Crash Course on Calculus

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Instructor's Handout

NOTE: This is truly a crash course. If you have never seen any calculus before, please let an instructor know so that they can provide consistent support through this packet.

Part 1: Limits

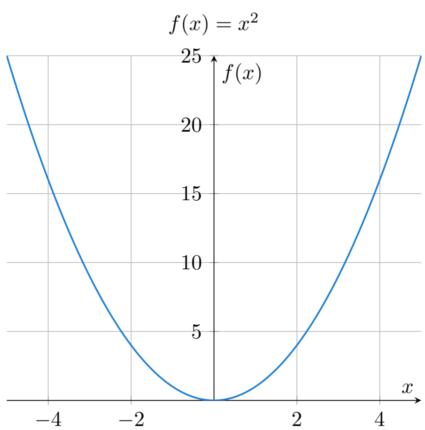
Definition 1: Limit
In an informal way, a *limit* is what a function $f(x)$ *approaches* as its input x approaches a particular value x_0. Mathematically, we denote this as

$$\lim_{x \to x_0} f(x),$$

which can be read as "the limit of $f(x)$ as x approaches x_0."

This is vague, so lets make it concrete with some examples.

Problem 2: Quadratic
One function that you already understand is $f(x) = x^2$. It's graph is given by:



We want to figure out what $f(x)$ approaches as x approaches 0. To do so, we're going to consider values of x that are increasingly close to 0. Then we're going to look for a nice trend.

What is $f(0.1)$?

How about $f(0.01)$? Now $f(0.001)$?

As x gets closer to 0, what value is $f(x)$ approaching?

Mathematically, we write this value as

$$\lim_{x \to 0} f(x) =$$

Solution

$f(0.1) = 0.01$
$f(0.01) = 0.0001.$
$f(0.001) = 0.000001.$
As x gets closer to 0, $f(x)$ is also getting closer to 0. So $\lim_{x \to 0} f(x) = 0$.

Problem 3: Using the graph
Alternatively, we can see this limit from the graph! Trace the graph of $f(x) = x^2$ as x gets closer to 0. What happens to the graph? What does it approach?

With this vague, intuitive process, we can calculate the limits of many different functions. for each of these, either calculate draw the graph and trace what happens or calculate a few values near the limit point.

Problem 4: Other examples
Calculate the following limits.

(a) $\lim_{x \to 1} 5x + 2 =$

Solution

7

(b) $\lim_{x \to 2} x^3 =$

Solution

8

(c) Use 5 to denote the constant function $f(x) = 5$. Then calculate $\lim_{x \to 2} 5 =$

Solution

5

Now, for all the previous problems, we could have cheated a little bit. We found $\lim_{x\to 2} x^3 = 8$, but we also could have noticed that $2^3 = 8$. So, in this particular case, $\lim_{x\to x_0} f(x) = f(x_0)$.

This is true for lots of functions, but is not true for all of them. The functions where it is true are called *continuous functions*. Let's calculate some examples where this isn't true.

Problem 5: Limits of discontinuous functions
Consider the limit

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-1}{x-1}.$$

If we try to plug $x = 1$ into this function, what do we find? Why is this an issue?

Solution

We find 0/0. This is an issue because we can't divide by 0.

Calculate $f(1.1)$, $f(1.01)$ and $f(1.001)$. What are these values approaching? What is the limit?

Solution

$f(1.1) = 2.1$
$f(1.01) = 2.01$
$f(1.001) = 2.001$
These are approaching 2!

Alternatively, we can try to simplify $f(x)$ first and then take the limit. Factor the numerator and simplify, then find the limit.

Solution

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x+1$$

so the limit is 2.

Note that this simplification is not technically mathematically valid because we took a function undefined at $x = 1$ and made it defined there. But for limits, we only care about points *near* $x = 1$, not $x = 1$ specifically. This simplification is valid for any $x \neq 1$.

Problem 6: (Challenge)
Numerically calculate the limit

$$\lim_{x \to 0} \frac{\sin(x)}{x},$$

by calculating the function at values of x increasingly close to $x = 0$. Feel free to use a calculator!

Solution

$\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Limits have a lot of nice properties, most of which we can determine from the intuition of a function "approaching" a value.

Problem 7: Addition
If we know that

$$\lim_{x \to x_0} f(x) = A \quad \text{and} \quad \lim_{x \to x_0} g(x) = B,$$

then we know that $f(x)$ approaches A as x nears x_0 and $g(x)$ approaches B as x nears x_0. What do we expect $f(x) + g(x)$ to approach?

Solution

$A + B$

Rewrite this statement without using A, B, just use $\lim_{x \to x_0} f(x)$ and $\lim_{x \to x_0} g(x)$:

$$\lim_{x \to x_0} \left[f(x) + g(x) \right] =$$

Solution

$$\lim_{x \to x_0} \left[f(x) + g(x) \right] = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$$

.

