Advanced 2

Manipulating Shapes

Prepared by Matthew Kowalski on October 9, 2025

Throughout this packet, there are a number of questions labeled **"open-ended."** These questions do not necessarily expect you to find a solution. Instead, they are intended to give you a chance to play with the problem without guidance. You can continue onto the next problem whenever you want.

There are also a number of questions labeled **"bonus"** or **"challenge"**. These are optional.

Part 1: Curves of Constant Width

The material and figures in this section are adapted from the forthcoming textbook "Masterclasses in Mathematics" by Ian Stewart and David Wood.

The wheel has been around for over 6000 years with little change to its design. It is a circle.
But can we use other shapes to roll? Might they be better?

The key characteristic we want out of a wheel is that it has a constant *diameter*. Namely, whenever it is touching the ground, its top point is always at the same height.

Problem 1: Diameter
What is the diameter of a circle of radius r?
How might you define the diameter of a more obscure shape?
Can a shape have more than one diameter?

Definition 2: Simple Closed Curve
A simple closed curve is anything you can draw without picking up your pencil and without crossing over your own line.

Definition 3: Support Lines
A support line for a simple closed curve is a line that meets the curve at one or more points, with the extra condition that the curve lies entirely on one side of that line (including points on the line).

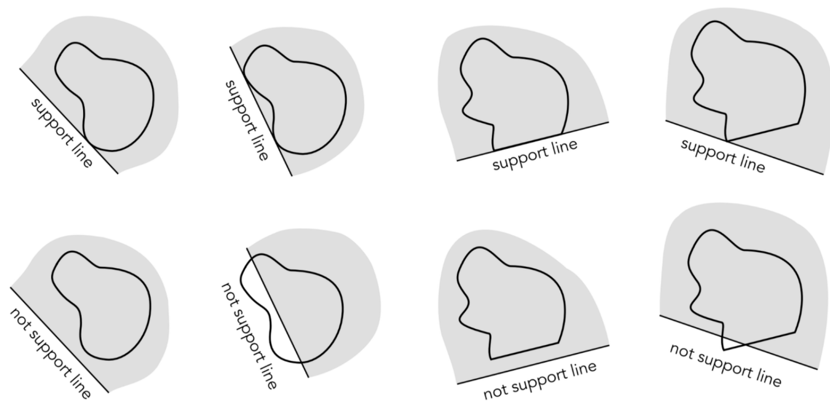


Figure 1: Some examples and non-examples of support lines.

Definition 4: Diameter
A *diameter* of a simple closed curve is the distance between two distinct parallel support lines on opposite sides. This diameter can depend on the direction in which the support lines are drawn.

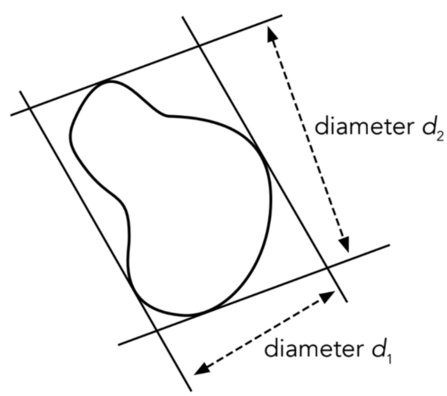


Figure 2: Two diameters of a typical curve.

Problem 5:
Briefly explain why pairs of parallel support lines exist. Given a direction, how could you find a pair of parallel support lines systematically?

Definition 6:
A curve has *constant diameter* if every diameter is the same, no matter which direction the parallel support lines point. For curves with constant diameter, we can then talk about *the diameter* without ambiguity.

Problem 7:
Draw an example of a simple closed curve with constant diameter.

Draw an example of a simple closed curve with non-constant diameter.

Problem 8: (Open-ended)
More than likely, in Problem 7, your example of a curve with constant diameter was a circle. Can you come up with any non-circle examples of a curve with constant diameter?
Hint: begin with a simple polygon. See if you can "fix" it to have constant diameter.

Problem 9: Reuleaux triangle
We'll now take you through a construction of some non-trivial shapes with constant diameter.

