Eigenproblems

Equations are just the boring part of mathematics. I attempt to see things in terms of geometry. —Stephen Hawking

Preliminaries

Recall that an *initial value problem* is an equation of the form

$$y'(t) = f(t, y), \quad \text{with} \quad y(t_0) = y_0. \tag{1}$$

Here y is an unknown function, $f(t, y)$ is some given function of t and y, and y_0 is the *initial value* at the initial time t_0. Generally speaking, an initial value problem will have just one solution $y(t)$.

We say that a function $y(t)$ satisfies (1)—or is a solution to (1)—if $y'(t) = f(t, y(t))$ is true for t near t_0 and if $y(t_0) = y_0$.

Often, we will consider a differential equation *without* an initial condition:

$$y'(t) = f(t, y).$$

Generally speaking, in this case, we will have *infinitely many* solutions $y(t)$. Often, these different solutions come from the $+c$ from some integral and appear like

$$y(t) = \cdots + c \quad \text{or} \quad y(t) = c \cdot (\ldots)$$

so make sure to keep track of those! We call solutions with arbitrary constants like this *general solutions*.

Part 1: Motivation

Problem 1: Rabbits and Foxes
Suppose that we have a population of clever foxes and a population of cute rabbits that live together in a forest. We'll let $f(t)$ denote the number of foxes at time t and $r(t)$ denote the number of rabbits.

Let's look at the rabbits. At any given moment, there is a chance that some rabbits are born, proportional to how many rabbits there are. Also, at any given moment, some rabbits are eaten by the foxes. The more foxes there are, the more rabbits that are being eaten. So we expect the number of rabbits to satisfy the differential equation

$$r'(t) = ar(t) + bf(t),$$

for some a, b.

Now let's look at the foxes. At any given moment, there are some foxes being born, again proportional to the number of foxes there are. Also, *the more food (rabbits) that there are, the more the foxes are able to reproduce.* Taking inspiration from the equation for $r'(t)$, write a reasonable equation that the fox population f might satisfy.

Problem 2: Exploring the problem
Together, the fox and rabbit populations form a system of differential equations:

$$\begin{cases} r'(t) &= ar(t) + bf(t), \quad \text{with} \quad r(0) = r_0 \\ f'(t) &= cr(t) + df(t), \quad \text{with} \quad f(0) = f_0. \end{cases}$$

Right now, we can't solve this problem outright. There are too many unknowns. So let's try some simple examples.
This problem uses the methods from our last ODE packet. Ask an instructor if you don't remember.

(1) Suppose that the rabbits and foxes can't find each other. That is, fix $b = c = 0$ and $a = d = 1$. Solve for $r(t)$ and $f(t)$.
Without a predator, what happens to the rabbit population as $t \to \infty$? Does this depend on r_0?

(2) Suppose instead that $a = d = 1$, $c = 0$, and $b = -1$. (Intuitively, the foxes are eating the rabbits, but that doesn't affect their birthrate.)

$$\begin{cases} r'(t) = r(t) - f(t), & r(0) = r_0 \\ f'(t) = f(t) & f(0) = f_0. \end{cases}$$

Hint: Solve for $f(t)$ first. Then assume that $r(t) = c_1 e^t + c_2 t e^t$ and solve for c_1, c_2. This is a method that we learned last time.

(3) Finally, suppose (strangely) that we consider the system:

$$\begin{cases} r'(t) = -f(t), & r(0) = 1 \\ f'(t) = r(t) & f(0) = 0. \end{cases}$$

(a) Using this system, compute $f''(t)$ in terms of $f(t)$ and verify that $f(t) = a\cos(t) + b\sin(t)$ is a general solution.
(b) Repeat this process for $r''(t)$ with different constants; i.e., $r(t) = c\cos(t) + d\sin(t)$.
(c) Using the system above and the initial conditions, solve for $f(t), r(t)$.

In this packet, we will fully characterize every solution to these systems. Amazingly, we will barely use any ODE methods at all! We just need some linear algebra.

