SCHRAMM LOEWNER EVOLUTION : SCALING LIMIT OF LOOP-ERASED RANDOM WALK

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Abstract. In this report we will present the basic definitions and results in the study of Schramm-Loewner evolution, specifically in its identity as the scaling limit of loop-erased random walks. Through this, we define various forms of random processes such as random walks, Brownian motion, loop-erased random walks, and self-avoiding walks and the common technique of defining a measure to analyze their limits. A brief discussion of open problems and the history of these topics is included.

1 Introduction

This paper aims to provide an introduction to Schramm Loewner evolution and the scaling limit of loop-erased random walks. The goal is to present the topics at a level accessible to students with a basic knowledge of complex analysis, measure theory, and statistics. For that reason, this paper will not go into great technical detail and will instead focus on the intuition and broad concepts.

The paper will begin with an overview of random walks and Brownian motion, with which the idea of a scaling limit will be introduced. From there, loop-erased random walks will be defined along with a measure on their space. From there, Schramm-Loewner evolution will be defined along with the Loewner equation and an overview of the proof that loop-erased random walks converge to Schramm-Loewner evolution will be provided. The paper will conclude with a section on self-avoiding random walks, which is an open problem.

The information in this paper is taken mostly from Oded Schramm's original paper on the topic[12], Schramm and Lawler's subsequent proof of the scaling limit[7], and Lawler's notes on these topics[5][6].

2 Simple Random Walks

To gain intuition on Brownian motion and to better understand the formulation of loop erased random walks, we first aim to define simple random walks. This section has the added benefit of introducing the construction of measures for random processes, which is central idea in this paper. For ease of understanding, we often drop the "simple" adjective of simple random walks, as it becomes confusing with the discussion of "simple" paths.

If confused on the later discussion of Brownian motion, the reader is encouraged to envision a corresponding situation with random walks. In the same vein, the reader is encouraged to think of Brownian motion as a continuous random walk, or conversely of random walks as discrete Brownian motion. Random walks have the advantage of being simple in concept and easy to physically construct.

Heuristically, a random walk is a path on a lattice $\mathbb{Z}^d$, such that each step is chosen randomly. We will restrict our attention to the lattice $\mathbb{Z} + i\mathbb{Z}$, but the definition holds with the obvious adjustments in any dimension. A random walk of length n is then a path consisting of n steps in this lattice. Using random variables, we formalize this as follows.

Definition 2.1. Define $X_1, X_2, \dots$ to be independent and identically distributed random variables with image in $\{\pm 1, \pm i\}$ and let there exist some $\omega_0 \in \mathbb{Z} + i\mathbb{Z}$. Then a **random walk** starting at ω_0, is an at most countable sequence of points $\omega = [\omega_0, \omega_1, \dots]$ such that

$$\omega_k = \omega_0 + \sum_{i=1}^{k} X_i$$

If specified, a random walk of finite length n is taken to be a finite sequence $\omega = [\omega_0, \omega_1, \dots, \omega_n]$. For a random walk ω, we let $|\omega| \in \mathbb{N} \cup \{\infty\}$ denote its length. The notation of ω denoting a random walk is chosen to align with Lawler's notes on SLE[5], and Schramm's original paper [12]. It should be noted here that no distribution has been inherently specified for the random variables X_j. In practice any distribution can be chosen, but in the absence of other information a uniform distribution is the default. In particular, this implies that $X_1, X_2, \dots$ have mean 0 and variance 1. For this paper, we will work with this specification.

Given a random walk ω, we can use linear interpolation to extend ω to a continuous path $\omega : [0, \infty) \to \mathbb{C}$ where $\omega(j) = \omega_j$ and likewise for finite walks. For the remainder of this paper we make this identification when needed without comment.

2.1 Measure on Random Walks

In the process of defining random walks, we defined a sequence of points $\omega_0, \omega_1, \dots$ where each point ω_k was determined as the sum of random variables. Because of this, we can take the points ω_k as random variables. By considering the distribution of these random variables, we can assign to each random walk a probability that it occurs and define a measure that assigns said probability to each path. Because the random process must always produce a path, this will be a probability measure.