Draw an equilateral triangle with side lengths d.
Verify that this is a curve with non-constant diameter. Where is the diameter not equal to d?
These are the problem regions that we need to address.

Label the corners of your triangle X, Y, Z. Connect Y to Z with a circular arc of radius d and center X. Similarly, connect Z to X with a circular arc centered at Y and connect X to Y with a circular arc centered at Z.

How did this process address our problem regions from before?
Note: for any pair of parallel support lines, one of the lines is touching a corner of the triangle.

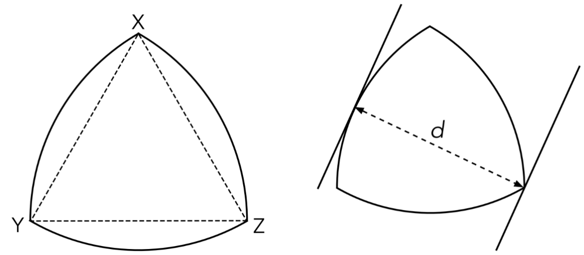


Figure 3: Reuleaux triangle: Curve of constant diameter constructed from an equilateral triangle.

Problem 10: Generalization
Draw a regular pentagon. Repeat the above process to create another (non-circular) curve of constant diameter.
Repeat this process for a heptagon.

Problem 11: Generalization
Does this process work for all polygons?
What happens if you try to do this process with a square? What fails?

Problem 12: Circumference
For your curves of constant diameter, calculate their circumference. How does this compare to the circumference of a circle?

Conjecture what the circumference of a curve of constant diameter d should be.
We will not prove this here.

Problem 13: Area
For the Reuleaux triangle, Problem 9, calculate its area. How does this compare to the area of a circle?
If you would like a big *challenge, you can try to calculate the area of the constant diameter pentagon.*

Conjecture what the area of a curve of constant diameter d should be.
Hint: Your conjecture should be a bound on the area, not a specific area.

Problem 14: Non-equilateral triangles

Draw a non-equilateral triangle with vertices X, Y, Z. See the figure for a visualization of the following process.

(a) Start with point X. Fix a large length ℓ. Draw a circular arc, centered at X, with radius ℓ, that sweeps from direction XY to XZ.

(b) Extend our curve with a circular arc, centered at Z, that extends the curve to direction YZ. Repeat this with another arc centered at Y.

(c) Extend our curve with a circular arc, centered at Y, that sweeps from direction YZ to YX. Repeat this with another arc centered at Z.

(d) Complete our closed curve by drawing a circular arc centered at X.

(e) The resulting curve has constant diameter. Why? Provide a brief explanation and convince yourself that this is true.

(f) Show that the diameter of this curve is $\ell + \overline{YZ} - \overline{XZ} - \overline{XY}$.

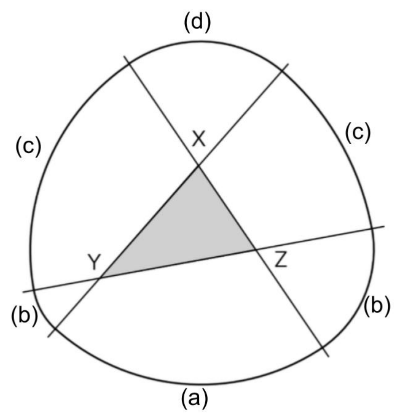


Figure 4: A curve of constant diameter constructed from a non-equilateral triangle.

Problem 15: (Bonus) Area and circumference
Draw a triangle with nice angles and side lengths. (For instance, a 45-45-90 triangle.)
Repeat the above process to create a curve with constant diameter. Verify your conjectures about the circumference and area of a curve with constant diameter.

Problem 16: (Bonus, challenge) Non-regular polygons
Draw a non-regular (convex) polygon with 5 sides. Can you generalize the process from Problem 14 to draw a curve of constant diameter?

