibly on a countable set) and therefore measurable.

Problem 3. Let $C([0,1])$ denote the Banach space of continuous functions on the interval $[0,1]$ endowed with the sup-norm. Let $\mathcal{F}$ be a σ-algebra on $C([0,1])$ so that for all $x \in [0,1]$, the map defined via

$$L_x(f) \;=\; f(x)$$

is $\mathcal{F}$-measurable. Show that $\mathcal{F}$ contains all open sets.

Solution. Since $C([0,1])$ is separable, every open set is a countable union of open balls, so it suffices to show that $\mathcal{F}$ contains every open ball. And every open ball is a countable union of closed balls, so it suffices to show $\mathcal{F}$ contains every closed ball. Fix $g \in C([0,1])$, $\epsilon > 0$, and let

$$E \;=\; \{f \in C([0,1]) : ||f-g||_{L^\infty} \leqslant \epsilon\}$$

be a closed ball. For each $q \in \mathbb{Q} \cap [0,1]$, let

$$E_q \;=\; \{f \in C([0,1]) : |f(q) - g(q)| \leqslant \epsilon\}.$$

Note that each $E_q \in \mathcal{F}$ because $E_q = L_q^{-1}(B(g(q), \epsilon))$ and $B(g(q), \epsilon)$ is a Borel set in $\mathbb{C}$. Now we claim that

$$E \;=\; \bigcap_{q \in \mathbb{Q}} E_q.$$

First, if $f \in E$, then $|f(x) - g(x)| \leqslant ||f-g||_{L^\infty} \leqslant \epsilon$ for all $x \in [0,1]$, so clearly $f \in E_q$ for every q, so $E \subseteq \bigcap_{q\in\mathbb{Q}} E_q$. Conversely, suppose $f \in E_q$ for every q. If we had $f \notin E$, then we would have $|f(x)-g(x)| > \epsilon$ for some $x \in [0,1]$, but since $|f-g|$ is continuous and $\mathbb{Q}$ is dense, this would imply the existence of $q \in [0,1]$ with $|f(q) - g(q)| > \epsilon$, a contradiction. So $E = \bigcap_{q\in\mathbb{Q}} E_q$, which expresses E as a countable intersection of elements of $\mathcal{F}$, so $E \in \mathcal{F}$. $\square$

Problem 4. For $n \geqslant 1$, let $a_n : [0,1) \to \{0,1\}$ denote the nth digit in the binary expansion of x, so that

$$x \;=\; \sum_{n \geqslant 1} a_n(x) 2^{-n} \quad \text{for all } x \in [0,1).$$

(We remove any ambiguity from this definition by requiring that $\liminf a_n(x) = 0$ for all $x \in [0,1)$.) Let $M([0,1))$ denote the Banach space of finite complex Borel measures on $[0,1)$ and define linear functionals L_n on $M([0,1))$ via

$$L_n(\mu) \;=\; \int_0^1 a_n(x)\, d\mu(x).$$

Show that no subsequence of the sequence L_n converges in the weak-$*$ topology on $M([0,1))^*$.

Solution. Let L_{n_k} be any subsequence of the L_n. To show that L_{n_k} is not weak-$*$ convergent, it suffices to find some $\mu \in M([0,1))$ such that $\{L_{n_k}(\mu)\}_{k=1}^\infty$ is not a convergent sequence in $\mathbb{C}$. Let

$$b \;=\; \sum_{k=1}^{\infty} (k \mod 2) \cdot 2^{-n_k},$$

i.e. b is the number in $[0,1)$ whose nth digit in binary is equal to 1 if $n = n_k$ for some odd k, and 0 otherwise. Now let $\mu = \delta_b$ be the point mass measure at b. Clearly $\mu \in M([0,1))$, and we have

$$L_{n_k}(\mu) \;=\; \int_0^1 a_{n_k}(x)\, d\mu(x) \;=\; a_{n_k}(b) \;=\; k \bmod 2.$$

So $\{L_{n_k}(\mu)\}_{k=1}^\infty$ is not a convergent sequence, so $\{L_{n_k}\}$ does not weak-$*$ converge. $\square$

Problem 5. Let $d\mu$ be a finite complex Borel measure on $[0,1]$ such that

$$\hat{\mu}(n) \;=\; \int_0^1 e^{2\pi i n x}\, d\mu(x) \;\to\; 0 \quad \text{as } n \to \infty.$$

Let $d\nu$ be a finite complex Borel measure on $[0,1]$ that is absolutely continuous with respect to $d\mu$. Show that

$$\hat{\nu}(n) \;\to\; 0 \quad \text{as } n \to \infty.$$

Solution. Since $d\nu$ is absolutely continuous with respect to $d\mu$, by the Radon-Nikodym theorem there is a function $f = \frac{d\nu}{d\mu} \in L^1(d\mu)$ such that

$$\hat{\nu}(n) \;=\; \int_0^1 e^{2\pi inx}\, d\nu(x) \;=\; \int_0^1 e^{2\pi inx} f(x)\, d\mu(x).$$

Fix $\epsilon > 0$. Since $d\mu$ is a finite Borel measure on a compact metric space, we know that the set of continuous functions is dense in $L^1(d\mu)$ with respect to the L^1 norm, so let g be a continuous function satisfying $||f-g||_{L^1} < \epsilon$. We also know that trigonometric polynomials are dense in the set of continuous functions with respect to the sup norm, so let P be a trigonometric polynomial such that $||g-P||_{L^\infty} < \epsilon$. Writing $P(x) = \sum_{m=-N}^{N} a_n e^{2\pi imx}$, we calculate

$$\lim_{n\to\infty} \int_0^1 e^{2\pi nx} P(x)\, d\mu(x) \;=\; \lim_{n\to\infty} \sum_{m=-N}^{N} a_n \int_0^1 e^{2\pi i(n+m)x}\, d\mu(x) \;=\; 0$$

by hypothesis. Thus, as soon as n is big enough so that

$$\left| \int_0^1 e^{2\pi nx} P(x)\, d\mu(x) \right| \;<\epsilon,$$

we have

$$\begin{aligned}
|\hat{\nu}(n)| \;&=\; \left| \int_0^1 e^{2\pi inx} f(x)\, d\mu(x) \right| \\
&\leqslant\; \left| \int_0^1 e^{2\pi inx}(f(x)-g(x))\, d\mu(x) \right| + \left| \int_0^1 e^{2\pi inx}(g(x)-P(x))\, d\mu(x) \right| + \left| \int_0^1 e^{2\pi inx} P(x)\, d\mu(x) \right| \\
&\leqslant\; \epsilon + \int_0^1 |f(x)-g(x)|\, d\mu(x) + \int_0^1 |g(x)-P(x)|\, d\mu(x) \\
&\leqslant\; \epsilon + \epsilon + \epsilon\mu[0,1],
\end{aligned}$$

which shows $\hat{\nu}(n) \to 0$ as $n \to \infty$. $\square$

Problem 6. Let $\overline{\mathbb{D}}$ be the closed unit disc in the complex plane, let $\{p_n\}$ be distinct points in $\mathbb{D}$ and let $r_n > 0$ be such that the discs $D_n = \{z : |z - p_n| \leqslant r_n\}$ satisfy

1. $D_n \subseteq \mathbb{D}$;
2. $D_n \cap D_m = \varnothing$ if $n \neq m$; and
3. $\sum r_n < \infty$.

Prove $X = \overline{\mathbb{D}} \backslash \bigcup_n D_n$ has positive area.

Solution. Let $f(x,y) = \sum_{i=1}^{\infty} \chi_{D_i}(x,y)$. Also let $u(x) = \sum_{i=1}^{\infty} \chi_{\pi(D_i)}(x)$ where π denotes projection onto the real axis. We have

$$\int_{-1}^{1} u(x)\, dx \;=\; \sum_{i=1}^{\infty} 2r_i \;<\; \infty$$

by hypothesis, so we conclude that $u(x) < \infty$ for a.e. $x \in (-1,1)$. For a fixed x, $u(x)$ counts the number of the D_i that intersect the line $\mathrm{Re}(z) = x$. Since the D_i are closed disjoint discs, $u(x) < \infty$ implies that the portion of the line $\mathrm{Re}(z) = x$ not contained in any of the D_i has positive (one-dimensional) Lebesgue measure. Let $m(x)$ denote the one-dimensional measure of the portion of the line $\mathrm{Re}(z) = x$ not contained in any of the D_i. Then the area of X is given exactly by $\int_{-1}^{1} m(x)\,dx$, and since m is a non-negative function which has a positive value for a.e. $x \in (-1, 1)$, this implies that $\int_{-1}^{1} m(x)\,dx > 0$. $\square$

Problem 7. Let $f(z)$ be a one-to-one continuous mapping from the closed annulus

$$\{1 \leqslant |z| \leqslant R\}$$

onto the closed annulus

$$\{1 \leqslant |z| \leqslant S\}$$

such that f is analytic on the open annulus $\{1 < |z| < R\}$. Prove $S = R$.

Solution. Let $A = \{z : 1 < |z| < R\}$ and $B = \{z : 1 < |z| < S\}$. We know that f maps ∂A to ∂B, so by composing f with an inversion if necessary we may assume that f maps the unit circle to itself. Since f is a nonvanishing analytic function in A, $\log|f|$ is harmonic in A and extends continuously to ∂A, and satisfies $\log|f(z)| = 0$ on $|z| = 1$ and $\log|f(z)| = \log(S)$ on $|z| = R$. Since A is a region on which the Dirichlet problem can be solved, $\log|f|$ is uniquely determined by its boundary values. Since $z \mapsto \log|z| \cdot \frac{\log(S)}{\log(R)}$ is another harmonic function on A with the same boundary values, we conclude that

$$\log|f(z)| \;=\; \log|z| \cdot \frac{\log(S)}{\log(R)}$$

for all $z \in A$. Therefore we have $|f(z)| = |z^\alpha|$ where $\alpha := \log(S)/\log(R)$. Since $f(z)$ and z^α are both analytic functions in the slit annulus $\tilde{A} := A\backslash[-R, -1]$, this implies that $f(z) = Cz^\alpha$ for some $|C| = 1$ (this is proven by applying the maximum principle to $f(z)/z^\alpha$ and $z^\alpha/f(z)$). But we know that f analytically continues to all of A, so by uniqueness of analytic continuation, z^α must also, which implies that α is a positive integer. But if $\alpha \geqslant 2$, then z^α is not one-to-one on A, so we must have $\alpha = 1$ and therefore $\log(R) = \log(S)$, so $R = S$. $\square$

Alternate solution. Suppose $R \neq S$. There is a mapping from $S_{1,R} = \{z : 0 < \mathrm{Re}(z) < \log(R)\}$ to the annulus $A_{1,R} = \{1 \leqslant |z| \leqslant R\}$ defined by $z \mapsto e^z$. Thus, a map f from $A_{1,R}$ to $A_{1,S}$ lifts to a map $\tilde{f}$ from $S_{1,R}$ to $S_{1,S}$ such that $\tilde{f}(z + 2\pi i) = \tilde{f}(z) + 2\pi i$. We may then rotate the strip so that it is horizontal and adjust $\tilde{f}$ so that $\tilde{f}(z + 2\pi) = \tilde{f}(z) + 2\pi$. Next, we map this strip to the upper half plane via $z \mapsto e^{z\pi/\log(R)}$. Thus f lifts to a conformal map h on the upper half plane satisfying

$$h(\lambda z) = \lambda' h(z)$$

with $\lambda = e^{2\pi^2/\log(R)} \neq e^{2\pi^2/\log(S)} = \lambda'$. As h must be a linear fractional transformation, we obtain a contradiction as there exists no such linear fractional transformation h. $\square$

Problem 8. Let $a_1, \ldots, a_n$ be $n \geqslant 1$ points in the disc $\mathbb{D}$ (possibly with repetitions), so that the function

$$B(z) \;=\; \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z}$$

has n zeros in $\mathbb{D}$. Prove that the derivative $B'(z)$ has $n - 1$ zeros in $\mathbb{D}$.

Solution. First assume that $B(0) \neq 0 \neq B'(0)$ and that B has no repeated roots. One can calculate that

$$\frac{B'(z)}{B(z)} \;=\; \sum_{j=1}^{n} \frac{1 - |a_j|^2}{(z - a_j)(1 - \overline{a_j} z)} \;=\; \frac{\sum_{j=1}^{n} \left[(1 - |a_j|^2) \prod_{i \neq j} (z - a_i)(1 - \overline{a_i} z)\right]}{\prod_{j=1}^{n} (z - a_j)(1 - \overline{a_j} z)}.$$

Since we assume B has no repeated roots, the zeros of B'/B are precisely the zeros of B'. Note that B'/B is a rational function with a numerator of degree $2(n-1)$, so it has $2(n-1)$ total zeros. With a lot of calculation, one can verify the identity

$$\overline{\frac{B'(1/\overline{z})}{B(1/\overline{z})}} \;=\; z^2\frac{B'(z)}{B(z)}.$$

This shows that for $z \neq 0$, $B'(z) = 0$ if and only if $B'(1/\overline{z}) = 0$. Since we assumed neither B nor B' vanish at 0, this implies that the zeros come in pairs $\{z, 1/\overline{z}\}$. Exactly one member of each pair is inside $\mathbb{D}$ and the other is outside $\mathbb{D}$, so since there are $2(n-1)$ total zeros of B', it must have $n-1$ zeros inside $\mathbb{D}$.

For the general case, it is a theorem that if B is any function of the given form with n factors, then there is a sequence B_k of functions of the given form, each with n factors, satisfying (a) $B_k \to B$ uniformly on $\overline{\mathbb{D}}$, (b) $B_k(0) \neq 0 \neq B_k'(0)$, and (c) B_k has no repeated roots. To see why this is true, note that $\frac{z-\alpha}{1-\overline{\alpha}z}$ converges uniformly on $\overline{\mathbb{D}}$ to $\frac{z-\beta}{1-\overline{\beta}z}$ as $\alpha \to \beta$. Therefore this is also true for products of functions of that form. Also note that $B_k(0)$ and $B_k'(0)$ are continuous functions of the roots $a_1, \ldots, a_n$. Therefore by just taking the original function B and perturbing its roots by sufficiently small amounts, we can guarantee that the new function has all of the desired properties and is still uniformly close to B.

So by the first part of this problem, we know that each B_k has exactly $n-1$ roots in $\mathbb{D}$. Since the convergence is uniform on $\overline{\mathbb{D}}$, we also know that $B_k' \to B'$ uniformly on $\overline{\mathbb{D}}$. Since each B_k has absolute value 1 on $\partial\mathbb{D}$, we then have that B_k'/B_k converges uniformly to B'/B on $\partial\mathbb{D}$, so by the argument principle

$$\#\text{ zeros of } B \text{ in } \mathbb{D} \;=\; \int_{\partial\mathbb{D}} \frac{B'}{B}\,dz \;=\; \lim_{k\to\infty}\int_{\partial\mathbb{D}} \frac{B_k'}{B_k}\,dz \;=\; \lim_{k\to\infty}(\#\text{ zeros of } B_k \text{ in } \mathbb{D}) \;=\; n-1. \quad \square$$

Problem 9a. Let $f(z)$ be an analytic function in the entire complex plane $\mathbb{C}$ and assume $f(0) \neq 0$. Let $\{a_n\}$ be the zeros of f, repeated according to their multiplicities. Let $R > 0$ be such that $|f(z)| > 0$ on $|z| = R$. Prove

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(Re^{i\theta})\right|\,d\theta \;=\; \log|f(0)| + \sum_{|a_n|<R} \log\frac{R}{|a_n|}.$$

Solution. Since f is not identically zero, there are only finitely many a_n satisfying $|a_n| < R$. Define

$$g(z) \;=\; \prod_{|a_n|<R} \frac{R(z-a_n)}{R^2 - \overline{a_n}z}.$$

Note that in the disc $|z| < R$, g has the same zeros as f, no poles, and $|g(z)| = 1$ for $|z| = R$. Therefore f/g is a nonvanishing holomorphic function in $|z| < R$, and $|f/g| = |f|$ on the boundary $|z| = R$. Therefore $\log|f/g|$ is a harmonic function in $|z| < R$, so we apply the mean value formula to obtain

$$\log\left|\frac{f(0)}{g(0)}\right| \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{f(Re^{i\theta})}{g(Re^{i\theta})}\right|\,d\theta \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(Re^{i\theta})\right|\,d\theta.$$

We also have

$$\log\left|\frac{f(0)}{g(0)}\right| \;=\; \log|f(0)| - \sum_{|a_n|<R} \log\left|\frac{R(0-a_n)}{R^2-0}\right| \;=\; \log|f(0)| + \sum_{|a_n|<R} \log\left|\frac{R}{a_n}\right|,$$

so combining this with the above equation gives the desired result. $\square$

Problem 9b. Prove that if there are constants C and λ such that $|f(z)| \leqslant Ce^{|z|^\lambda}$ for all z, then

$$\sum\left(\frac{1}{|a_n|}\right)^{\lambda+\epsilon} \;<\; \infty$$

for all $\epsilon > 0$.

Solution. Let $N(R) = \#\{n : |a_n| < R\}$. Applying part (a) with $2R$ in place of R we get

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(2Re^{i\theta})|\,d\theta \;=\; \log|f(0)| + \sum_{|a_n|<2R} \log\left(\frac{2R}{|a_n|}\right) \;\leqslant\; \log|f(0)| + \sum_{|a_n|<R} \log\left(\frac{2R}{|a_n|}\right) \;\leqslant\; \log|f(0)| + N(R)\log(2).$$

By the hypothesis on the growth rate of f, we also have

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(2Re^{i\theta})|\,d\theta \;\leqslant\; (2R)^\lambda + \log(C),$$

so combining the two estimates gives $(2R)^\lambda + \log(C) \geqslant \log|f(0)| + N(R)\log(2)$, which implies that

$$N(R) \;\leqslant\; \frac{(2R)^\lambda - \log(C) - \log|f(0)|}{\log(2)} \;\leqslant\; K(2R)^\lambda$$

for some constant K and R sufficiently large. Let M be big enough so that the above estimate holds whenever $R \geqslant 2^{M-1}$. It suffices to show that

$$\sum_{|a_n|\geqslant 2^{M-1}} \left(\frac{1}{|a_n|}\right)^{\lambda+\epsilon} \;<\; \infty$$

for any $\epsilon > 0$. We estimate

$$\begin{aligned}\sum_{|a_n|\geqslant 2^{M-1}} \left(\frac{1}{|a_n|}\right)^{\lambda+\epsilon} \;&=\; \sum_{r=M}^{\infty} \sum_{2^{r-1}\leqslant|a_n|<2^r} \left(\frac{1}{|a_n|}\right)^{\lambda+\epsilon} \;\leqslant\; \sum_{r=M}^{\infty} (N(2^r) - N(2^{r-1})\left(\frac{1}{2^{r-1}}\right)^{\lambda+\epsilon} \\ &\leqslant\; \sum_{r=M}^{\infty} \frac{N(2^r)}{(2^{r-1})^{\lambda+\epsilon}} \;\leqslant\; K\sum_{r=M}^{\infty} \frac{(2^{r+1})^\lambda}{(2^{r-1})^{\lambda+\epsilon}} \;=\; K\cdot 2^{2\lambda+\epsilon}\sum_{r=M}^{\infty}(2^{-\epsilon})^r \;<\; \infty. \quad \square\end{aligned}$$

Problem 10. Let $a_1, \ldots, a_n$ be $n \geqslant 1$ distinct points in $\mathbb{C}$ and let $\Omega = \mathbb{C}\backslash\{a_1, \ldots, a_n\}$. Let $H(\Omega)$ be the vector space of real-valued harmonic functions on Ω and let $R(\Omega) \subseteq H(\Omega)$ be the space of real parts of analytic functions on Ω. Prove the quotient space $\frac{H(\Omega)}{R(\Omega)}$ has dimension n, find a basis for this space, and prove it is a basis.

Solution. We claim that the functions $f_i = \log|z - a_i|$ form a basis for this space. We will work with a homology basis $\gamma_1, \ldots, \gamma_n$ for Ω, consisting of small counterclockwise circles around each point. For a function $u \in H(\Omega)$ be arbitrary, we let $*du = -u_y dx + u_x dy$ denote the conjugate differential for u. Recall that the periods of $*du$ with respect our homology basis are defined to be the real numbers $\int_{\gamma_i} u$. (See section 6.1 in Ahlfors.)

The harmonic function $a(z) = \log|z|$ defined on $\mathbb{C}\backslash\{0\}$ has conjugate differential $d\theta$, and so the period of $*da$ on a counterclockwise circle about the origin is 2π. Alternatively one can see this by setting $f = a_x - ia_y$ (which is analytic) and then writing $f\,dz = da + i * da$. The differential da is exact, and we can compute that $f(z) = \frac{1}{z}$. Thus the integral of $i * dv$ around a counterclockwise circle is $2\pi i$, and we again get a period of 2π. Note that the period of $*da$ around any cycle homologous to 0 is 0, since the integral of $f dz$ around such a cycle is 0. Therefore by translating, we see that the period of $*df_i$ along γ_j is $2\pi\delta_{ij}$.

If $u \in R(\Omega)$ then u has a harmonic conjugate v and $*du = dv$, which is exact. Thus each period of u is 0. If $\sum_{i=1}^n a_i f_i \in R(\Omega)$, then it must have period 0 about each cycle. By linearity of periods, this can only happen if each a_i is 0. So our f_i's are independent.

Let $g \in H(\Omega)$ be arbitrary, with $*dg$ having periods p_i on γ_i. Set

$$\widetilde{g} = g - \frac{1}{2\pi}\sum_{i=1}^n p_i f_i,$$

so that $*d\tilde{g}$ has period 0 on each γ_i. We claim that $\tilde{g}$ lies in $R(\Omega)$, which will imply that the f_i's span. Indeed we have that $*d\tilde{g}$ is exact and so we may integrate $*d\tilde{g}$ to obtain a harmonic conjugate for $\tilde{g}$. More precisely, set $f(z) = \tilde{u}_x - i\tilde{u}_y$. Then $f dz = du + i * du$ is exact on Ω and so f has an anti-derivative $F = U + iV$ on Ω. It's easy to verify that U and u agree up to constants, so V is a harmonic conjugate for u.

Problem 11. Let $1 \leqslant p < \infty$ and let $U(z)$ be a harmonic function on the complex plane $\mathbb{C}$ such that

$$\iint\limits_{\mathbb{R}\times\mathbb{R}} |U(x+iy)|^p \, dx \, dy \;<\; \infty.$$

Prove that $U(z) = 0$ for all $z = x + iy \in \mathbb{C}$.