Formally, we first define the space of random walks that we are considering. This can be random walks without constraints, however it will be useful to allow conditions on the random walks. A wide variety of conditions can be required, but we restrict our attention to fixed starting points in specified regions.

Let D be a nonempty simply connected open strict subset of $\mathbb{C}$. We approximate D by an adjusted lattice G consisting of the interior vertices $D \cap (\mathbb{Z} + i\mathbb{Z})$ and the boundary vertices where the edges of $\mathbb{Z} + i\mathbb{Z}$ intersect the boundary of D. we then define $\Lambda^*_{a,D}$ for $a \in D$ to be the collection of random walks in D that start at a', where a' is a fixed vertex in $(\mathbb{Z} + i\mathbb{Z})$ closest to a, and end at the boundary vertices of G.

We can then define a measure on $\Lambda^*_{a,D}$ by assigning to each walk in $\Lambda^*_{a,D}$ the probability that it occurs randomly. We denote this measure by $\mu^*_{a,D}$ and extend this measure to the space of paths in D by setting $\mu^*_{a,D}$ to 0 outside of $\Lambda^*_{a,D}$. Because a random walk must be created, it follows that $\mu^*_{a,D}$ is a probability measure.

In practice, when the space D is large relative to $\mathbb{Z} + i\mathbb{Z}$ and a is far from ∂D this measure often takes the form $\mu^*_{a,D}(\omega) \approx (2d)^{-|\omega|}$ where d is the dimension of space. For our purposes, this implies that our measure is approximately defined by $\omega \mapsto 4^{-|\omega|}$. With this measure constructed, we can discuss the limit of random walks by considering weak convergence, or convergence in distribution.

3 Brownian Motion

To discuss the limit of simple random walks and to formally describe Schramm-Loewner Evolution, we must first define Brownian motion. For intuition's sake, it should be noted that Brownian motion can thought of as a random walk on a very fine grid.

Formally, Brownian motion is defined as follows.

Definition 3.1. A Brownian motion $B^{x_0} : [0, \infty) \to \mathbb{R}$ with starting point $x_0 \in \mathbb{R}$ is a random, almost surely continuous function such that $B^{x_0}(0) = x_0$ almost surely and such that for any sequence $t_1 < t_1 < \cdots < t_n$, the increments $B^{x_0}(t_{i+1}) - B^{x_0}(t_i)$ are independent normally distributed random variables with mean 0 and variance $t_{i+1} - t_i$. Formally, this is called a Wiener process.[13]

The inclusion of almost surely in this statement is a byproduct of probability theory. In practice, and especially when Brownian motions are considered as limits of random walks, this can be understood as continuous.

3.1 Important Properties

There are two properties of Brownian motion that will be used to show the result on the scaling limit of LERWs. These are the Markov proper and conformal invariance.

A common and important property of Brownian motion is the Markov property. Intuitively, the Markov property states that Brownian motion is memoryless in the fact that only the present state will affect future states. For Brownian motion, this is formally stated as follows

Theorem 3.2. *Let $B(t)$ be a Brownian motion. Then the process $B(s+t)$ for $t \in [0, \infty)$ is a Brownian motion started at $B(s)$ and is independent of the process $B(t)$ for $t \in [0, s)$.*

The proof of this fact is immediate from the definition of Brownian motion.

A perhaps more important but far less trivial property of Brownian motion in two dimensions is conformal invariance. Informally, conformal invariance asserts that conformal maps take Brownian motions in one space to Brownian motions in another. Formally, this is stated as follows.

Theorem 3.3. [13] (Levy's Theorem) : *Let $\phi : U \to V$ be a conforal map between open subsets of the complex plane and let $B : [0, T) \to U$ be a Brownian motion with initial point z_0. Define $\tau : [0, T) \to [0, \infty)$ by*

$$\tau(t) = \int_0^T |\phi'(B(s))|^2 ds$$

and define $T' = \lim_{t \to T} \tau(t)$. Then τ is a homeomorphism from $[0, T)$ to $[0, T')$ and

$$\tilde{B}(\tau(t)) = \phi(B(t))$$

is a Brownian motion in V with initial point $\phi(z_0)$.

