Advanced 2

Special Relativity

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All diagrams are adapted from Izaak Neutelings, see
`https://tikz.net/relativity_minkowski_diagram/`

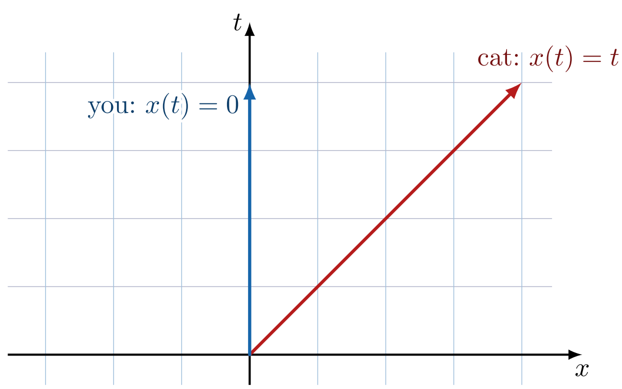
Part 1: Spacetime Diagrams

We are going to derive the principles and consequences of special relativity using basic geometry.
To help with our visualization, we will be using spacetime diagrams (called *Minkowski diagrams*).

To make our models simpler, we will only be considering *one spatial dimension.*

We plot space, which we denote by x, as the horizontal axis, and time, which we denote by t as the vertical axis. For a given object, we can then plot its position at any given time.
This will give a (potentially curvy) line that we call the object's *world line.*

Example 1:
Suppose that at time $t = 0$, you are standing still with your cat at your feet. Your cat walks away from you at speed 1. We can represent this with a spacetime diagram:



NOTE: The horizontal axis is space and the vertical axis is time. We are only working with one spatial dimension.

Problem 2:
Suppose that you are standing still at time $t = 0$ and your many pets lie at your feet.

- Your cat, unhappy that she is not fed, begins walking away to your right at speed 2.
- Your dog, distracted by a squirrel, walks away to your left at speed 1.
- Your hamster, just wanting to feel included, waits a second and then follows the dog at speed 2. After reaching your dog, your hamster turns around and sprints after the cat at speed 3.

Draw this situation in the provided spacetime diagram.

Problem 3: Event

Any single point (x, t) on a spacetime diagram is considered an *event* because it describes a time and place. For instance, what is the event that corresponds to your hamster catching up to your dog?

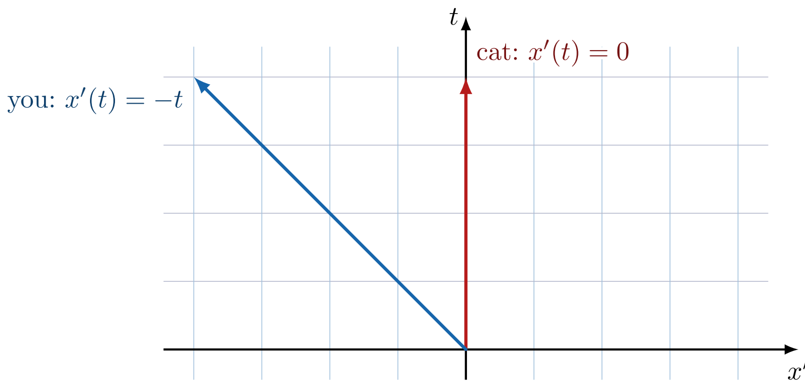
Problem 4:

Suppose that the situation of Problem 2 occurred while you were riding on a train moving to the right at speed 1. Everything occurs relative to you in the same way. Draw the same diagram in this new situation. Are any of your pets staying still in this new situation?

Part 2: Galilean Relativity

Much like you can watch your pets scatter from the perspective of a train, we can watch the world from anyone's perspective. When we shift perspective like this, just using our normal intuition, we call this *Galilean relativity*.