Solution. Let q be the conjugate exponent, so $1/p + 1/q = 1$. Since U is harmonic on all of $\mathbb{C}$, for any $r > 0$ and any $z \in \mathbb{C}$ we have the mean value property

$$U(z) \;=\; \frac{1}{\pi r^2} \iint\limits_{B(z,r)} U(x+iy) \, dx \, dy.$$

By Hölder's inequality we have

$$\begin{aligned} |U(z)| \;&\leqslant\; \frac{1}{\pi r^2} \iint\limits_{B(z,r)} |U(x+iy)| \, dx \, dy \;=\; \frac{1}{\pi r^2} \left(\iint B(z,r) |U(x+iy)|^p \, dx \, dy \right)^{1/p} \left(\iint\limits_{B(z,r)} 1 \, dx \, dy \right)^{1/q} \\ &\leqslant\; \frac{(\pi r^2)^{1/q}}{\pi r^2} \left(\iint\limits_{\mathbb{R}\times\mathbb{R}} |U(x+iy)|^p \, dx \, dy \right)^{1/p} \;\leqslant\; C r^{2(1/q - 1)} \;=\; C r^{-2/p} \end{aligned}$$

for some constant $C < \infty$. This holds for any $r > 0$, so we can take $r \to \infty$ and conclude that $U(z) = 0$ (because $-2/p < 0$). $\square$

Problem 12. Let $0 < \alpha < 1$ and let $f(z)$ be an analytic function on the unit disc $\mathbb{D}$. Prove that if

$$|f(z) - f(w)| \;\leqslant\; C|z - w|^\alpha$$

for all $z, w \in \mathbb{D}$ and some constant $C \in \mathbb{R}$, then there is a constant $A = A(C) < \infty$ such that

$$|f'(z)| \;\leqslant\; A(1 - |z|)^{\alpha - 1}.$$

Solution. Fix $z \in \mathbb{D}$. Then for any $r > 0$ we have

$$\int_{|w-z|=r} \frac{1}{(w-z)^2} \, dw \;=\; 0,$$

so by the Cauchy integral formula we can write

$$f'(z) \;=\; \frac{1}{2\pi i} \int_{|w-z|=r} \frac{f(w)}{(w-z)^2} \, dw \;=\; \frac{1}{2\pi i} \int_{|w-z|=r} \frac{f(w) - f(z)}{(w-z)^2} \, dw.$$

Therefore taking absolute values inside we get

$$|f'(z)| \;\leqslant\; \frac{1}{2\pi} \cdot 2\pi r \cdot \frac{1}{r^2} \cdot \sup_{|w-z|=r} |f(z) - f(w)| \;\leqslant\; \frac{1}{r} C r^\alpha \;=\; C r^{1-\alpha}.$$

This is true for any r for which $B(z,r) \subseteq \mathbb{D}$, so pick $r = \frac{1-|z|}{2}$, then we get

$$|f'(z)| \;\leqslant\; A(1 - |z|)^{\alpha - 1}. \quad \square$$

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Problem 1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is non-decreasing. Show that if $A \subseteq \mathbb{R}$ is a Borel set, then so is $f(A)$.

Solution. Let $\mathcal{F} = \{A \subseteq \mathbb{R} : f(A) \text{ is Borel}\}$. It suffices to show that $\mathcal{F}$ is a σ-algebra containing all closed intervals. It's clear that $\varnothing \in \mathcal{F}$. Since f is non-decreasing, it is continuous except for at most countably many jump discontinuities. Thus $f(\mathbb{R})$ is a countable union of intervals, so it's Borel, so $\mathbb{R} \in \mathcal{F}$. Suppose $A \in \mathcal{F}$. Note that $f(A)$ and $f(A^c)$ have at most countably many elements in common and that $f(\mathbb{R}) = f(A) \cup f(A^c)$, so we can write $f(A^c) = f(\mathbb{R}) \backslash f(A) \cup$ (countable set), so $f(A^c)$ is Borel and thus $A^c \in \mathcal{F}$. Finally, if $A_1, A_2, \ldots \in \mathcal{F}$, then we have $f\left(\bigcup A_n\right) = \bigcup f(A_n)$, so it's Borel, so $\bigcup A_n \in \mathcal{F}$. Thus $\mathcal{F}$ is a σ-algebra. If $[a, b]$ is a closed interval, then by the same argument as above, since f is non-decreasing, $f([a, b])$ is an at most countable union of intervals, so it's Borel. Therefore $\mathcal{F}$ contains all closed intervals so we're done. $\square$

Problem 2. Let $\{f_n\}$ denote a bounded sequence in $L^2([0,1])$. Suppose the sequence also converges almost everywhere. Show that then $\{f_n\}$ converges in the weak topology on $L^2([0,1])$.

Solution. Say that $||f_n||_{L^2} \leqslant M$ for all n and that $f_n \to f$ almost everywhere. Then also $|f_n|^2 \to |f|^2$ almost everywhere, so by Fatou's lemma we have

$$\int |f|^2 \;=\; \int \liminf_{n\to\infty} |f_n|^2 \;\leqslant\; \liminf_{n\to\infty} \int |f_n|^2 \;\leqslant\; M^2,$$

so also $f \in L^2$ and $||f||_{L^2} \leqslant M$. To show that $f_n \to f$ weakly in L^2, we need to show that $\phi(f_n) \to \phi(f)$ for every $\phi \in (L^2)^*$, and by L^p-L^q duality, this is the same as showing that $\int f_n g \to \int fg$ for every $g \in L^2$. Fix $g \in L^2$ and $\epsilon > 0$. Since $|g|^2$ is integrable, let $\delta > 0$ be such that $\lambda(E) < \delta$ implies $\int_E |g|^2 < \epsilon$ (here λ denotes Lebesgue measure). By Egorov's theorem, we can find a set $E \subseteq [0,1]$ such that $f_n \to f$ uniformly on E^c and $\lambda(E) < \delta$. Let n be big enough so that $|f_n - f| < \epsilon / ||g||_{L^2}$ on E^c. Then we have

$$\begin{aligned}
\int |f_n g - fg| \;&=\; \int_A |f_n g - fg| + \int_{A^c} |f_n g - fg| \;=\; \int_A |g||f_n - f| + \int_{A^c} |g||f_n - f| \\
&\leqslant \left(\int_A |g|^2\right)^{1/2} \left(\int_A |f_n - f|^2\right)^{1/2} + \left(\int_{A^c} |g|^2\right)^{1/2} \left(\int_{A^c} |f_n - f|^2\right)^{1/2} \\
&\leqslant \epsilon^{1/2} \left(\int_{[0,1]} 4(|f_n|^2 + |f|^2)\right)^{1/2} + ||g||_{L^2} \left(\int_{[0,1]} \epsilon^2 / ||g||_{L^2}^2\right)^{1/2} \\
&\leqslant \epsilon^{1/2} (8M^2)^{1/2} + \epsilon.
\end{aligned}$$

This shows that $\int |f_n g - fg| \to 0$ as $n \to \infty$, which implies the desired result. $\square$

Problem 3. Let $\{\mu_n\}$ denote a sequence of Borel probability measures on $\mathbb{R}$. For $n \in \mathbb{N}$ and $x \in \mathbb{R}$ we define

$$F_n(x) \;:=\; \mu_n((-\infty, x]).$$

Suppose the sequence $\{F_n\}$ converges uniformly on $\mathbb{R}$. Show that then for every bounded continuous function $f : \mathbb{R} \to \mathbb{R}$, the numbers

$$\int_{\mathbb{R}} f(x)\, d\mu_n(x)$$

converge as $n \to \infty$.

Solution. Let $\mathcal{F}$ denote the set of linear combinations of characteristic functions of disjoint intervals of the form $(a, b]$, where a may be $-\infty$ and b may be ∞. First we show the result holds for elements of $\mathcal{F}$. Let

$g = \sum_{k=1}^N \alpha_k \chi_{(a_k,b_k]}$. Then we have (with the convention that $F_n(\infty) = 1$ and $F_n(-\infty) = 0$)

$$\begin{aligned} \left| \int g\, d\mu_n - \int g\, d\mu_m \right| &= \left| \sum_{k=1}^N \alpha_k (F_n(b_k) - F_n(a_k)) - \sum_{k=1}^N \alpha_k (F_m(b_k) - F_m(a_k)) \right| \\ &\leqslant \sum_{k=1}^N |\alpha_k| \left(|F_n(b_k) - F_m(b_k)| + |F_n(b_k) - F_n(a_k)| \right). \end{aligned}$$

Fix $\epsilon > 0$. Since the sequence $\{F_n\}$ converges uniformly, pick n, m big enough so that $||F_n - F_m||_{L^\infty} < \epsilon/(2\sum |\alpha_k|)$. Then the above estimate implies that for all such n, m, we have $\left|\int g\, d\mu_n - \int g\, d\mu_m\right| < \epsilon$. So the numbers $\{\int g\, d\mu_n\}$ form a Cauchy sequence in $\mathbb{R}$ and therefore converge. This establishes the result for elements of $\mathcal{F}$.

Now let f be any bounded continuous function $\mathbb{R} \to \mathbb{R}$. On any compact interval, f can be approximated in the L^∞ norm by functions in $\mathcal{F}$. So just work on a compact interval that is big enough so that almost all of the mass of the μ_n is inside that interval (this can be made precise using the fact that the F_n converge uniformly on $\mathbb{R}$, but I don't have time to write it down right now). Fix $\epsilon > 0$ and pick $g \in \mathcal{F}$ such that $||f - g||_{L^\infty} < \epsilon$. Then for n, m big enough, we have

$$\begin{aligned} \left| \int f\, d\mu_n - \int f\, d\mu_m \right| &\leqslant \left| \int f\, d\mu_n - \int g\, d\mu_n \right| + \left| \int g\, d\mu_n - \int g\, d\mu_m \right| + \left| \int g\, d\mu_m - \int f\, d\mu_m \right| \\ &\leqslant \int |f - g|\, d\mu_n + \int |f - g|\, d\mu_m + \epsilon \\ &\leqslant \epsilon\mu_n(\mathbb{R}) + \epsilon\mu_m(\mathbb{R}) + \epsilon \;=\; 3\epsilon, \end{aligned}$$

which establishes the desired result. $\square$

Problem 4. Consider the Banach space $V = C([-1,1])$ of all real-valued continuous functions on $[-1,1]$ equipped with the supremum norm. Let $B = \{f \in V : ||f||_{L^\infty} \leqslant 1\}$ be the closed unit ball in V. Show that there exists a bounded linear functional $\Lambda : V \to \mathbb{R}$ such that $\Lambda(B)$ is an open subset of $\mathbb{R}$.

Solution. Define $\Lambda : V \to \mathbb{R}$ by

$$\Lambda(f) \;=\; -\int_{-1}^0 f(x)\, dx + \int_0^1 f(x)\, dx.$$

It is clear that $|\Lambda(f)| \leqslant 2\, ||f||_{L^\infty}$ for all $f \in V$, so Λ is a bounded linear functional. Since Λ is continuous and B is a connected set, $\Lambda(B)$ is a connected subset of $\mathbb{R}$ and is therefore an interval. We claim that $\Lambda(B)$ is the open interval $(-2, 2)$.

Let f_n be the function which is equal to -1 for $x \in [-1, -1/n]$, equal to 1 for $x \in [1/n, 1]$, and linear on $[-1/n, 1/n]$. Note that each $f_n \in B$, and we calculate $\Lambda(f_n) = 2 - 1/n$. Since $\Lambda(B)$ is an interval in $\mathbb{R}$, this implies that $(-2, 2) \subseteq \Lambda(B)$. We now just need to check that Λ never achieves the values ± 2. But note that we have $|\Lambda(f)| \leqslant \int_{-1}^1 |f(x)|\, dx \leqslant 2$. But the second inequality is strict for all f which are not identically ± 1. Since $\Lambda(\pm 1) = 0$, this shows that in fact the strict inequality $|\Lambda(f)| < 2$ holds for all $f \in B$, so we conclude that $\Lambda(B) = (-2, 2)$. $\square$

Problem 5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a bounded and measurable function satisfying $f(x+1) = f(x)$ and $f(2x) = f(x)$ for almost every $x \in \mathbb{R}$. Show that then there exists a constant $c \in \mathbb{R}$ such that $f(x) = c$ for almost every $x \in \mathbb{R}$.

Solution. Let Z be the measure zero set of bad points for which the given property doesn't hold. Let $\widetilde{Z}$ be the set of all points in $\mathbb{R}$ which are reachable from a point in Z by a finite sequence of the operations $x \mapsto x+1$, $x \mapsto x-1$, $x \mapsto 2x$, or $x \mapsto x/2$. Then $\widetilde{Z}$ is just a countable union of translates and dilates of Z, so $\widetilde{Z}$ also has measure zero. We will show that f is constant on the complement of $\widetilde{Z}$. By construction of $\widetilde{Z}$, for any $x \notin \widetilde{Z}$ we have $2^{-n}(2^n x + 1 + 2^n m) = x + m + 2^{-n} \notin \widetilde{Z}$ for all integers n, m. Let Q be the set of numbers of the form $m + 2^{-n}$ for $n, m \in \mathbb{Z}$.

Let $x_0, y_0 \notin \widetilde{Z}$ and fix $\epsilon > 0$. Since f is bounded, it is locally integrable. Therefore by the Lebesgue differentiation theorem we can pick $r > 0$ such that

$$\left| f(x_0) - \frac{1}{2r}\int_{x_0-r}^{x_0+r} f(t)\,dt \right| \;<\; \epsilon, \qquad \left| f(y_0) - \frac{1}{2r}\int_{y_0-r}^{y_0+r} f(t)\,dt \right| \;<\; \epsilon.$$

Also, since f is bounded we can find $\delta > 0$ such that for any set $A \subseteq \mathbb{R}$, $\lambda(A) < \delta$ implies $\int_A |f(t)|\,dt < \epsilon r$ (here λ denotes Lebesgue measure). We can pick a number $q \in Q$ such that $|(x_0 + q) - y_0| < \delta/2$. Then, since $f(t+q) = f(t)$ for all $t \notin \widetilde{Z}$, which is almost every t, we have the estimate

$$\begin{aligned}
\left| \frac{1}{2r}\int_{x_0-r}^{x_0+r} f(t)\,dt - \frac{1}{2r}\int_{y_0-r}^{y_0+r} f(t)\,dt \right| \;&=\; \frac{1}{2r}\left| \int_{x_0+q-r}^{x_0+q+r} f(t)\,dt - \int_{y_0-r}^{y_0+r} f(t)\,dt \right| \\
&=\; \frac{1}{2r}\left| \int_{[x_0+q-r,x_0+q+r]\Delta[y_0-r,y_0+r]} f(t)\,dt \right| \;<\; \epsilon/2.
\end{aligned}$$

So combining the above three inequalities with the triangle inequality gives $|f(x_0) - f(y_0)| < (2 + 1/2)\epsilon$, and taking $\epsilon \to 0$ shows that $f(x_0) = f(y_0)$, so f is constant on the complement of $\widetilde{Z}$. $\square$

Alternative Solution. Let E be the measure zero set on which $f(x) \neq f(2x)$. Then $f(x) = f(2x)$ for all $x \in E^c$, and so $f(2^k x) = f(x)$ for all $x \in E^c$ and $k \in \mathbb{N}$. Since we are only trying to show that f is constant almost everywhere, we can discard E. So, we can suppose $f(2^k x) = f(x)$ for all x. Moreover, $f(x+1) = f(x)$ for almost all x means f can be considered as a function on $S^1 = \mathbb{R}/\mathbb{Z} = [0,1)$. As a bounded measurable function on S^1, f is in $L^1(S^1)$, and so has Fourier coefficients $\hat{f}(k)$ for all $k \in \mathbb{Z}$. An elementary theorem says that $L^1(S^1)$ functions are determined by their Fourier coefficients. Therefore, to show f is constant, it is enough to show that every nonzero Fourier coefficient of f vanishes (since then f will have the same Fourier coefficients as the constant function $x \mapsto \hat{f}(0)$).

Now, for any $k \in \mathbb{N}$, and any $n \in \mathbb{Z}$,

$$\begin{aligned}
\hat{f}(n) &= \int_0^1 f(x)e^{-2\pi nix}\,dx \\
&= \int_0^1 f(2^k x)e^{-2\pi nix}\,dx \\
&= 2^{-k}\int_0^{2^k} f(y)e^{-2\pi in2^{-k}y}\,dy \\
&= 2^{-k}\sum_{j=0}^{2^k-1}\int_0^1 f(y)e^{-2\pi in2^{-k}(y+j)}\,dy \\
&= c_{k,n}\cdot 2^{-k}\int_0^1 f(y)e^{-2\pi in2^{-k}y}\,dy,
\end{aligned}$$

where $c_{k,n}$ is the constant

$$c_{k,n} = \sum_{j=0}^{2^k-1} e^{-2\pi in2^{-k}j}.$$

But, if $n2^{-k}$ is not an integer, then

$$c_{k,n} = \frac{(e^{-2\pi in2^{-k}})^{2^k} - 1}{e^{-2\pi in2^{-k}} - 1} = \frac{e^{-2\pi in} - 1}{e^{-2\pi in2^{-k}} - 1} = 0,$$

and so $\hat{f}(n) = 0$ in this case. But if $n \neq 0$, then of course there is some $k \in \mathbb{N}$ with $n2^{-k} \notin \mathbb{Z}$. Consequently $\hat{f}(n) = 0$ if $n \neq 0$, which completes the proof. $\square$

Problem 6. Let $f \in L^2(\mathbb{C})$. For $z \in \mathbb{C}$ we define

$$g(z) \;=\; \int_{\{w \in \mathbb{C}: |w-z| \leqslant 1\}} \frac{|f(w)|}{|z-w|}\, dA(w)$$

where dA denotes integrations with respect to Lebesgue measure on $\mathbb{C}$. Show that then $|g(z)| < \infty$ for almost every $z \in \mathbb{C}$ and that $g \in L^2(\mathbb{C})$.

Solution. Let $C = \int_{|u| \leqslant 1} \frac{1}{|u|}\, dA(u) < \infty$. We have

$$\begin{aligned} |g(z)|^2 \;&=\; \left(\int_{|w-z|\leqslant 1} \frac{|f(w)|}{|w-z|}\, dA(w) \right)^2 \;\leqslant\; \left(\int_{|w-z|\leqslant 1} \frac{|f(w)|^2}{|w-z|}\, dA(w) \right) \left(\int_{|w-z|\leqslant 1} \frac{1}{|w-z|}\, dA(w) \right) \quad \text{by Cauchy-Schwarz} \\ &\leqslant\; C \cdot \int_{|w-z|\leqslant 1} \frac{|f(w)|^2}{|w-z|}\, dA(w). \end{aligned}$$

Therefore we can estimate

$$\begin{aligned} \int_{\mathbb{C}} |g(z)|^2\, dA(z) \;&\leqslant\; C \int_{\mathbb{C}} \int_{|w-z|\leqslant 1} \frac{|f(w)|^2}{|w-z|}\, dA(w)\, dA(z) \\ &\leqslant\; C \int_{\mathbb{C}} |f(w)|^2 \int_{|z-w|\leqslant 1} \frac{1}{|z-w|}\, dA(z)\, dA(w) \quad \text{by Tonelli} \\ &\leqslant\; C^2\, ||f||^2_{L^2(\mathbb{C})} \;<\; \infty. \end{aligned}$$

This shows both that $|g(z)| < \infty$ for almost every $z \in \mathbb{C}$ and $g \in L^2(\mathbb{C})$. $\square$

Alternate solution. Change variables to polar coordinates to write

$$g(z) \;=\; \int_0^1 \int_0^{2\pi} \frac{|f(z + re^{i\theta}|}{r} \cdot r\, d\theta\, dr \;=\; \int_0^1 \int_0^{2\pi} |f(z + re^{i\theta})|\, d\theta\, dr.$$

Then by Minkowski's integral inequality we can estimate

$$\begin{aligned} ||g||_{L^2(\mathbb{C})} \;&=\; \left(\int |g(z)|^2\, dA(z) \right)^{1/2} \;=\; \left(\int \left(\int_0^1 \int_0^{2\pi} |f(z + re^{i\theta})|\, d\theta\, dr \right)^2 dA(z) \right)^{1/2} \\ &\leqslant\; \int_0^1 \int_0^{2\pi} \left(\int |f(z + re^{i\theta}|^2\, dA(z) \right)^{1/2} d\theta\, dr \;<\; \infty \end{aligned}$$

by the assumption that $f \in L^2(\mathbb{C})$. $\square$

Problem 7. Prove that there exists a meromorphic function f on $\mathbb{C}$ with the following properties.

1. $f(z) = 0$ if and only if $z \in \mathbb{Z}$.
2. $f(z) = \infty$ if and only if $z - 1/3 \in \mathbb{Z}$.
3. $|f(x + iy)| \leqslant 1$ for all $x \in \mathbb{R}$ and all $y \in \mathbb{R}$ with $|y| \geqslant 1$.

Solution. Let $f(z) = \frac{1}{2} \frac{\sin(\pi z)}{\sin(\pi(z-1/3))}$. It's clear that f is meromorphic with $f(z) = 0$ if and only if $z \in \mathbb{Z}$ and $f(z) = \infty$ if and only if $z - 1/3 \in \mathbb{Z}$. Now we just estimate

$$\begin{aligned} 2|f(x+iy)| \;&=\; \left| \frac{\exp(i\pi z) - \exp(-i\pi z)}{\exp(i\pi(z-1/3)) - \exp(-i\pi(z-1/3))} \right| \;\leqslant\; \frac{|\exp(i\pi z)| + |\exp(-i\pi z)|}{||\exp(i\pi(z-1/3))| - |\exp(-i\pi(z-1/3))||} \\ &=\; \frac{\exp(-\pi y) + \exp(\pi y)}{|\exp(-\pi y) - \exp(\pi y)|} \;\leqslant\; 2 \quad \text{when } |y| \geqslant 1. \quad \square \end{aligned}$$

Problem 8. Show that a harmonic function $u : \mathbb{D} \to \mathbb{R}$ is uniformly continuous if and only if it admits the representation

$$u(z) \;=\; \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) f(e^{i\theta})\,d\theta, \quad z \in \mathbb{D},$$

with $f : \partial\mathbb{D} \to \mathbb{R}$ continuous.

Solution. It is a standard fact that u is uniformly continuous on $\mathbb{D}$ if and only if it admits a continuous extension to $\partial\mathbb{D}$. First suppose that u admits a continuous extension to $\partial\mathbb{D}$. Then the Poisson integral formula is exactly the representation

$$u(z) \;=\; \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) u(e^{i\theta})\,d\theta$$

(To prove the Poisson integral formula, you simply apply the regular mean value formula to u composed with the conformal map $w \mapsto \frac{w+z}{1+\bar{z}w}$ and simplify the change of variables. Not sure if proving that would be required for this problem or not).

Conversely, suppose u has the above representation. We just need to show that the continuous function $f : \partial\mathbb{D} \to \mathbb{R}$ continuously extends u. Fix $e^{i\theta_0} \in \partial\mathbb{D}$. We need to show that $u(z) \to f(e^{i\theta_0})$ as $z \to e^{i\theta_0}$ in $\mathbb{D}$. Fix $\epsilon > 0$. Pick δ_1 such that $|\theta - \theta_0| < \delta_1$ implies $|f(e^{i\theta}) - f(e^{i\theta_0})| < \epsilon$ (by continuity of f). Also, since $\partial\mathbb{D}$ is compact, let $M = \max_{\theta\in[0,2\pi]} |f(e^{i\theta})|$. Now we can pick $\delta > 0$ to be small enough so that

$$|z - e^{i\theta_0}| < \delta \quad \text{and} \quad |\theta - \theta_0| \geqslant \delta_1 \quad \text{imply} \quad \frac{1-|z|^2}{|e^{i\theta}-z|^2} = \operatorname{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) < \frac{\epsilon}{2M}.$$

Then for all $|z - e^{i\theta_0}| < \delta$, we have the estimate (using the fact that $\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\,d\theta = 2\pi$ for any $z \in \mathbb{D}$)

$$\begin{aligned}
|u(z) - f(e^{i\theta_0})| \;&=\; \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2} f(e^{i\theta})\,d\theta - \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2} f(e^{i\theta_0})\,d\theta\right| \\
&\leqslant\; \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2} |f(e^{i\theta}) - f(e^{i\theta_0})|\,d\theta \\
&\leqslant \frac{1}{2\pi}\left(\int_{|\theta-\theta_0|<\delta_1} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\epsilon\,d\theta + \int_{|\theta-\theta_0|\geqslant\delta_1} \frac{\epsilon}{2M} 2M\,d\theta\right) \\
&\leqslant\; \frac{\epsilon}{2\pi}\,(2\pi + 2\pi) \;=\; 2\epsilon.
\end{aligned}$$

This shows that $u(z) \to f(e^{i\theta_0})$ as $z \to e^{i\theta_0}$ so f is a continuous extension of u to $\partial\mathbb{D}$ and we are done. $\square$

Problem 9. Consider a map $F : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ with the following properties.