Part 2: Kakeya Needle Problem

In 1917, Kakeya proposed a simple problem:

What is the smallest area required to rotate a needle (line segment of length 1) by $180°$ *in the plane?*

This exact statement was solved in 1927, but refinements of this statement are still being researched. One refinement, the so-called Kakeya conjecture, was solved by a former UCLA professor just this year! In this section, we'll explore Kakeya's original statement.

In our area, we can *rotate* or *translate* the needle. We are not allowed to pick it up. Below is a (very inefficient) area showing how we might move our needle.

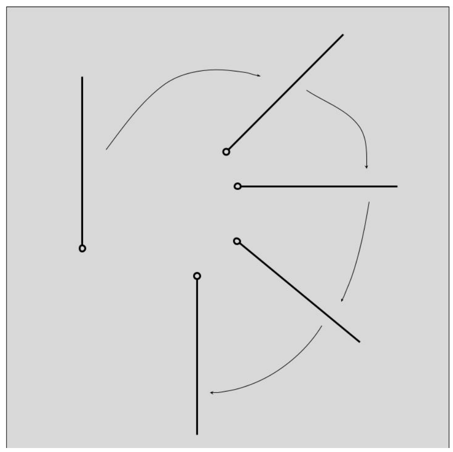


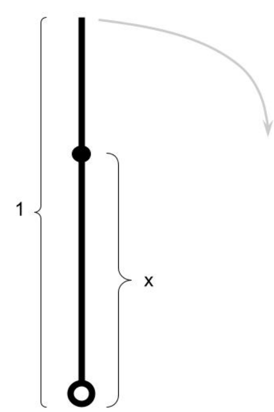
Figure 5: A very inefficient region for the Kakeya needle problem.

Problem 17:
Try to construct a region within which you can rotate the needle $180°$. Can you think of any such regions? What are their areas?
Hint: there's no need to be fancy here. We're just looking for a starting point.
(Open-ended) What's the smallest region you can find?

Problem 18: *Rough* upper bound
Starting with the needle straight up, rotate the needle 180° about one of its ends.
What region is swept out by this rotation? What is the area of this region?

Problem 19: Rough upper bound
What happens if you fix a different point of the needle and rotate about that point? What region is swept out by this rotation? What is the area of this region?
Choose a point distance x from the end of the needle and rotate the needle 180° *about this point.*

Using this method, what is the minimum area required? Which rotation point gives this area?



Problem 20: Better upper bound
Instead of picking one point to rotate about, suppose we change the point midway.

Choose one end of the needle. Rotate the needle $60°$ around this end. Then, rotate another $60°$ about the other end of the needle. Repeat this process, alternating which end we rotate about, until the needle has rotated a full $180°$. What area was swept out?

Repeat this process, but instead rotating $90°$ each time. Did the area increase or decrease?
What happens if we try a smaller angle like $45°$ or $30°$? *No need to find exact numbers for this. Just get a sense of whether the area is increasing or decreasing.*

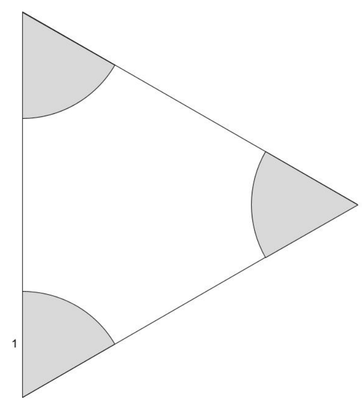
The key to this problem is to re-use space. In Problem 18, the needle passed through each point of our region only once. However, in Problem 20 the needle rotates through *most* points 3 times. Let's try to do better by introducing some translations.

Problem 21: Best convex bound

Start with the needle straight up and rotate clockwise by $60°$ around the bottom of the needle. Instead of rotating again, shift the needle in the direction it is pointing by some fixed amount. *(We'll find the optimal amount soon. For now, shift by a large amount, say* 2*. See the figure.)* Now rotate another $60°$ around the top of the needle before shifting again. Continue this process until the needle rotates a full $180°$.

When we shift the needle a large amount, what shape is made? If we make this shift smaller, what happens to the region that we sweep out?