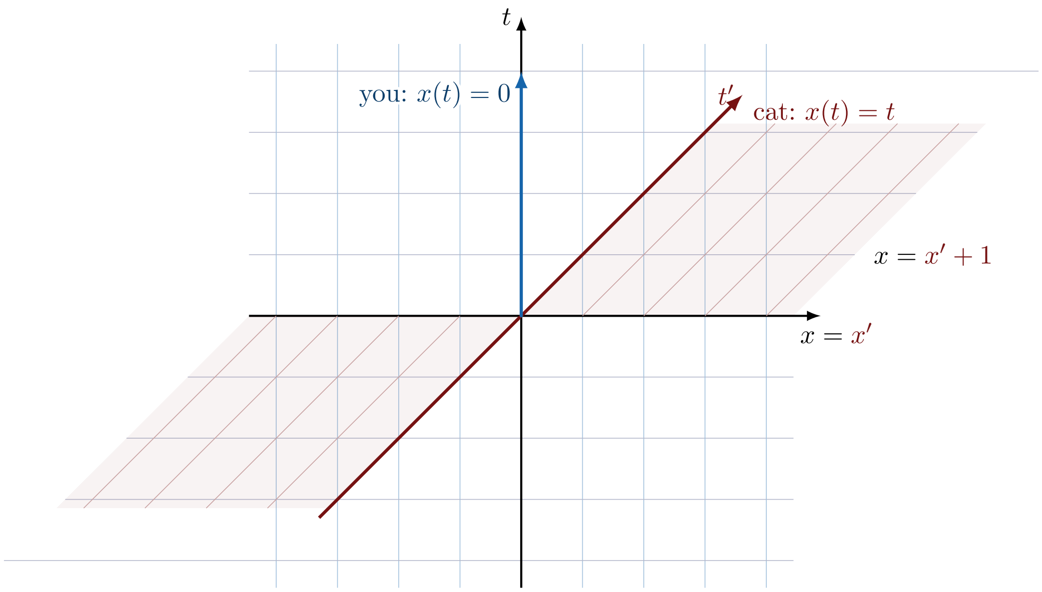
Example 5:
Consider the situation of Example 1 again, but now from the perspective of your cat. From your cat's perspective, she's the one staying still and you're the one walking away, only now you're walking away at speed 1 to the left. We'll denote our new spatial variable with x'.



Problem 6:
Draw the situation from Problem 2 in your cat's perspective.
What if we drew the situation from Problem 4 in your cat's perspective?
Would there be any change when the cat is on the train? Why or why not?

Definition 7: Reference Frame
When we view the world from the perspective of different objects, we say that we are working in different *reference frames*. The original Example 1, where you are stationary, is your reference frame. The new plot in Example 5, where your cat is stationary, is your cat's reference frame.

If we want to compare what is happening in multiple reference frames at once, we can graph multiple spacetime grids on one plot. If we overlay the cat's reference frame onto your reference frame, we can visualize everything in Example 1 as:



Here x, t are the spacetimes coordinates in your perspective and x', t' are the spacetime coordinates in your cat's perspective. Note that $t = t'$ for any point while $x = x' + t$.

Problem 8:
What does it mean for two events to lie on the same vertical (blue) line from your perspective?

What does it mean for two events to lie on the same slanted (red) line from your cat's perspective?

Problem 9:
In the situation from Problem 2, when will your hamster catch up to your cat?
Choose the most convenient reference frame to work in, you shouldn't have to do much math.

Part 3: Special Relativity

Galilean relativity is nice until we start going really, *really* fast. Since most of us are terribly slow, we can use it without any issues. However, in reality, things are much weirder. In particular, there is a maximum speed: the speed of light,

$$c = 299,792,458\,\frac{m}{s}.$$

Nothing can move faster than the speed of light and **in every reference frame, light will move at this speed**.

Let's see if this is consistent with Galilean relativity. We are going to making things easier for ourselves now and change units. Instead of measuring time t, we are now going to measure ct.

Problem 10:
Suppose you are sitting still and you send one photon to your right. Draw this photon on a spacetime diagram, with horizontal axis x and vertical axis ct.

Problem 11:
Suppose you are now sitting on a train that is moving to the right at $c/2$ and again send one photon to your right. Draw this diagram in the reference frame of the ground.

Draw your (Galilean) reference frame on top of this diagram. What is the speed of the photon in your reference frame? Is that a problem?

Problem 12:
Clearly, Galilean relativity and the absolute speed of light do not mix well together. Having noticed this, you are now in the same boat as early 20th century physicists. Can you brainstorm any ways to fix Galilean relativity to account for this absolute speed of light?