1. For each fixed $z \in \mathbb{C}$ the map $w \mapsto F(z,w)$ is injective.
2. For each fixed $w \in \mathbb{C}$ the map $z \mapsto F(z,w)$ is holomorphic.
3. $F(0,w) = w$ for $w \in \mathbb{C}$.

Show that then

$$F(z,w) \;=\; a(z)w + b(z)$$

for $z, w \in \mathbb{C}$, where a and b are entire functions with $a(0) = 1$, $b(0) = 0$, and $a(z) \neq 0$ for $z \in \mathbb{C}$.

Solution. Define $G(z,w) = \frac{F(z,w)-F(z,0)}{F(z,1)-F(z,0)}$. We claim that $G(z,w) = w$ for all z, w. Then we can just take $a(z) = F(z,1) - F(z,0)$ and $b(z) = F(z,0)$ and we will be done. By the injectivity condition, the denominator of $G(z,w)$ is never 0, so for each fixed w, $z \mapsto G(z,w)$ is an entire function. Also note that $G(0,w) = w$ and that $G(z,0) = 0$ for all z and $G(z,1) = 1$ for all z. So the desired condition is verified for $w = 0,1$. Fix $w \neq 1$. Then by the injectivity condition, if $G(z,w) = 1$ for any z, then $w = 1$, and if $G(z,w) = 0$ for any z, then $w = 0$. So $z \mapsto G(z,w)$ is an entire function that misses both 0 and 1, so by Picard's little theorem, $z \mapsto G(z,w)$ is constant. Then the fact that $G(0,w) = w$ implies that $G(z,w) = w$ for all z, so we are done. $\square$

Problem 10. Let $\{f_n\}$ be a sequence of holomorphic functions on $\mathbb{D}$ with the property that

$$F(z) \;:=\; \sum_{n=1}^{\infty} |f_n(z)|^2 \;\leqslant\; 1$$

for all $z \in \mathbb{D}$. Show that the series defining $F(z)$ converges uniformly on compact subsets of $\mathbb{D}$ and that F is subharmonic.

Solution. Since f_n is holomorphic, $|f_n|^2$ is subharmonic. Therefore each $g_N := \sum_{n=1}^{N} |f_n|^2$ is also subharmonic, and we have that g_N increases monotonically to F pointwise. Notice that if subharmonic were replaced by harmonic, we would be done automatically by Harnack's Principle. The following argument is just a modification of the proof of Harnack to work for subharmonic functions, where we rely heavily on the fact that F is bounded and that the g_N are partial sums rather than general subharmonic functions (it's not true in general that an increasing limit of subharmonic functions converges locally uniformly to another subharmonic function).

First, suppose we knew that $g_N \to F$ locally uniformly on $\mathbb{D}$. Then since each g_N is continuous, F also is, and for any disc $B(z_0, r) \subseteq \mathbb{D}$, we have

$$F(z_0) \;=\; \lim_{N\to\infty} g_N(z_0) \;\leqslant\; \lim_{N\to\infty} \frac{1}{2\pi}\int_0^{2\pi} g_N(z_0 + re^{i\theta})\,d\theta \;=\; \frac{1}{2\pi}\int_0^{2\pi} F(z_0 + re^{i\theta})\,d\theta$$

by the monotone convergence theorem (or by uniform convergence on compact sets). So F is continuous and satisfies the sub mean value property, so it is subharmonic.

Now we show local uniform convergence. Fix a compact set $K \subseteq \mathbb{D}$ and $\epsilon > 0$. By compactness, there is a radius $r > 0$ such that $B(z,r) \subseteq \mathbb{D}$ for any $z \in K$. Also by compactness, we can cover K with finitely many balls $B(w_1, r/2) \cup \ldots \cup B(w_k, r/2)$. For any $z \in K$,

$$\lim_{N\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \left(F\left(z + \frac{r}{2}e^{i\theta}\right) - g_N\left(z + \frac{r}{2}e^{i\theta}\right)\right) d\theta \;=\; 0$$

again by the monotone convergence theorem (this is where we need the fact that F is bounded). So let N be large enough so that

$$\max_{1\leqslant j\leqslant k} \frac{1}{2\pi}\int_0^{2\pi} \left(F\left(z_j + \frac{r}{2}e^{i\theta}\right) - g_N\left(z_j + \frac{r}{2}e^{i\theta}\right)\right) d\theta \;<\; \epsilon.$$

Now for any $M > N$, $g_M - g_N = \sum_{n=N+1}^{M} |f_n|^2$ is still a positive subharmonic function (this is where we need the fact that the g_N are partial sums). Therefore it satisfies the "sub Poisson integral formula" (regular Poisson integral formula but with a $\leqslant$ instead of $=$). For any $z \in K$, we have $z \in B(z_j, r/2)$ for some j, so we apply the sub Poisson formula on $B(z_j, r)$ to obtain

$$\begin{aligned} g_M(z) - g_N(z) \;&\leqslant\; \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - |z - z_j|^2}{|(z_j + re^{i\theta}) - z|^2}\left(g_M(z_j + re^{i\theta}) - g_N(z_j + re^{i\theta})\right) d\theta \\ &\leqslant\; \frac{r + |z - z_j|}{r - |z - z_j|}\frac{1}{2\pi}\int_0^{2\pi} (g_M - g_N)(z_j + re^{i\theta})\,d\theta \\ &\leqslant\; \frac{r + r/2}{r - r/2}\cdot\frac{1}{2\pi}\int_0^{2\pi} (F - g_N)(z_j + re^{i\theta})\,d\theta \;<\; 3\epsilon. \end{aligned}$$

This shows that the sequence g_N is uniformly Cauchy on K and therefore converges uniformly to F on K, so $g_N \to F$ locally uniformly on $\mathbb{D}$ and we are done. $\square$

Problem 11. Let $f : \mathbb{D} \to \mathbb{C}$ be an injective and holomorphic function with $f(0) = 0$ and $f'(0) = 1$. Show that then

$$\inf\{|w| : w \notin f(\mathbb{D})\} \leqslant 1$$

with equality if and only if $f(z) = z$ for all $z \in \mathbb{D}$.

Solution. We analyze the situation when $\inf\{|w| : w \notin f(\mathbb{D})\} \geqslant 1$. Then $\mathbb{D} \subseteq f(\mathbb{D})$, and since f is injective, it has a holomorphic inverse $g : \mathbb{D} \to \mathbb{D}$ on the disk. It's clear that $g(0) = 0$ and $g'(0) = 1$, so by the Schwarz lemma (and the fact that $g'(0) = 1$) we must have $g(z) = z$. Thus $f(z) = z$ as well. The original statement follows.

Problem 12. Let f, g, and h be complex-valued functions on $\mathbb{C}$ with

$$f = g \circ h.$$

Show that if h is continuous, and both f and g are holomorphic, then h is holomorphic as well.

Solution. Let B (for bad) be the set of points z for which $g'(h(z)) = 0$. For $z \in \mathbb{C}\backslash B$, we can find an analytic local inverse g_U^{-1} for g on a neighborhood of U of $h(z)$. Thus on U, we can write $h = g_U^{-1} \circ f$, which implies that h is analytic at z. So h is analytic on $\mathbb{C}\backslash B$.

Since g is non-constant, we must have $g'(z) = 0$ only on a discrete set. Furthermore, h is continuous, so in fact B is discrete. But h is continuous so by Riemann's theorem on removable singularities, h must be analytic.

Remark. It's not true in general that the preimage of a discrete set under a continuous function is also discrete (a constant function is a counterexample), so that step takes a bit more work. Let Z denote the zeros of g' and suppose that $h^{-1}(Z)$ has a limit point. Take a convergent sequence z_n with $\{h(z_n)\} \subseteq Z$, so it's discrete. The set $\{h(z_n)\}$ can't be infinite, because its also discrete, so the limit would have to be infinity, but z_n converges to a non-infinite limit z_∞, which is impossible by the continuity of h. So $\{h(z_n)\}$ is a finite set, meaning that there is some subsequence $\{z_{n_k}\}$ converging to z_∞ on which h is constant. But then f is also constant on $\{z_{n_k}\}$, and since f is holomorphic this implies f is a constant, which is a contradiction.

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Problem 1. Suppose $f \in L^1(\mathbb{R})$ satisfies

$$\limsup_{h\to 0} \int_{\mathbb{R}} \left| \frac{f(x+h) - f(x)}{h} \right| dx = 0.$$

Show that $f = 0$ almost everywhere.

Solution. Let $F(x) = \int_{-\infty}^{x} |f(t)|\, dt$. We then consider the difference quotient

$$\begin{aligned}\left|\frac{F(x+h) - F(x)}{h}\right| &= \frac{1}{|h|}\left|\int_{-\infty}^{x} |f(t+h)| - |f(t)|\, dt\right| \\ &\leqslant \int_{-\infty}^{x} \left|\frac{f(t+h) - f(t)}{h}\right| \\ &\leqslant \int_{\mathbb{R}} \left|\frac{f(t+h) - f(t)}{h}\right| dx.\end{aligned}$$

By hypothesis, this last quantity tends to 0 as $h \to 0$. So F is differentiable with derivative 0, and is therefore constant. It follows (by continuity from below) that $\int_{\mathbb{R}} |f(t)|\, dt = 0$, and so $f = 0$ a.e.

Alternate solution. Let $F(x) = \int_{-\infty}^{x} f(t)\, dt$. Since f is integrable, by the Lebesgue differentiation theorem we have that for a.e. $x \in \mathbb{R}$,

$$f(x) \;=\; \lim_{h\to 0} \frac{1}{h} \int_{x}^{x+h} f(t)\, dt \;=\; \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}.$$

So for any two Lebesgue points $x > y$, we have

$$\begin{aligned}|f(x) - f(y)| \;&=\; \lim_{h\to 0} \left| \frac{F(x+h) - F(x)}{h} - \frac{F(y+h) - F(y)}{h} \right| \;=\; \lim_{h\to 0} \left| \int_{y+h}^{x+h} \frac{f(t)}{h}\, dt - \int_{y}^{x} \frac{f(t)}{h}\, dt \right| \\ &=\; \lim_{h\to 0} \left| \int_{y}^{x} \frac{f(t+h) - f(t)}{h}\, dt \right| \;\leqslant\; \limsup_{h\to 0} \int_{\mathbb{R}} \left| \frac{f(t+h) - f(t)}{h}\, dt \right| \;=\; 0.\end{aligned}$$

So f is constant a.e., and since f is also integrable we must have $f = 0$ a.e. $\square$

Problem 2. Given $f \in L^2(\mathbb{R})$ and $h > 0$ we define

$$Q(f,h) \;=\; \int_{\mathbb{R}} \frac{2f(x) - f(x+h) - f(x-h)}{h^2} f(x)\, dx.$$

(a) Show that

$$Q(f,h) \geqslant 0 \quad \text{for all } f \in L^2(\mathbb{R}) \text{ and all } h > 0.$$

(b) Show that the set

$$E \;=\; \{f \in L^2(\mathbb{R}) : \limsup_{h\to 0} Q(f,h) \leqslant 1\}$$

is closed in $L^2(\mathbb{R})$.

Solution.

(a) It suffices to show that

$$\int_{\mathbb{R}} 2f(x)^2 dx \geqslant \int_{\mathbb{R}} f(x)(f(x+h) - f(x-h))dx.$$

Indeed by Cauchy-Schwarz

$$\begin{aligned}\int_{\mathbb{R}} f(x)(f(x+h)-f(x-h))dx &\leqslant \|f\|_2 \cdot \|f(x+h)-f(x-h)\|_2 \\ &\leqslant \|f\|_2 \cdot (\|f(x+h)\|_2 + \|f(x-h)\|_2) \\ &= \|f\|_2 (\|f\|_2 + \|f\|_2) \\ &= 2\|f\|_2^2,\end{aligned}$$

as desired.

(b) Let $g(x) = 2f(x) - f(x+h) - f(x-h)$. Note $g \in L^2$. Using the form of Plancherel that says $\langle f, g\rangle = \left\langle \widehat{f}, \widehat{g}\right\rangle$, we can rewrite

$$Q(f,h) \;=\; \int_{\mathbb{R}} \frac{2\widehat{f}(u) - e^{ihu}\widehat{f}(u) - e^{-ihu}\widehat{f}(u)}{h^2}\overline{\widehat{f}(u)}\,du \;=\; \int_{\mathbb{R}} \frac{2-2\cos(hu)}{h^2}\left|\widehat{f}(u)\right|^2 du.$$

Now let f_n be a sequence in E with $f_n \to f$ in L^2. By passing to a subsequence if necessary, we may also assume that $f_n \to f$ almost everywhere. By Plancherel, we also have $\widehat{f_n} \to \widehat{f}$ in L^2, and by passing to a further subsequence if necessary we can also assume $\widehat{f_n} \to \widehat{f}$ almost everywhere. Then by Fatou's lemma, since $1 - \cos(hu) \geqslant 0$ for all h, u, for each n we have

$$\begin{aligned} 1 &\geqslant \limsup_{h\to 0} \int_{\mathbb{R}} \frac{2-2\cos(hu)}{h^2}\left|\widehat{f_n}(u)\right|^2 du \;\geqslant\; \liminf_{h\to 0}\int_{\mathbb{R}} \frac{2-2\cos(hu)}{h^2}\left|\widehat{f_n}(u)\right|^2 du \\ &\geqslant \int_{\mathbb{R}} \liminf_{h\to 0} \frac{2-2\cos(hu)}{h^2}\left|\widehat{f_n}(u)\right|^2 du \;=\; \int_{\mathbb{R}} u^2 \left|\widehat{f_n}(u)\right|^2 du.\end{aligned}$$

Then by applying Fatou's lemma again, this time in n, we have

$$\int_{\mathbb{R}} u^2\left|\widehat{f}(u)\right|^2 du \;=\; \int_{\mathbb{R}} \liminf_{n\to\infty} u^2\left|\widehat{f_n}(u)\right|^2 du \;\leqslant\; \liminf_{n\to\infty}\int_{\mathbb{R}} u^2\left|\widehat{f_n}(u)\right|^2 du \;\leqslant\; 1,$$

so $u \mapsto u^2\left|\widehat{f}(u)\right|^2$ is integrable. Note we have the estimate

$$\frac{2-2\cos(hu)}{h^2} \;=\; u^2\frac{2-2\cos(hu)}{(hu)^2} \;\leqslant\; 5u^2$$

for all $h, u \in \mathbb{R}$ because $t \mapsto \frac{2-2\cos(t)}{t^2}$ is bounded by 5 for all real t. Therefore we have

$$\frac{2-2\cos(hu)}{h^2}\left|\widehat{f}(u)\right|^2 du \;\leqslant\; 5u^2\left|\widehat{f}(u)\right|^2 du$$

for all $h, u \in \mathbb{R}$, where the function on the right is integrable, so by the dominated convergence theorem we have

$$1 \;\geqslant\; \int_{\mathbb{R}} u^2\left|\widehat{f}(u)\right|^2 du \;=\; \int_{\mathbb{R}} \lim_{h\to 0}\frac{2-2\cos(hu)}{h^2}\left|\widehat{f}(u)\right|^2 du \;=\; \lim_{h\to 0}\int_{\mathbb{R}} \frac{2-2\cos(hu)}{h^2}\left|\widehat{f}(u)\right|^2 du \;=\; \lim_{h\to 0} Q(f,h),$$

so $f \in E$ and thus E is closed in L^2. $\square$

Problem 3. Suppose $f \in L^1(\mathbb{R})$ satisfies

$$\limsup_{\epsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)f(y)|}{|x-y|^2+\epsilon^2}\,dx\,dy \;<\; \infty.$$

Show that $f = 0$ almost everywhere.

Solution. By applying monotone convergence to the limit (after using Tonelli's theorem to convert the double integral into an integral over $\mathbb{R}^2$), we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)f(y)|}{|x-y|^2}\,dx\,dy < \infty.$$

If f is not zero almost everywhere, then f has a Lebesgue point a with $|f(a)| > 0$. We have

$$\int_{a-r}^{a+r}\int_{a-r}^{a+r} \frac{|f(x)f(y)|}{|x-y|^2}\,dx\,dy \geqslant \int_{a-r}^{a+r}\int_{a-r}^{a+r} \frac{|f(x)f(y)|}{(2r)^2}\,dx\,dy = \left(\frac{1}{2r}\int_{a-r}^{a+r} |f(x)|\,dx\right)^2.$$

By the Lebesgue differentiation theorem, the right side tends to $f(a)^2$ as $r \to 0^+$. On the other hand, the left-most integral must tend to 0, since the integrand is in L^1 (in fact L^1_{loc} is enough). This is a contradiction, so we must have $f = 0$ a.e.

Problem 4.

(a) Fix $1 < p < \infty$. Show that

$$f \mapsto [Mf](x,y) \;=\; \sup_{r>0,\rho>0} \frac{1}{4r\rho}\int_{-r}^{r}\int_{-\rho}^{\rho} f(x+h, y+\ell)\,dh\,d\ell$$

is bounded on $L^p(\mathbb{R}^2)$.

(b) Show that

$$[A_r f](x,y) \;=\; \frac{1}{4r^3}\int_{-r}^{r}\int_{-r^2}^{r^2} f(x+h, y+\ell)\,dh\,d\ell$$

converges to f a.e. in the plane as $r \to 0$.

Solution.

(a) For $g : \mathbb{R} \to \mathbb{R}$, let

$$Mg(x) := \sup_{r>0} \frac{1}{2r}\int_{-r}^{r} |g(x+h)|\,dh$$

be the usual maximal operator. For $x \in \mathbb{R}$, define $f_x(y) := f(x,y)$. Since $f \in L^p(\mathbb{R}^2)$, $f_x \in L^p(\mathbb{R})$ for a.e. $x \in \mathbb{R}$ (this is proved by Tonelli's theorem). Therefore by the usual Hardy-Littlewood maximal theorem, we have

$$\int |Mf_x(y)|^p\,dy \;\lesssim\; \int |f_x(y)|^p\,dy$$

for a.e. $x \in \mathbb{R}$. Now, for each $y \in \mathbb{R}$, define $g_y(x) := Mf_x(y)$. Tonelli's theorem and the above inequality show that $g_y \in L^p(\mathbb{R})$ for a.e. $y \in \mathbb{R}$:

$$\begin{aligned}\int\left(\int |g_y(x)|^p\,dx\right)dy \;&=\; \iint |Mf_x(y)|^p\,dy\,dx \\ &\lesssim\; \iint |f_x(y)|^p\,dy\,dx \;=\; ||f||^p_{L^p(\mathbb{R}^2)} \;<\; \infty.\end{aligned}$$

Therefore using Hardy-Littlewood again we have

$$\int |Mg_y(x)|^p\,dx \;\lesssim\; \int |g_y(x)|^p\,dx$$

for a.e. $y \in \mathbb{R}$. Now note that we have

$$\begin{aligned}
[Mf](x,y) &\leqslant \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} \sup_{\rho>0} \frac{1}{2\rho} \int_{-\rho}^{\rho} |f(x+h, y+\ell)| \, d\ell \, dh \quad \text{by Tonelli} \\
&= \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} M f_{x+h}(y) \, dh \\
&= M g_y(x).
\end{aligned}$$

So by the above work we conclude that

$$\iint |[Mf](x,y)|^p \, dx \, dy \;\leqslant\; \iint |Mg_y(x)|^p \, dx \, dy \;\lesssim\; \iint |g_y(x)|^p \, dx \, dy \;\lesssim\; ||f||^p_{L^p(\mathbb{R}^2)} . \quad \square$$

(b) We mimic the proof of the Lebesgue differentiation theorem. Define

$$T_r f(x,y) \;:=\; \frac{1}{4r^3} \int_{-r}^{r} \int_{-r^2}^{r^2} |f(x,y) - f(x+h, y+\ell)| \; dh \, d\ell, \qquad Tf(x,y) \;:=\; \limsup_{r\to 0} T_r f(x,y).$$

It suffices to show that $Tf = 0$ a.e., and for that it suffices to show that for any fixed $\alpha > 0$, $\lambda\{(x,y) : Tf(x,y) \geqslant \alpha\} = 0$ (where λ denotes 2-dimensional Lebesgue measure). Fix $\alpha > 0$ and $\epsilon > 0$. Note that the desired result is obviously true for continuous functions. Since continuous functions are dense in L^p, write $f = g + u$ where g is continuous and $||u||_{L^p} < \epsilon$. The operator T_r is subadditive, so $T_r f \leqslant T_r g + T_r u$, and taking $r \to 0$ gives that $Tf \leqslant Tu$.

We now estimate the quantity $\lambda\{(x,y) : Tu(x,y) \geqslant \alpha\}$. Notice that

$$T_r u(x,y) \;\leqslant\; \frac{1}{4r^3} \int_{-r}^{r} \int_{-r^2}^{r^2} (|u(x,y)| + |u(x+h, y+\ell)|) \; dh \, d\ell \;\leqslant\; |u(x,y)| + [Mu](x,y).$$

So $\{(x,y) : Tu(x,y) \geqslant \alpha\} \subseteq \{(x,y) : |u(x,y)| \geqslant \alpha/2\} \cup \{(x,y) : Mu(x,y) \geqslant \alpha/2\}$, which implies that

$$\begin{aligned}
\lambda\{(x,y) : Tu(x,y) \geqslant \alpha\} &\leqslant \lambda\{(x,y) : |u(x,y)| \geqslant \alpha/2\} + \lambda\{(x,y) : Mu(x,y) \geqslant \alpha/2\} \\
&\leqslant \frac{||u||^p_{L^p}}{(\alpha/2)^p} + \frac{||Mu||^p_{L^p}}{(\alpha/2)^p} \quad \text{by Chebyshev's inequality} \\
&\leqslant \frac{\epsilon^p 2^p}{\alpha^p} + \frac{C^p \epsilon^p 2^p}{\alpha^p} \quad \text{where } C \text{ is the constant from part (a) on the boundedness of } f \mapsto [Mf].
\end{aligned}$$

Since $Tf \leqslant Tu$, we also have $\lambda\{(x,y) : Tf(x,y) \geqslant \alpha\} \leqslant \frac{\epsilon^p 2^p}{\alpha^p} + \frac{C^p \epsilon^p 2^p}{\alpha^p}$. Now the left side does not depend on ϵ, so we can take $\epsilon \to 0$ and conclude that $\lambda\{(x,y) : Tf(x,y) \geqslant \alpha\} = 0$. $\square$

Problem 5. Let μ be a real-valued Borel measure on $[0,1]$ such that

$$\int_0^1 \frac{1}{x+t} d\mu(t) = 0$$

for all $x > 1$. Show that $\mu = 0$.

Solution. Let S denote the real span of the functions of the form $\frac{1}{x+t}$ for $x > 1$ in $C([0,1])$. We apply Stone-Weirstrass to show that S is dense in $C([0,1])$. For $x_0 \neq x_1 > 1$, we have

$$\frac{1}{x_0+t} \cdot \frac{1}{x_1+t} = \frac{1}{x_1 - x_0} \left(\frac{1}{x_0+t} - \frac{1}{x_1+t} \right),$$

which lies in S. We also have that

$$\frac{1}{x+t} \cdot \frac{1}{x+t+\epsilon} \to \frac{1}{(x+t)^2}$$

uniformly on $[0,1]$ as $\epsilon \to 0^+$. Thus $\frac{1}{(x+t)^2}$ lies in $\overline{S}$ for $t > 1$. Therefore the product of any two elements in S lies in $\overline{S}$. This implies that $\overline{S}$ is closed under multiplication. Indeed if f and g lie in $\overline{S}$ then we have sequences $f_i \to f$ and $g_i \to g$ uniformly with $f_i, g_i \in S$. Since f and g are bounded on $[0,1]$, we have that $f_i g_i \to fg$ uniformly, and so $fg \in \overline{S}$.