We do not provide a proof here for the sake of brevity, but the reader is encouraged to consult Berestycki and Norris's notes on SLE [1] and Terrence Tao's notes on the subject on his blog [13].

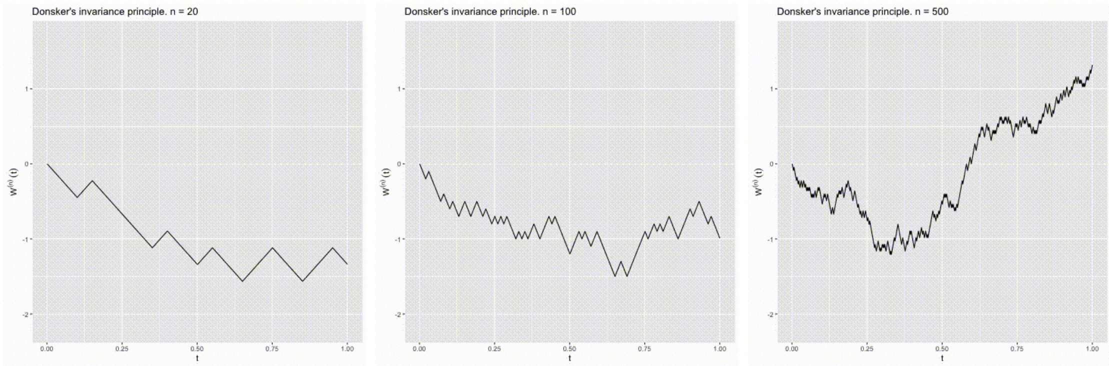

FIGURE 1. Convergence of simple random walk to Brownian motion, sampled at $n = 20, 100, 500$. Taken from Wikimedia Commons, 2019 [9]

3.2 Measure on Brownian Motion

In the same way that we defined a measure on the space of random walks adhering to some set of constraints, we can also define a measure on the space of Brownian motions. To do so, we consider a Brownian motion B. The point $B(t)$ is a random variable and so can be given a distribution that depends on our constraints. We can then extend this and assign a distribution to the motion as a whole. This distribution defines a measure on the space of Brownian motions and can be extended to the space of all paths adhering to the same constraints in the obvious way.

Unlike simple random walks, measures on Brownian motions are zero for any specific path. Because of this, it is only interesting to consider the measure of an uncountable set of paths.

3.3 Limit of Simple Random Walks

Let $\omega = [\omega_0, \omega_1, \dots]$ denote a random walk on $\mathbb{Z}^d$ from a fixed starting point ω_0. From ω, we can define a diffusively rescaled random walk $W^{(n)} : [0, \infty) \to \mathbb{R}^d$ by

$$B^{(n)}(t) = \frac{\omega_{\lfloor nt \rfloor}}{\sqrt{n}}$$

As shown in Donsker's theorem [3], which can be thought of as akin to the central limit theorem, $B^{(n)}(t)$ converges in distribution to a normally distributed random variable $B(t)$ as $n \to \infty$. By definition, this is equivalently saying that rescaled random walks converge in distribution to Brownian motion. Because the measure on random walks and Brownian motion were defined in terms of these distributions, this implies that the measure on simple random walks converges weakly to the measure on Brownian measure. The visual idea of this convergence is shown in figure ??.

4 LOOP-ERASED RANDOM WALK

The scaling limit of simple random walks has long been known to be Brownian motion, and in fact, it is an intuitive, almost obvious result. However, by restricting the random paths that we consider, the scaling limit becomes much more interesting. To that end,