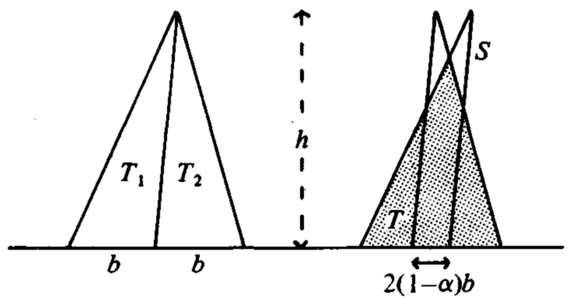
By choosing the proper shift, we can make the region an equilateral triangle. What is the area of this triangle? How does this compare to our earlier regions?



As it turns out, this triangle is the best you can do with a convex region!

Problem 22: (Bonus, challenge)
In 1927, Besicovitch solved the Kakeya needle problem by showing that you can construct a region with arbitrarily small area! The construction of this is fairly complicated, but relies on the idea we developed earlier: re-using as much space as possible. We won't detail the full construction here; this is just the general idea. Feel free to ask your instructor for more details!

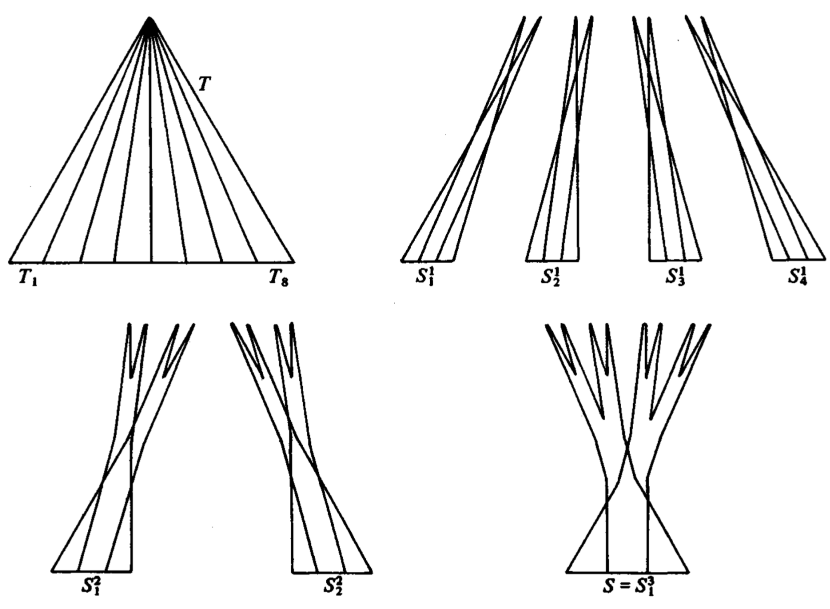
Let T_1, T_2 be adjacent triangles with bases of length b on a line L and heights h, as shown in the figure. If we slide T_2 a distance $2(1-\alpha)b$ to the left, we can overlap the triangles a small amount.
What is the total area of T_1 and T_2 originally?
What is the difference in area between these two regions? (The area of the intersection.)
What α minimizes the area of our final region? How does the area of S compare to T_1 and T_2?



Problem 23: (Bonus, challenge) continued
If we begin with an equilateral triangle of height 1, note that we can rotate a needle 60° within the triangle. By doing this overlapping procedure, we can still rotate the needle 60°, now we have to do it in two 'steps'. We can rotate 30° within T_1 and 30° within T_2 and we've reduced the area we need!

However, in order to rotate the full 60°, we need to be able to move our needle between T_1 and T_2 without using two much extra area. How can you move the needle from T_1 to T_2 using an arbitrarily small area? Draw the resulting region.
Hint: use the Kakeya squeegee problem from the warm-up.

To get our area even lower (arbitrarily low, in fact), we need to decompose our triangle even further. See the following figure for how we do that. With the Kakeya squeegee problem, this allows us to rotate our needle 60° in an arbitrarily small area. All we need to do then is connect three of these so-called *Perron trees* and we have solved the Kakeya problem.



