One Day Explanation of Ordinary Differential Equations

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Instructor's Handout

Among all of the mathematical disciplines the theory of differential equations is the most important... It furnishes the explanation of all those elementary manifestations of nature which involve time. —Sophus Lie

Notation

We're going to use two different notations for derivatives,

$$\frac{df}{dx} = f'(x).$$

These notations mean the exact same thing, the left one just emphasizes that it is a derivative *with respect to* x, whereas the right one says that it is a derivative *with respect to whatever the input is.*

Similarly, we could have

$$\frac{dg}{dt} = g'(t),$$

where now the input variable is t and the derivative is taken with respect to t.

Part 1: Carbon Dating and Skydiving

How can scientists tell that a fossilized tree trunk is 38,000 years old? The answer lies in *radioactive decay*, and the math behind it gives us our first glimpse of *differential equations.*

Every living thing contains a small amount of carbon-14, a radioactive form of carbon created when cosmic rays strike the atmosphere. While a plant or animal is alive, it constantly exchanges carbon with the environment, keeping the same ratio of carbon-14 as the air around it.

Once it dies, that exchange stops and the carbon-14 inside begins to decay. After 5,700 years, only half of the carbon-14 remains. After another 5,700 years, only a quarter remains, and so on.

Problem 1: Carbon Dating a Sample
Suppose that you find an old log and want to know its age. You know that a log of that size should contain 128 lbs of carbon-14.

(a) You measure that the log now only contains 64lbs of carbon-14. How old is the log?

Solution

5700 years old if you measure 64 lbs.

(b) Instead, suppose the log now only contains 16lbs of carbon-14. How old is the log?

Solution

$5700 * 3$ years old if you measure 16 lbs.

(c) What if you measured 30 lbs of carbon-14? How old is the log?

Solution

$5700 * \log_2(128/30)$ years old if you measure 30 lbs.

(d) Let's come up with a general formula. Suppose that there are $C(t)$ lbs of carbon-14 in the log at time t. We'll let $t = 0$ denoting the time that the log died.
If you measure $C(t)$ lbs of carbon-14, how can you calculate the time t since the log died? That is, write an expression for t given $C(t)$.

Solution

$t = 5700 * \log_2(128/C(t))$

(e) Rearrange this to get a function for the amount of carbon-14 in the log as a function of t. That is, solve for $C(t)$.

Solution

$C(t) = 128 * 2^{t/5700}$

In reality, the scientists won't automatically know the half-life of the material they're studying. Instead, they need to observe the material in their lab and use that to figure out the function $C(t)$. Let's try to do that.

Problem 2: Experimental Derivation
You find the same log as before, but you don't know the half-life of carbon-14. Instead, you have some basic chemistry knowledge.

Every moment, any carbon-14 atom has a set probability to decay. This means that the decay rate of your sample should be proportional to how much carbon it currently has. Mathematically, this means

$$\frac{dC}{dt} = kC(t).$$

(a) Suppose briefly that you measure $k = 1$. Find a function $C(t)$ such that $C'(t) = C(t)$.

Solution

$C(t) = e^t$.

(b) Let k be an arbitrary constant again. Taking inspiration from the previous part, find a function $C(t)$ such that $C'(t) = kC(t)$.

Solution

$C(t) = e^{kt}$.

(c) We already know that at $t = 0$, $C(0) = 128$. (This is called an initial condition.) Find a function $C(t)$ such that $C'(t) = kC(t)$ and $C(0) = 128$.
Hint: You can multiply your C from the previous part by any constant and it will still satisfy $C' = kC$. Why?

Solution

$C(t) = 128e^{kt}$.

(d) You experimentally measure that $k = \ln(2)/5700$. Find the function $C(t)$ given $C(0) = 128$ and $C'(t) = kC(t)$. Does this match up with the previous problem?

Solution

$C(t) = 128e^{\ln(2)t/5700} = 128 * 2^{t/5700}$.

Carbon dating (despite its importance) is a pretty straightforward differential equation. Indeed, as we saw in Problem 1, we don't really need to solve a differential equation at all!
Let's try something a little more complicated.

Problem 3: Skydiving

When you jump out of a plane, something strange happens. Initially, you accelerate faster and faster as gravity pulls down on you. However, after a few seconds, you start to slow down and you reach *terminal velocity*. Any ideas why?

Solution

I expect that most students should be able to identify air resistance as the culprit.