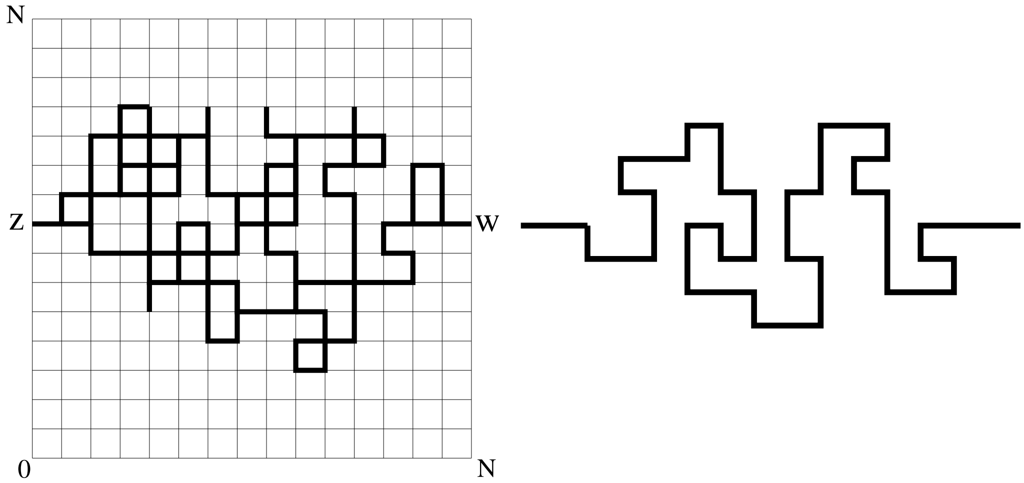

FIGURE 2. Example of Loop-Erased Random Walk, before and after loop erasure. Credit to Lawler for the images.[5]

we consider LERWs, a process that creates random walks without self-intersections by erasing loops from simple random walks.

We first detail how a LERW is constructed. Let $\omega = [\omega_0, \dots, \omega_n]$ be a random walk in $\mathbb{Z} + i\mathbb{Z}$. We construct its loop-erasure $LE(\omega)$ as follows. Define

$$j_0 = \max\{i : \omega_i = \omega_0\}$$

We then iteratively construct j_k for $k > 0$ as

$$j_k = \max\{i : \omega_i = \omega_{j_{k-1}+1}$$

Repeating this process until $j_\ell = n$ for some $\ell \geq 0$. We then let $LE(\omega) = [\omega_{j_0}, \dots, \omega_{j_\ell}]$. With this construction, $LE(\omega)$ is a subpath of ω without self-intersections. This process is illustrated in figure **??**. [5]

We can further restrict this process by specifying the domain in which we are working or the LERWs that we care about. For instance, we may require that the LERWs be contained in $(\mathbb{Z}+i\mathbb{Z}) \cap \mathbb{U}$, as we will do later with our scaling argument, or we may require that the LERWS have fixed start and end point. This is equivalent to constraining the random walks that are fed into the loop-erasure process.

4.1 Construction of a Measure

Let D be a domain as in the random walk case and $\Lambda_{a,D}$ denote the space of LERWs starting at $a \in D$ and ended at the boundary vertices of D. Much like in the random walk case, this constraint can in practice be anything, but we are concerned with this constraint specifically.

On $\Lambda^*_{a,D}$ we defined a measure $\mu^*_{a,D} : \Lambda^*_{a,D} \to [0, \infty)$ such that $\mu^*_{a,D}$ assigns to each $\omega \in \Lambda^*_{a,D}$ the probability that said random walk occurs. We then define a probability measure $\mu_{a,D} : \Lambda_{a,D} \to [0, 1]$ by

$$\mu(\eta) = \sum_{\substack{\omega \in \Lambda^*_{a,D} \\ LE(\omega)=\eta}} \mu^*(\omega) \tag{4.1}$$

As was done for random walks, we extend $\mu_{a,D}$ to all paths in D. This is equivalent to the formulation given in section 2 of Schramm's original paper on SLE [12], which treats LE as a random closed subset of $\overline{D}$ and then finds the measure μ as the probability distribution of LE.

As any easier formulation, we may instead simply assign the measure $4^{-|\omega|}$ to each $\omega \in \Lambda^*$, and then form μ by equation 4.1. Though different initially, this will converge to the same measure when the scaling limit is taken.[5]

5 Schramm-Loewner Evolution

Schramm Loewner evolution is a process of generating random curves in complex space by driving the Loewner equation with Brownian motion. As we are concerned with the scaling limit of LERW, we focus on radial SLE, which generates curves in the unit disk.