Problem 8: Multiplication

If we know that

$$\lim_{x \to x_0} f(x) = A \quad \text{and} \quad \lim_{x \to x_0} g(x) = B,$$

then we know that $f(x)$ approaches A as x nears x_0 and $g(x)$ approaches B as x nears x_0. What do we expect $f(x) \cdot g(x)$ to approach?

Solution

AB

Rewrite this statement without using A, B, just use $\lim_{x \to x_0} f(x)$ and $\lim_{x \to x_0} g(x)$:

$$\lim_{x \to x_0} \big[f(x)g(x)\big] =$$

Solution

$$\lim_{x \to x_0} \big[f(x)g(x)\big] = \Big[\lim_{x \to x_0} f(x)\Big]\Big[\lim_{x \to x_0} g(x)\Big].$$

Problem 9: Multiplication by a constant

Let c be a constant. Using the previous problem, write

$$\lim_{x \to x_0} cf(x)$$

in terms of $\lim_{x \to x_0} f(x)$.

Solution

$$\lim_{x \to x_0} cf(x) = c \lim_{x \to x_0} f(x).$$

(Based on problems 8 and 9, we see that the limit is a linear operator.)

Part 2: Derivatives

The derivative tells us the *instantaneous* rate of change of a function $f(x)$. It is like taking the slope of a function, but at one specific point.

Problem 10: Slope
What is the slope of a line $f(x)$ if you know that $f(1) = 5$ and $f(5) = 25$? What formula did you use to calculate this?

Solution

4
rise over run

We do the same thing to define the derivative of a function, only we want to the "run" to be infinitesimally small. In this way, the derivative gives us an *instantaneous* slope.

Definition 11: Derivative
We define the *derivative* of a function $f(x)$ at a point x, denoted $f'(x)$, to be the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Problem 12: Constant function
Let's calculate the derivative of a constant function, $f(x) = c$ for some fixed c.

Set up the formula for $f'(x)$ and simplify the numerator as far as you can:

Solution

$$f'(x) = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h}$$

If you evaluate this limit, What do you find for the derivative of f? How does this compare to the slope of the line f?

Solution

$f'(x) = 0$, this is the slope of the line

Problem 13: Line
Let's calculate the derivative of a line, $f(x) = ax + b$.

Set up the formula for $f'(x)$ and simplify the numerator as far as you can:

Solution

$$f'(x) = \lim_{h \to 0} \frac{a(x+h) + b - ax - b}{h} = \lim_{h \to 0} \frac{ah}{h}$$

Cancel any h terms that you can and evaluate the limit. What do you find for the derivative of f? How does this compare to the slope of the line f?

Solution

$f'(x) = a$, this is the slope of the line

Problem 14: Quadratic
Let's calculate the derivative of a quadratic, $f(x) = x^2$.

Set up the formula for $f'(x)$ and simplify the numerator as much as possible.

Solution

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2hx + h^2}{h}$$

Simplify the fraction by dividing the numerator by h. Then evaluate the limit. What is the derivative of $f(x) = x^2$?

Solution

$f'(x) = 2x$.

Problem 15: Sum of functions
Using the limit definition of a derivative and Problem 8, find the derivative of $f(x) + g(x)$. That is, calculate the following in terms of $f'(x)$ and $g'(x)$:

$$(f+g)'(x) = \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$$

Solution

$= f'(x) + g'(x)$

This is called the *sum* rule. We can also consider the derivative of the product of two functions. This formula is more complicated, so we'll give you it.

Theorem 16: Product rule
The derivative of the *product* of two functions $f(x)g(x)$ is given by

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Note that the derivative 'hits' every term in the product individually.

Problem 17: (Challenge)
Use the limit definition of the derivative to prove the product rule.
(Only do this if you are looking for a challenge.)

Problem 18: Multiplication by a constant
Given a constant c and a function $f(x)$, use the product rule to calculate the derivative of the function $cf(x)$.

Solution

$(cf)'(x) = cf'(x)$.

With the sum rule, this means that the derivative is a linear operator.

Problem 19: Power rule

We want to show that

$$(x^n)' = nx^{n-1}$$

for any integer n. We will only prove this for $n \geq 0$, but you can assume it for $n < 0$ as well.

Start with $n = 0$, $n = 1$, and $n = 2$. Show that this formula for each of these cases. *Hint: you've seen these already.*

Now suppose that the formula holds for some n. We want to induct and show that it also holds for $n + 1$. Decompose $x^{n+1} = x^n x$ and then use the product rule to show that

$$x^{n+1} = (n + 1)x^n.$$

By induction, this is enough to know that the formula works for all n!