Part 2: Eigenvectors and Eigenvalues

Consider the system of linear ODEs:

$$\begin{cases} x'(t) = ax(t) + by(t) \\ y'(t) = cx(t) + dy(t). \end{cases}$$

Let $\mathbf{x}(t)$ be the *vector* solution given by

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad \text{with} \quad \mathbf{x}'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}.$$

We define a matrix A as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{so that} \quad \mathbf{x}' = A\mathbf{x}.$$

Let's look for some solutions to this vector system.

Problem 3: Guess a Solution
Sometimes, all you can do is guess. Taking inspiration from our last ODEs packet, *guess* that our solution $\mathbf{x}(t)$ takes the form:

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v},$$

where $\mathbf{v}$ is some fixed vector (constant in t). If $\mathbf{x}$ is a solution to our system, solve for a relationship between λ, $\mathbf{v}$, and A.

Definition 4: Eigenvectors and eigenvalues
Given a matrix A, we say that a nonzero vector $\mathbf{v}$ is an *eigenvector* with *eigenvalue* λ if

$$A\mathbf{v} = \lambda v. \tag{2}$$

We say that λ is an eigenvalue of A, if (2) holds for *some* $\mathbf{v} \neq \mathbf{0}$.

Problem 5: Simple Examples
Find *all* of the eigenvalues of the following matrices. For each eigenvalue, find (at least) one nonzero eigenvector.

(1)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(2)

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(3)

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(4)

$$E = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Problem 6: A Non-example
Consider the matrix:

$$D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(1) Show that D does not have any *real* eigenvalues.
Hint: Suppose that $D\mathbf{v} = \lambda\mathbf{v}$ for some real λ and $\mathbf{v} \neq 0$, then show that $\mathbf{v} = 0$ or $\lambda^2 = -1$.

(2) Show that D has complex eigenvalues $\pm i$.
Hint: Your eigenvectors will also have to be complex.

Finding these eigenvalues and eigenvectors can get very annoying, very quickly. Instead, we'll use the determinant to do our work for us.

Part 3: Determinants

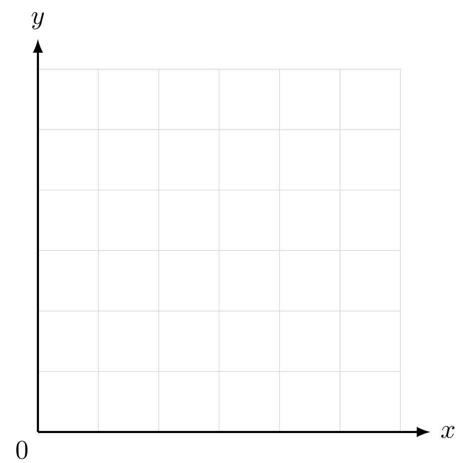
Problem 7: Geometric Interpretation
Consider the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

(1) The four points (vectors),

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

form a unit square in the plane. Compute the image of each of these points under A and draw the resulting parallelogram. (For your picture, you may assume that $a > c > 0$ and $d > b > 0$.)



(2) Compute the area of the resulting parallelogram in terms of a, b, c, d.

Definition 8: Determinant
Given a 2×2 matrix, we define the *determinant* to be the quantity:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

This is exactly the area that we found in the last part. The determinant determines how much a matrix stretches or squishes area.

The benefit of the determinant is that it determines whether a matrix A sends any nonzero vectors $\mathbf{v}$ to $\mathbf{0}$. That is, whether there exists $\mathbf{v} \neq \mathbf{0}$ such that $A\mathbf{v} = \mathbf{0}$. This inspires the following definition:

Definition 9: Kernel
Given a matrix A, we define the *kernel* of A to be

$$\ker A = \{\mathbf{v} : A\mathbf{v} = \mathbf{0}\}.$$

That is, the kernel of A is the set of all vectors which A sends to $\mathbf{0}$.

Problem 10: Basic Exercise
Prove that $\mathbf{0} \in \ker A$ for any matrix A.

Problem 11: Zero Determinant implies Nonzero Kernel
Prove that if $\det A = 0$ then $\ker A \neq \{\mathbf{0}\}$.
Hint: Given the previous problem, this is the same as showing that there exists $\mathbf{v} \neq 0$ *such that* $\mathbf{v} \in \ker A$.