Suppose we know that air resistance pushes back on the skydiver, exerting a force $F_a = kv^2$ upwards on the skydiver, where v is the skydiver's speed and k is a constant. We also know that gravity pulls downwards on the skydiver with a constant force $F_g = mg$, where m is the skydiver's mass. Write an equation for the skydiver's acceleration v', in terms of their speed v.

Hint: Newton's 2^{nd} *law says that* $v' = F_{total}/m$*, where* F_{total} *is the total force on the skydiver. Remember that air resistance and gravity are in different directions! Gravity accelerates the skydiver, while air resistance decelerates the skydiver.*

Solution

From the hint, $mv' = mg - kv^2 \implies v' = g - \frac{k}{m}v^2$.

From this equation, can you determine her top speed? What about an equation for the speed of the skydiver $v(t)$ over time?

Over the next few parts, we'll slowly build up the tools we need to solve this skydiving puzzle. We'll begin by defining what differential equations actually are and what it means to "solve" one. Then we'll practice solving the simplest types by integrating, learn how to separate variables, and see how to reason about solutions even when we can't find formulas. By the end, you'll be able to predict the motion of our skydiver, along with many other systems that change over time.

Part 2: Initial Value Problems

Science is a differential equation. Religion is a boundary condition.

—Alan Turing

Inspired by the previous section, we define an initial value problem as follows.

Definition 4: First Order Initial Value Problem (IVP)

An *initial value problem* is a mathematical expression of the form

$$\frac{df}{dt} = F(t, f) \quad \text{with} \quad f(t_0) = f_0, \tag{1}$$

where f is an unknown function, F is a given function of t and f, t_0 is the initial time, and f_0 is a constant, called the initial value. This is called *first order* because it only involves the first derivative.

For example, from the previous part, an initial value problem might be

$$C'(t) = \frac{\ln(2)}{5700} C(t) \quad \text{with} \quad C(0) = 128.$$

Here C is the unknown function, the initial value is 128 at time $t_0 = 0$, and $F(t, f) = \frac{ln(2)}{5700} f$.

Definition 5: Solution to an IVP

We say that a function $f(t)$ is a solution to (1) or satisfies (1) if $f(0) = f_0$ and $f'(t) = F(t, f)$ is true.

Problem 6: Some practice

Determine whether the following functions are solutions to the given IVP:

(a) Is $\quad f(t) = e^t \quad$ a solution to $\quad f' = f, \quad$ with $\quad f(0) = 1$?

Solution

yes.

(b) Is $\quad f(t) = t^2 \quad$ a solution to $\quad \frac{df}{dt} = 2\sqrt{f}, \quad$ with $\quad f(0) = 0$?

Solution

yes.

(c) Is $\quad f(t) = \frac{1}{t} \quad$ a solution to $\quad f' + f^2 = 0, \quad$ with $\quad f(1) = 4$?

Solution

no, $f(1) = 1 \neq 4$

(d) Is $\quad f(t) = \sin(t) \quad$ a solution to $\quad f' = \cos(t), \quad$ with $\quad f(0) = 0$?

Solution

yes

Part 3: Integrate Both Sides

The simplest initial value problem is set up as

$$y'(t) = f(t), \quad \text{with} \quad y(t_0) = y_0.$$

Notice that the right hand side does not depend on y, only on t. For these problems, we can just integrate both sides and find the answer.

Problem 7: An example
Consider the initial value problem

$$y'(t) = t \quad \text{with} \quad y(1) = 2.$$

(a) Ignore the initial value for now and just focus on $y' = t$. Integrate both sides with respect to t to find an expression for $y(t)$.
Don't forget the $+c$*!*

Solution

$y(t) = \frac{1}{2}t^2 + c.$

(b) The $+c$ is an arbitrary constant, it can be anything. Choose c appropriately so that $y(t)$ solves the initial value problem.

Solution

$y(t) = \frac{1}{2}t^2 + \frac{3}{2}$

Problem 8: More examples
Solve the following initial value problems.

(a) $y'(t) = \cos(t)$, $y(0) = 5$

Solution

$y(t) = \sin(t) + 5.$

(b) $y'(t) = e^{2t}$, $y(0) = \frac{1}{2}$

Solution

$y(t) = \frac{1}{2}e^{2t}$

(c) $y'(t) = e^t \cos(e^t)$, $y(0) = 0$.

Solution

$y(t) = \sin(e^t) - \sin(1)$.