Hence $\overline{S}$ is an algebra. It's clear that $\overline{S}$ separates points, and that there is no point x_0 such every function in $\overline{S}$ vanishes at x_0. Thus $\overline{S} = C([0,1])$.

So we have that $\int_0^1 f(t)d\mu(t) = 0$ for all f in S, and by density for all f in $C([0,1])$. Note that μ is a finite measure, otherwise $\int_0^1 \frac{1}{2+t}$ would be either ∞ or $-\infty$. By the Riesz representation theorem, we must have $\mu = 0$.

Remark. We used a slighly non-standard (although well-known) version of Stone-Weirstrass here. It's easy to avoid this, and instead show that the constant function 1 lies in $\overline{S}$. For instance, the functions $\frac{x}{x+t}$ converge uniformly to 1 on $[0,1]$ as $x \to \infty$.

Alternate Solution. Let $a_k = \int_0^1 t^k d\mu(t)$. For $x \in (0,1)$ we have

$$0 = \int_0^1 \frac{1}{1/x+t} d\mu(t) = \int_0^1 \frac{x}{1+tx} d\mu(t) = \int_0^1 \left(\sum_{k=0}^{\infty} (-1)^k t^k x^{k+1} \right) d\mu(t) = \sum_{k=0}^{\infty} (-1)^k a_k x^{k+1},$$

where swapping the order of summation and integration can be justified by Fubini-Tonelli, after noting that μ is finite (to prove Fubini-Tonelli for signed measures, one looks at a Jordan decomposition and applies Fubini separately to each piece). This latter sum is a power a series in x which is identically 0 for $x \in (0,1)$, so each a_k must equal 0. By taking linear combinations of the a_k, we see that $\int p(x)d\mu(t) = 0$ for any polynomial p. But polynomials are dense in $C([0,1])$, and so $\mu = 0$ by the Riesz representation theorem.

Problem 6. Let $\mathbb{T}$ denote the unit circle in the complex plane and let $\mathcal{P}(\mathbb{T})$ denote the space of Borel probability measures on $\mathbb{T}$ and $\mathcal{P}(\mathbb{T} \times \mathbb{T})$ denote the space of Borel probability measures on $\mathbb{T} \times \mathbb{T}$. Fix $\mu, \nu \in \mathcal{P}(\mathbb{T})$ and define

$$\mathcal{M} \;=\; \left\{ \gamma \in \mathcal{P}(\mathbb{T} \times \mathbb{T}) : \iint\limits_{\mathbb{T}\times\mathbb{T}} f(x)g(y)\, d\gamma(x,y) = \int_{\mathbb{T}} f(x)\, d\mu(x) \cdot \int_{\mathbb{T}} g(y)\, d\nu(y) \quad \text{for all } f, g \in C(\mathbb{T}) \right\}.$$

Show that $F : \mathcal{M} \to \mathbb{R}$ defined by

$$F(\gamma) \;=\; \iint\limits_{\mathbb{T}\times\mathbb{T}} \sin^2\left(\frac{\theta-\phi}{2}\right) d\gamma(e^{i\theta}, e^{i\phi})$$

achieves its minimum on $\mathcal{M}$.

Solution (trick). Note that $\sin^2\left(\frac{\theta-\phi}{2}\right) = \frac{1}{2}(1 - \cos\theta\cos\phi + \sin\theta\sin\phi)$, which is just a sum of three functions of the form $f(\theta)g(\phi)$ where each $f, g \in C(\mathbb{T})$. So by definition of $\mathcal{M}$, $F(\gamma)$ is actually independent of γ, so F is constant on $\mathcal{M}$ and therefore obviously achieves its minimum. □

Alternate solution (idea generalizes to other similar problems). Let $I = \inf_{\gamma \in M} F(\gamma)$. Let γ_n be a sequence of measures in $\mathcal{M}$ such that $F(\gamma_n) \to I$ as $n \to \infty$. Since $\mathbb{T} \times \mathbb{T}$ is compact, one version of the Riesz representation theorem says that the space of complex Borel measures on $\mathbb{T} \times \mathbb{T}$ is isomorphic to $C(\mathbb{T} \times \mathbb{T})^*$, and the operator norm of a measure is its total variation. Therefore $\mathcal{P}(\mathbb{T} \times \mathbb{T})$ is a subset of the unit ball in $C(\mathbb{T} \times \mathbb{T})^*$. By the Banach-Alaoglu theorem, this unit ball is weak-$*$ compact, and since $C(\mathbb{T} \times \mathbb{T})$ is separable, it is actually sequentially compact. Thus there is a subsequence $\{\gamma_{n_k}\}$ that weak-$*$ converges to some complex Borel measure γ in the unit ball of $C(\mathbb{T} \times \mathbb{T})^*$.

We claim that γ is the minimizer of F. We need to verify that $\gamma \in \mathcal{M}$ and that $F(\gamma) = I$. Note that γ is a probability measure because

$$\gamma(\mathbb{T} \times \mathbb{T}) \;=\; \iint\limits_{\mathbb{T}\times\mathbb{T}} 1\, d\gamma \;=\; \lim_{n\to\infty} \iint\limits_{\mathbb{T}\times\mathbb{T}} 1\, d\gamma_n \;=\; 1$$

by weak-$*$ convergence because 1 is continuous. To show that $\gamma \in \mathcal{M}$, let $f, g \in C(\mathbb{T})$ be fixed. Then the function $(x, y) \mapsto f(x)g(y)$ is in $C(\mathbb{T} \times \mathbb{T})$, so by weak-$*$ convergence we have

$$\iint\limits_{\mathbb{T}\times\mathbb{T}} f(x)g(y)\, d\gamma(x,y) \;=\; \lim_{n\to\infty} \iint\limits_{\mathbb{T}\times\mathbb{T}} f(x)g(y)\, d\gamma_n(x,y) \;=\; \int_{\mathbb{T}} f(x)\, d\mu(x) \cdot \int_{\mathbb{T}} g(y)\, d\nu(y).$$

Thus $\gamma \in \mathcal{M}$. To show that $F(\gamma) = I$, just note that $\sin^2\left(\frac{\theta-\phi}{2}\right)$ is also continuous on $\mathbb{T} \times \mathbb{T}$, so weak-$*$ convergence implies $F(\gamma) = \lim_{n\to\infty} F(\gamma_n)$. $\square$

Problem 7. Let $F : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be jointly continuous and holomorphic in each variable separately. Show that $z \mapsto F(z, z)$ is holomorphic.

Solution. Fix an $R > 0$ then if $|w| < R$ then we have from the Residue Theorem that

$$F(z,w) = \frac{1}{2\pi i}\int_{|\xi|=R} \frac{F(\xi,w)}{(\xi - z)} d\xi$$

Now using the residue formula again gives

$$F(\xi,w) = \frac{1}{2\pi i}\int_{|\eta|=R} \frac{F(\xi,\eta)}{(\eta - w)} d\eta$$

so it follows

$$F(z,w) = -\frac{1}{4\pi^2}\int_{|\xi|=R}\int_{|\eta|=R} \frac{F(\xi,\eta)}{(\xi - z)(\eta - w)} d\xi d\eta$$

i.e.

$$F(z,z) = -\frac{1}{4\pi^2}\int_{|\xi|=R}\int_{|\eta|=R} \frac{F(\xi,\eta)}{(\xi - z)(\eta - w)} d\xi d\eta$$

Now let $R \subset B_R(0)$ be a reactangle, then

$$\int_{\partial R} F(z,z)dz = \frac{1}{4\pi^2}\int_{\partial R}\int_{\theta=0}^{2\pi}\int_{\varphi=0}^{2\pi} \frac{F(Re^{i\varphi}, Re^{i\theta})}{(Re^{i\varphi} - z)(Re^{i\theta} - z)} R^2 e^{i(\varphi+\theta)} d\varphi d\theta dz$$

$$= \frac{1}{4\pi^2}\int_{\theta=0}^{2\pi}\int_{\varphi=0}^{2\pi}\int_{\partial R} \frac{F(Re^{i\varphi}, Re^{i\theta})}{(Re^{i\varphi} - z)(Re^{i\theta} - z)} R^2 e^{i(\varphi+\theta)} d\varphi d\theta dz = 0$$

since the integrand is holomorphic in z and the swap of integrals is justified by continuity of the integrand which on $z \in B_R(0)$ (just parametrize R and convert the contour integral into a regular integral then undo the parametrization on the last step for the rectangle). And as $F(z, z)$ is continuous, we deduce from Morrera's that $F(z, z)$ is holomorphic on $B_R(0)$. And as R was arbitrary we conclude $F(z, z)$ is entire. $\square$

Problem 8. Determine the supremum of

$$\left|\frac{\partial u}{\partial x}(0,0)\right|$$

among all harmonic functions $u : \mathbb{D} \to [0, 1]$.

Solution. The answer is $2/\pi$. Since $\mathbb{D}$ is simply connected, any such u is the real part of an analytic function $f = u + iv : \mathbb{D} \to S := \{z \in \mathbb{C} : 0 \leqslant \mathrm{Re}(z) \leqslant 1\}$. Adding a pure imaginary constant doesn't change anything, so we can assume $f(0)$ is real. We have $f' = u_x + iv_y$, so we want to bound $\mathrm{Re}(f'(0))$. Since we can pre-compose f with a rotation without changing the absolute value of f' or changing the codomain of f, this is the same as bounding $|f'(0)|$. This shows that the desired supremum is the same as the supremum of $|f'(0)|$ over all $f : \mathbb{D} \to S$ holomorphic with $f(0) \in \mathbb{R}$. Let f be such a function. Let $T : S \to \mathbb{D}$ be the conformal map given by

$$T(z) \;=\; \frac{\exp(i\pi z) - i}{\exp(i\pi z) + i}.$$

Let $\alpha = T(f(0))$ and let $\psi(z) = \frac{z-\alpha}{1-\overline{\alpha}z}$ be the automorphism of $\mathbb{D}$ that sends α to 0. Then $g = \psi \circ T \circ f$ is a holomorphic function $\mathbb{D} \to \mathbb{D}$ with $g(0) = 0$. So by the Schwarz lemma we have $|g'(0)| \leqslant 1$. Now we compute

$$\begin{aligned} |g'(0)| &= |\psi'(\alpha)||T'(f(0))||f'(0)| = \frac{1}{1-|\alpha|^2}|T'(f(0))||f'(0)| \geqslant |T'(f(0))||f'(0)| \\ &\geqslant 2\pi \left|\frac{\exp(i\pi f(0))}{(\exp(i\pi f(0)) + i)^2}\right| = \left|\frac{2\pi}{2i + 2i\,\mathrm{Im}(\exp(i\pi f(0)))}\right| \geqslant \frac{\pi}{2} \end{aligned}$$

because $\exp(i\pi f(0))$ lies on the top half of the unit circle because $f(0) \in [0,1]$. Therefore we conclude

$$1 \geqslant |g'(0)| \geqslant \frac{\pi}{2}|f'(0)|,$$

which shows that $2/\pi$ is an upper bound for the desired quantity. Now taking

$$f(z) = T^{-1}(z) = \frac{1}{i\pi}\log\left(\frac{i+iz}{1-z}\right),$$

where the log here is well-defined because $\frac{i+iz}{1-z} \in \mathbb{H}$ for all $z \in \mathbb{D}$, it's easy to calculate that $|f'(0)| = 2/\pi$, so it must be the supremum and it's actually attained. $\square$

Alternate solution. Fix $0 < R < 1$ and define $u_R(z) := u(Rz) \in C(\overline{D})$ and is harmonic on $\mathbb{D}$. So by Poisson Integral Formula we have for $0 < r < 1$ that

$$u_R(re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{|e^{i\theta} - re^{i\varphi}|^2} u(Re^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2} u(Re^{i\theta})d\theta$$

so now we have by identifying $re^{i\varphi} = r\cos(\varphi) + ir\sin(\varphi) = x + iy$ that for $x > 0$

$$u_R(x,0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-x^2}{1-2x\cos(\theta)+x^2} u(Re^{i\theta})d\theta$$

Hence

$$\frac{\partial u_R}{\partial x}(x,0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{-2x(1-2x\cos(\theta)+x^2) - (-2\cos(\theta)+2x)(1-x^2)}{(1-2x\cos(\theta)+x^2)^2} u(Re^{i\theta})d\theta$$

where differentiating under the integral sign is fine since $R < 1$ so the Poisson Kernel is smooth in $\overline{B_R(0)}$. In particular,

$$\frac{\partial u_R}{\partial x}(0,0) = \frac{1}{2\pi}\int_0^{2\pi} 2\cos(\theta) u_R(Re^{i\theta})d\theta$$

so we have from $0 \leqslant u \leqslant 1$ that

$$\frac{\partial u_R}{\partial x}(0,0) \leqslant \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 2\cos(\theta)d\theta = \frac{2}{\pi}$$

and

$$\frac{\partial u_R}{\partial x} = R\frac{\partial u}{\partial x} \Rightarrow \frac{\partial u}{\partial x}(0,0) \leqslant \frac{2}{\pi}$$

so define $f : S^1 \to [0,1]$ via

$$f = \begin{cases} 1 \text{ on } \theta \in [-\pi/2, \pi/2] \\ 0 \text{ else} \end{cases}$$

then

$$u(z) := \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta} - z|} f(e^{i\theta})d\theta$$

is the desired harmonic function with these properties. Note that the usual proof of if $f \in C(\partial\mathbb{D})$ then the usual proof of

$$\lim_{r\to 1} u(re^{i\theta}) := \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|} f(e^{i\theta})d\theta = f(e^{i\theta})$$

extends to bounded functions at every point of continuity of f. Therefore, $u(re^{i\theta}) \to \chi_{[-\pi/2,\pi/2]}$ everywhere except at $\theta = \pm\pi/2$ i.e. u obtains the boundary data f a.e., so by our above computation our function u obtains the boundary data. $\square$

Alternate solution. By the mean value property we have for any $0 < r < 1$

$$\frac{\partial u}{\partial x}(0,0) = \frac{1}{\pi r^2}\int_{B_r(0)} \frac{\partial u}{\partial x}(z)dA(z) = \frac{1}{\pi r^2}\int_{\partial B_r(0)} un_1 d\sigma$$

where n_1 is the first component of the unit normal on $\partial B_r(0)$ i.e. x/r.

$$= \frac{1}{\pi r^2}\int_{\theta=0}^{2\pi} ru(re^{i\theta})\cos(\theta)d\theta = \frac{1}{\pi r}\int_{\theta=0}^{2\pi} u(re^{i\theta})\cos(\theta)d\theta$$

Since $0 \leqslant u \leqslant 1$ we deduce

$$\frac{\partial u}{\partial x}(0,0) \leqslant \frac{1}{\pi r}\int_{\theta=-\pi/2}^{\pi/2} \cos(\theta)d\theta = \frac{2}{\pi r}$$

So sending $r \to 1$ we deduce that

$$\frac{\partial u}{\partial x}(0,0) \leqslant \frac{2}{\pi}$$

And from construction we see that the max must be 1 on $[-\pi/2, \pi/2]$ and 0 else, so we use the same function as before. $\square$

Problem 9. Consider the formal product

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^z\left(1-\frac{z}{n}\right).$$

(a) Show that the product converges for any $z \in (-\infty, 0)$.

(b) Show that the resulting function extends from this interval to an entire function of $z \in \mathbb{C}$.

Solution.

(a) Notice if $a_n := (1+\frac{1}{n})^z(1-\frac{z}{n})$ then $a_n \to 1$ as $n \to \infty$, so by looking at the tail of the product if necessary, we can assume $a_n \in B_{1/2}(1)$. This means we can define the standard complex logarithim i.e. $\log(z)$ where $\theta \in (-\pi, \pi]$ and $\log(a_n)$ is well defined. So now by taking logs we see that by taking limits

$$\log(\prod_{n=1}^{\infty} a_n) = \sum_{n=1}^{\infty}\log(a_n) = \sum_{n=1}^{\infty}\log(a_n - 1 + 1) = \sum_{n=1}^{\infty}(a_n - 1) + O((a_n-1)^2)$$

so the product converges iff

$$\sum_{n=1}^{\infty}(a_n - 1)$$

converges. Now observe

$$\sum_{n=1}^{\infty}|(1+\frac{1}{n})^z(1-\frac{z}{n}) - 1|$$

so observe that

$$\frac{}{x}(1+x)^z = z(1+x)^{z-1}$$

so Taylor Expansion at $x = 0$ gives

$$(1+x)^z = 1 + zx + O(x^2)$$

so we have

$$(1+\frac{1}{n})^z(1-\frac{z}{n}) = (1-\frac{z}{n})(1+\frac{z}{n}) + O_{|z|}(1/n^2) = 1 - \frac{z^2}{n^2} + O_{|z|}(1/n^2)$$

Therefore,

$$\sum_{n=1}^{\infty} |(1+\frac{1}{n})^z(1-\frac{z}{n} - 1)| = \sum_{n=1}^{\infty} \frac{|z|^2}{n^2} + O_{|z|}(1/n^2)$$

which converges. $\square$

(b) For the second part, notice that for any $z \in \mathbb{C}$ we can define the complex log for a_n so we have $(1+1/n)^z = \exp(z\log(1+1/n))$ is a well defined holomorphic function (where we are using the standard log branch). And by our earlier computations the sum converges locally uniformly on compact subsets of $\mathbb{C}$ so

$$\prod_{n=N}^{M} a_n = \exp(\sum_{n=N}^{M} (a_n - 1) + O((a_n-1)^2)) = \exp(\sum_{n=N}^{M} O(\frac{|z|}{n^2}))$$

so the product converges locally uniformly, hence the limit is holomorphic. $\square$

Problem 10. Let $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and let $\Omega = \mathbb{C}^*\backslash\{0,1\}$. Let $f : \Omega \to \Omega$ be a holomorphic function.

(a) Prove that if f is injective then $f(\Omega) = \Omega$.

(b) Make a list of all such injective functions f.

Solution. Part (a) follows from part (b) by just examining the list of all possible functions and observing that each of them is surjective. For part (b) we first consider the same problem on a modified region $\widetilde{\Omega} := \mathbb{C}^*\backslash\{0,\infty\}$. Let $g : \widetilde{\Omega} \to \widetilde{\Omega}$ be injective and holomorphic. First we show that the injectivity implies that when considered as a function on all of $\mathbb{C}^*$, g has at worst simple poles at 0 and ∞ (i.e. g has either a removable singularity or a simple pole at 0 and ∞). Essential singularities are impossible by the big Picard theorem. To show that higher order poles are impossible, suppose g has a pole of order $\geqslant 2$ at 0 (the argument for ∞ is the same). Then $1/g$ has a zero of order $\geqslant 2$ at 0. Let γ be a small circle around the origin; then the argument principle says that $(1/g)(\gamma)$ winds twice around 0. Thus there is a neighborhood U of 0 such that $(1/g)(\gamma)$ winds at least twice around every point of U, and by the argument principle again, this means that g achieves every value in U at least twice inside of γ. This contradicts g being injective unless it happens to be the case that every value in U is achieved by g at one point with multiplicity 2. But this is impossible because if $g(z_0) = w_0$ with multiplicity 2, then g' vanishes at z_0. So if the above situation happened, then g' would be identically zero on $(g')^{-1}(U)$, which is an open set, so by uniqueness of analytic continuation this would imply that g' is identically zero, which is also a contradiction. Thus we conclude that g has at worst simple poles at 0 and ∞.

Therefore we have the representation $g(z) = a/z + b + cz$ for some $a, b, c \in \mathbb{C}$. But note that by hypothesis, $g(z)$ is never 0 for $z \in \widetilde{\Omega}$. The equation $a/z + b + cz = 0$ always has a nonzero, non-infinite solution if $a \neq 0 \neq c$, so we must have $a = 0$ or $c = 0$. And in either case, we must then also have $b = 0$ to avoid achieving 0. So the only possible functions g are $g(z) = az$ and $g(z) = a/z$ with $a \neq 0$.

Now let $f : \Omega \to \Omega$ be injective and holomorphic. This induces an injective holomorphic function $g = T^{-1}fT : \widetilde{\Omega} \to \widetilde{\Omega}$ where $T(z) = z/(z+1)$ is an automorphism of $\mathbb{C}^*$ sending 0 to 0 and ∞ to 1. Therefore by the above we have

$$g(z) = \frac{f(z/(z+1))}{f(z/(z+1)) - 1} = az \text{ or } \frac{a}{z}.$$

After simplifying everything and changing variables $w = z/(z+1)$ we find that the only possibilities for f are

$$f(w) \; = \; 1 + \frac{w-1}{(a-1)w+1}, \qquad f(w) \; = \; 1 + \frac{w}{(a-1)w-a} \quad \text{for some } a \neq 0.$$

Since az and a/z are both surjective as maps $\widetilde{\Omega} \to \widetilde{\Omega}$, and we got the possibilities for f by composing with conformal maps, it's clear that both of these possibilities are surjective as maps from $\Omega \to \Omega$. $\square$

Comment Instead of using the big Picard theorem as above, we can cite the much simpler Casorati-Weierstrass theorem.

Alternate solution. Let

$$\varphi(z) := \frac{z}{z-1}$$

then φ is a Mobius Transformation such that $\varphi(0) = 0$, $\varphi(1) = \infty$, and $\varphi(\infty) = 1$. In particular, if $U := \mathbb{C}\backslash\{0\}$ then $\varphi(U) =$ so

$$g(z) := \varphi^{-1} \circ f \circ \varphi : U \to U$$

And since φ is conformal it suffices to show that if g is injective then it is surjective.

Now g is a holomorphic function on $\mathbb{C}\backslash\{0\}$. By Great Picard Theorem the singularity at 0 is either removable or a pole.

Removable Singularity Case If $g(z)$ has a removable singularity at 0 then g extends to be an injective entire function. Indeed, if $g(0) = g(w)$ for some $w \neq 0$, the open mapping theorem implies that g is not injective on $\mathbb{C}\backslash\{0\}$ (since a small ball around 0 and around w maps to a ball around $g(0) = g(w)$). Therefore, $g(z)$ is an injective entire function so g is a linear function i.e. $g(z) = az + b$, so it follows that g is surjective. And as $g(z) \neq 0$ for $z \neq 0$ it follows that $b = 0$ so $g(z) = az$.

Pole Case If $g(z)$ has a pole at $z = 0$, then $1/g$ has a removable singularity at $z = 0$ and since g does not map to 0, so we see $1/g$ is an injective holomorphic function on $\mathbb{C}\backslash\{0\}$ that has a removable singularity at $z = 0$. By applying our previous case, we deduce that $1/g(z) = az + b \Rightarrow g(z) = \frac{1}{az+b}$. As $g(z)$ has a pole at $z = 0$ we must have $b = 0$ and $a \neq 0$, so $g(z) = 1/(az)$ which is a Mobius Transformation and $g(0) = \infty$ with $g(\infty) = 0$, so it follows from Mobius Transformations being conformal that g is surjective on U.

By our previous considerations we must have

$$g(z) = 1/(az) \text{ or } az$$

for some $a \in \mathbb{C}$ that is not 0. So by undoing our mobius transformations we deduce that

$$f(z) = \varphi \circ (1/az) \circ \varphi^{-1} \text{ or } \varphi \circ (az) \circ \varphi^{-1}. \quad \square$$

Problem 11. For $R > 1$ let A_R be the annulus $\{1 < |z| < R\}$. Assume there is a conformal mapping F from A_{R_1} onto A_{R_2}. Prove that $R_1 = R_2$.