We can also consider the derivative of the composition of two functions, i.e. $f(g(x))$. This is also complicated, so we will give you the formula again:

Theorem 20: Chain Rule
The derivative of the composition of two functions is given by

$$(f(g(x)))' = f'(g(x))g'(x).$$

Note that the derivative *works its way inwards*, hitting f on the outside first and then hitting g on the inside.

Problem 21: Chain rule proof (Challenge)
Use the limit definition of the derivative to prove the chain rule.
(Only do this if you are looking for a challenge.)

Finally, we can also consider the derivative of the quotient of two functions, i.e. $f(x)/g(x)$. This is also complicated, so we will give you the formula again:

Theorem 22: Quotient Rule
The derivative of the quotient of two functions is given by

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - g'(x)f(x)}{(g(x))^2}.$$

Problem 23:
Rewrite

$$\frac{f(x)}{g(x)} = f(x) * (g(x))^{-1}.$$

Using the product rule and the chain rule, with the fact that $(x^{-1})' = -x^{-2}$ to show the quotient rule.

There are a few other derivatives that are important to know, but we don't have time to derive them

Theorem 24: Common Derivatives

(a) *Power rule*

$$(x^n)' = nx^{n-1}$$

(b) *Exponentials*

$$(e^x)' = e^x$$

(c) *Trig functions*

$$(\sin(x))' = \cos(x), \quad \text{and} \quad (\cos(x))' = -\sin(x)$$

Problem 25: Logarithms
We can use the derivative of the exponential to find the derivative of a logarithm. Let $f(x) = \ln(x)$ and recall that

$$f(e^x) = x.$$

Use the chain rule to write out the derivative of both sides of the equation. Simplify the expression to find a formula for $f'(e^x)$.

Solution

$f'(e^x)e^x = 1 \implies f'(e^x) = \frac{1}{e^x}$.

Replace all instances of e^x with x to find an expression for $f'(x)$ and hence the derivative of $\ln(x)$.

Solution

$f'(x) = \frac{1}{x}$.

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Problem 26: More derivative practice
Using only the sum, product, chain, and quotient rules along with the derivatives you've already seen to calculate the derivative of the following:

(a) $f(x) = (\sin(x))^2$.

Solution

$f'(x) = 2\sin(x)\cos(x)$

(b) $f(x) = x\ln(x)$.

Solution

$f'(x) = \ln(x) + 1$

(c) $f(x) = x^3 + 5x^2 + 2x + 7$.

Solution

$f'(x) = 3x^2 + 10x + 2.$

Part 3: Integrals

We're going to give only a quick version of the integral. Integrals are a very important and deep subject in mathematics. You will learn much more about integrals in a calculus course!

Definition 27: Integral
The integral of a function is denoted

$$\int f(x)dx,$$

which can be read as "the integral of f with respect to x."

The integral of a function is its inverse-derivative. That means that the integral of $f'(x)$ is $f(x)+c$ where c is a arbitrary constant. Mathematically, this is denoted by

$$\int f'(x)dx = f(x) + c$$

The constant c comes from the fact that $(f(x)+c)' = f'(x)$.

Equivalently, the derivative is an inverse-integral. Mathematically, this is denoted as

$$\left(\int f(x)dx\right)' = f(x).$$

Problem 28: Some Examples
Using the derivatives and derivative rules that you've seen so far, calculate the following integrals. Don't forget the $+c$ terms!

(a) $\displaystyle\int e^x dx =$

Solution

$e^x + c.$

(b) $\displaystyle\int 3x^2 dx =$

Solution

$x^3 + c.$

(c) $\int \frac{1}{x} dx =$

Solution

$\ln(x) + c.$

(a) $\int \left(2x^2 + 5x\right) dx =$

Solution

$\frac{2}{3}x^3 + \frac{5}{2}x + c.$

Problem 29: Integrals of Sums
Suppose that we want to take the integral of the sum of two functions. Using the sum rule for derivatives, what should the integral be?
That is, rewrite the following in terms of $\int f(x)dx$ and $\int g(x)dx$.

$$\int \left(f(x) + g(x)\right) dx \ =$$

Solution

$\int f(x)dx + \int g(x)dx.$

Problem 30: U-substitution (challenging)
Using the chain rule for inspiration, evaluate the following integrals. Don't forget the $+c$!

(a) $\displaystyle\int 2xe^{x^2}\,dx =$

Solution

$e^{x^2} + c.$

(b) $\displaystyle\int 2\sin(x)\cos(x)dx =$

Solution

$\sin^2(x) + c.$

(c) $\displaystyle\int \cos(x)e^{\sin(x)}dx =$

Solution

$e^{\sin(x)} + c.$

(a) $\displaystyle\int 3(x\sin^2(x))^2\left[\sin^2(x)+2x\sin(x)\cos(x)\right]dx =$

Solution

$(x\sin^2(x))^3 + c.$