Intuitively, SLE creates curves by taking Brownian motion on the unit circle and then pushing said motion inwards towards zero in a conformal way. In practice, this arises by creating an evolving random family of conformal maps that satisfy Loewner's equation according to some Brownian motion on the unit disk.

Schramm-Loewner evolution was first proposed by Oded Schramm in his 1999 paper as the scaling limit of LERWs and uniform spanning trees.[12] Since then, it has been conjectured and shown to be the scaling limit of a handful of random processes, including self-avoiding walks, and critical percolation.

5.1 Formulation

We first establish the notion of capacity. Let $\mathbb{U}$ denote the unit disk $\mathbb{U} = \{z : |z| < 1\}$ and suppose that $D \subset \mathbb{U}$ is a simply connected open subset of $\mathbb{U}$ containing 0. The Riemann mapping theorem then implies that there exists a conformal map $\psi : D \to \mathbb{U}$. By requiring that $\psi(0) = 0$ and $\psi'(0) \in \mathbb{R}_{>0}$, we gain uniqueness. With these requirements, let $\psi_D : D \to \mathbb{U}$ denote the conformal map. Because $D \subset \mathbb{U}$, arguing by the Schwarz lemma it follows that $\psi'_D(0) \geq 1$. We then call $\log \psi'_D(0)$ the *capacity* of $\overline{\mathbb{U}} \setminus D$ from 0.

To define Schramm-Loewner evolution, we first motivate Loewner equation's. Suppose that there exists some continuous simple curve $\eta : [0, \infty] \to \overline{\mathbb{U}}$ with $\eta(0) \in \partial\mathbb{U}$. For $t \in [0, \infty]$, we define $U_t = \mathbb{U} \setminus \eta[0, t]$ to be the complement of η up to t in the unit disk. Because $\eta[0, t]$ is compact and η is simple, U_t is a simply connected open subset of $\mathbb{U}$. Therefore, the earlier reasoning implies that there exists a conformal map $\psi_{U_t} : U_t \to \mathbb{U}$. Let $g_t = \psi_{U_t}$. By reparameterizing η, we can ensure that $g'_t(0) = \exp(t)$. This is called parameterizing by capacity.

With this evolving family of conformal maps, we define the driving function of the curve η to be the limit

$$W(t) = \lim_{z \to \eta(t)} g_t(z)$$

Because g_t is a conformal map from U_t to $\mathbb{U}$, it follows that $W(t)$ exists for all t and $W : [0, \infty) \to \partial\mathbb{U}$. It can also be shown that $W(t)$ is continuous, though the derivation distracts from the purpose of the paper. For a thorough derivation of these properties, the reader is encouraged to consult Pommerenke's "Boundary Behaviour of Conformal Maps".[10]

Intuitively, the function W classifies how the path η changes as it traverses $\mathbb{U}$. By construction, the map g_t maps the complement of the segment $\eta[0, t]$ back to the unit disk $\mathbb{U}$. The endpoint of this segment, $\eta(t)$ will be taken to a point on $\partial\mathbb{U}$ after a limit is taken. In this way, changes in η can be projected onto the boundary of $\mathbb{U}$ and collected into the driving function $W(t)$. This process is shown beautifully in Henry Jackson-Flux's animation of SLE_4 at link[4]. This animation deals specifically with Chordal SLE, which is SLE in the upper half plane rather than the unit disk. However, the evolving upper half plane on the right half of the video gives strong intuition for the driving function when the intersection of the path with the real line is followed.

With this driving function, Loewner's theorem states that g_t satisfies Loewner's differential equation

$$\partial_t g_t(z) = -g_t(z)\frac{g_t(z)+W(t)}{g_t(z)-W(t)} \tag{5.1}$$

By construction, it follows that g_t also satisfies the initial condition $g_0 = \mathrm{id}_{\mathbb{U}}$. [7]