Problem 9: Every problem like this ever (Challenge)
Only do this problem if you know definite integrals.
Solve the initial value problem

$$y'(t) = f(t), \quad \text{with} \quad y(t_0) = y_0,$$

in general by integrating both sides from t_0 to t. Leave your answer in terms of $\int_{t_0}^{t} f(s)ds$.

Solution

$y(t) = y(t_0) + \int_{t_0}^{t} f(s)ds$.

Intuitively, this makes sense! Think about this like we're driving, with position $y(t)$. At any given time t, our velocity is $y'(t) = f(t)$.

Initially, at time t_0, we start at y_0 with velocity $f(t_0)$. If we want to find our position at the "next" moment of time, i.e. $t_0 + ds$ for some extremely small ds, then we would expect to travel $f(t_0)ds$ in that amount of time. This would imply $y(t_0 + ds) \approx y(t_0) + f(t_0)ds$.

To find our position at a later time t, we would need to keep adding these little bits of distance up. So at time $t_0 + 2ds$, we would expect to be at position $y(t_0) + f(t_0)ds + f(t_0 + ds)ds$. Doing this all the way up to t and then taking the limit as ds gets infinitesimally small, we would find the integral!

Part 4: Separable Equations

A separable equation is a first order differential equation that can be written as

$$y' = f(t)g(y).$$

So-called because the right hand side can be *separated* into a t-dependent term and a y-dependent term. Let's try to solve one of these.

Problem 10: An example
Consider the initial value problem

$$\frac{dy}{dt} = \frac{y}{t} \quad \text{with} \quad y(1) = -2.$$

(a) Ignore the initial condition for now. Group every term with a y on the left hand side and every term with a t on the right hand side. This includes treating $\frac{dy}{dt}$ like a fraction and moving the dt to the right hand side.
Yes, we're cheating. But today, if it's good enough for the physicists then it's good enough for us.

Solution

$\frac{1}{y}dy = \frac{1}{t}dt$

(b) Add an integral sign to both sides and integrate. Because of the dy and dt, the left hand side will be integrated with respect to y and the right hand side will be integrated with respect to t.
Don't forget the $+c$!

Solution

$\ln(y) = \ln(t) + c.$

(c) Move all of the $+c$ terms onto the right hand side and combine them into one constant. Then solve for $y(t)$ as a function of t. (You can make e^c an arbitrary constant too.)

Solution

$y(t) = ct.$

(d) Choose the constant appropriately so that $y(t)$ satisfies $y(1) = -2$.

Solution

$y(t) = -2t.$

Problem 11: More examples
Solve the following initial value problems

(a) $\frac{dy}{dt} = e^{-y}(2t-4)$, $y(5) = 0$.

Solution

$y = \ln(x^2 - 4x - 4)$.

(b) $y'(t) = 6y^2 t$, $y(1) = \frac{1}{25}$.

Solution

$y(t) = \frac{1}{28-3x^2}$.

(c) $y' = e^{t+y}$, for $y(0) = 0$.

Solution

$y(t) = -\ln(2 - e^t)$.

Problem 12: Back to the skydiver (Challenge)
We now know enough to solve for the velocity of the skydiver! Recall that a skydiver's velocity satisfies the initial value problem

$$mv' = mg - kv^2, \quad \text{with} \quad v(0) = 0.$$

Solve this equation to find the skydiver's velocity $v(t)$. What is her terminal velocity?
NOTE: This is still a challenging problem. We'll learn to solve this qualitatively in the next section.
Hint: The hyperbolic tangent function $\tanh(x)$ *satisfies*

$$\tanh(x) = \frac{e^{2x} - 1}{e^{2x} + 1} \quad \textit{and} \quad \frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1 - x^2}.$$

Solution

$$\frac{v'}{g - \frac{k}{m}v^2} = 1 \implies \sqrt{\frac{mg}{k}}\tanh^{-1}\left(\sqrt{\frac{k}{mg}}v\right) = t + c \implies v = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{k}{mg}}(t+c)\right)$$

and $c = 0$ because $\tanh(0) = 0$. Also, $\tanh(x)$ asymptotically approaches 1, so the terminal velocity is $\sqrt{mg/k}$.
This is challenging, it is okay for the students to skip this.

Problem 13: Rigorous example (Challenge)
If the idea of separating dy and dt makes you uneasy, splitting the $\frac{dy}{dt}$ into a fraction can be justified with a careful u-substitution. Try that out on the following example:

$$y' = y^2 \sin(t), \quad \text{for} \quad y(0) = 1.$$

This time, move everything with a y to the left hand side and everything with a t to the right. *Don't move the dt, that's not allowed.* Instead, integrate both sides with respect to t and then make a u-substitution for the left integral.