Solution. See Spring 2017 #7.

Problem 12. Let $f(z)$ be bounded and holomorphic on the unit disc $\mathbb{D}$. Prove that for any $w \in \mathbb{D}$ we have

$$f(w) \; = \; \frac{1}{\pi} \int_{\mathbb{D}} \frac{f(z)}{(1-\bar{z}w)^2} \, dA(z),$$

where $dA(z)$ means integration with respect to Lebesgue measure.

Solution. Consider f as an element of the Bergman space $A^2(\mathbb{D}) := \left\{ f : \mathbb{D} \to \mathbb{C} \text{ holomorphic} : \int_{\mathbb{D}} |f(z)|^2 \, dA(z) < \infty \right\}$. This is a Hilbert space with inner product

$$\langle f, g \rangle \;=\; \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA(z)$$

and orthonormal basis $\left\{ z \mapsto \sqrt{\frac{n+1}{\pi}} z^n \right\}_{n=0}^{\infty}$ (It's easy to check that these are actually an inner product and orthonormal basis). For each fixed $w \in \mathbb{D}$, we first show the map $f \mapsto f(w)$ is a bounded linear functional on A^2. We have

$$|f(w)| \;=\; \left| \frac{1}{\pi \left(\frac{1-|w|}{2}\right)^2} \int_{B(w,(1-|w|)/2)} f(z)\, dA(z) \right| \;\lesssim\; \left(\int_{B(w,(1-|w|)/2)} |f(z)|^2 \, dA(z) \right)^{1/2} \;\leqslant\; ||f||_{A^2}$$

where the equality is by the mean value property of holomorphic functions and the first inequality is by Cauchy-Schwarz. Thus $f \mapsto f(w)$ is bounded, and it's clearly linear.

Thus by the Riesz representation theorem, for each $w \in \mathbb{D}$ there is a function $g_w \in A^2$ such that

$$f(w) \;=\; \langle f, g_w \rangle \;=\; \int_{\mathbb{D}} f(z)\overline{g_w(z)}\, dA(z)$$

for all $f \in A^2$. So we just need to show that $g_w(z) = \frac{1}{\pi(1-\overline{w}z)^2}$. By definition of the functions g_w, for any z we have

$$g_w(z) \;=\; \langle g_w, g_z \rangle \;=\; \sum_{n=0}^{\infty} \langle g_w, e_n \rangle \overline{\langle g_z, e_n \rangle} \quad \text{by Parseval (where } \{e_n\} \text{ is the orthnormal basis mentioned above)}$$

$$= \sum_{n=0}^{\infty} \overline{\langle e_n, g_w \rangle} \langle e_n, g_z \rangle \;=\; \sum_{n=0}^{\infty} \overline{e_n(w)} e_n(z) \;=\; \sum_{n=0}^{\infty} \frac{1}{\pi}(n+1)(\overline{w}z)^n \;=\; \frac{1}{\pi(1-\overline{w}z)^2}. \quad \square$$

Alternative Solution

If $w = 0$ this is the mean value property for analytic functions, so assume $w \neq 0$. Let

$$dz = dx + idy, \; d\overline{z} = dx - idy;$$

then

$$d\overline{z} \wedge dz = 2idx \wedge dy.$$

Also let

$$\frac{\partial g}{\partial z} = \frac{1}{2}\left(\frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y} \right),$$

$$\frac{\partial g}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y} \right),$$

for any function g. Then

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = \frac{\partial g}{\partial z}dz + \frac{\partial g}{\partial \overline{z}}d\overline{z}.$$

Now, since f is analytic, we have

$$\frac{\partial}{\partial \overline{z}} \left\{ \frac{f(z)}{1 - w\overline{z}} \right\} = \frac{wf(z)}{(1-w\overline{z})^2}.$$

Thus, the 2-form in the integrand equals

$$\frac{f(z)dx \wedge dy}{(1-w\overline{z})^2} = \frac{1}{2i}dF,$$

where F is the 1-form

$$F = \frac{f(z)dz}{w(1 - w\overline{z})}.$$

Therefore, by Stokes' theorem,

$$\frac{1}{\pi}\int_{\mathbb{D}} \frac{f(z)dx \wedge dy}{(1 - w\overline{z})^2} = \frac{1}{2\pi i}\int_{\mathbb{D}} dF = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} F = \frac{1}{2\pi i w}\int_{\partial\mathbb{D}} \frac{f(z)dz}{1 - w\overline{z}}$$

$$= \frac{1}{2\pi i w}\int_{\partial\mathbb{D}} \frac{zf(z)}{z - w} dz = \frac{1}{w} w f(w) = f(w),$$

by the Cauchy integral formula.

In general, if $f : \mathbb{D} \to \mathbb{C}$ is analytic and bounded, let $f_r(z) = f(z)$ for $0 < r < 1$. Then f_r is analytic on the larger disc $D(0, 1/r)$ and hence by the above

$$f_r(w) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{f_r(z)}{(1 - w\overline{z})^2} dA(z).$$

By continuity, $f_r(w) \to f(w)$ as $r \to 1$. Moreover, $f_r \to f$ pointwise on $\mathbb{D}$, and since f, f_r are bounded, the dominated convergence theorem implies

$$f(w) = \lim_{r\to 1} f_r(w) = \lim_{r\to 1} \frac{1}{\pi}\int_{\mathbb{D}} \frac{f_r(z)}{(1 - w\overline{z})^2} dA(z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{f(z)}{(1 - w\overline{z})^2} dA(z).$$

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Problem 1. Let $\{f_n\}$ be a sequence of real-valued Lebesgue measurable functions on $\mathbb{R}$, and let f be another such function. Assume that
(a) $f_n \to f$ Lebesgue almost everywhere
(b) $\int |x||f_n(x)|\,dx \leqslant 100$ for all n, and
(c) $\int |f_n(x)|^2\,dx \leqslant 100$ for all n.
Prove that $f_n \in L^1$ for all n, that $f \in L^1$, and that $||f_n - f||_{L^1} \to 0$. Also show that neither assumption (b) nor assumption (c) can be omitted while making these deductions.

Solution. To show that $f_n \in L^1$, note that

$$\int_{\mathbb{R}} |f_n| \;=\; \int_{|x|\leqslant 1} |f_n| + \int_{|x|>1} |f_n| \;\leqslant\; \left(\int_{|x|\leqslant 1} |f_n|^2\right)^{1/2} 2^{1/2} + \int_{|x|>1} |x||f_n(x)| \;\leqslant\; C \;<\; \infty$$

for some constant C independent of n by hypotheses (b) and (c). Now to show that $f \in L^1$, note that by Fatou's lemma we have

$$\int |f| \;=\; \int \liminf_{n\to\infty} |f_n| \;\leqslant\; \liminf_{n\to\infty} \int |f_n| \;\leqslant\; C \;<\; \infty.$$

Now we show $f_n \to f$ in L^1. First we need two "uniformity" estimates:

$$\int_{|x|>R} |f_n| \;\leqslant\; \int_{|x|>R} \frac{|x|}{R}|f_n| \;\lesssim\; \frac{1}{R}$$
$$\int_E |f_n| \;\leqslant\; m(E)^{1/2}\left(\int_E |f_n|^2\right)^{1/2} \;\lesssim\; m(E)^{1/2}.$$

where the implied constant is independent of n in both. By the same Fatou's lemma argument, the above estimates also hold for f. Let $\epsilon > 0$. Let R be big enough so that $\int_{|x|>R} |f_n| < \epsilon$ for all n and $\int_{|x|>R} |f| < \epsilon$. By Egorov's theorem, there is a set $E \subseteq \{|x| \leqslant R\}$ on which $f_n \to f$ uniformly, and by the second estimate above we may pick $m(E^c)$ to be small enough so that $\int_{E^c} |f_n|, \int_{E^c} |f_n| < \epsilon$. Then we have

$$\begin{aligned}
\int |f_n - f| \;&=\; \int_{|x|>R} |f_n - f| + \int_E |f_n - f| + \int_{E^c} |f_n - f| \\
&\leqslant\; \int_{|x|>R} |f_n| + \int_{|x|>R} |f| + \int_E |f_n - f| + \int_{E^c} |f_n| + \int_{E^c} |f| \\
&<\; 4\epsilon + \int_E |f_n - f|.
\end{aligned}$$

Taking $n \to \infty$, since we have uniform convergence on E, gives

$$\limsup_{n\to\infty} |f_n - f| \;<\; 4\epsilon.$$

This holds for any $\epsilon > 0$, so the result follows. $\square$

Problem 2. Let (X, ρ) be a compact metric space which has at least two points, and let $C(X)$ be the space of continuous functions $X \to \mathbb{R}$ with the uniform norm. Let D be a dense subset of X and for each $y \in D$ define $f_y \in C(X)$ by $f_y(x) = \rho(x, y)$. Let A be the subalgebra of $C(X)$ generated by the collection $\{f_y : y \in D\}$.
(a) Prove that A is dense in $C(X)$ under the uniform norm.
(b) Prove that $C(X)$ is separable.

Solution. (a) By one version of the Stone-Weierstrass theorem, it's enough to check that A separates points (for all $x \neq y \in X$ there exists $f \in A$ with $f(x) \neq f(y)$) and is nonvanishing (for all $x \in X$ there exists $f \in A$ with $f(x) \neq 0$). Both of these are easily verified because X has at least two points by hypothesis. For separating points, given $x \neq y$ let $f = f_y$. For nonvanishing, given x let $f = f_y$ for any $y \neq x$. $\square$

(b)

Problem 3. Let (X, ρ) be a compact metric space and let $P(X)$ be the set of all Borel probability measures on X. Assume $\mu_n \to \mu$ in the weak-$*$ topology on $P(X)$. Prove that $\mu_n(E) \to \mu(E)$ whenever E is a Borel susbet of X such that $\mu(\overline{E}) = \mu(E^\circ)$, where $\overline{E}$ is the closure and E° is the interior.

Solution. Applying the portmanteau theorem twice, since E° is open and $\overline{E}$ is closed, we have

$$\mu(E^\circ) \;\leqslant\; \liminf_{n\to\infty} \mu_n(E^\circ) \;\leqslant\; \liminf_{n\to\infty} \mu_n(E) \;\leqslant\; \limsup_{n\to\infty} \mu_n(E) \;\leqslant\; \limsup_{n\to\infty} \mu_n(\overline{E}) \;\leqslant\; \mu(\overline{E})$$

But by hypothesis, $\mu(E^\circ) = \mu(\overline{E})$, so every inequality in the chain is actually an equality. Since $\mu(E)$ also necessarily fits somewhere in between $\mu(E^\circ)$ and $\mu(\overline{E})$, which are equal, we conclude

$$\liminf_{n\to\infty} \mu_n(E) \;=\; \limsup_{n\to\infty} \mu_n(E) \;=\; \mu(E). \quad \square$$

Problem 4. Let $\mathbb{T}$ be the unit circle in the complex plane and for each $\alpha \in \mathbb{T}$ define the rotation map $R_\alpha : \mathbb{T} \to \mathbb{T}$ by $R_\alpha(z) = \alpha z$. A Borel probability measure μ on $\mathbb{T}$ is called α-invariant if $\mu(R_\alpha(E)) = \mu(E)$ for all Borel sets $E \subseteq \mathbb{T}$.
(a) Let m be Lebesgue measure on $\mathbb{T}$. Show that for every $\alpha \in \mathbb{T}$, m is α-invariant.
(b) Prove that if α is not a root of unity, then the set of powers $\{\alpha^n : n \in \mathbb{Z}\}$ is dense in $\mathbb{T}$.
(c) Prove that if α is not a root of unity, then m is the only α-invariant Borel probability measure on $\mathbb{T}$.

Solution. Throughout, we identify $\mathbb{T}$ with the interval $[0, 1)$ in the natural way, so "α is not a root of unity" is replaced by "α is irrational".

(a) When viewed as a map on $[0, 1)$, $R_\alpha(x) = x + \alpha \pmod 1$. We know that Lebesgue measure is translation invariant, so R_α is measure preserving when considered as a map $[0, 1) \to \mathbb{R}$. But in the case where $E \subseteq [0, 1)$ has $R_\alpha(E) \cap [1, \infty) \neq \varnothing$, $R_\alpha(E)$ may be reassembled as a subset of $[0, 1)$ by just translating $R_\alpha(E) \cap [1, \infty)$ to the left by 1, which still preserves Lebesgue measure. Thus R_α preserves m. $\square$

(b) **Method 1.** It's enough to show $\{n\alpha : n \geqslant 0\}$ is dense in $\mathbb{T}$. Since α is irrational, the orbit contains infinitely many distinct points. Therefore by the pigeonhole principle, for every $\epsilon > 0$ there exist some $n < m$ such that $||n\alpha - m\alpha||_{\mathbb{T}} < \epsilon$ ($||\cdot||_{\mathbb{T}}$ denotes "mod 1" distance). Therefore the rotation $x \mapsto (m - n)\alpha$ is a rotation by less than ϵ, so $\{j(m - n)\alpha : j \geqslant 0\}$ is a subset of the orbit such that every point of $\mathbb{T}$ is at most ϵ away from some $j(m - n)\alpha$. Such subsets exist for any $\epsilon > 0$, so the orbit is dense. $\square$

(b) **Method 2.** It's enough to show $\{n\alpha : n \geqslant 0\}$ is dense in $\mathbb{T}$. In fact we show a stronger result which is the equidistribution theorem, i.e. for any $0 \leqslant a < b \leqslant 1$,

$$\lim_{N\to\infty} \frac{\#\{n : a \leqslant n\alpha \leqslant b\}}{N} \;=\; b - a.$$

For any $f \in L^1(\mathbb{T})$, set

$$A_N f \;:=\; \frac{1}{N} \sum_{n=0}^{N-1} f(n\alpha), \qquad I(f) \;:=\; \int_{\mathbb{T}} f \, dm.$$

The first step is to show that for $f \in C(\mathbb{T})$, $A_N f \to I(f)$ as $N \to \infty$. It's easy to see that this property is linear and behaves well under L^∞ approximation, so since trig polynomials are dense in $C(\mathbb{T})$, it's enough to show that this result holds for $f(x) = \exp(2\pi ikx)$ for any $k \in \mathbb{Z}$. We calculate directly

$$A_N f \;=\; \frac{1}{N}\sum_{n=0}^{N-1} \exp(2\pi ik\alpha)^n \;=\; \frac{1}{N}\begin{cases} N & k=0 \\ \frac{1-\exp(2\pi iNk\alpha)}{1-\exp(2\pi ik\alpha)} & k \neq 0 \end{cases} \;=\; \begin{cases} 1 & k=0 \\ O_k(1/N) & k \neq 0 \end{cases}$$

because $\exp(2\pi ik\alpha) \neq 1$ for all $k \neq 0$ because α is irrational. Thus

$$\lim_{N\to\infty} A_N f \;=\; \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases} \;=\; I(f).$$

To finish the proof, we want to apply this convergence to the characteristic function $\chi_{[a,b]}$, but it's not continuous, so we have to approximate. Take sequences f_k, g_k of continuous functions satisfying $0 \leqslant g_k \leqslant \chi_{[a,b]} \leqslant f_k \leqslant 1$ with f_k and g_k both converging Lebesgue almost everywhere to $\chi_{[a,b]}$. Then we have

$$A_N g_k \;\leqslant\; A_N \chi_{[a,b]} \;\leqslant\; A_N f_k, \qquad I(g_k) \;\leqslant\; I(\chi_{[a,b]}) \;\leqslant\; I(f_k).$$

Taking $N \to \infty$ then gives

$$I(g_k) \;\leqslant\; \liminf_{N\to\infty} A_N \chi_{[a,b]} \;\leqslant\; \limsup_{N\to\infty} A_N \chi_{[a,b]} \;\leqslant\; I(f_k),$$

and by the Dominated Convergence Theorem taking $k \to \infty$ gives

$$I(\chi_{[a,b]}) \;\leqslant\; \liminf_{N\to\infty} A_N \chi_{[a,b]} \;\leqslant\; \limsup_{N\to\infty} A_N \chi_{[a,b]} \;\leqslant\; I(\chi_{[a,b]}),$$

so they are all equal, as desired. This finishes the proof because $\lim_{N\to\infty} A_N\chi_{[a,b]}$ is exactly the expression on the left side and $I(\chi_{[a,b]})$ is exactly the expression on the right side of the desired equation. $\square$

(c) **Method 1.** It's enough to show that $\int f\,d\mu = \int f\,dm$ for all $f \in C(\mathbb{T})$. Write

$$\begin{aligned}\int f(x)\,d\mu(x) - \int f(z)\,dm(z) \;&=\; \int\int (f(x) - f(z))\,dm(z)\,d\mu(x) \;=\; \int\int (f(x) - f(x+z))\,dm(z)\,d\mu(x) \\ &=\; \int\int (f(x) - f(x+z))\,d\mu(x)\,dm(z)\end{aligned}$$

where the last equality is by Fubini and the second to last equality is by the translation invariance of m. So it suffices to show that $\int (f(x) - f(x+z))\,d\mu(x) = 0$ for each fixed $z \in \mathbb{T}$. By the density from part (b), there is a subsequence $n_j\alpha \to z$ as $j \to \infty$. Thus since f is continuous and $\mathbb{T}$ is compact, we have $f(x+n_j\alpha) \to f(x+z)$ uniformly over $x \in \mathbb{T}$ as $j \to \infty$. Therefore, since we are assuming μ is invariant under rotations by α, we have

$$\int (f(x) - f(x+z))\,d\mu(x) \;=\; \int f(x)\,d\mu(x) - \int f(x+z)\,d\mu(x) \;=\; \int f(x+n_j\alpha)\,d\mu(x) - \int f(x+z)\,d\mu(x)$$

for every j, and taking $j \to \infty$ makes the right side equal to 0 because the convergence is uniform and f is continuous. $\square$

(c) **Method 2 (motivated by ergodic theory).** Suppose α is irrational. Then if f is a trig polynomial, the same direct calculation from part (b) shows that

$$A_N f(x) \;:=\; \frac{1}{N}\sum_{n=0}^{N-1} f(x+n\alpha) \to \int_{\mathbb{T}} f\,dm$$

as $N \to \infty$ for any fixed $x \in \mathbb{T}$. Let μ be any R_α-invariant measure. Then since trig polynomials are bounded, the Dominated Convergence Theorem gives

$$\int A_N f\,d\mu \to \int \left(\int f\,dm\right)\,d\mu \;=\; \int f\,dm.$$

But since μ is R_α-invariant, the left side is equal to $\int f\, d\mu$ for all N. Thus $\int f\, d\mu = \int f\, dm$ for all trig polynomials f, and by density they are equal for all $f \in C(\mathbb{T})$, so by the Riesz representation theorem $\mu = m$. $\square$

Problem 5. Let $\{f_n\}$ be a sequence of continuous real-valued functions on $[0,1]$ and suppose $f_n(x)$ converges to another real valued function $f(x)$ at **every** $x \in [0,1]$.
(a) Prove that for every $\epsilon > 0$ there is a dense subset $D_\epsilon \subseteq [0,1]$ such that if $x \in D_\epsilon$ then there are an open interval $I \ni x$ and a positive integer N_x such that for all $n > N_x$, $\sup_{y \in I} |f_n(y) - f(y)| \leqslant \epsilon$.
(b) Prove that f cannot be the characteristic function $\chi_{\mathbb{Q} \cap [0,1]}$.

Solution. (a) As per the hint, we consider the closed sets

$$F_{N,\epsilon} = \{y \in [0,1] : |f_n(y) - f_m(y)| \leqslant \epsilon, m, n > N\}.$$

Since $f_n(y)$ converges pointwise for all y,

$$\bigcup_{N \in \mathbb{N}} F_{N,\epsilon/2} = [0,1].$$

Let

$$D_\epsilon = \bigcup_{N \in \mathbb{N}} \mathrm{Int}(F_{N,\epsilon/2})$$

where $\mathrm{Int}(A)$ is the interior of A. Since each element x in D_ϵ is contained in some open interval $I \subset F_{N_x,\epsilon/2}$ such that if $y \in I$, then

$$|f(y) - f_n(y)| \leqslant |f_n(y) - f_m(y)| + |f(y) - f_m(y)| < \epsilon$$

where m (depending on y) is chosen to be greater than N_x such that $|f_m(y) - f(y)| < \frac{\epsilon}{2}$. It remains to show that D_ϵ is dense. To show that it is dense, observe that for any interval $[a,b]$

$$\bigcup_{N \in \mathbb{N}} [a,b] \cap F_{N,\epsilon/2} = [a,b]$$

so by the Baire category theorem, some $F_{N,\epsilon/2}$ has nonempty interior in $[a,b]$. This implies that $\mathrm{Int}(F_{N,\epsilon/2}) \cap [a,b]$ is nonempty. Hence, D_ϵ is dense. $\square$

(b) Note that the definition of D_ϵ implies that it is open. For $\epsilon = \frac{1}{n}$, we may take

$$S = \bigcap_{n=1}^{\infty} D_{1/n}.$$

We claim that S consists of continuity points of f. Let $\epsilon > 0$ and $x \in S$. Choose n such that $3n^{-1} < \epsilon$. Then $x \in D_{1/n}$ so there exists a δ-neighborhood around x such that if y is in that δ neighborhood,

$$|f_n(y) - f(y)| \leqslant \frac{\epsilon}{3}$$

$$|f_n(y) - f_n(x)| \leqslant \frac{\epsilon}{3}$$

for some $n > N$. Hence

$$|f(y) - f(x)| \leqslant |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f(y) - f_n(y)| < \epsilon$$

as desired. By the Baire category theorem, S is dense. As $\chi_{\mathbb{Q} \cap [0,1]}$ is not continuous anywhere in $[0,1]$ anywhere, it follows that it cannot be a pointwise limit of continuous functions. $\square$

Problem 6. Let $f \in L^2(\mathbb{R})$ and assume the Fourier transform satisfies $\left|\widehat{f}(\xi)\right| > 0$ for Lebesgue almost every $\xi \in \mathbb{R}$. Prove the set of finite linear combinations of the translates $f_y(x) = f(x-y)$ is norm dense in

$L^2(\mathbb{R})$.

Solution. See Spring 2012 # 6.

Problem 7. Let $f(z)$ be an analytic function on the entire complex plane $\mathbb{C}$ such that the function $U(z) = \log|f(z)|$ is Lebesgue area integrable. Prove f is constant.

Solution. See Spring 2013 # 7.

Problem 8. Let $\mathcal{D}$ be the space of analytic function $f(z)$ on the unit disc $\mathbb{D}$ such that $f(0) = 0$ and $\int_{\mathbb{D}} |f'(z)|^2\,dx\,dy < \infty$.
(a) Prove $\mathcal{D}$ is complete in the norm

$$||f|| = \left(\int_{\mathbb{D}} |f'(z)|^2\,dx\,dy\right)^{1/2}.$$

(b) Give a necessary and sufficient condition on the coefficients a_n for the function $f(z) = \sum_{n\geqslant 1} a_n z^n$ to belong to $\mathcal{D}$.