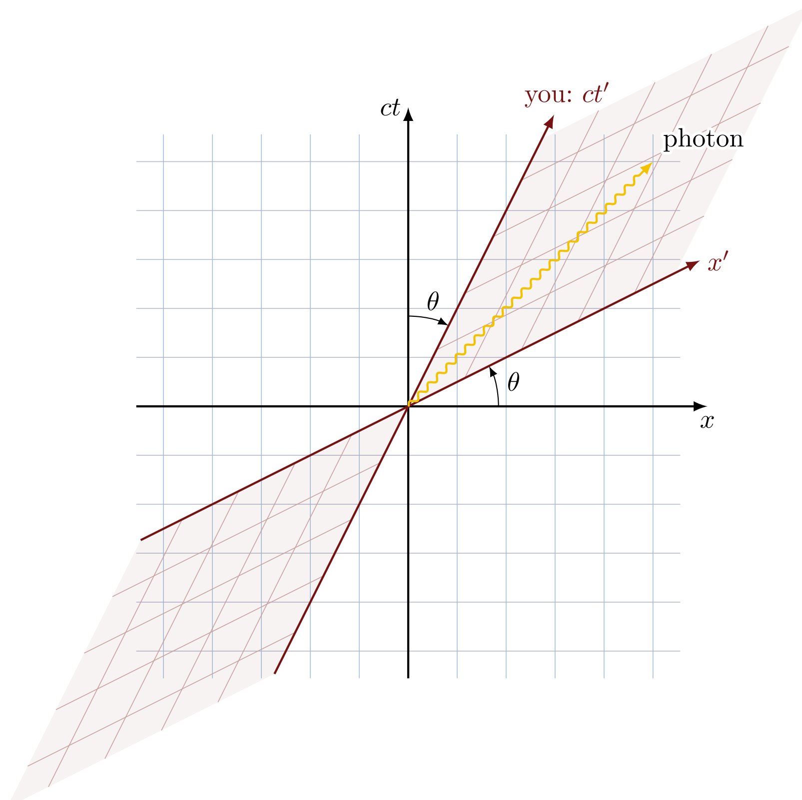
Hint 1: Try different methods of drawing the axes of your reference frame that would maintain the speed of light in both the rest frame and in your reference frame.

Hint 2: The photon worldline always bisects the angle between the space and time axes.
Is there a way that you can make that happen in your reference frame?

Don't worry if you don't have any ideas! It took physicists a while to figure this out. Whenever you want to move on, we have the solution on the new page.

Definition 13: Lorentz Boost
Looking at our spacetime diagram from Problem 10, we see that the photon worldline bisects the angle between the ct axis and x axis. So if we want to maintain this speed in all reference frames, we just need to make sure that photons bisect our new time axis ct' and our new space axis x'.

In order to do this, we're going to to rotate our space axis x' by the same angle that our ct' axis is rotated. Rotating both axes like this is called a *Lorentz boost* and is best visualized in the following diagram:



As before, the slanted (red) axes are your reference frame as you're moving.
Note that you (and anything moving the same speed as you) are stationary in this reference frame.

Remark 14:
In this diagram, we've not only rotated your space axis (x'), we've also adjusted the scale of ct' and x' relative to the rest frame. This scaling comes from physical experiments which we will conduct later.

Problem 15:
Please verify that in the diagram above, if you shoot a photon behind you, it still moves at speed c in both your reference frame and the ground's reference frame. You can do this by drawing directly on the diagram.

Problem 16:
A caveat to Lorentz boosts is that we cannot boost to reference frames which are at the speed of light or faster. Based on the diagram given, why can't we do that?
"If my calculations are correct, when this baby hits c, you're gonna see some serious stuff."

Problem 17:
This diagram implies some strange things. We'll spend the next sections discussing some consequences of this, but take a minute to note anything weird that you notice.

In particular, look at a unit of time (or a unit of length) in your frame vs the rest frame.

Which is longer, one unit of time in your reference frame or in the rest frame?

Which is longer, one unit of distance in your reference frame or in the rest frame?

Consider the implications of a slanted space line. What does it mean if two events both lie on this line?

Part 4: Simultaneity

Let's look now at the consequences of a rotated spatial coordinate. To help ourselves switch between different perspectives, we'll bring in some professionals: Alice and Bob.

