Less than 33 Miniatures

Based on the textbook "Thirty-three Miniatures: Mathematical and Algorithmic Applications of Linear Algebra" by Jiri Matousek

The appearance of linear-algebraic methods is often unexpected.
— Jiri Matousek

Miniature 1: Fibonacci Numbers, Quickly

The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined by the relations

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n \text{ for } n \geq 0.$$

Obviously, F_n can be calculated using roughly n arithmetic operations. Using linear algebra, can we do it more quickly?

Problem 1:
Write an algorithm that uses linear algebra to compute the n^{th} Fibonacci number in $O(\log n)$ time.

Steps for Problem 1:

- *Encode two Fibonacci numbers* F_n, F_{n+1} *as a vector* $v_n = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$.
- *Write down a matrix* M *that returns the next vector* v_{n+1} *given the previous vector* v_n*; that is, find* M *such that* $Mv_n = v_{n+1}$.
- *Write a formula for* v_n *given* $v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
- *Describe an algorithm that computes* v_n *(and hence finds* F_{n+1}*) using only* $O(\log n)$ *matrix multiplications.*

Miniature 2: Fibonacci Numbers, the Formula

Problem 2:
Derive an explicit formula for the n^{th} Fibonacci number F_n.

Steps for Problem 2:

- *Let W be the vector space of all sequences $(u_0, u_1, u_2, \ldots)$ of real numbers — with coordinate-wise addition and multiplication by real numbers — which satisfy the Fibonacci recursion relationship:*

$$W = \big\{(u_0, u_1, u_2, \ldots) \subset \mathbb{R} \;\big|\; u_{n+2} = u_{n+1} + u_n\big\}.$$

- *What is the dimension of this vector space? Write down a basis.*
- *Find all possible values of τ such that $v_n = \tau^n$ is a sequence in W.*
- *Verify that these special sequences form a basis for W.*
- *Write the Fibonacci sequence $F = (F_0, F_1, F_2, F_3, \ldots) = (0, 1, 1, 2, \ldots)$ as a linear combination of these special sequences. Then find an explicit formula for the n^{th} Fibonacci number F_n.*

Miniature 3: The Clubs of Oddtown

There are n citizens living in Oddtown. Their main occupation was forming various clubs, which at some point started threatening the very survival of the city. In order to limit the number of clubs, the city council decreed the following innocent-looking rules:

- Each club has to have an odd number of members.
- Every two clubs must have an even number of members in common.

Problem 3:
Under these rules, show that it is impossible to form more than n clubs.

Steps for problem 3:

- *Encode the club participation of Oddtown in a matrix: Suppose there are m clubs and n citizens. Define an $m \times n$ matrix A where $A_{ij} = 1$ if citizen j is in club i and $A_{ij} = 0$ if citizen j is not in club i. For the remainder of this problem, treat this as a matrix with entries in $\mathbb{F}_2$.*
- *Look at the matrix product AA^T. What do the rules of Oddtown tell you about AA^T?*
- *What is the rank of the matrix AA^T? How does this compare to the number of clubs?*
- *How does the rank of AA^T compare to the rank of A? Use this to conclude that the maximal number of clubs is n.*

Miniature 6: Odd Distances

Problem 4:
Show there are no 4 points in the plane such that the distance between each pair is an odd integer.

Steps for problem 4:

- *Let* $\mathbf{0}$ *be one of the points and* a, b, c *the other three. Suppose that the distance between each pair of points is odd. Write down this condition in terms of* a, b, c*. We will be showing that this creates a contradiction.*
- *Show that if* m *is an odd integer, then* $m^2 = 1 \pmod 8$*.*
- *Derive the polarization identity*

  $$\|u + v\|^2 = \|u\|^2 + \|v\|^2 + 2\langle u, v\rangle$$

  and use it to show that

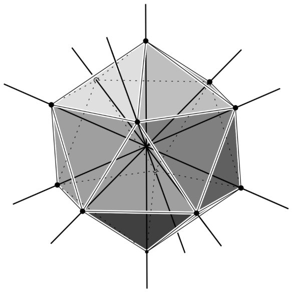
  $$2\langle a, b\rangle = 2\langle a, c\rangle = 2\langle b, c\rangle = 1 \pmod 8).$$

  .
- *Define a matrix* A *with columns* a, b, c*. What is the maximum rank of* A*?*
- *Write a formula for the product* $B = A^T A$ *in terms of* $\|a\|^2, \langle a, b\rangle, \langle a, c\rangle, \ldots$ *etc. Using your previous results, compute* $2B \pmod 8$*.*
- *What is the determinant of* $2B \pmod 8$*? What does this tell you about the rank of* B*? Use this to find a contradiction.*

Miniature 9: Equiangular Lines

What is the largest number of lines in $\mathbb{R}^3$ such that the (nonzero) angle between every two of them is the same?

Everybody knows that in $\mathbb{R}^3$ there cannot be more than three mutually orthogonal lines, but the situation for angles other than 90 degrees is more complicated. For example, the six longest diagonals of the regular icosahedron (connecting pairs of opposite vertices) are equiangular:



As we will prove, this is the largest number one can get.

Problem 5:
Show that the largest number of equiangular lines in $\mathbb{R}^3$ is 6, and in general, there cannot be more than $\binom{d+1}{2}$ equiangular lines in $\mathbb{R}^d$.