Can you give an intuitive, geometric explanation for this result given the area interpretation of the determinant?

Problem 12: Nonzero Kernel implies Zero Determinant
Prove that if $\ker A \neq \{\mathbf{0}\}$ then $\det A = 0$.

Problem 13: Eigenvalue Implies Zero Determinant
Suppose that a matrix A has eigenvalue λ. Show that $\det(A - \lambda I) = 0$, where I is the *identity matrix*:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Problem 14: Zero Determinant Implies Eigenvalue
Suppose that $\det(A - \lambda I) = 0$. Prove that λ is an eigenvalue of A.

Problem 15: Some Examples

The last two problems tell us that if we want to find the eigenvalues of a matrix, we just need to look at the zeros of $\det(A - \lambda I)$. From there, we can find the eigenvectors manually. Let's try that on a few examples. For each of the following matrices, find *all* of the eigenvalues, and at least one eigenvector per eigenvalue:

(1)

$$A = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$$

(2)

$$B = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

(3)

$$C = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix}$$

(4) *Silly Example*: Find the eigenvalue of D and find two *linearly independent* eigenvectors. (That is, find two eigenvectors which are not parallel.)

$$D = \begin{bmatrix} 37 & 0 \\ 0 & 37 \end{bmatrix}$$

Problem 16: A Trick for Eigenvectors

Suppose that we know that λ is an eigenvalue of A. If

$$A - \lambda I = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

show that $\begin{bmatrix} -b \\ a \end{bmatrix}, \begin{bmatrix} -d \\ c \end{bmatrix}$ are both eigenvectors with eigenvalue λ. In particular, *we only care about the values of one row.* This works for complex values too.

Part 4: Distinct Real Eigenvalues

Let's go back to systems of linear ODEs. Consider the following system:

$$\begin{cases} x'(t) = ax(t) + by(t) \\ y'(t) = cx(t) + dy(t). \end{cases} \tag{3}$$

Again, we write this with vectors and matrices as:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{so that} \quad \mathbf{x}' = A\mathbf{x}.$$

Earlier, we saw that if we can find an eigenvalue λ with eigenvector $\mathbf{v}$ for the matrix A, then

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$$

is a solution to our system.

Problem 17: Linear Independence
Suppose that we can find two real eigenvalues $\lambda_1 \neq \lambda_2$ of A with nonzero eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ respectively. Show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.
(That is, show that $\mathbf{v}_1 \neq c\mathbf{v}_2$ for any constant c.)

Problem 18: General Solution
In the previous packet, we saw that if we can find two linearly independent solutions to a homogeneous second-order ODE:

$$x_1(t), x_2(t) \quad \text{solve} \quad x'' + ax' + bx = 0,$$

then the general solution can be written as

$$x(t) = a_1 x_1(t) + a_2 x_2(t),$$

where a_1, a_2 are arbitrary constants to be determined by the initial condition.

The exact same is true of our systems. If

$$\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1, \mathbf{x}_2 = e^{\lambda_2 t}\mathbf{v}_2 \quad \text{solve} \quad \mathbf{x}' = A\mathbf{x},$$

where $\lambda_1 \neq \lambda_2$ are real and $\mathbf{v}_1, \mathbf{v}_2 \neq \mathbf{0}$, then the general solution is given by

$$\mathbf{x}(t) = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 = a_1 e^{\lambda_1 t}\mathbf{v}_1 + a_2 e^{\lambda_2 t}\mathbf{v}_2.$$

Find the general solution to the following systems:

(1)

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \mathbf{x}$$

(2)

$$\mathbf{x}' = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix} \mathbf{x}$$

Problem 19: Initial Value Problem
Given your general solutions to the previous problem, find *the* solutions to the following initial value problems:

(1)

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

(2)

$$\mathbf{x}' = \begin{bmatrix} -5 & 1 \\ 4 & -2 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