Part 3: Moving Couch Problem

*"Pivot! PIVOT! **PIVOT!**"*

Suppose that you are trying to move your sofa into your new college dorm. The hallway has a 90° turn that you need to pivot the sofa through. What is the biggest sofa that you can get into your dorm?

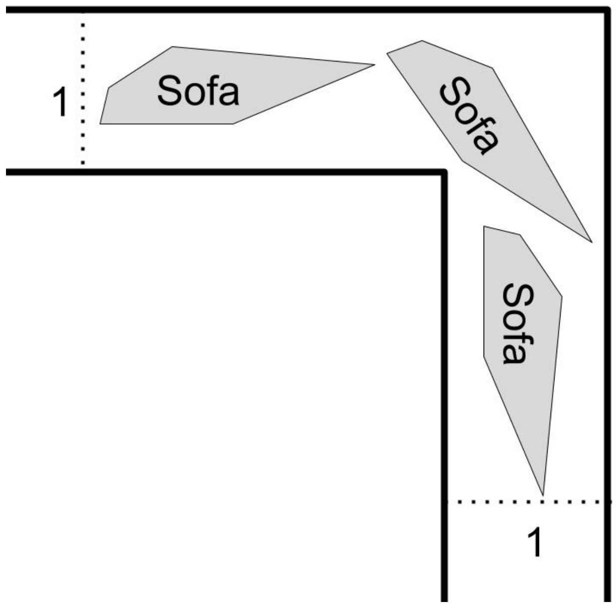


Figure 6: A very inefficient example of a sofa that can moved through the corner.

We'll assume that the sofa is *connected*, meaning that it is not broken into multiple pieces.

Problem 24: (Open-ended)
Try to brainstorm some possible sofas that you can move through the corner.

- Can you find a sofa with area 1?
- How about a sofa with area $\pi/2$?

Problem 25: Lower bound
Construct a rectangular sofa with area 1 that you can move through the corner.
Hint: this sofa doesn't have to rotate, it can just slide.

Problem 26: Refined lower bounds
What if our couch is a semicircle of radius r? How large can r be? What area does this give?

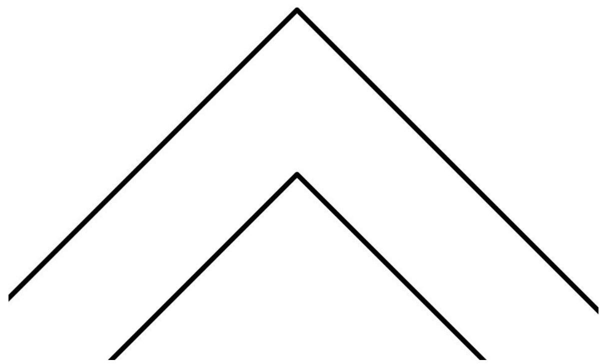
Problem 27: Upper bound
Consider a sofa with maximum width w and maximum length ℓ. To find an upper bound for the area of a sofa, we're going to bound w and ℓ.

(a) What is the maximum possible width w of the sofa?

(b) Suppose that our sofa is convex. What is the maximum possible length ℓ of the sofa?
Hint: consider the situation where the sofa is rotated $45°$. *What happens to the base of the sofa? This gives a* rough *upper bound for the maximum area.*

(c) (Challenge) Consider the situation where the sofa is rotated $45°$ more closely. If the sofa is convex, what region of the corner must it lie within? What is an upper bound for the area of a convex sofa?
The figure in the following problem may help.

Problem 28: (Challenge)
Suppose that we no longer force our sofa to be convex. You know that the width of the sofa is $w = 1$ and the sofa is connected. When the sofa is rotated $45°$ (halfway through the corner), what region(s) might the sofa lie in? Use this to find an upper bound on the area of the sofa.
Hint: Using the fact that the maximum width is 1, draw horizontal strips on the provided figure and look at the overlap with the corner.



Problem 29: (Open-ended)
The largest sofa size is 2.219531... (only proved in 2024!) with a shape similar to a telephone. Can you get any closer to this optimal value? How might you proceed?