Solution. (a) Let f_n be a Cauchy sequence in $\mathcal{D}$. Then by definition, f_k' is a Cauchy sequence in $L^2(\mathbb{D})$. Since L^2 is known to be complete, there is some g with $f_k' \to g$ in $L^2(\mathbb{D})$. We need to show that g is holomorphic, and for this we use the standard trick. Fix $0 < r < 1$, then for any $|z| \leqslant r$ and any $f \in \mathcal{D}$ we have

$$|f'(z)| = \left|\int_{B(z,(1-r)/2)} f'(w)\,dA(w)\right| \leqslant \int_{B(z,(1-r)/2)} |f'(w)|\,dA(w) \lesssim_r \left(\int_{B(z,(1-r)/2)} |f'(w)|^2\,dA(w)\right)^{1/2} \leqslant ||f||_{\mathcal{D}},$$

so $||f'||_{L^\infty\left(\overline{B(0,r)}\right)} \lesssim_r ||f||_{\mathcal{D}}$. Thus, since f_n is a Cauchy sequence in $\mathcal{D}$, f_n' is a uniformly Cauchy sequence on $\overline{B(0,r)}$. Since $L^\infty\left(\overline{B(0,r)}\right)$ is complete, we see that f_n' converges uniformly to some limit function on $\overline{B(0,r)}$. This holds for any $r < 1$, so f_n' has a locally uniform limit on $\mathbb{D}$. But since $f_n' \to g$ in $L^2(\mathbb{D})$, it has a subsequence converging pointwise to g, so in fact $f_n' \to g$ locally uniformly on $\mathbb{D}$, which implies g is holomorphic. Let G be the unique primitive of g with $G(0) = 0$. Then $||f_n - G||_{\mathcal{D}} = ||f_n' - g||_{L^2(\mathbb{D})} \to 0$, so $\mathcal{D}$ is complete. $\square$

(b) We have $f'(z) = \sum_{n\geqslant 1} n a_n z^{n-1}$. Write this as $f'(re^{i\theta}) = \sum_{n\geqslant 1} n a_n r^{n-1} e^{i(n-1)\theta}$ and then we have

$$\left|f'(re^{i\theta})\right|^2 = \sum_{n,k\geqslant 1} nka_n\overline{a_k} r^{n+k-2} e^{i(n-k)\theta},$$

so

$$\begin{aligned}
\int_{\mathbb{D}} |f'(z)|^2\,dx\,dy &= \int_0^1 \int_0^{2\pi} \sum_{n,k\geqslant 1} nka_n\overline{a_k} r^{n+k-2} e^{i(n-k)\theta}\, r\,d\theta\,dr \\
&= \int_0^1 \sum_{n,k\geqslant 1} nka_n\overline{a_k} r^{n+k-1} \int_0^{2\pi} e^{i(n-k)\theta} \quad \text{because the series converges uniformly on compact sets} \\
&= \int_0^1 \sum_{n\geqslant 1} n^2 |a_n|^2 r^{2n-1}\,dr \quad \text{by orthonormality} \\
&= \sum_{n\geqslant 1} n^2 |a_n|^2 \int_0^1 r^{2n-1}\,dr \quad \text{by the Monotone Convergence Theorem} \\
&= \frac{1}{2}\sum_{n\geqslant 1} n\,|a_n|^2.
\end{aligned}$$

Thus a necessary and sufficient condition is that $\sum_{n\geqslant 1} n\,|a_n|^2 < \infty$. $\square$

Problem 9. Consider the meromorphic function $g(z) = -\pi z\cot(\pi z)$ on the entire plane $\mathbb{C}$.
(a) Find all poles of g and determine the residue of g at each pole.
(b) In the Taylor series representation $\sum_{k=0}^{\infty} a_k z^k$ of $g(z)$ about $z = 0$, show that for each $k \geqslant 1$

$$a_{2k} \;=\; \sum_{n\geqslant 1} \frac{2}{n^{2k}}.$$

Solution. See Spring 2013 # 11.

Problem 10. For $-1 < \beta < 1$ evaluate

$$\int_0^\infty \frac{x^\beta}{1+x^2}\,dx.$$

Solution. See Spring 2014 # 11.

Problem 11. An **analytic Jordan curve** is a set of the form $\Gamma = f(\{|z| = 1\})$ where f is analytic and one to one on an annulus $\{r < |z| < 1/r\}$, $0 < r < 1$. Let $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere, let $N < \infty$, and let $\Omega \subseteq \mathbb{C}^*$ be a domain for which $\partial\Omega$ has N connected components, none of which are single points. Prove there is a conformal mapping from Ω onto a domain bounded by N pairwise disjoint analytic Jordan curves.

Solution.

Problem 12. If $\alpha \in \mathbb{C}$ satisfies $0 < |\alpha| < 1$ and if $n \geqslant 1$, show that the equation $e^z(z-1)^n = \alpha$ has exactly n simple roots in the half plane $\{\mathrm{Re}(z) > 0\}$.

Solution. By Rouche's theorem, since $|\alpha| < |e^z(x-1)^n|$ on the circle $\{|z-1| = 1\}$, $e^z(x-1)^n = 0$ and $e^z(x-1)^n - \alpha = 0$ have the same number of solutions (counting multiplicity) in the disk $\{|z-1| = 1\}$, which is n. Every solution in $\{\mathrm{Re}(z) > 0\}$ lies in this disk, as for all other points z

$$|e^z(z-1)^n| \geqslant 1 > |\alpha|.$$

Therefore we need only to show that all solutions in this disk are simple roots. This can be done by taking the derivative:

$$\frac{d}{dz}\left[e^z(z-1)^n - \alpha\right] = e^z(z-1)^n + ne^z(z-1)^{n-1} = (z-1+n)e^z(z-1)^{n-1}.$$

This is zero only when $z = 1-n$ or $z = 1$. The first value is not in $\{\mathrm{Re}(z) > 0\}$, while the second value can never solve $e^z(z-1)^n = \alpha$ when $|\alpha| > 0$. $\square$

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Problem 1. Let $f \in C^2(\mathbb{R})$ be a real-valued function that is uniformly bounded on $\mathbb{R}$. Prove that there exists a point $c \in \mathbb{R}$ such that $f''(c) = 0$.

Solution. If f'' never vanishes, then by the intermediate value theorem it must be either always positive or always negative. Let us assume without loss of generality (otherwise we could replace f by $-f$) that $f'' > 0$ on all of $\mathbb{R}$. This implies that the derivative f' is strictly increasing on all of $\mathbb{R}$. Let us also assume now that there is a point $p \in \mathbb{R}$ for which $f'(p) > 0$. In that case, for any $x > p$ we have by the mean value theorem

$$f(x) - f(p) \;\geqslant\; \left(\inf_{t\in[p,x]} f'(t)\right)(x-p) \;\geqslant\; f'(p)(x-p)$$

and this grows to ∞ as $x \to \infty$, contradicting the hypothesis that f is bounded.

In case there is no such p, there must be $q \in \mathbb{R}$ such that $f'(q) < 0$ and then we repeat the same argument going backwards in x: if $x < q$, then $f(q) - f(x) \leqslant (\sup_{t\in[x,q]} f'(t))(q-x) \leqslant f'(q)(q-x)$ which tends to $-\infty$ as $x \to -\infty$. $\square$

Problem 2. Let μ be a Borel probability measure on $[0,1]$ that has no atoms. Let also $\mu_1, \mu_2, \ldots$ be Borel probability measures on $[0,1]$. Assume that $\mu_n \to \mu$ in the weak-$*$ topology. Denote $F(t) = \mu[0,t]$ and $F_n(t) = \mu_n[0,t]$ for $t \in [0,1]$. Prove that F_n converges uniformly to F.

Solution. First we establish pointwise convergence. Since $[0,t]$ is a closed set, the portmanteau theorem gives

$$F(t) \;=\; \mu[0,t] \;\geqslant\; \limsup_{n\to\infty} \mu_n[0,t] \;=\; \limsup_{n\to\infty} F_n(t)$$

for each $t \in [0,1]$. Since μ is atomless, we also have $F(t) = \mu[0,t)$ for each t, and since $[0,t)$ is an open set in $[0,1]$ the portmanteau theorem also gives

$$F(t) \;=\; \mu[0,t) \;\leqslant\; \liminf_{n\to\infty} \mu_n[0,t) \;\leqslant\; \liminf_{n\to\infty} F_n(t).$$

for each t. These two inequalities together show that $F_n \to F$ pointwise.

Also notice that because μ is atomless, F is continuous and therefore uniformly continuous because $[0,1]$ is compact. Now we upgrade the pointwise convergence to uniform convergence via an elementary lemma:
Claim: Suppose F_n and F are functions $[0,1] \to [0,1]$ satisfying:

- F_n, F are increasing.
- $F_n(0) = F(0) = 0$ and $F_n(1) = F(1) = 1$.
- F is uniformly continuous.
- $F_n \to F$ pointwise.

Then $F_n \to F$ uniformly.
Proof: Let $\epsilon > 0$. Let $\delta > 0$ be small enough as in the definition of uniform continuity of F. Let $0 < x_1 < x_2 < \ldots < x_m < 1$ be points with $|x_{i+1} - x_i| < \delta$. Then we may pick N large so that $|F_n(x_i) - F(x_i)| < \epsilon$ for all $n > N$ and all x_i. Now for any $n > N$ and any $t \in [0,1]$, we have $t \in [x_i, x_{i+1})$ for some i, and we estimate

$$\begin{aligned}
|F_n(t) - F(t)| &\leqslant |F_n(t) - F_n(x_i)| + |F_n(x_i) - F(x_i)| + |F(x_i) - F(t)| \\
&\leqslant \left|F_n(x_{i+1}) - F_(x_i)\right| + \epsilon + \epsilon \\
&\leqslant \left|F_n(x_{i+1}) - F_(x_{i+1})\right| + |F_n(x_i) - F(x_i)| + 2\epsilon \;\leqslant\; 4\epsilon
\end{aligned}$$

The inequality in the first term passing from the first to second line above is due to the fact that F_n is increasing. Thus $F_n \to F$ uniformly. $\square$

Problem 3. (a) Let f be a positive continuous function on $\mathbb{R}$ such that $\lim_{|t|\to\infty} f(t) = 0$. Show that the set $\{hf : h \in L^1(\mathbb{R}, m), ||h||_{L^1} \leqslant K\}$ is a closed nowhere dense set in $L^1(\mathbb{R}, m)$, for any $K \geqslant 1$ (m denotes Lebesgue measure on $\mathbb{R}$).

(b) Let $\{f_n\}$ be a sequence of positive continuous functions on $\mathbb{R}$ such that for each n we have $\lim_{|t|\to\infty} f_n(t) = 0$. Show that there exists $g \in L^1(\mathbb{R}, m)$ such that $g/f_n \notin L^1(\mathbb{R}, m)$ for all n.

Solution. (a) Let E_K be the set in question, it can also be written $E_K = \{g \in L^1 : ||g/f||_{L^1} \leqslant K\}$. Fix $\epsilon > 0$, we will construct $\widetilde{g} \in L^1$ such that $||g - \widetilde{g}||_{L^1} < \epsilon$ and $||\widetilde{g}/f||_{L^1} > K$, thus achieving the desired conclusion. First, pick M large enough so that $f(x) < \epsilon/K$ for all $|x| \geqslant M$. Then define $\widetilde{g}(x) = g(x) + \epsilon 1_{[M,M+1]}(x)\,\mathrm{sgn}(g(x))$, where $\mathrm{sgn}(y) := 1$ if $y \geqslant 0$ and -1 if $y < 0$. We have defined $\widetilde{g}$ so that $|\widetilde{g}(x)| = |g(x)| + \epsilon 1_{[M,M+1]}(x)$ for each $x \in \mathbb{R}$. Now we estimate

$$||\widetilde{g} - g||_{L^1} \;=\; \left|\left|\epsilon 1_{[M,M+1]}(x)\,\mathrm{sgn}(g(x))\right|\right|_{L^1} \;=\; \epsilon,$$

and

$$\int |\widetilde{g}(x)|/f(x)\,dx \;=\; \int |g(x)|/f(x)\,dx + \epsilon\int_M^{M+1} \frac{1}{f(x)}\,dx \;>\; K$$

as desired. $\square$

(b) Let $E_{K,n}$ be the set E_K above corresponding to the function f_n. Part (a) showed that each $E_{K,n}$ is nowhere dense, and it's clear that each $E_{K,n}$ is closed in the L^1 norm. Since L^1 is a complete metric space, the Baire category theorem implies that

$$L^1 \supsetneq \bigcup_{n\geqslant 1}\bigcup_{K\geqslant 1} E_{K,n}.$$

Therefore there exists $g \in L^1$ with the property that for each n, g is not in $E_{K,n}$ for any K, which means that for each n, $||g/f_n||_{L^1} > K$ for every K, which means that $g/f_n \notin L^1$ for any n, as desired. $\square$

Problem 4. Let V be the subspace of $L^\infty([0,1], m)$ (where m is Lebesgue measure) defined by

$$V \;=\; \left\{f \in L^\infty([0,1], m) : \lim_{n\to\infty} n\int_{[0,1/n]} f\,dm \text{ exists}\right\}.$$

(a) Prove that there exists $\varphi \in L^\infty([0,1], m)^*$ (i.e. a continuous functional on $L^\infty([0,1], m)$) such that $\varphi(f) = \lim_{n\to\infty} n\int_{[0,1/n]} f\,dm$ for every $f \in V$.

(b) Show that, given any $\varphi \in L^\infty([0,1], m)^*$ satisfying the condition in (a) above, there exists no $g \in L^1([0,1], m)$ such that $\varphi(f) = \int fg\,dm$ for all $f \in L^\infty([0,1], m)$.

Solution. (a) If we verify that φ is continuous and linear on the subspace V, then the conclusion follows immediately from the Hahn-Banach theorem. It is obviously linear. And an easy estimate shows that

$$\left|\lim n\int_0^{1/n} f - \lim n\int_0^{1/n} g\right| \;\leqslant\; \lim n\int_0^{1/n} |f - g|,$$

so it's also clearly continuous in the L^∞ norm. $\square$

(b) First note that $C([0,1]) \subseteq V$ and that $\varphi(f) = f(0)$ for all continuous f. So to show no such g exists it's enough to show the non-existence of a $g \in L^1([0,1], m)$ such that $\int fg = f(0)$ for all continuous f. To show this, suppose g were such a function. Fix any $0 < a < b < 1$ and let f_ϵ be a continuous function satisfying $f_\epsilon = 1$ on $[a,b]$, $f_\epsilon = 0$ outside of $(a-\epsilon, b+\epsilon)$, and $0 \leqslant f_\epsilon \leqslant 1$ everywhere. In particular, $f(0) = 0$, so $\int gf_\epsilon = 0$. But as $\epsilon \to 0$, f_ϵ converges pointwise to the indicator function $1_{[a,b]}$, so by the dominated convergence theorem $\int_a^b g = 0$. This holds for any $0 < a < b < 1$ which implies that g is identically zero, a contradiction. $\square$

Problem 5. (a) Prove that $L^p([0,1],m)$ are separable Banach spaces for $1 \leqslant p < \infty$ but $L^\infty([0,1],m)$ is not (where m is Lebesgue measure).

(b) Prove that there exists no linear bounded surjective map $T : L^p([0,1],m) \to L^1([0,1],m)$ if $p > 1$.

Solution. (a) The fact that L^p is a Banach space is a standard theorem (called the Riesz-Fischer theorem I think). To show that they are separable, first note that the continuous functions are dense in L^p. This is another standard fact, which is proved by applying Lusin's theorem and then the inner regularity property of Lebesgue measure and then Urysohn's lemma. Finally, we know that $C([0,1])$ is separable by, for example, taking as a countable dense subset the family of all piecewise linear functions on $[0,1]$ with slope changes only at points with rational coordinates.

To see that L^∞ is not separable, consider the family of indicator functions $\{1_{[0,r]}\}_{0 \leqslant r \leqslant 1}$. It's clear that there are uncountably many of these and that $\left|\left|1_{[0,r} - 1_{[0,s]}\right|\right|_{L^\infty} = 1$ for any distinct r, s, so L^∞ can not be separable. $\square$

(b) Suppose such a T did exist. Then the adjoint map $T^* : (L^1)^* \cong L^\infty \to (L^p)^* \cong L^{p'}$ would be a bounded linear *injective* map from a non-separable space into a separable space (by part (a)). This is impossible, essentially by the open mapping theorem. If $A : X \to Y$ is a linear bounded injective map, then consider it as a linear bounded bijection $A : X \to A(X)$. Then the open mapping theorem says that $A^{-1} : A(X) \to X$ is also bounded, which is equivalent to saying that $||Ax||_Y \geqslant \epsilon\,||x||_X$ for all $x \in X$. Therefore an uncountable discrete set in X would get translated to an uncountable discrete set in Y, so it's impossible to have a bounded linear injective map from a non-separable space into a separable one. $\square$

Problem 6. Let H be a Hilbert space and $\{\xi_n\}$ a sequence of vectors in H such that $||\xi_n|| = 1$ for all n.
(a) Show that if ξ_n converges weakly to a vector $\xi \in H$ with $||\xi|| = 1$, then $\lim_{n\to\infty} ||\xi_n - \xi|| = 0$.

(b) Show that if $\lim_{n,m\to\infty} ||\xi_n + \xi_m|| = 2$, then there exists a vector $\xi \in H$ such that $\lim_{n\to\infty} ||\xi_n - \xi|| = 0$.

Solution. (a) We have

$$||\xi_n - \xi||^2 \;=\; \langle \xi_n - \xi, \xi_n - \xi\rangle \;=\; ||\xi_n||^2 + ||\xi||^2 - \langle \xi_n, \xi\rangle - \langle \xi, \xi_n\rangle \;=\; 2 - \langle \xi_n, \xi\rangle - \overline{\langle \xi_n, \xi\rangle}.$$

By weak convergence, we know $\langle \xi_n, \xi\rangle \to \langle \xi, \xi\rangle = 1$ as $n \to \infty$, so $||\xi_n - \xi||^2 \to 0$ as desired. $\square$

(b) Use the parallelogram law:

$$||\xi_n + \xi_m||^2 + ||\xi_n - \xi_m||^2 \;=\; 2\,||\xi_n||^2 + 2\,||\xi_m||^2 \;=\; 4.$$

By the hypothesis this implies that $||\xi_n - \xi_m||^2$ tends to 0 as $n, m \to \infty$, so $\{\xi_n\}$ is a Cauchy sequence, and since Hilbert spaces are complete it must be a convergent sequence. $\square$

Problem 7. Let $f : \mathbb{C} \to \mathbb{C}$ be entire and non-constant, and set

$$T(r) \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log_+ \left|f(re^{i\theta})\right|\, d\theta.$$

Here $\log_+ s = \max(\log s, 0)$. Show that $T(r) \to \infty$ as $r \to \infty$.

Solution. First assume that f does not vanish at 0. Then we can apply Jensen's formula, which says

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\theta})\right|\, d\theta \;=\; \log|f(0)| + \sum_a \log\left(\frac{r}{|a|}\right)$$

where the sum is over all $a \in \mathbb{C}$ such that $|a| \leqslant r$ and $f(a) = 0$ (Note there are only finitely many such a for each fixed r). Also note that $T(r)$ is greater than the left side above. If f has only finitely many zeros, then for r big enough the sum on the right side has a constant number of terms, and each term increases like $\log(r)$ as $r \to \infty$. If f has infinitely many zeros, then the sum on the right side keeps adding more terms as $r \to \infty$, and each term is at least 1. Thus $T(r) \to \infty$ as $r \to \infty$.

Now consider the case where $f(0) = 0$. Then let m be the order of vanishing at 0 and apply the above argument to $g(z) = z^{-m} f(z)$. Then we have

$$T(r) \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log_+ |r^m g(re^{i\theta})|\, d\theta \;=\; m\log(r) + \frac{1}{2\pi}\int_0^{2\pi} \log_+ |g(re^{i\theta})|\, d\theta.$$

This expression is greater than the corresponding $T(r)$ expression for the non-vanishing-at-0 function g, which we already proved goes to ∞, so we're done. $\square$

Problem 8. Show that

$$\sin z - z\cos z \;=\; \frac{z^3}{3}\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\lambda_n^2}\right), \quad z \in \mathbb{C},$$

where λ_n is a sequence in $\mathbb{C}$, $\lambda_n \neq 0$ for all n, such that

$$\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|^2} \;<\; \infty.$$

Solution. The function $f(z) = \sin z - z\cos z$ is of (exponential) order 1, since

$$|f(z)| = O(e^{|z|^\alpha})$$

for all $\alpha > 1$. By Hadamard's Factorization Theorem,

$$f(z) = z^m e^{p(z)} \prod_{z_k} E_1\left(\frac{z}{z_k}\right)$$

where m is the oder of the zero at $z = 0$, $p(z)$ is a polynomial of degree at most 1, $\{z_k\}$ is an enumeration of the zeros of $f(z)z^{-m}$ (with mulitplicity) and E_1 is the first Weierstrass elementary factor:

$$E_1(z) = (1 - z)e^z.$$

The theorem also gives that

$$\sum_{z_n} \frac{1}{|z_n|^2} < \infty$$

and that the infinite product converges absolutely.

Now we work to get the above product in the desired form. Notice that f is an odd funciton. Therefore, if $w \neq 0$ is a zero of f, then $-w$ is a zero with the same multiplicity. We can then relabel the zeros of f to be $\{\lambda_n\}_{n \in \mathbb{Z}\setminus\{0\}}$ where $-\lambda_n = \lambda_{-n}$. Since the infinite product converges absolutely, we are allowed to rearrange the terms. Putting the n-th term with the $(-n)$-th term gives

$$f(z) = z^m e^{p(z)} \prod_{n=1}^{\infty}\left(1 - \frac{z}{\lambda_n}\right) e^{z/\lambda_n}\left(1 - \frac{z}{-\lambda_n}\right) e^{z/-\lambda_n} = z^m e^{p(z)} \prod_{n=1}^{\infty}\left(1 - \frac{z}{\lambda_n^2}\right).$$

We can also write

$$\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|^2} = \frac{1}{2}\sum_{z_n} \frac{1}{|z_n|^2} < \infty.$$

Therefore we need only to show that $m = 3$ and $p(x) = -\ln(3)$.

To determine m, we need only to compute the first few derivatives of f:

$$\begin{aligned} f(z) = \sin z - z\cos z &\implies f(0) = 0 \\ f'(z) = z\sin z &\implies f'(0) = 0 \\ f''(z) = \sin z + z\cos z &\implies f''(0) = 0 \\ f'''(z) = 2\cos z - z\sin z &\implies f'''(0) = 2. \end{aligned}$$

Therefore $m = 3$. Lastly, we turn to $p(z)$. We can write $p(z) = Az + B$. We know that f is odd, and we can see by our product we have found already that $f(x)e^{-p(z)}$ is an odd function. Therefore $e^{p(z)}$ is even, which means $A = 0$. To determine B, we use our expansion to compute

$$f'''(z) = 6e^B + z(\cdots)$$

where $(\cdots)$ is a holomorphic function we have factored a z out of. Since $f'''(0) = 2$, $B = -\ln(3)$, as desired.