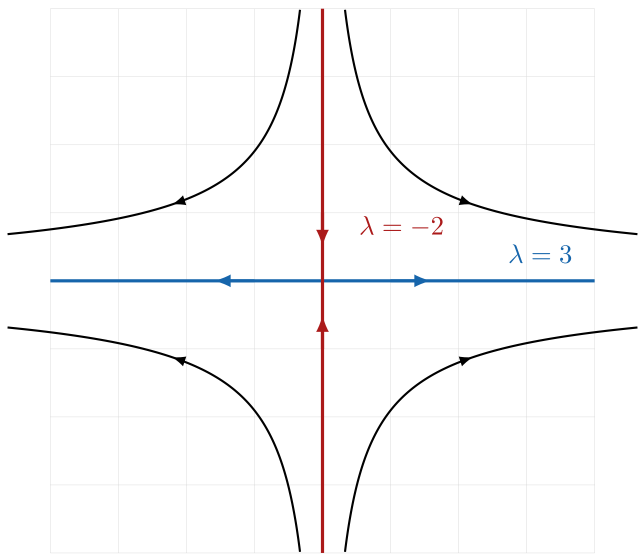
Part 5: Sketching Solutions

Problem 20:
Often, we care less about the exact solutions to a system of ODEs and more about the qualitative behavior. To see this behavior, it is useful to (roughly) sketch the general solutions.

For example, consider the simple system of ODEs:

$$\mathbf{x}' = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x} \quad \text{with solution} \quad \mathbf{x}(t) = ae^{3t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + be^{-2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We can sketch the general solution as



This sketch tells us the general behavior of solutions, if we fix an initial condition.

A: Which solution lines correspond to the fundamental solutions?

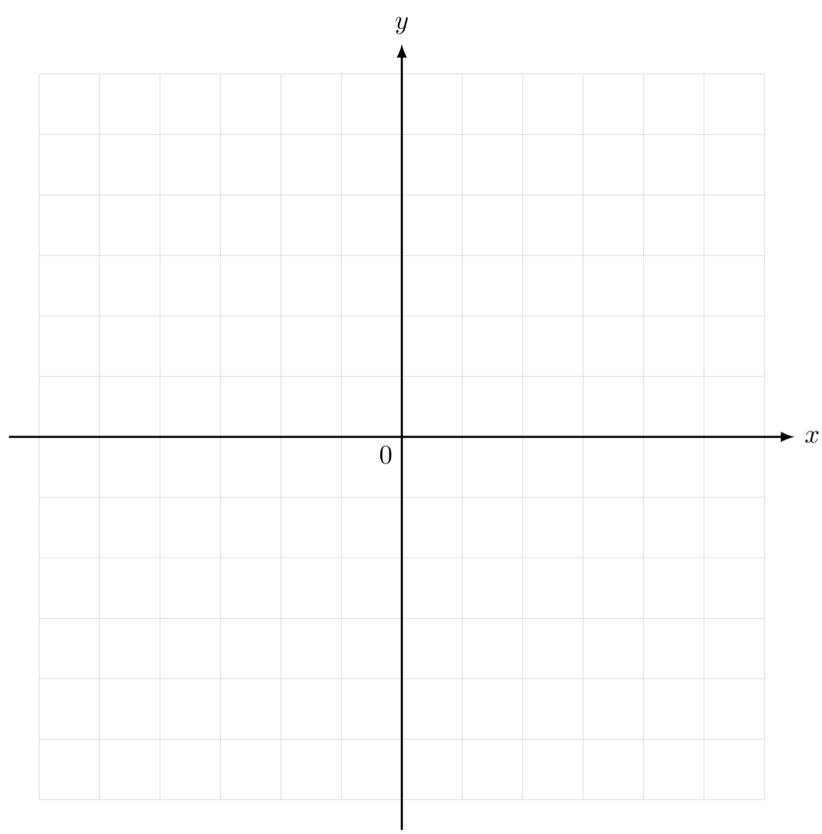
B: If the initial condition is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, what path does the solution follow? What happens as $t \to \infty$? What about $t \to -\infty$?