This process works in the opposite direction as well. Suppose that $W : [0,\infty) \to \partial U$ is a continuous function. Then for all $z \in \overline{\mathbb{U}}$, there exists some g_t that satisfies Loewner's differential equation, equation 5.1, with the initial condition $g_0 = \mathrm{id}_{\mathbb{U}}$. Define $K_t = \mathbb{U} \setminus D_t$ where D_t is the domain of definition of g_t. If W arises in the earlier process from a simple path η, then we can recover η by $\eta = g_t^{-1} \circ W$ and so $K_t = \eta[0,t]$. However, for arbitrary continuous $W : [0,\infty) \to \mathbb{U}$, it is not guaranteed that K_t defines a path, let alone a simple one. Regardless, by this association, any simple path in $\mathbb{U}$ starting on the boundary can be identified with a driving function $W(t)$. [7]

5.2 Radial SLE

By restricting the type of driving function, we can guarantee that K_t defines a path almost surely. This process defined radial Schramm-Loewner evolution. To that end, let $B : [0,\infty) \to \mathbb{R}$ be a Brownian motion. We then define radial SLE_κ to be the process $(K_t, t \geq 0)$ generated by the driving function $W(t) = \exp(iB(\kappa t))$. Conventionally, the starting point $B(0)$ is taken to be uniformly random in $[0, 2\pi]$.

As shown in Schramm and Rohdes, the process K_t almost surely defines a continuous curve γ such that $\mathbb{U}\backslash K_t$ is is the component of $\mathbb{U}\backslash\eta[0,t]$ containing 0. Further classification depends on the parameter κ. For $\kappa \in [0,4]$ it has been shown that K_t is almost surely a simple curve. For $\kappa \in (4,8)$ it has been shown that K_t is almost surely not a simple path. Finally, for $\kappa > 8$, it has been shown that the path γ is almost surely space-filling. [11]

6 Scaling Limit of LERW

We seek to provide a rough overview of the proof that LERWs converge in distribution to SLE_2. This fact was first conjectured by Schramm in his 1999 paper "Scaling limits of loop-erased random walks and uniform spanning trees" [12] and later proven by the 2003 paper by Lawler, Schramm, and Werner "Conformal invariance of planar loop-erased random walks and uniform spanning trees" [7].

Specifically, it has been shown that LERWs from 0 to $\partial\mathbb{U}$ converge in distribution to SLE_2. Intuitively, this makes some sense. Under a limit, LERWs should approach simple paths that start at 0 and approach a random point on the boundary of $\mathbb{U}$. Schramm-Loewner evolution then works in the reverse direction and creates a random path from the boundary of $\mathbb{U}$ to 0.

In essence, the argument that the limit of LERW exists follows from weak convergence on compact spaces. Let $\Lambda_{a,D}$ be defined as it is in the previous section and let $\Lambda_{a,D}^\delta$ denote the loop-erased random walks on the lattice $\delta(\mathbb{Z}+i\mathbb{Z})$ constructed in the same manner with probability measure $\mu^{\delta}_{a,D} : \Lambda^{\delta}_{a,D} \to [0,1]$. By endowing this space with the Hausdorff metric, it can be shown that the space of paths in D is almost surely compact. Then because the space of Borel probability measures on a compact space is compact in the weak topology, there exists a subsequence $\delta_j \to 0$ such that $\mu^{\delta_j}_{a,D}$ converges weakly to some probability measure $\mu_{a,D}$ on the space of continuous paths in D. As LERWs contain no self-intersections, this resulting probability measure should be supported on the space of simple paths in D. It can then be shown that when restricted to the case of LERW starting at some point and ending on the boundary of D, that this subsequential convergence is upgraded to convergence. [12]

The proof of the identity of the scaling limit centers around the conformal invariance of the limit of LERWs. Specifically, this is stated as follows

Theorem 6.1. *Let D be a simply connected, open strict subset of $\mathbb{C}$ and let there exist some $a \in D$. Then the scaling limit of LERW from a to ∂D exists. Additionally, if $f : D \to D'$ is a conformal map on $D' \subset \mathbb{C}$ then $f_*\mu_{a,D} = \mu_{f(a),D'}$.*

At the time of Schramm's original paper on the topic, this theorem was only conjecture supported by simulations and the work of Rick Kenyon. Schramm continued to prove that LERWs converge to SLE_2 assuming this conjecture. [12] Intuitively, this conformal invariance of the limit of LERW can be used to prove that the limit of LERW must satisfy the Loewner equation. It then remains to show that the driving function of said process must be Brownian motion.