Problem 18 to Problem 21 can all be completed on one spacetime diagram. Feel free to use the one provided below, or draw your own if it gets too crowded.

Problem 18:
Draw a spacetime diagram from Alice's reference frame or use the one provided. What do the horizontal gridlines represent?

If two events (remember, points in spacetime) lie on the same horizontal line, what does that imply about the events?

Problem 19:
Suppose that Bob is walking to the right at speed $c/3$, relative to Alice. Add Bob's worldline to your spacetime diagram. Superimpose Bob's reference frame onto the diagram

What do the "horizontal" lines in Bob's reference frame represent to Bob?

Problem 20:
Suppose that Alice is passing time, snapping to some music with her arms out.
Both of her arms are length 1 and she snaps both hands every unit time. So there is a snap at location $x = \pm 1$ at $ct = 1$, $ct = 2$, and so on.

Draw this situation on your spacetime diagram.

Consider this from Bob's perspective. What does Bob hear? (We'll assume that sound propagates instantly, so if a sound occurs at $ct' = 3.5$ then Bob hears it instantly at $ct' = 3.5$.)

Problem 21:
Bob is deeply annoyed at Alice because she keeps. *snapping. out.* ***of. TIME***.

In a fit of frustration he starts running faster and faster to the right. Eventually, he notices that the snaps are changing. What happens to the timing of the snaps? Do the snaps ever come back into time? At what speed is Bob running if they do?

Problem 22:
Done with music, Alice and Bob decide to have a race and they agree to race from $x = 0$ to $x = 2$. Suppose that Alice can run at an impressive speed of $c/2$ while Bob can only run at a measly speed of $c/4$.

A: Who wins the race?

B: Is there any reference frame in which Bob wins?

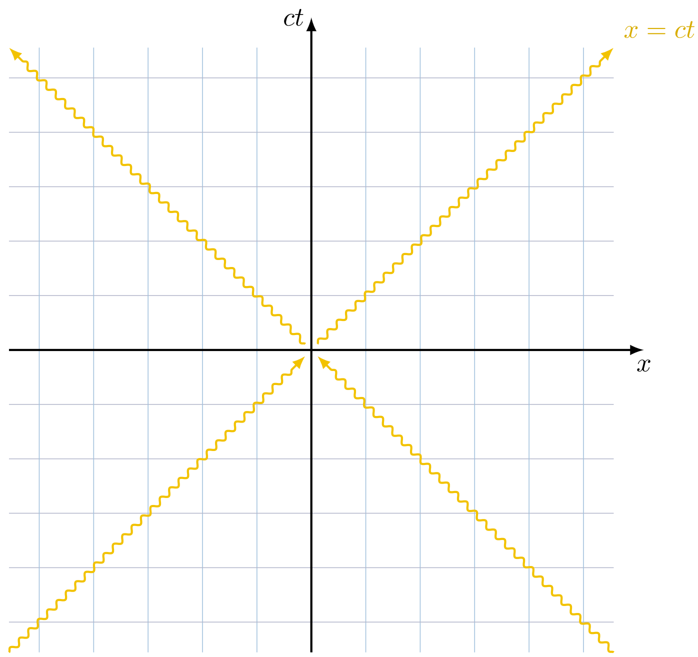
C: Suppose instead that Alice starts at $x = 0$ and is racing to $x = 2$, while Bob starts at $x = 8$ and is racing to $x = 10$. Is there now a reference frame where Bob wins? Why?
Hint: Plot these events on a spacetime diagram.

Problem 23:
This weirdness with simultaneity might make us question what the past, present, and future are. To make sense of this, suppose that we want to define the *present* to be every event which is simultaneous to you, right here, right now — a.k.a. the spacetime point $(0,0)$ — in at least one reference frame.

On the given spacetime diagram, draw the region which represents the present.

Draw a region which represents the future: events that are later in time in every reference frame.

Draw a region which represents the past: events that are earlier in time in every reference frame.



Another way to think of this is thinking of causality. In a reference frame, the *future* is every event that can be affected by an event at $(0,0)$. The *past* is every event that could have affected an event at $(0,0)$. The *present* is every event that is causally independent of $(0,0)$.