C: Consider the intermediate solutions (those in black). If the initial condition is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, what path does the solution follow? As $t \to \infty$, which solution does this approach? What about as $t \to -\infty$?

Problem 21:
Consider the following system of linear ODEs:

$$\mathbf{x}' = \begin{bmatrix} 7 & -1 \\ -1 & 7 \end{bmatrix}.$$

We are going to sketch the possible behaviors of the general solution on the following plot:



A: Find the eigenvalues and eigenvectors of the given matrix. Write down the two fundamental solutions $\mathbf{x}_1, \mathbf{x}_2$ to the system of ODEs.

B: Sketch the paths of the fundamental solutions for all t. Also sketch the *negatives* of those paths; that is, $-\mathbf{x}_1$ and $-\mathbf{x}_2$.

C: For the intermediate regions, we only care about the rough behavior of solutions. Consider the particular solution:

$$\mathbf{x}(t) = e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{8t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

This corresponds to the initial condition $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$. As $t \to \infty$, which term is larger, e^{8t} or e^{6t}? What does this imply about the behavior of this solution as $t \to \infty$?

As $t \to -\infty$, which term is larger, e^{6t} or e^{8t}? What does this imply about the behavior of this solution as $t \to -\infty$?

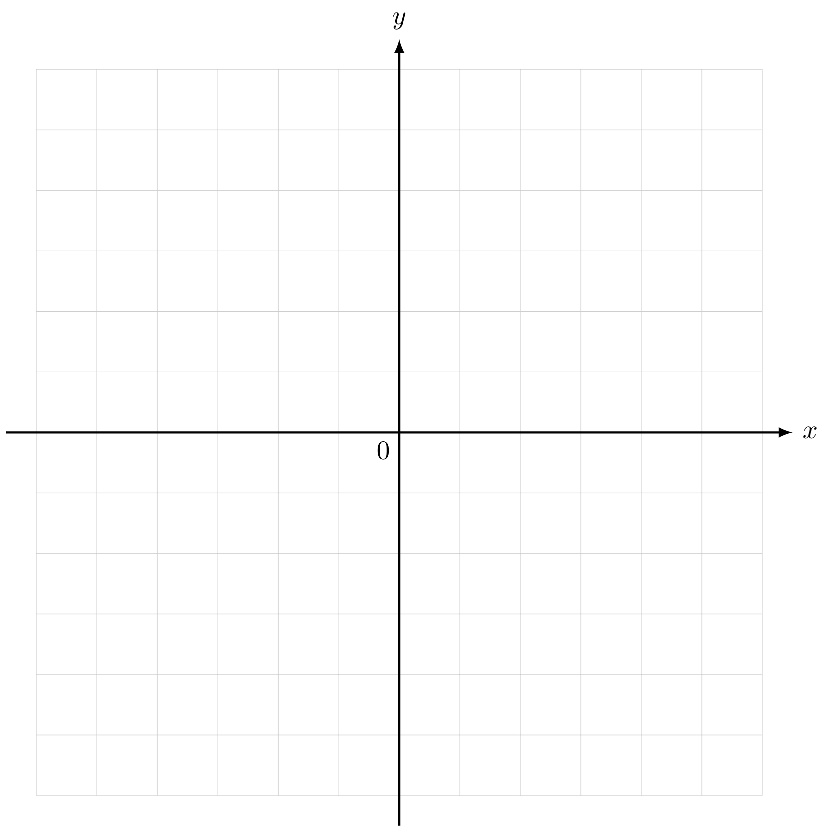
Using these asymptotic behaviors, sketch the solution.

D: The previous part shows us that a solution will flow from a smaller eigenvalue to a larger. Use this principle to complete our sketch.

Problem 22:
Consider the following system of linear ODEs:

$$\mathbf{x}' = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}.$$

Following the previous steps, sketch the solutions to this system of ODEs:



Problem 23: Properties of Solutions
Recall the Picard–Lindelöf theorem: If $f(t,y)$ and $\frac{\partial f}{\partial y}$ are continuous near (t_0, y_0), then the initial value problem

$$y' = f(t,y), \quad \text{with} \quad y(t_0) = y_0$$

has a unique solution (at least for a short time).