Solution

$$\begin{aligned}
\int y^{-2}y'dt &= \int \sin(t)dt \\
-y^{-1} &= -\cos(t) + c \\
y(t) &= \frac{1}{\cos(t) + c} \\
y(t) &= \frac{1}{\cos(t)}
\end{aligned}$$

Part 5: Sketching Autonomous Equations

Despite technically being able to solve for the velocity of a skydiver in problem 12, it is still very challenging! The equation is difficult to solve and uses fancy functions that, once we have the solution, make it challenging to determine the terminal velocity. Instead of going through that process, let's develop a theory of autonomous equations to determine what the skydiver experiences *qualitatively*.

Definition 14: Autonomous
A differential equation is called *autonomous* if it is of the form

$$\frac{dy}{dt} = f(y).$$

Notably, the equation only depends on y, not on the independent variable t.

For these equations, it is easy to sketch what the solutions look like! (Though not always easy to actually find the solutions.) You'll notice that we didn't give an initial condition for the ODE above. That's because we will sketch what happens to solutions as the initial value changes.

Problem 15: Stationary Solutions
Consider the autonomous differential equation

$$y' = y^3 - y^2 - 6y.$$

We want to find *stationary solutions* of the form $y(t) = y_0$, where y_0 is a constant.

(a) If $y(t) = y_0$, what is $y'(t)$?

Solution

0.

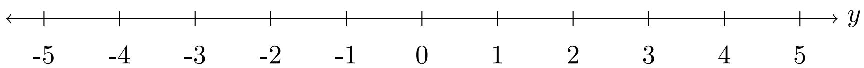
(b) Based on the given differential equation, for what values of y is $y' = 0$?
What are the stationary solutions of this differential equation?

Solution

$y = 0, -2, 3.$

Problem 16: Phase diagrams

We want to figure out (qualitatively) what happens to the solutions that aren't stationary. To do so, we're going to draw a *phase diagram* on the following number line.



(a) Draw open circles at each of the stationary solutions.

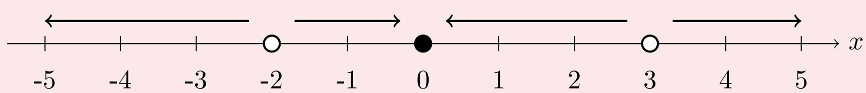
(b) For each region between the stationary solutions, determine whether solutions in that region will be moving to the right or to the left. Then, draw an arrow in the relevant direction. *Hint: check whether $y' > 0$ or $y' < 0$ for y in that region.*

(c) If the solutions flow towards the stationary solution, then the solution is called *stable*: if you nudge the initial condition away from the stationary point, the solution will come back.

If the solutions flow away from the stationary solution, then the solution is called *unstable*: if you nudge the initial condition away from the stationary point, the solution will run away.

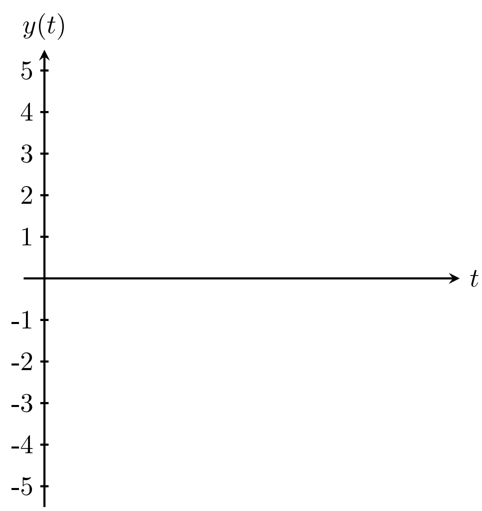
For each stationary solution, label it as stable or unstable. For stable solutions, fill in the circle on your phase diagram. For unstable solutions, leave the circle open.

Solution



Problem 17: Sketching Solutions

We're now going to use our phase diagram to sketch various solutions to the given differential equation on the graph below.



(a) For each stationary solution, draw the graph of the solution.