Problem 24: Bell's Spaceship Paradox
Suppose that we have two spaceships, distance L apart, tied together with floss of length L. The floss is so weak that any stretching at all will cause it to disintegrate.

The spaceships are at rest and then simultaneously accelerate to speed $c/2$.

Draw the spacetime diagram for this situation. Include the reference frame of the spaceships *after* they start moving (i.e. the reference frame moving at speed $c/2$).
What happens to the floss in the boosted frame? Does it break? Why?

Problem 25: Bell's Spaceship Paradox (continued)
The same outcome has to occur in all reference frames, so we know that the floss breaks in the stationary reference frame. However, in the stationary reference frame, the two spaceships accelerate at the same time, so our explanation no longer seems accurate.

Can you come up with a hypothesis for why the floss breaks in the stationary reference frame?

Hint: The floss breaks if the ships are farther apart than the length of the floss.

Part 5: Proper Time and Proper Length

We mentioned that the space and time axes also get rescaled during a Lorentz boost. Let's figure out by how much! To do this, we're going to use the only consistent tool at our disposal, the speed of light.

Problem 26: Time dilation
Suppose that Alice remains stationary. In her reference frame, Bob moves to the right at speed v. Every quantity in Alice's reference frame will be denoted normally, i.e. x, t. Every quantity in Bob's reference frame will be given a prime, i.e. x', t'.

Bob holds onto a very special clock. The clock consists of a light bulb, a mirror distance L away from the light (perpendicular to the direction Bob is moving) and a photosensor.

A: Draw the experimental setup. Note again that the light bulb and mirror are separated in the y direction, not the x.

B: In Bob's reference frame (where the clock isn't moving), how long does it take the light to be emitted, travel to the mirror, bounce back, and be reabsorbed? Write this as t'.

C: In Alice's reference frame, she still sees the light move, reflect off the mirror, then come back. How long does this take?
Note: The mirror and photosensor are moving at speed v during this process. Also, the light is now moving at an angle, but it still moves at c.

D: Combine these two formulas to find a relationship between t and t'.
If one unit of time elapses in Bob's reference frame, does more or less time elapse in Alice's reference frame? Who ages faster?

Problem 27: Space dilation

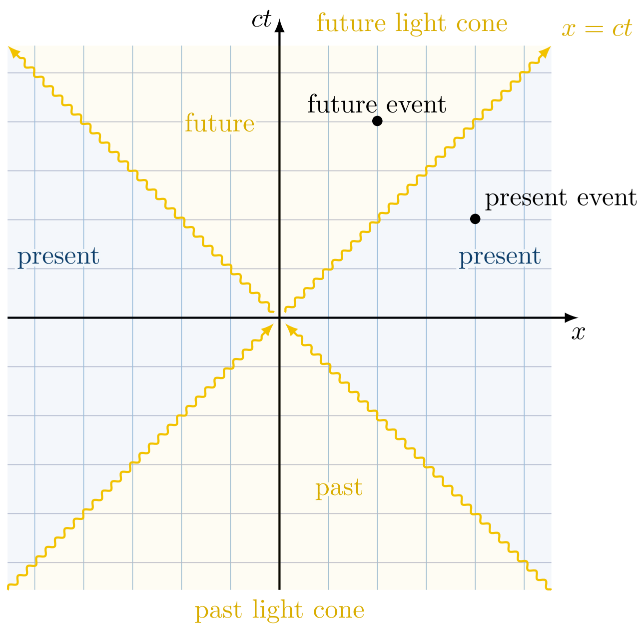
Since the speed of light is always constant, we can use time dilation to see what happens to our notion of distance. Suppose that a photon is emitted from one location and absorbed in another.

A: Suppose that in Alice's reference frame, the photon takes time t to be emitted and absorbed. In this time, what distance x does the photon travel?

B: Suppose that Bob is traveling at speed v relative to Alice. Using time dilation, how long does the photon take to be emitted and absorbed in Bob's reference frame? What distance x' does the photon travel in that time?
This will give you a formula relating distance x' in Bob's reference frame to distance x in Alice's reference frame.