Problem 9. Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$ and let $A(\mathbb{D})$ be the space of functions holomorphic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$. Let

$$U \;=\; \{f \in A(\mathbb{D}) : |f(z)| = 1 \text{ for all } z \in \partial\mathbb{D}\}.$$

Show that $f \in U$ if and only if f is a finite Blaschke product,

$$f(z) \;=\; \lambda \prod_{j=1}^{N} \frac{z - a_j}{1 - \overline{a_j}z},$$

for some $a_j \in \mathbb{D}$, $1 \leqslant j \leqslant N < \infty$ and $|\lambda| = 1$. $\square$

Solution. Note that f can have only finitely many zeros in D, so denote them $a_1, \ldots, a_N$. Then we observe that

$$g(z) \;:=\; f(z) \prod_{j=1}^{N} \frac{1 - \overline{a_j}z}{z - a_j}$$

is a non-vanishing analytic function in $\mathbb{D}$. Also, it is easy to check that each Blaschke factor of the form $z \to \frac{1-\overline{a_j}z}{z-a_j}$ preserves the unit circle, so g still has the property that $|g(z)| = 1$ if $|z| = 1$. Now g is a non-vanishing analytic function of $\mathbb{D}$ that preserves the unit circle, so it can be extended to all of $\mathbb{C}$ by reflecting over the unit circle: Define $h(z)$ on all of $\mathbb{C}$ by

$$h(z) \;:=\; \begin{cases} g(z) & |z| \leqslant 1 \\ \frac{1}{\overline{g(1/\overline{z})}} & |z| > 1 \end{cases}.$$

It's clear that h is analytic in $|z| < 1$ and also in $|z| > 1$. It's also clear that h is continuous on the unit circle. To show that h is also analytic on the unit circle, one can use Morera's theorem as in the proof of the Schwarz reflection principle (details omitted). Thus h is entire, and since g is continuous and non-vanishing on the compact set $\overline{\mathbb{D}}$, h is bounded as well, and therefore h is constant, so g is constant, and that constant must be unimodular by the condition $|g(z)| = 1$ for all $|z| = 1$. Thus f has the desired form. $\square$

Alternatively, one could apply the maximum modulus principle to both g and $1/g$. Since $|g| = 1$ on $\partial\mathbb{D}$, we must have $|g| \leqslant 1$ throughout $\mathbb{D}$. But since g is nonvanishing, $1/g$ is also analytic in $\mathbb{D}$ and satisfies the same boundary condition, so also $|1/g| \leqslant 1$ throughout $\mathbb{D}$. Thus $|g| = 1$ everywhere on $\mathbb{D}$, so the open mapping theorem implies that g must indeed be constantly equal to some unimodular λ as desired. $\square$

Problem 10. For $a > 0, b > 0$, evaluate the integral

$$\int_0^\infty \frac{\log x}{(x+a)^2 + b^2}\, dx.$$

Solution. Choose a branch of the complex logarithm that has the undefined axis at the nonnegative real axis. Consider a keyhole contour C around $[0, \infty)$. Consider the integral

$$\int_C \frac{\log(x)^2}{(x+a)^2+b^2}dx$$

As the radius of the outer arc of the keyhole increases, $x\log(x)/((x+a)^2+b^2) \to 0$. The inner arc can be written as

$$\int_{0+\epsilon}^{2\pi-\epsilon} \ell \frac{e^{i\theta}(\log(\ell)^2+(i\theta)^2)}{(\ell e^{i\theta}+a)^2+b^2}$$

where ℓ is the radius of the inner arc and ϵ is a small positive number. As $\ell\log(\ell) \to 0$ as $\ell \to 0$, the integral of the inner arc goes to zero as the radius of the inner arc decreases. By the Cauchy residue theorem,

$$\frac{\pi}{b}(\log(-a+bi)^2 - \log(-a-bi)^2) = \int_C \frac{\log(x)^2}{(x+a)^2+b^2}dx = \int_0^\infty \frac{\log(x)^2}{(x+a)^2+b^2} - \frac{(\log(x)+2\pi i)^2}{(x+a)^2+b^2}$$

But the right hand side is just

$$-4\pi i \int_0^\infty \frac{\ln(x)}{(x+a)^2+b^2}dx$$

where ln is the standard logarithm. The answer is thus

$$\frac{1}{4bi}(\ln(a+bi)^2 - \ln(a-bi)^2). \quad \square$$

Alternate solution. Let $t = \frac{a^2+b^2}{x}$. Then $dx = -\frac{a^2+b^2}{t^2}dt$. Notice that

$$t^2\left(\left(\frac{a^2+b^2}{t}+a\right)^2+b^2\right) = (a^2+b^2+at^2)^2+(bt)^2 = (a^2+b^2)((a+t)^2+b^2).$$

The integrand becomes

$$-\frac{\log(a^2+b^2)-\log(t)}{(a+t)^2+b^2}.$$

and the integral becomes

$$I = \int_0^\infty \frac{\log(x)}{(x+a)^2+b^2}dx = \int_0^\infty \frac{\log(a^2+b^2)-\log(t)}{(a+t)^2+b^2}dt = \int_0^\infty \frac{\log(a^2+b^2)}{(a+t)^2+b^2}dt - I.$$

Hence

$$I = \frac{\log(a^2+b^2)\left(\frac{\pi}{2}-\arctan\left(\frac{a}{b}\right)\right)}{2b}. \quad \square$$

Problem 11. Let $u \in C^\infty(\mathbb{R})$ be smooth and 2π-periodic. Show that there exists a bounded holomorphic function f_+ in the upper half-plane $\mathrm{Im} z > 0$ and a bounded holomorphic function f_- in the lower half-plane $\mathrm{Im} z < 0$ such that

$$u(x) \;=\; \lim_{\epsilon\to 0^+}(f_+(x+i\epsilon) - f_-(x-i\epsilon)), \quad x \in \mathbb{R}.$$

Solution. The function u is 2π-periodic, so we can find a Fourier series expansion for it. That is,

$$u(x) = \frac{1}{2}a_0 + \sum_{n=1}^\infty a_n\cos(nx) + \sum_{n=1}^\infty b_n\sin(nx)$$

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} u(x)dx$$
$$a_n = \frac{1}{\pi}\int_0^{2\pi} u(x)\cos(nx)dx$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} u(x)\sin(nx)dx.$$

Since u is smooth, we can use integration by parts twice to bound

$$|a_n|, |b_n| \leqslant \frac{1}{\pi n^2}\int_0^{2\pi} |u''(x)|dx.$$

From this we see that a_n and b_n are absolutely convergent series.

Now, define

$$f(z) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n e^{inz} - i\sum_{n=1}^{\infty} b_n e^{inz}.$$

This is holomorphic on $\mathbb{C}$ since on every compact set it is the absolutely convergent sum of holomorphic functions. In the top half plane, f is bounded above as

$$|f(z)| \leqslant \frac{1}{2}|a_0| + \sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} |b_n|$$

when $\mathrm{Im}(z) \geqslant 0$.

Let $f_+(z) = \frac{1}{2}f(z)$ and $f_-(z) = \frac{1}{2}\overline{f(\overline{z})}$. The second function is also holomorphic and bounded in the lower half plane. The desired limit is just

$$\lim_{\varepsilon\to 0^+} \mathrm{Re}(f(x+i\varepsilon))$$

which we see by construction is $u(x)$. $\square$

Alternate solution. As u is smooth and 2π-periodic, $u \circ \frac{\log}{i} : S^1 \to \mathbb{R}$ is smooth. We may solve the Dirichlet problem (by e.g. integrating against the Poisson kernel)

$$\Delta v = 0, v|_{\partial D} = u$$

where D is the unit disk. Let f be a holomorphic function on D with imaginary part equal to v. Then $g_+ := f \circ e^{2\pi i x} : \mathbb{H} \to \mathbb{C}$ where $\mathbb{H}$ is the upper half plane. We may then use the Schwarz reflection principle to define a holomorphic function $g_-(z) := \overline{g_+(\overline{z})}$. Then noticing that f_+ and f_- are both periodic modulo 2π, it suffices to check the limit in the problem for $x \in [0, 2\pi)$. Since $g_+(z) - g_-(\overline{z}) = 2i\mathrm{Im}(g_+)(z) = 2iv(\log(z)/i)$. We thus have

$$\lim_{\epsilon\to 0^+}(g_+(x+i\epsilon) - g_-(x-i\epsilon)) = \lim_{\epsilon\to 0^+} 2iv(\log(x+i\epsilon)/i) = 2iu(x)$$

so taking $f_+ = g_+/2i$ and $f_- = g_-/2i$, we obtain the desired result. $\square$

Problem 12. Let H be the vector space of entire functions $f : \mathbb{C} \to \mathbb{C}$ such that

$$\int_{\mathbb{C}} |f(z)|^2\, d\mu(z) \ < \ \infty.$$

Here $d\mu(z) = e^{-|z|^2} d\lambda(z)$ where $d\lambda(z)$ is Lebesgue measure on $\mathbb{C}$.

(a) Show that H is a closed subspace of $L^2(\mathbb{C}, d\mu)$.

(b) Show that for all $f \in H$ we have

$$f(z) \;=\; \frac{1}{\pi}\int_{\mathbb{C}} f(w)e^{z\overline{w}}\,d\mu(w), \qquad z \in \mathbb{C}.$$

Hint: Show that the normalized monomials

$$e_n(z) \;=\; \frac{1}{(\pi n!)^{1/2}} z^n$$

form an orthonormal basis for H.

Solution. (a) Suppose $\{f_n\}$ is a sequence in H converging in $L^2(\mu)$ to some $f \in L^2(\mu)$. We need to show that f is entire. To do that, it's enough to show that $f_n \to f$ uniformly on compact sets. Let R be fixed and fix a point z with $|z| \leqslant R$, and estimate (using the mean value property)

$$\begin{aligned}
|f_n(z) - f_m(z)| \;&=\; \left|\int_{|w-z|\leqslant 1} (f_n(w) - f_m(w))\,d\lambda(w)\right| \;\leqslant\; \int_{|w-z|\leqslant 1} |f_n(w) - f_m(w)|\,d\lambda(w) \\
&\leqslant\; \left(\int_{|w-z|\leqslant 1} |f_n(w) - f_m(w)|^2\,d\lambda(w)\right)^{1/2} \pi^{1/2} \\
&\lesssim\; \left(e^{(R+1)^2}\int_{|w-z|\leqslant 1} |f_n(w) - f_m(w)|^2 e^{-|w|^2}\,d\lambda(w)\right)^{1/2} \\
&\lesssim_R\; ||f_n - f_m||_{L^2(\mu)}\,.
\end{aligned}$$

This shows that the sequence f_n is uniformly Cauchy on each compact set. Thus the sequence f_n must have a uniform limit on each compact set, and since we already know $f_n \to f$ in L^2 this limit must be f, as desired. $\square$

(b) Following the hint, we calculate

$$\begin{aligned}
\int_{\mathbb{C}} z^n\overline{z^m}\,d\mu(z) \;&=\; \int_{\mathbb{C}} z^n\overline{z}^m e^{-|z|^2}\,d\lambda(z) \;=\; \int_0^\infty\int_0^{2\pi} (re^{i\theta})^n(r^{-i\theta})^m e^{-r^2} r\,d\theta\,dr \\
&=\; \int_0^\infty r^{n+m} re^{-r^2}\int_0^{2\pi} e^{i\theta(n-m)}\,d\theta\,dr.
\end{aligned}$$

This equals 0 if $n \neq m$, establishing the "ortho" part of orthonormal. We continue in the case $n = m$:

$$=\; 2\pi\int_0^\infty r^{2n} re^{-r^2}\,dr.$$

Integrating by parts will turn this integral into

$$2\pi n\int_0^\infty r^{2(n-1)} re^{-r^2}\,dr,$$

so we get a recurrence relation. Letting $I(n) = \int_0^\infty r^{2n} re^{-r^2}\,dr$, we have shown $I(n) = nI(n-1)$, and a simple calculation shows $I(0) = \int_0^\infty re^{-r^2}\,dr = 1/2$. Therefore $I(n) = (1/2)n!$, so

$$\int_{\mathbb{C}} |z^n|^2\,d\mu(z) \;=\; 2\pi I(n) \;=\; \pi n!,$$

which shows that e_n is an orthonormal family. To show it is actually a basis, just recall that any entire function f has a power series expansion, $f(z) = \sum_{n \geqslant 0} a_n z^n$. This is a statement of pointwise convergence, but the series also converges in $L^2(\mu)$ because

$$\left\| \sum_{n \geqslant N} a_n z^n \right\|_{L^2(\mu)}^2 = \sum_{n \geqslant N} |a_n|^2 \, ||z^n||_{L^2(\mu)}^2$$

is the tail of the convergent series

$$||f||_{L^2(\mu)}^2 = \left\| \sum_{n \geqslant 0} a_n z^n \right\|_{L^2(\mu)}^2 = \sum_{n \geqslant 0} |a_n|^2 \, ||z^n||_{L^2(\mu)}^2 .$$

and therefore tends to 0 as $N \to \infty$.

Now we address the original question. Let $f \in H$. From Plancherel's (Parseval's?) theorem we know that we can write $f = \sum_{n \geqslant 0} \langle f, e_n \rangle e_n$ where the convergence is in $L^2(\mu)$. But in the previous paragraph we have just shown that the pointwise power series representation of f also converges in $L^2(\mu)$ and therefore the Plancherel representation of f, since the basis elements e_n are multiples of the monomials z^n, must be the same as the power series representation of f, and therefore it holds pointwise. Therefore for each $z \in \mathbb{C}$ we have

$$\begin{aligned} f(z) &= \sum_{n \geqslant 0} \langle f, e_n \rangle e_n(z) = \sum_{n \geqslant 0} \frac{1}{\pi n!} \left(\int_{\mathbb{C}} f(w) \overline{w}^n \, d\mu(w) \right) z^n \\ &= \frac{1}{\pi} \int_{\mathbb{C}} f(w) \sum_{n \geqslant 0} \frac{1}{n!} (z\overline{w})^n \, \mu(w) = \frac{1}{\pi} \int_{\mathbb{C}} f(w) e^{z\overline{w}} \, d\mu(w) \end{aligned}$$

as desired. Interchaning the sum and the integral is allowed in this situation by the dominated convergence theorem because

$$\sum_{n \geqslant 0} \left| f(w)(z\overline{w})^n \frac{1}{n!} \right| = \frac{1}{\pi} |f(w)| \left| e^{|z||w|} \right|$$

is in $L^1(\mu)$ (as a function of w, for z fixed) by Cauchy-Schwarz:

$$\int_{\mathbb{C}} |f(w)| \left| e^{|z||w|} \right| \, d\mu(w) \leqslant \left(\int_{\mathbb{C}} |f(w)|^2 \, d\mu(w) \right)^{1/2} \left(\int_{\mathbb{C}} (e^{|z|})^{2|w|} e^{-|w|^2} \, d\lambda(w) \right)^{1/2} < \infty. \quad \square$$

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Problem 1. Given σ-finite measures $\mu_1, \mu_2, \nu_1, \nu_2$ on a measurable space $(X, \mathcal{X})$, suppose that $\mu_1 \ll \nu_1$ and $\mu_2 \ll \nu_2$. Prove that the product measures $\mu_1 \otimes \mu_2$ and $\nu_1 \otimes \nu_2$ on $(X \times X, \mathcal{X} \otimes \mathcal{X})$ satisfy $\mu_1 \otimes \mu_2 \ll \nu_1 \otimes \nu_2$ and the Radon-Nikodym derivatives obey

$$\frac{d(\mu_1 \otimes \mu_2)}{d(\nu_1 \otimes \nu_2)}(x, y) \;=\; \frac{d\mu_1}{d\nu_1}(x)\frac{d\mu_2}{d\nu_2}(y)$$

for $\nu_1 \otimes \nu_2$ almost every $(x, y) \in X \times X$.

Solution. The second part implies the first part in the following sense. Let $f_1 = \frac{d\mu_1}{d\nu_1}$ and $f_2 = \frac{d\mu_2}{d\nu_2}$ and define $f(x, y) = f_1(x)f_2(y)$ on $X \times X$. First of all it's clear that $f \in L^1(X \times X, \nu_1 \otimes \nu_2)$. If we prove that

$$\int g(x, y)f(x, y)\, d(\nu_1 \otimes \nu_2)(x, y) \;=\; \int g(x, y)\, d(\mu_1 \otimes \mu_2)(x, y)$$

for any measurable g on $X \times X$, then it follows that $\mu_1 \otimes \mu_2 \ll \nu_1 \otimes \nu_2$ and that f is the Radon-Nikodym derivative $\frac{\mu_1 \otimes \mu_2}{\nu_1 \otimes \nu_2}$.

To prove that identity, first note that if g is of the form $g(x, y) = g_1(x)g_2(y)$, then

$$\begin{aligned}
\int g(x, y)f(x, y)\, d(\nu_1 \otimes \nu_2)(x, y) \;&=\; \int g_1(x)f_1(x)g_2(y)f_2(y)\, d(\nu_1 \otimes \nu_2)(x, y) \\
&=\; \int g_1(x)f_1(x)\, d\nu_1(x) \cdot \int g_2(y)f_2(y)\, d\nu_2(y) \;=\; \int g_1(x)\, d\mu_1(x) \cdot \int g_2(y)\, d\mu_2(y) \\
&=\; \int g_1(x)g_2(y)\, d(\mu_1 \otimes \mu_2)(x, y) \;=\; \int g(x, y)\, d(\mu_1 \otimes \mu_2)(x, y).
\end{aligned}$$

So the desired identity holds for functions of that form, and by linearity it holds also for any linear combinations of functions of that form. Then by the monotone convergence theorem, the family of functions for which the identity holds is closed under monotone increasing pointwise limits. Thus by the monotone class theorem the identity holds for all measurable g, so we are done. $\square$

Problem 2. Let μ be a finite Borel measure on $\mathbb{R}$ with $\mu\{x\} = 0$ for all $x \in \mathbb{R}$ and let $\varphi(t) = \int_{\mathbb{R}} e^{itx}\, d\mu(x)$. Prove that

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\varphi(t)|^2\, dt \;=\; 0.$$

Solution. We start with a direct calculation:

$$\begin{aligned}
\frac{1}{2T}\int_{-T}^{T} |\varphi(t)|^2\, dt \;&=\; \frac{1}{2T}\int_{-T}^{T} \varphi(t)\overline{\varphi(t)}\, dt \\
&=\; \frac{1}{2T}\int_{-T}^{T}\int_{\mathbb{R}} e^{itx}\, d\mu(x) \int_{\mathbb{R}} e^{-ity}\, d\mu(y)\, dt \\
&=\; \frac{1}{2T}\int_{T}^{-T}\int_{\mathbb{R}^2} e^{it(x-y)}\, d\mu(x)\, d\mu(y)\, dt \\
&=\; \frac{1}{2T}\int_{\mathbb{R}^2}\int_{T}^{-T} e^{it(x-y)}\, dt\, d\mu(x)\, d\mu(y) \\
&=\; \int_{\mathbb{R}^2} f_T(x, y)\, d\mu(x)\, d\mu(y)
\end{aligned}$$

where $f_T(x, y)$ is defined to be 1 if $x = y$ and $\frac{\sin(T(x-y))}{T(x-y)}$ if $x \neq y$. This expression for f_T follows by computing the inner integral above using basic calculus. Now we note that $|f_T(x, y)| \leqslant 1$ for all T, x, y, and since μ is a finite measure we may apply the dominated convergence theorem to conclude that

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\varphi(t)|^2\,dt \;=\; \int_{\mathbb{R}^2} \lim_{T\to\infty} f_T(x,y)\,d\mu(x)\,d\mu(y) \;=\; \int_{\mathbb{R}^2} 1_{x=y}\,d\mu(x)\,d\mu(y).$$

So we will be done as soon as we show that the hypothesis that μ is non-atomic implies that $(\mu\times\mu)\{(x,y) : x=y\} = 0$. Let S_δ be the strip $|x-y|\leqslant\delta$. Then by Fubini's theorem we can calculate

$$\begin{aligned}
(\mu\times\mu)\{x=y\} &= \lim_{\delta\to 0}(\mu\times\mu)(S_\delta)\\
&= \lim_{\delta\to 0}\int_{\mathbb{R}^2} 1_{|x-y|\leqslant\delta}\,d\mu(x)\,d\mu(y)\\
&= \lim_{\delta\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}} 1_{|x-y|\leqslant\delta}\,d\mu(y)\,d\mu(x)\\
&= \lim_{\delta\to 0}\int_{\mathbb{R}} \mu[x-\delta,x+\delta]\,d\mu(x).
\end{aligned}$$

Since μ is a finite measure the integrand is bounded uniformly by some $C<\infty$ for all δ, x, so again by the dominated convergence theorem we conclude

$$(\mu\times\mu)\{x=y\} \;=\; \int_{\mathbb{R}} \lim_{\delta\to 0}\mu[x-\delta,x+\delta]\,d\mu(x) \;=\; \int_{\mathbb{R}} \mu\{x\}\,d\mu(x) \;=\; 0. \quad \square$$

Problem 3. Consider a measure space $(X,\mathcal{X})$ with σ-finite measure μ and let $p\in(1,\infty)$. Let $L^{p,\infty}$ be the set of measurable $f : X\to\mathbb{R}$ with $[f]_p = \sup_{t>0} t\mu\{|f|>t\}^{1/p}$ finite. Let

$$||f||_{p,\infty} \;=\; \sup_{E\in\mathcal{X},\,\mu(E)\in(0,\infty)} \frac{1}{\mu(E)^{1-1/p}}\int_E |f|\,d\mu.$$

Prove that there exist $c_1, c_2\in(0,\infty)$ – which may depend on p and μ – such that for all $f\in L^{p,\infty}$,

$$c_1[f]_p \;\leqslant\; ||f||_{p,\infty} \;\leqslant\; c_2[f]_p.$$

Solution. For $t>0$, let $E_t = \{|f|>t\}$. Then for each t we have

$$||f||_{p,\infty} \;\geqslant\; \frac{1}{\mu(E_t)^{1-1/p}}\int_{E_t}|f|\,d\mu \;\geqslant\; \frac{t\mu(E_t)}{\mu(E_t)^{1-1/p}} \;=\; t\mu(E_t)^{1/p}.$$

Taking the sup over $t>0$ yields $||f||_{p,\infty}\geqslant [f]_p$.

For the reverse inequality, let E be any set with $0<\mu(E)<\infty$. Then we can calculate

$$\begin{aligned}
\frac{1}{\mu(E)^{1-1/p}}\int_E |f|\,d\mu &= \frac{1}{\mu(E)^{1-1/p}}\int_0^\infty \mu\{x\in E : |f(x)|>t\}\,dt \;\leqslant\; \frac{1}{\mu(E)^{1-1/p}}\int_0^\infty \min(\mu(E),\mu\{|f|>t\})\,dt\\
&\leqslant \frac{1}{\mu(E)^{1-1/p}}\int_0^\infty \min(\mu(E),[f]_p^p/t^p)\,dt\\
&= \frac{1}{\mu(E)^{1-1/p}}\int_0^r \mu(E)\,dt + \frac{1}{\mu(E)^{1-1/p}}\int_r^\infty \frac{[f]_p^p}{t^p}\,dt \qquad \text{where } r=\frac{[f]_p}{\mu(E)^{1/p}}\\
&= \frac{r\mu(E)}{\mu(E)^{1-1/p}} + \frac{[f]_p^p}{\mu(E)^{1-1/p}}\left(\frac{1}{p}r^{-p+1}\right)\\
&= \left(1+\frac{1}{p}\right)[f]_p.
\end{aligned}$$

Then taking the sup over all such E gives $||f||_{p,\infty} \leqslant (1 + 1/p)[f]_p$. $\square$

Problem 4. Let $A \subseteq \mathbb{R}$ be measurable with positive Lebesgue measure. Prove that the set $A - A = \{z - y : z, y \in A\}$ has non-empty interior. Hint: consider the function $\varphi(x) = \int 1_A(x + y)1_A(y)\,dy$.