(b) For each intermediate region, draw one representative solution. Remember that y' tells you the slop of the solution. What do you think these solutions should look like? Why? *Hint: For each region, start close to the unstable stationary solution and then asymptotically approach the stable stationary solution.*

Problem 18: Back to the skydiver (Challenge)
Recall again that the skydiver's velocity satisfies the initial value problem

$$mv' = mg - kv^2, \quad \text{with} \quad v(0) = 0.$$

Follow the above process to analyze this differential equation. Assume that $m, g, k > 0$.
Ignoring the initial value for now, what are the stationary solutions for this differential equation? Draw a phase diagram like you did above.

Solution

The stationary solutions are $v = \pm\sqrt{mg/k}$. The phase diagram is unstable at $-\sqrt{mg/k}$ and stable at $\sqrt{mg/k}$.
It is okay if they assumed that $v \geq 0$.

Since the skydiver starts at initial velocity 0, what speed does she asymptotically approach as $t \to \infty$? What does this tell you about the terminal velocity?

Solution

the terminal velocity is $\sqrt{mg/k}$.

What would happen if the skydiver somehow got above her terminal velocity? Why?

Solution

Her velocity would come back down because the terminal velocity is stable.

Problem 19: Another example
Repeat the above process to sketch solutions for the following differential equation:

$$y' = y^2 - 2y + 1.$$

What is the stability of the stationary solutions? Stable, unstable, or neither?

Solution

Neither, $y = 1$ is stable from the left $y < 1$ and unstable to the right $y > 1$.

Problem 20: Actually solving the ODE (Challenge)
If you want a challenge, we're now going to solve the differential equation. To do so, we need to use the method of partial fractions.

(a) Solve for constants A, B, C such that

$$\frac{1}{y^3 + y^2 - 6y} = \frac{A}{y+2} + \frac{B}{y} + \frac{C}{y-3}$$

Solution

$A = \frac{1}{10}$, $B = -\frac{1}{6}$, $C = \frac{1}{15}$

(b) Using the previous problem and separating the equation, solve for an implicit relationship between y and t.
This should not be an explicit function $y(t) = ...$ Instead, you should have some messy *combination of y's on the left hand side and some combination of t's on the right.*

Solution

$(y+2)^{\frac{1}{10}} y^{-\frac{1}{6}} (y-3)^{\frac{1}{15}} = ce^t$

Part 6: Existence and Uniqueness of Solutions

So far, we've seen how to *solve* simple differential equations. But before solving, it's worth asking an even more basic question: *Does a solution even exist? And if so, is it unique?*

Not every differential equation behaves nicely! Let's look at some examples.

Problem 21: Solutions don't always exist
Consider the initial value problem:

$$y'(t) = \frac{1}{t}, \quad \text{with} \quad y(0) = 0.$$

Does a solution exist? Why or why not?
Hint: Assume you can solve it and then find an issue with the solution.

Solution

This is a bit of a silly example, but any ODE without a solution is silly.
If we try to solve, we find $y(t) = \ln(t) + c$ which doesn't exist at $t = 0$, let alone equaling 0 at $t = 0$.

Problem 22: Be careful
Consider the initial value problem:

$$y' = y^2 \sin(t), \quad \text{for} \quad y(0) = 0.$$

Try to find a solution. *A solution exists*, but you'll need to be very careful to find it.

Solution

If students try to separate and solve, they won't find the solution. Instead, they need to notice that $y = 0$ is a solution.

Problem 23: Solutions are not always unique
Take a look at the initial value problem:

$$y' = y^{\frac{1}{3}}, \quad y(0) = 0.$$

Solve this IVP by treating it like a separable equation. This will give one solution.
It turns out that there are many more! Find another solution. If you want a challenge, try to find infinitely many solutions.
Hint: we know that $y(0) = 0$. What is $y'(0)$? Intuitively, what do you think $y(t)$ is for really small t?

Solution

$\frac{3}{2}y^{2/3} = t + c \implies y = (\frac{2t}{3} + c)^{3/2} \implies y = (\frac{2t}{3})^{3/2}$
Another solution is just $y(t) = 0$. Since $y(0) = 0$ and $y'(0) = 0$, we might expect that y just stays at 0 forever!
For infinitely many solutions, we can stay at 0 for arbitrarily long and then swap to $y = (\frac{2t}{3})^{3/2}$. This gives infinitely many solutions of the form

$$y(t) = 0 \quad \text{for} \quad t \leq t_0$$
$$y(t) = (\frac{2(t - t_0)}{3})^{3/2} \quad \text{for} \quad t \geq t_0$$

Problem 24: Finite-Time Blowup
Consider the initial value problem:

$$y' = y^2, \quad \text{with} \quad y(0) = 1.$$

Solve this IVP by separating variables. What do you notice about the solution? Does $y(t)$ exist for all t or does it stop existing at a point?