Because a matrix is always (uniformly) continuous, for systems of linear ODEs, we have an even stronger result: For any initial condition $(t_0, \mathbf{x}_0)$ the system of ODEs

$$\mathbf{x}' = A\mathbf{x}, \quad \text{with} \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

has a unique solution $\mathbf{x}(t)$ that exists for all t.

Using this theorem, answer the following questions.

A: For a point $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ in the plane, can we guarantee that *some* solution goes through that point?

B: When sketching the general solution, is it possible for two particular solutions to cross paths?

Part 6: Complex Eigenvalues

Problem 24: Simple Example — Purely Imaginary
Consider the system of ODEs:

$$\mathbf{x}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}.$$

We have already shown that this matrix has eigenvalues $\pm i$ with eigenvectors $\begin{bmatrix} \pm i \\ 1 \end{bmatrix}$. This tells us that we should have a general solution:

$$\mathbf{z}(t) = \alpha e^{it} \begin{bmatrix} i \\ 1 \end{bmatrix} + \beta e^{-it} \begin{bmatrix} -i \\ 1 \end{bmatrix},$$

where α, β are complex. *However,* we were given a real-valued system, so we want a real-valued solution. To find the general real-valued solution from our complex-valued one, we will use *Euler's identity*:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

To save you a lengthy calculation, we want to show that the real-valued general solution can be written as

$$\mathbf{x}(t) = (a\cos t + b\sin t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + (-a\sin(t) + b\cos(t)) \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where a, b are real.

A: Given the general solution above, write down two fundamental real-valued solutions to our system of ODEs.

B: Verify that these are indeed solutions to the system of ODEs. (Plug them in.)

C: Verify that these fundamental solutions are linearly independent.
(Show that $\mathbf{x}_1(t) \neq c\mathbf{x}_2(t)$ for any constant c.)

Problem 25: More Complicated
Consider the system of ODEs:
$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{x}.$$

(1) Find the eigenvalues and eigenvectors for this matrix. Write down the *complex-valued* general solution for this equation.

(2) We claim that the *real-valued* general solution is given by
$$\mathbf{x}(t) = e^t \left((a\cos t + b\sin t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + (-a\sin t + b\cos t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).$$
Repeat the process of the previous problem to show that this is a general solution to our system of ODEs.
   (a) Identify two fundamental solutions from the general solution.
   (b) Verify that these are indeed solutions to the system.
   (c) Verify that these solutions are linearly independent.

Problem 26: General Case

With these two previous examples, we have found that if we have:

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad \text{eigenvalues: } a \pm ib, \quad \text{eigenvectors: } \mathbf{v}_1 \pm i\mathbf{v}_2,$$

where $\mathbf{v}_1, \mathbf{v}_2$ are real vectors, then the *real-valued* general solution is given by:

$$\mathbf{x}(t) = e^{at}\big((c_1 \cos bt + c_2 \sin bt)\mathbf{v}_1 + (-c_1 \sin bt + c_2 \cos bt)\mathbf{v}_2\big).$$

Use this to find the general solutions of the following systems of ODEs:

(1)

$$\mathbf{x}' = \begin{bmatrix} 3 & -13 \\ 5 & 1 \end{bmatrix} \mathbf{x}.$$

(2)