Steps for problem 5:

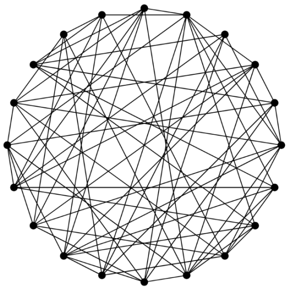
- *Suppose that you have a configuration of* n *lines with unit directions* $v_1, \ldots, v_n \in \mathbb{R}^d$. *Suppose that the angle between any two distinct lines is a fixed* θ, *with* $0 < \theta < \frac{\pi}{2}$. *What does this condition tell you about* $\langle v_i, v_j \rangle$ *for* $i \neq j$*?*
- *Consider the* $d \times d$ *matrices* $M_i = v_i v_i^T$. *Show that* M_i *is symmetric for each* i.
- *Show that the vector space of symmetric* $d \times d$ *matrices has dimension* $\binom{d+1}{2}$.
- *If we can show that the matrices* M_i *are linearly independent, then we are done. (Why?) To show this, suppose that we have a linear combination of* M_i *such that*

  $$\sum_i a_i M_i = 0.$$

  Multiply this sum on the left by v_j^T *and on the right by* v_j. *Write this sum entirely in terms of the coefficients* a_i *and* $\cos^2 \theta$.
- *Using the fact that* $\cos^2 \theta < 1$, *show that* $a_i = 0$ *for all* i. *(This is challenging. Feel free to ask your instructor.)*

Miniature 10: Where is the Triangle?

Does a given graph contain a triangle, i.e., three vertices u, v, w, every two of them connected by an edge? This question is not entirely easy to answer for graphs with many vertices and edges. For example, where is a triangle in this graph?



An obvious algorithm for finding a triangle inspects every triple of vertices, and thus it needs roughly n^3 operations for an n-vertex graph (there are $\binom{n}{3}$ triples to look at, and $\binom{n}{3}$ is approximately $n^3/6$ for large n). Can we do any better with linear algebra?

Problem 6:
Given a graph G with n vertices, use the fact that an $n \times n$ matrix can be squared in $O(n^{2.371339})$ time to write an $O(n^{2.371339})$ algorithm that determines whether G contains a triangle.

Steps for problem 6:

- *Let A be the adjacency matrix of G. Consider $B = A^2$. Using the definition of matrix multiplication, what does the entry B_{ij} tell us about the graph G?*
- *In terms of the entries A_{ij} and B_{ij}, what does it mean for G to contain a triangle?*
- *Using the fast matrix multiplication as a black box, write an algorithm that checks if G contains a triangle.*

Miniature 12: Tiling a Rectangle by Squares

Problem 7:
Show that a rectangle R with side lengths 1 and x, where x is irrational, cannot be tiled by finitely many squares.

Steps for problem 7:

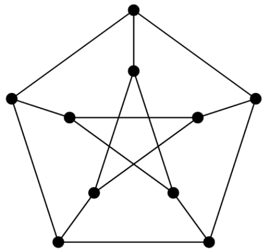
- *Suppose that a tiling exists consisting of squares* $Q_1, \ldots, Q_n$. *Let* s_i *be the side length of* Q_i.
- *Consider the real numbers* $\mathbb{R}$ *as a vector space over the field* $\mathbb{Q}$. *Let* $V \subset \mathbb{R}$ *be the subspace generated by* $x, s_1, s_2, \ldots, s_n$.
- *We can define a linear function* f *such that* $f(1) = 1$ *and* $f(x) = -1$. *(This is possible because* $x, 1$ *are linearly independent over* $\mathbb{Q}$*).*
- *For a rectangle* A *with side lengths* a *and* b*, define* $v(A) = f(a)f(b)$. *Using the linearity of* f*, prove that*

  $$v(R) = \sum_{i=1}^{n} v(Q_i).$$

  (This is the most challenging part. Draw a picture and don't be afraid to ask your instructor for help.)
- *Using the definition of* v, f, Q_i*, and* R*, find a contradiction.*

Miniature 13: Three Petersens Are Not Enough

The famous Petersen graph



has 10 vertices of degree 3. The complete graph K_{10} has 10 vertices of degree 9. Yet, it is not possible to cover all edges of K_{10} by three copies of the Petersen graph.

Problem 8:

Show that there are no three subgraphs of K_{10}, each isomorphic to the Petersen graph, that together cover all edges of K_{10}.

Steps for problem 8:

- *Suppose for the sake of contradiction that K_{10} can be covered by three copies of the Petersen graph: P, Q, and R.*
- *Let A be the adjacency matrix of K_{10} and A_P, A_Q, and A_R be the adjacency matrices of the subgraphs P, Q, and R respectively. Show that*

$$A = A_P + A_Q + A_R.$$

- *Write down an adjacency matrix for the Petersen graph. Show that this adjacency matrix has eigenvalue 1 with geometric multiplicity 5. Because A_P is equivalent to any adjacency matrix of the Petersen graph, this implies that A_P has eigenvalue 1 with geometric multiplicity 5.*
- *Suppose that x is an eigenvalue of A_p with eigenvalue 1; that is, $A_p x = x$. Show that $\langle \mathbf{1}, x \rangle = 0$ where $\mathbf{1} = (1, \ldots, 1)^T$. The same is true of A_Q.*
- *Consider the subspace $\mathbf{1}^\perp = \{x : \langle \mathbf{1}, x \rangle = 0\} \subset \mathbb{R}^{10}$. What is the dimension of this subspace?*
- *Using the dimension of $\mathbf{1}^\perp$, prove that A_Q and A_P share a nonzero eigenvector v with eigenvalue 1.*
- *Prove that v is an eigenvector of A_R with eigenvalue -3. Using the fact that R is a Petersen graph, show that this is a contradiction.*