Solution. Without loss of generality we can assume that A has finite Lebesgue measure. First note that φ is non-negative. Next, note that $\varphi(x) > 0$ implies that $\{y : y, x + y \in A\}$ has positive Lebesgue measure, so in particular is not empty, which implies $x \in A - A$. Now consider the integral

$$\begin{aligned}\int \varphi(x)\,dx \;&=\; \int_{\mathbb{R}}\int_{\mathbb{R}} 1_A(x+y)1_A(y)\,dy\,dx \;=\; \int_{\mathbb{R}} 1_A(y)\int_{\mathbb{R}} 1_A(x+y)\,dx\,dy \\ &=\; \int 1_A(y)m(A-y)\,dy \;=\; m(A)\int 1_A(y)\,dy \;=\; m(A)^2 \;>\; 0.\end{aligned}$$

This implies that $\phi(x) > 0$ on a positive measure set of x. However, this doesn't yet imply that $\phi(x) > 0$ for all x in some open interval, but if we show also that φ is continuous, then that conclusion follows and we'll be done. To show that, we estimate

$$\begin{aligned}|\varphi(x) - \varphi(z)| \;&=\; \left|\int 1_A(x+y)1_A(y)\,dy - \int 1_A(z+y)1_A(y)\,dy\right| \;\leqslant\; \int 1_A(y)\,|1_A(x+y) - 1_A(z+y)|\;dy \\ &\leqslant\; \left(\int 1_A(y)^2\,dy\right)^{1/2}\left(\int |1_A(x+y) - 1_A(z+y)|^2\;dy\right)^{1/2} \;=\; m(A)^{1/2}\,||\tau_x 1_A - \tau_z 1_A||_{L^2}\,,\end{aligned}$$

where $\tau_x : L^2 \to L^2$ is the translation operator, $(\tau_x f)(t) = f(x + t)$. It's well known that for any fixed $f \in L^2$, $x \mapsto \tau_x f$ is a continuous map $\mathbb{R} \to L^2$, a fact that is proved by first considering compactly supported smooth functions and then using density. Thus φ is continuous. $\square$

Problem 5. Prove the following claim. Let H be a Hilbert space with the scalar product of x and y denoted by (x, y) and let $A, B : H \to H$ be linear operators with $(Bx, y) = (x, Ay)$ for all x, y. Then A and B are both bounded.

Solution. Suppose B were discontinuous. Then it is unbounded, so we can find a sequence $x_n \in H$ such that $\|x_n\| = 1$ and $\|Bx_n\| > n$. By the converse to Uniform Boundedness Principle, since the set of bounded linear functionals $(Bx_n, \cdot)$ is unbounded in norm, there is some $y \in H$ such that $\sup_n |(Bx_n, y)| = \infty$. Using our condition on B, this says that $\sup_n |(x_n, Ay)| = \infty$. But this is impossible, as we can bound $|(x_n, Ay)|$ above by $\|Ay\|$. This is a contradiction, so B must be continuous. Repeating this argument with A and B switched shows that A is continuous as well. $\square$

Problem 6. Recall that $\ell^\infty(\mathbb{N})$ is a Banach space with respect to the norm $||x||_\infty = \sup_{n\geqslant 1} |x_n|$.

(a) Prove that there exists a continuous linear function ϕ on $\ell^\infty(\mathbb{N})$ such that

$$\phi(x) \;=\; \lim_{n\to\infty} x_n$$

whenever the limit exists.

(b) Prove that this ϕ is not unique.

Solution. (a) The set C of all sequences in $\ell^\infty(\mathbb{N})$ for which the limit exists is closed. This is because if $x^{(m)}$ is a sequence of sequences converging to a limit x, then

$$\limsup_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n^{(m)} + \|x - x^{(m)}\|_\infty = \|x - x^{(m)}\|_\infty + \lim_{n\to\infty} x_n^{(m)}$$

and

$$\liminf_{n\to\infty} x_n \geqslant \liminf_{n\to\infty} x_n^{(m)} - \|x - x^{(m)}\|_\infty = -\|x - x^{(m)}\|_\infty + \lim_{n\to\infty} x_n^{(m)}$$

for every m. Letting m go to infinity shows that $\limsup x_n$ and $\liminf x_n$ are equal.

The set C is closed and the operator L defined on C that sends a sequence to its limit is linear, both by the standard properties of sequential limits. It is also bounded, because the limit of a sequence can be no more than its supremum. By the Hahn-Banach Theorem, we can extended the operator L to be a continuous linear functional on all of $\ell^\infty(\mathbb{N})$, which we can call ϕ. $\square$

(b) Let z be the sequence defined by $z_n = (-1)^n$ and let D be the span of C and z, which is also a closed vector space. We can extend L to be a bounded linear operator L' on D by choosing a value for $L'(z_n)$. Again, we can use Hahn-Banach to extend L' to ϕ on $\ell^\infty(\mathbb{N})$. Any two different choices for $L'(z_n)$ necessarily give different choices for ϕ. $\square$

Problem 7. Let $J \subseteq \mathbb{R}$ be a compact interval and let μ be a finite Borel measure whose support lies in J. For $z \in \mathbb{C}\backslash J$ define

$$F_\mu(z) \;=\; \int_{\mathbb{R}} \frac{1}{z-t}\, d\mu(t).$$

Prove that the mapping $\mu \mapsto F_\mu$ is one-to-one.

Solution. Suppose $F_\mu = F_\nu$. To conclude that $\mu = \nu$ it's enough to show that $\int f\, d\mu = \int f\, d\nu$ for all continuous $f : J \to \mathbb{C}$, and in fact we just need to show it for a dense set of such f. To do this we use the Stone-Weierstrass theorem. Let $\mathcal{F}$ be the closed (in the sup norm) linear span of the family $\{t \mapsto \frac{1}{z-t} : z \in \mathbb{C}\backslash J\}$. By definition, $\mathcal{F}$ is closed under linear combinations. We also know that $\mathcal{F}$ is closed under multiplication because of partial fraction decomposition: $\frac{1}{(z-t)(w-t)}$ can always be written as $\frac{A}{z-t} + \frac{B}{w-t}$ for some complex scalars A, B. It's clear that $\mathcal{F}$ separates points of J because each function $t \mapsto \frac{1}{z-t}$ is injective, and it's also clear that $\mathcal{F}$ is closed under conjugation because $\overline{\frac{1}{z-t}} = \frac{1}{\bar{z}-t}$. All that remains is to show that $\mathcal{F}$ contains a nonzero constant. Let M be a real number that is much larger than either of the endpoints of J and consider the element of $\mathcal{F}$ $t \mapsto \frac{M}{M-t}$. It's clear that

$$\sup_{t\in J}\left|\frac{M}{M-t} - 1\right| \to 0$$

as $M \to \infty$ and therefore the constant function 1 belongs to $\mathcal{F}$, so by the Stone-Weierstrass theorem, $\mathcal{F}$ is equal to all of $C(J, \mathbb{C})$. But by hypothesis, $\int f\, d\mu = \int f\, d\nu$ for each f of the form $t \mapsto \frac{1}{z-t}$, and since $\mathcal{F}$ is the closed linear span of such functions, we in fact have $\int f\, d\mu = \int f\, d\nu$ for all $f \in \mathcal{F} = C(J, \mathbb{C})$ and therefore $\mu = \nu$. $\square$

Problem 8. A function $f : \mathbb{C} \to \mathbb{C}$ is entire and has the property that $|f(z)| = 1$ when $|z| = 1$. Prove that $f(z) = az^n$ for some integer $n \geqslant 0$ and some $a \in \mathbb{C}$ with $|a| = 1$.

Solution. Consider the behavior of f just on the unit disc. Referring to Spring 2019 #9, we see that f can be written as a Blaschke product

$$f(z) \;=\; \lambda \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z}$$

where the a_j are the zeros of f in the unit disc. Now we can extend this formula to all of $\mathbb{C}$ by noting that $\lambda \prod (z - a_j)$ and $f(z) \prod (1 - \overline{a_j} z)$ are two entire functions which agree on the unit disc and applying uniqueness of analytic continuation. Now if any of the a_j were nonzero, then the Blaschke product formula for f would imply that f has a pole at $z = 1/\overline{a_j}$, which is impossible. Therefore all of the a_j must be zero, so the Blaschke product reduces to $f(z) = \lambda z^n$. $\square$

Problem 9. Determine the number of zeros of the polynomial

$$P(z) \;=\; z^6 - 6z^2 + 10z + 2$$

in the annulus $1 < |z| < 2$. Prove your claim.

Solution. Let $f(z) = z^6$ and let $\tilde{f}(z) = 10z$. On the circle $|z| = 2$, we have $|f(z)| = 2^6 = 64$ and $|f(z) - P(z)| = |6z^2 - 10z - 2| \leqslant 6(2^2) + 10(2) + 2 = 46 < 64$, so by Rouche's theorem, f and P have the same number of zeros inside $|z| < 2$, which is six. Now on the cricle $|z| = 1$, we have $|\tilde{f}(z)| = 10$ and $|\tilde{f}(z) - P(z)| = |-z^6 + 6z^2 - 2| \leqslant 1 + 6 + 2 = 9 < 10$, so again by Rouche's theorem we know $\tilde{f}$ and P have the same number of zeros in $|z| < 1$, which is one. Therefore P has five zeros in the annulus $1 < |z| < 2$. □

Problem 10. Evaluate
$$\lim_{x\to\infty} \int_0^x \sin(t^2)\, dt.$$

Solution. Write $\sin(t^2) = \frac{e^{it^2} - e^{-it^2}}{2i}$. For $R > 0$, let C_R denote the positively oriented contour which is the boundary of the set $S = \{z \in \mathbb{C} : |z| < R : 0 \leqslant \arg(z) \leqslant \frac{\pi}{4}\}$. By Cauchy's theorem,

$$0 = \int_{C_R} e^{it^2}\, dt = \int_0^R e^{it^2} dt + Ri \int_0^{\frac{\pi}{4}} e^{iR^2 e^{2i\theta}} e^{i\theta} d\theta - \frac{(1+i)\sqrt{2}}{2} \int_0^R e^{-t^2} dt = A + B + C$$

For R sufficiently large, we have

$$\begin{aligned}
|B| &\leqslant R \int_0^{\frac{\pi}{4}} e^{-R^2 \sin(2\theta)} d\theta \\
&= R \int_0^{1/R^{3/2}} e^{-R^2 \sin(2\theta)} d\theta + R \int_{R^{-3/2}}^{\frac{\pi}{4}} e^{-R^2 \sin(2\theta)} d\theta \\
&\leqslant R^{-1/2} + Re^{-2R^{1/2}} \frac{\pi}{4} \to_{R\to\infty} 0.
\end{aligned}$$

Hence
$$\lim_{R\to\infty} \int_0^R e^{it^2} dt - \frac{(1+i)\sqrt{2}}{2} \lim_{R\to\infty} \int_0^R e^{-t^2} dt = 0.$$

But by a calculation due to Gauss (Gaussian integral)
$$\lim_{R\to\infty} \int_0^R e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$$

and hence
$$\lim_{R\to\infty} \int_0^R e^{it^2} dt = \frac{(1+i)\sqrt{2\pi}}{4}$$

By using the contour corresponding to the boundary of the set $T = \{z \in \mathbb{C} : \frac{7\pi}{4} \leqslant \arg(z) \leqslant 2\pi\}$, we can argue similarly that
$$\lim_{R\to\infty} \int_0^R e^{-it^2} dt = \frac{(1-i)\sqrt{2\pi}}{4}$$

and hence
$$\lim_{R\to\infty} \int_0^R \sin(t^2) dt = \frac{\sqrt{2\pi}}{4}. \quad \square$$

Problem 11. Find a conformal map of the domain
$$D \;=\; \{z \in \mathbb{C} : |z-1| < \sqrt{2}, |z+1| < \sqrt{2}\}$$

onto the open unit disc centered at the origin. It suffices to write this map as a composition of explicit conformal maps.

Solution. First note the intersection points of the two circles $|z-1| = \sqrt{2}$, $|z+1| = \sqrt{2}$ are $\pm i$. Start by applying $z \mapsto \frac{z-i}{z+i}$. This takes i to 0, $-i$ to ∞, and maps circles to circles, so it will map both circles $|z-1| = \sqrt{2}$ and $|z+1| = \sqrt{2}$ to lines through the origin. We can calculate that it maps $1 + \sqrt{2}i$ to a real multiple of $1-i$ and $-1-\sqrt{2}i$ to a real multiple of $1+i$, and therefore the boundary the region gets mapped to the "X" shape $\{x = y\} \cup \{x = -y\}$. To figure out where the interior goes, just choose some interior points and see where they go, and we find that our original region gets mapped to the region $\{y > x\} \cap \{y < -x\}$. Once this is done it can be transformed into the unit disc using mostly standard maps. First apply $z \mapsto -z$ to flip everything, and then apply the principal branch ($\arg(z) \in (-\pi, \pi)$) of $z \mapsto z^{2/3}$ to transform it into the right half-plane. Finally apply $z \mapsto iz$ and then $z \mapsto \frac{z-i}{z+i}$ to get to the unit disc. $\square$

Problem 12. Show that

$$F(z) \;=\; \int_1^\infty \frac{t^z}{\sqrt{1+t^3}}\,dt$$

is well defined (by the integral) and analytic in $\{z : \mathrm{Re}(z) < 1/2\}$ and admits a meromorphic continuation to the region $\{z : \mathrm{Re}(z) < 3/2\}$.

Solution. If $\mathrm{Re}(z) < \frac{1}{2}$ then

$$|F(z)| \leqslant \int_1^\infty \left|\frac{t^z}{\sqrt{1+t^2}}\right| \leqslant \int_1^\infty \frac{t^{\mathrm{Re}(z)}}{\sqrt{1+t^3}}.$$

Since the integrand on the right is asymptotically $t^{\Re z - 3/2}$, the integral for F converges absolutely. We can differentiate under the integral sign (which is valid since the integrand is continuously differentiable) to get

$$F'(z) = \int_1^\infty \frac{t^z \ln t}{\sqrt{1+t^3}} dt$$

which is well-defined by the same argument as above. Thus F is analytic.

Using integration by parts with the factors $\frac{1}{z-1/2}t^{z-1/2}$ and $\frac{1}{\sqrt{1+t^{-3}}}$ we compute

$$F(z) = \frac{1}{z-1/2} - \frac{3}{2z-1}\int_1^\infty \frac{t^{z-9/2}}{(1+t^{-3})^{3/2}} dt.$$

This integral converges absolutely for $\mathrm{Re}(z) < \frac{7}{2}$. Again if we take the derivative of the integrand, we only pick up an additonal factor of ln, so the function is meromorphic in this region. $\square$

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Problem 1. Assume $f \in C_c^\infty(\mathbb{R})$ satisfies

$$\int_{\mathbb{R}} e^{-tx^2} f(x)\,dx \;=\; 0 \qquad \text{for any } t \geqslant 0.$$

Show that $f(x) = -f(-x)$ for any $x \in \mathbb{R}$.

Solution. Say the support of f is contained in $[-M, M]$. Let $g(x) = f(x) + f(-x)$ and note that it is enough to show that $g = 0$ on the interval $[0, M]$. It's clear that

$$\int_0^M g(x)e^{-tx^2}\,dx \;=\; \int_{\mathbb{R}} f(x)e^{-tx^2}\,dx \;=\; 0$$

for all $t \geqslant 0$. Let $\mathcal{F} = \operatorname{span}\{x \mapsto e^{-tx^2} : t \geqslant 0\} \subseteq C([0, M], \mathbb{R})$. We see that $\mathcal{F}$ is closed under linear combinations by definition, and closed under multiplication because $e^{-tx^2}e^{-sx^2} = e^{-(t+s)x^2}$. We also see that $\mathcal{F}$ contains a nonzero constant by taking $t = 0$. Finally, it's clear that $\mathcal{F}$ separates points because each $x \mapsto e^{-tx^2}$ is strictly decreasing on $[0, M]$. Therefore the Stone-Weierstrass theorem implies that $\mathcal{F}$ is dense in $C([0, M], \mathbb{R})$, so by the equation above and a standard argument it follows that $\int_0^M g(x)\phi(x)\,dx = 0$ for all continuous ϕ and therefore $g = 0$ on $[0, M]$ as desired. $\square$

Problem 2. Assume $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of differentiable functions satisfying

$$\int_{\mathbb{R}} |f_n(x)|\,dx \;\leqslant\; 1 \qquad \text{and} \qquad \int_{\mathbb{R}} |f_n'(x)|\,dx \;\leqslant\; 1.$$

Assume also that for any $\epsilon > 0$ there exists $R(\epsilon) > 0$ such that

$$\sup_n \int_{|x| \geqslant R(\epsilon)} |f_n(x)|\,dx < \epsilon.$$

Show that there exists a subsequence of $\{f_n\}$ that converges in $L^1(\mathbb{R})$.

Solution.

Problem 3. Prove that $L^\infty(\mathbb{R}^n) \cap L^3(\mathbb{R}^n)$ is a Borel subset of $L^3(\mathbb{R}^n)$.

Solution. We can write

$$L^\infty \cap L^3 \;=\; \bigcup_{M=1}^{\infty} \{f \in L^3 : |f| \leqslant M \text{ a.e.}\}.$$

We claim that each of these sets is closed in the L^3 topology. Suppose $\{f_n\}$ is a sequence satisfying $|f_n| \leqslant M$ a.e. for all n and suppose $f_n \to f$ in L^3. Then there is a subsequence converging to f almost everywhere, so also $|f| \leqslant M$ a.e. Therefore $L^\infty \cap L^3$ is a countable union of closed sets, so it's Borel. $\square$

Problem 4. Fix $f \in L^1(\mathbb{R})$. Show that

$$\lim_{n \to \infty} \int_0^2 f(x) \sin(x^n)\,dx \;=\; 0.$$

Solution.

Problem 5. Rigorously determine the infimum of

$$\int_{-1}^1 \Big| P(x) - |x| \Big|^2\,dx$$

over all choices of polynomials P of degree at most three.

Solution. Consider the Hilbert space $H = L^2([-1,1])$ and the (finite-dimensional and therefore closed) subspace M_3 of polynomials of degree at most three. Let $f(x) = |x|$. We are trying to calculate $\inf_{P \in M_3} ||P - f||_{L^2}^2$. It is a standard fact about Hilbert spaces that this is uniquely attained by the orthogonal projection of f onto M_3. To find this we can just choose an orthonormal basis $\{e_0, e_1, e_2, e_3\}$ for M_3 and then the orthogonal projection is given by $P^* := \sum_i \langle f, e_i \rangle e_i$, and the squared distance is given by $||f - P^*||_{L^2}^2 = ||f||_{L^2}^2 - \sum_i |\langle f, e_i \rangle|^2$. We will choose for our basis the normalized Legendre polynomials:

$$e_0 = \frac{1}{\sqrt{2}}, \quad e_1 = \sqrt{3/2} \cdot x, \quad e_2 = \sqrt{5/8} \cdot (3x^2 - 1), \quad e_3 = \sqrt{7/8} \cdot (5x^3 - 3x)$$

and then it is straightforward to calculate that the orthogonal projection is given by $P^*(x) = 1/2 + (5/16)(3x^2 - 1)$, and the squared distance is $1/96$. $\square$

Problem 6. Define a sequence of linear functionals on $L^\infty(\mathbb{R})$ as follows:

$$L_n(f) \;=\; \frac{1}{n!} \int_0^\infty x^n e^{-x} f(x)\, dx.$$

(a) Prove that no subsequence of this sequence converges in weak-$*$.
(b) Explain why this does not contradict the Banach-Alaoglu theorem.

Solution.

Problem 7. Let $\mathcal{F}_M$ be the set of functions holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ that satisfy

$$\int_0^{2\pi} |f(e^{it})|\, dt \;\leqslant\; M \;<\; \infty.$$

Show that every sequence $\{f_n\}$ in $\mathcal{F}_M$ contains a subsequence that converges uniformly on compact subsets of $\mathbb{D}$.

Solution.

Problem 8. For each $z \in \mathbb{C}$, let

$$F(z) \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{(z/2)^{2n}}{(n!)^2}.$$

(a) Show that F is an entire function and satisfies $|F(z)| \leqslant e^{|z|}$.
(b) Show that there is an infinite collection of numbers $a_n \in \mathbb{C}$ so that

$$F(z) \;=\; \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{a_n^2}\right)$$

and the product converges uniformly on compact subsets of $\mathbb{C}$.

Solution. (a) To show that F is entire, it's enough to compute the radius of convergence of the power series

$$\frac{1}{R} \;=\; \limsup_{n \to \infty} \left(\frac{1}{2^{2n}(n!)^2}\right)^{1/n}.$$

This limit is easily evaluated to be 0, for example by using Stirling's formula. Therefore the radius of convergence is infinite so F is entire. For the upper bound, we have

$$e^{|z|} \;=\; \sum_{n \geqslant 0} \frac{|z|^n}{n!} \;\geqslant\; \sum_{n \geqslant 0} \frac{|z|^{2n}}{(2n)!} \;\geqslant\; \sum_{n \geqslant 0} \frac{|z|^{2n}}{2^{2n}(n!)^2} \;=\; F(z).$$

The last inequality follows from the elementary inequality $2^{2n}(n!)^2 \geqslant (2n)!$ for all $n \geqslant 1$, which is easily proved by induction. $\square$

(b) The upper bound from the previous part implies that the order of F is at most one, and by Hadamard's theorem this implies the genus of F is also at most one. Let $\{b_n\}$ be an enumeration of all the zeros of F. Note that $F(0) \neq 0$. Then by Weierstrass's theorem we know F has a product expansion of the form either

$$F(z) \;=\; e^{g(z)} \prod_{n\geqslant 1} \left(1 - \frac{z}{b_n}\right) \exp(z/b_n) \qquad \text{or} \qquad F(z) \;=\; C \prod_{n\geqslant 1} \left(1 - \frac{z}{b_n}\right)$$

depending on if the genus is 0 or 1, where $g(z)$ is a polynomial of degree at most 1. The theorem also guarantees that this product converges uniformly on compact sets. Now note that F is an even function, so all of its zeroes come in $\pm$ pairs. Therefore if $\{a_n\}$ is an enumeration of one choice from each pair of zeros of F, the product can be arranged as

$$F(z) \;=\; e^{g(z)} \prod_{n\geqslant 1} \left(1 - \frac{z}{a_n}\right)\left(1 + \frac{z}{a_n}\right) \;=\; e^{g(z)} \prod_{n\geqslant 1} \left(1 - \frac{z^2}{a_n^2}\right).$$

It remains to show that $g(z)$ is necessarily identically zero. We know that F is an even function, and the infinite product is also an even function, so $e^{g(z)}$ is also even. Since g is a polynomial of degree at most 1 this implies that g is constant. Finally, we note that $F(0) = 1$ and therefore $g(z)$ must be identically zero, obtaining the desired expression. $\square$

Problem 9. Let $f \in L^1(\mathbb{C}) \cap C^1(\mathbb{C})$ where L^1 is with respect to two dimensional Lebesgue measure and C^1 is in the real variable sense. Show that the integral

$$u(z) \;=\; -\frac{1}{2\pi} \iint\limits_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z}\, d\lambda(\zeta)$$

(where λ is two-dimensional Lebesgue measure) defines a C^1 function on the whole complex plane that satisfies

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) u(x+iy) \;=\; f(x+iy).$$

Solution.

Problem 10. Evaluate the improper Riemann integral

$$\int_0^\infty \frac{x^2-1}{x^2+1} \frac{\sin x}{x}\, dx.$$

Solution.

Problem 11. Let $\mathbb{T} = \partial\mathbb{D}$ be the unit circle and let $K \subsetneq \mathbb{T}$ be a compact proper subset.
(a) Show that there is a sequence of polynomials $P_n(z)$ so that $P_n(z) \to \overline{z}$ uniformly on K.
(b) Show that there is no sequence of polynomials $P_n(z)$ for which $P_n(z) \to \overline{z}$ uniformly on $\mathbb{T}$.

Solution.

Problem 12. Let u be a continuous subharmonic function on $\mathbb{C}$ that satisfies

$$\limsup_{|z|\to\infty} \frac{u(z)}{\log|z|} \;\leqslant\; 0.$$

Show that u is constant on $\mathbb{C}$.

Solution.