$$\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} \mathbf{x}$$

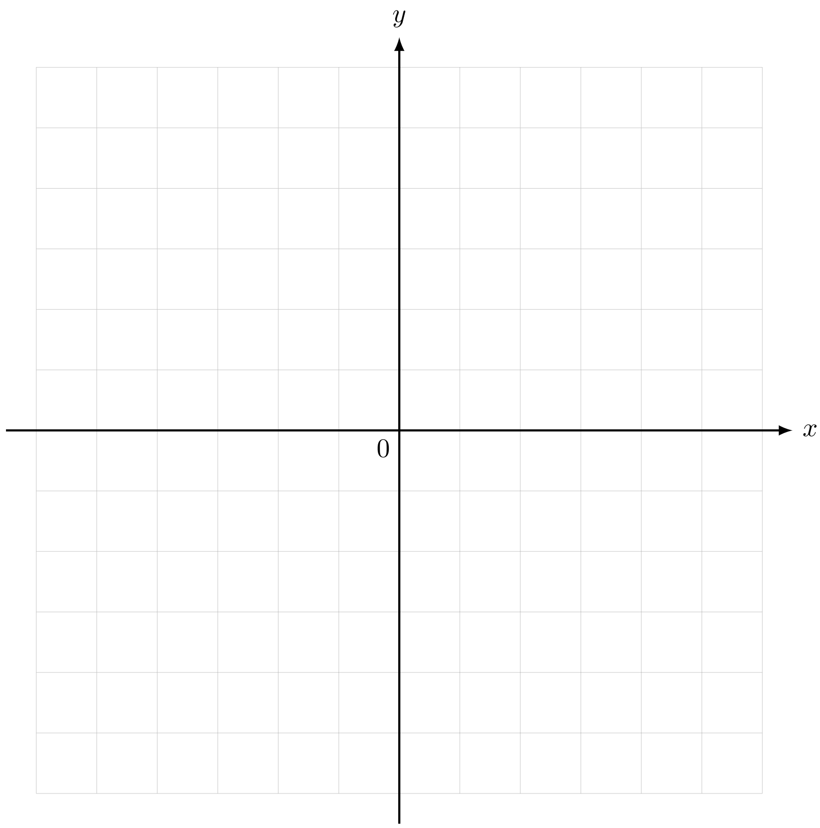
Part 7: Sketching Complex Eigenvalues

Let's try sketching some solutions!

Problem 27: A Simple Example
Consider our earlier example:

$$\mathbf{x}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{with solution} \quad \mathbf{x}(t) = (a\cos t + b\sin t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + (-a\sin(t) + b\cos(t))\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We're going to sketch the solutions to this system of ODEs on the following plot:

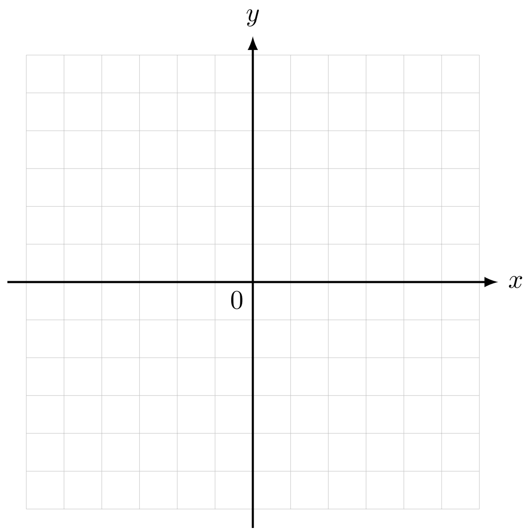


A: First, fix $a = 1$ and $b = 0$. Sketch the solution on the given plot. (This should be a familiar function.)
B: Which direction is the solution travelling? *Hint:* Calculate $\mathbf{x}'(0)$. This will give a direction.
C: Choosing other values of a, b, fill in the plot with a few solutions. All of the solutions should look very similar.

Problem 28: More Complicated
Consider our earlier example:

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{with solution} \quad \mathbf{x}(t) = e^t\left((a\cos t + b\sin t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + (-a\sin(t) + b\cos(t))\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right).$$

We're going to sketch the solutions to this system of ODEs on the following plot:



(1) The imaginary part of our eigenvalues tells us that solutions will rotate around the origin. Which direction will they rotate?

Hint: Fix an initial condition $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and calculate $\mathbf{x}'(0)$.

(2) In the last problem, we saw that the imaginary part of the eigenvalue causes our solutions to rotate around the origin. What does the component e^t do to our solutions? What happens as $t \to \infty$? What about $t \to -\infty$?