Remark 28:
Switching back and forth between reference frames gets confusing quickly, especially when you are consideration time and space dilation. To avoid this, we will define a new notion of time and space that doesn't care about which reference frame we're in. Since this will be universal, we'll call these *proper time*, denoted by τ, and *proper distance*, denoted by χ.

Recall our notion of past and future from earlier:



Problem 29: Proper Time
For any event (x, ct) in the past or future, we can choose a reference frame to make it occur at $x' = 0$.

A: What velocity do we need to boost to so that the event (x, ct) now occurs at $x' = 0$?

B: After boosting to this reference frame, what time ct' does the event occur at?
This will be our proper time $c\tau$.

Problem 30: Proper Distance
Similarly, for any event (x, ct) in the present, we can choose a reference frame to make it occur at $t' = 0$.

A: What velocity do we need to boost to so that the event (x, ct) now occurs at $t' = 0$?

B: After boosting to that velocity, what spatial position x' does the event occur at? This will be our proper distance χ.

Defined in this way, proper time and proper distance give a quick formula to figure out how old an object is, or how far an object traveled, in its own reference frame!

We'll now use this proper time and proper space to quickly solve one of the most famous paradoxes in special relativity.

Problem 31: Twin paradox
Suppose that you have two twins, Mark and Lucas, born at the exact same instant and location. One day, Mark sends Lucas off in a spaceship traveling at $c/2$ to a planet that is one lightyear away ($c * (1 \text{ year})$). Upon reaching the planet, Lucas realizes that he's all alone! Saddened, he turns around immediately, traveling back at speed $c/2$.

The question is: Which twin is older? We'll break this down step by step.

A: From Mark's perspective, what time does Lucas reach the planet? What time does he return?

B: From Lucas' perspective, how long does it take him to reach the planet? Hint: use proper time.

C: How long does it take him to return?

D: Which twin is older?

Problem 32: Twin paradox (continued)
The weird part is, from Lucas' perspective, it was Mark who flew off and then came back! So why is Mark not younger than Lucas? Let's break this down step by step.

E: To help answer this, draw a spacetime diagram of this situation. In Lucas' reference frames (one for going out to the planet and one for coming back), draw lines of constant time. What happens to Mark from Lucas' perspective?
Hint: changing speed is weird. Focus on where Lucas changes speed.

F: We said that when Lucas landed on the planet, he immediately took off back towards Earth. From Mark's perspective, how long was Lucas on that planet?

G: What is physically different between Lucas' perspective and Mark's perspective?

Part 6: Length Contraction

With proper time and distance done, we can now tackle length contraction easily! Length contraction is weird because different parts of our object will now be experiencing different times.

Problem 33:
Suppose that you (at rest) see a rod of length L moving at speed v.

A: Using the provided grid, draw a spacetime diagram where the left side of the rod is at $x = 0$ at $t = 0$.

B: When we switch to the rod's reference frame, space gets rotated. Draw a line in the rod's reference frame which represents the rod at time $t' = 0$, when the left side is at $x' = 0$. Call the right side of the rod at this time P.

C: Switching back to your reference frame, what are the spacetime coordinates (ct, x) of P?

D: Use the formula for proper distance to compute the length L' of the rod in its own reference frame.

E: Which is larger, L' or L? Do moving objects shrink?

Problem 34: Ladder paradox

Aiden and Matt have a ladder of length $2L$ that they are trying to squeeze into a barn of length L. Suppose that the barn has a front door and back door which can be open/closed simultaneously.

Now, Aiden is particular smart, so he gives Matt the ladder and has Matt run at the barn at speed $v = \sqrt{3}c/2$. Does the ladder fit in the barn? We'll analyze this in the new few questions.

A: What is the length of the ladder from Aiden's perspective when Matt is running? Does the ladder fit in the barn?

B: As soon as Matt and the ladder are inside the barn, Aiden quickly closes and opens the doors of the barn. Success!
However, consider this from Matt's perspective. From Matt's perspective, he's holding a ladder of length $2L$ and a barn is flying at him at speed $v = \sqrt{3}c/2$. By length contraction, what is the length of the barn?

C: Despite the barn being too short, we know that the ladder has to fit! Using the provided grid, draw a spacetime diagram of the situation. Include Matt's reference frame on your diagram.

D: From Matt's perspective, why don't the doors of the barn crush the ladder?