To show that the driving process must be Brownian motion, it is shown that the limit of LERW has a Markov property. To that end, let ω be a LERW from 0 to δD and let ω' be a subpath that extends from some $q \in \omega$ to ∂D. Then the distribution of $\omega - \omega'$ is the same as that of the LERW from 0 to $\delta(D - \omega')$, conditioned to hit q. When we then take the limit of LERWs, this transforms into a Markov property for the driving function of SLE when the conformal map from $\delta(D - \omega')$ to $\mathbb{U}$ is applied. The Markov property of the driving force combined with the conformal invariance can then be used to show that it must be Brownian motion. [12]

A separate argument that we avoid here then shows that it must in fact be SLE_2.

7 SELF-AVOIDING WALK (SAW)

Though the scaling limit of LERW has been proven to be SLE_2, Schramm conjectured that a similar result could be shown for a similar type of random walk, a self-avoiding walk. This conjecture was presented and explored in the 2002 paper by Lawler, Schramm, and Werner "On the scaling limit of planar self-avoiding walk".[8]

Simply, a self-avoiding walk is a random walk ω with the additional constraint that $\omega(j) \neq \omega(i)$ for $i \neq j$. With the linear interpolation, this implies that ω is a simple curve. Formally, we define this as follows.

Definition 7.1. A self-avoiding walk (SAW) of length n in $\mathbb{Z} + i\mathbb{Z}$ is a finite sequence $\omega = [\omega_0, \dots, \omega_n] \subset \mathbb{Z} + i\mathbb{Z}$ such that $\omega_i \neq \omega_j$ for $i \neq j$.

As with the previous walks and random processes, our analysis of SAWs is far more concerned with their space and measure. To that end, we let Ω_n denote the sets of all SAWs of length n and we let Ω^*_n denote the restriction of Ω_n to the walks starting at the origin. It should be noted that Ω_n has countably infinite elements, as there are countably infinite choices of starting point, but Ω^*_n has finite elements by simple combinatorics.

Consider the number of SAWs of length n, starting at the origin. It is clear that given a SAW of length $n+m$, a splitting and translation yields two separate SAWs of length n and m respectively. It then follows that $\#\Omega^*_{n+m} \leq (\#\Omega^*_n)(\#\Omega^*_m)$, where $\#$ is used to indicate cardinality. Taking a logarithm and applying Fekete's Subadditive Lemma, it follows that $\#\Omega^*_n \sim \beta^n$ for some β, known as the connective constant. The connective constant β is dependent on the lattice and is only rigorously known to be between 2.6 and 2.7.[8] Recently, the connective constant for the hexagonal lattice was found to be $\sqrt{2+\sqrt{2}}$ by Dominil-Copin and Smirnov [2].

Combining the spaces of finite SAWs, we define $\Omega = \cup_{n=1}^{\infty}\Omega_n$ and $\Omega^* = \cup_{n=1}^{\infty}\Omega^*_n$. We then define a measure μ_{SAW} on Ω by that $\mu_{SAW} : \omega \in \Omega_n \mapsto \beta^{-n}$. It should be noted that μ_{SAW} can be viewed equivalently as a measure on $\Omega_n, \Omega^*, \Omega^*_n$. While not initially a probability measure, it can be shown that μ_{SAW} will converge to a probability measure under a scaling limit.

To taking a scaling limit, we create a finer lattice and finer SAWs. To that end, let Λ be a lattice in $\mathbb{C}$. For $\delta > 0$, we let ω^δ be a SAW on the lattice $\delta\Lambda$ which has been reparameterized so that

$$\omega^\delta(j\delta^{1/\nu}) = \delta\omega(j)$$

Taking the limit as $\delta \to 0$, we should arrive at a probability measure on the space of simple paths once again. [8]

It was conjectured in 2003 that under this measure, SAWs would converge in distribution to $\text{SLE}_{8/3}$. Currently, it has been shown that if the scaling limit exists and is conformally covariant, then the scaling limit is $\text{SLE}_{8/3}$. However, the existence of said limit and conformal covariance has not been shown.

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