Solution

$-y^{-1} = t + c \implies y = \frac{1}{c-t} \implies y = \frac{1}{1-t}$
This stops existing at $t = 1$.

Problem 25: Just weird
Consider the initial value problem

$$y' = \frac{1}{y}, \quad \text{with} \quad y(0) = 0.$$

Treating this like a separable equation, try to find a solution, is it possible?
What is $y'(0)$?
Does $y(t)$ exist for all t?

Solution

Some students may protest that $y'(0) = \infty$, which is also an issue. If you try to solve the equation, you find $y = \sqrt{2t + c} = \sqrt{2t}$. This is kinda fine. It exists at $t = 0$, and satisfies the initial condition. However, it doesn't exist for $t < 0$ and $y'(0) = \infty$. So can we even really say that $y'(0) = \frac{1}{y}$? Kinda yes kinda no.

Questions like this prompt mathematicians to make themselves *very* clear by what a "solution" actually is and to really check when it exists and when it's unique.
Particularly for partial differential equations, this is an active area of research!

For the curious, the standard existence and uniqueness theorem for ODEs is the following:

Theorem 26: Existence and Uniqueness (Picard-Lindelöf)
If $f(t, y)$ and its partial derivative $\frac{\partial f}{\partial y}$ are both continuous near the point (t_0, y_0), then the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0$$

has a unique solution (at least for some short interval around t_0).

This result guarantees that "nice" equations behave nicely.

Part 7: Linear Equations

Definition 27: Linear differential equation
A differential equation is called *linear* if it is of the form

$$y' + a(t)y = f(t).$$

Problem 28: Integrating factor
Given a linear differential equation:

$$y' + a(t)y = f(t).$$

We want to multiply this equation by an *integrating factor* $\mu(t)$,

$$\mu(t)y'(t) + \mu(t)a(t)y(t) = \mu(t)f(t),$$

so that the left hand side can be written as a product rule:

$$\mu y' + \mu a y = (yg)',$$

for some function $g(t)$.

(a) Using the product rule, match terms to determine the function g and a differential equation that μ must solve.

Solution

$$\mu y' + \mu a y = (yg)' = gy' + g'y$$

so $g = mu$ and $\mu' = g' = a\mu$.

(b) Solve the differential equation for μ to show that

$$\mu(t) = e^{\int a(t)dt}.$$

Solution

$\mu' = a\mu$ is separable.

Once we have this μ, we can solve the differential equation! Let's see it in action.

Problem 29: An example
Consider the following linear differential equation

$$y'(t) + \cos(t)y(t) = 0, \quad \text{with} \quad y(0) = 5$$

(a) Find $a(t)$ and then calculate the integrating factor $\mu(t) = \exp(\int a(t)dt)$.
You can ignore the $+c$ here.

Solution

$\mu(t) = e^{\sin(t)}$

(b) Multiply both sides of the equation by μ and recognize the left hand side as the product rule for $(\mu y)'$. This is now an equation that you can solve!

Solution

$y(t) = ae^{-\sin(t)} = 5e^{-\sin(t)}$

Problem 30: Another example
Solve the following first order linear differential equation

$$\sin(t)y' + y\cos(t) = 1, \quad \text{with} \quad y(\pi/2) = 5.$$

Make sure to put it in the right form first!

Solution

An integrating factor isn't strictly necessary for this. The left hand side is already written as a product rule.
$(\sin(t)y)' = 1 \implies y = \frac{c}{\sin(t)} = \frac{5}{\sin(t)}$.

Problem 31: Another another example
We're going to solve the following first order linear differential equation in two different ways.

$$(1-t^2)y'(t) + 2ty(t) = 0, \quad \text{with} \quad y(2) = 1$$

(a) Solve the ODE as a linear equation. Make sure to get it in the correct form first!

Solution

$y' + \frac{2t}{1-t^2}y(t) = 0$
$\mu = \frac{1}{1-t^2}$
$(\frac{y}{1-t^2})' = 0$
$y = c(1-t^2) = -\frac{1}{3}(1-t^2)$

(b) Solve the ODE as a separable equation.

Solution

$\frac{y'}{y} = \frac{-2t}{1-t^2} \implies \ln(y) = \ln(1-t^2) + c \implies y(t) = c(1-t^2) \implies y(t) = -\frac{1}{3}(1-t^2)$.