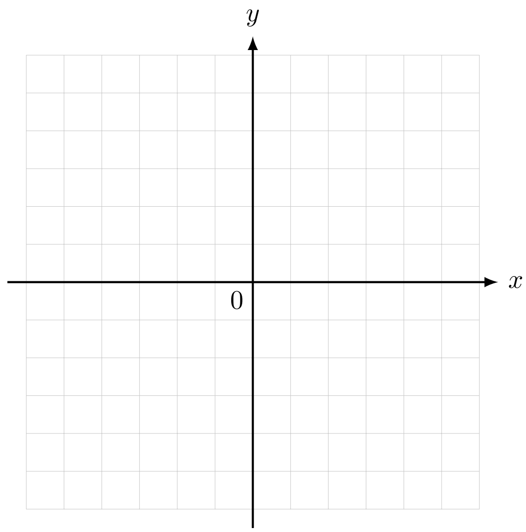
(3) If a solution begins at some $\mathbf{x}_0 \neq \mathbf{0}$, will it ever reach $\mathbf{0}$? Either as $t \to \infty$ or as $t \to -\infty$?

(4) Combining the previous parts, roughly sketch a few sample solutions to this system of ODEs.

Problem 29: (Challenge) The Shape of Spirals
Consider the example:

$$\mathbf{x}' = \begin{bmatrix} -4 & -17 \\ 2 & 2 \end{bmatrix} \mathbf{x} \quad \text{with solution} \quad \mathbf{x}(t) = e^{-t}\left((a\cos 5t + b\sin 5t)\begin{bmatrix} -3 \\ 2 \end{bmatrix} + (-a\sin 5t + b\cos 5t)\begin{bmatrix} 5 \\ 0 \end{bmatrix} \right).$$

We're going to sketch the solutions to this system of ODEs on the following plot:



(1) In what direction do our solutions spiral? (Clockwise or counter-clockwise?)

(2) What is the period of rotation for our solution? That is, how long does it take a solution to complete a full revolution?

(3) Suppose that $\mathbf{x}(0) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. (That is, $a = 1, b = 0$.) At $t = \pi/10$, in what direction will our solution lie? That is, ignore the component e^{-t} in our solution.

(4) After the same amount of time, in what direction will the our solution lie? That is, in what direction does our solution lie at time $t = \frac{2\pi}{10}$? How about $t = \frac{3\pi}{10}$ and $t = \frac{4\pi}{10}$?

(5) Because these are all the same length of time, this implies that our solution moves *slowly* from $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} -5 \\ 0 \end{bmatrix}$ and *quickly* from $\begin{bmatrix} -5 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Because it is moving at a uniform speed inwards toward the origin, this implies that our solution will be *stretched* like an ellipse. Determine in what direction the solution is stretched and sketch a solution on the given plot.

Problem 30: More Problems

For each of the following systems, find the eigenvalues and eigenvectors and sketch the solutions. You do not need to write down any general solutions, but feel free to if it is helpful.

(1)

$$\mathbf{x}' = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \mathbf{x}.$$

(2)

$$\mathbf{x}' = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \mathbf{x}.$$

(3)

$$\mathbf{x}' = \begin{bmatrix} \pi & 0 \\ 0 & \pi \end{bmatrix} \mathbf{x}.$$

Part 8: Higher Dimension Problems (Challenge)

For an $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix},$$

we define the derivative recursively via

$$\det A = a_{11} \det A_{11} - a_{21} \det A_{22} + \cdots - (-1)^n a_{n1} \det A_{n1}$$

where A_{ij} is the matrix A without its i^{th} row and j^{th} column; e.g.,

$$A_{11} = \begin{bmatrix} a_{22} & \dots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \dots & a_{nn} \end{bmatrix}.$$

For example, the determinant of a 3×3 matrix is

$$\det \begin{bmatrix} a & b & c \\ d & f & g \\ h & j & k \end{bmatrix} = afk + bgh + cdj - agj - bdk - cfh.$$

Problem 31: (Challenge)
Generalizing the methods in this packet, find the general solution to the following systems of linear ODEs.

(1)

$$\mathbf{x}' = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \mathbf{x}.$$

(2)

$$\mathbf{x}' = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & -2 \end{bmatrix} \mathbf{x}.$$

(3)

$$\mathbf{x}' = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x}.$$