Constant Coefficient Conundrums
or A Guide to Guessing

Prepared by Matt on November 17, 2025

Instructor's Handout

In order to solve this differential equation you look at it 'til a solution occurs to you. —George Pólya

Preliminaries

Even though we learned a lot of techniques last week, we're not going to use the majority of them. All that we really need to know is that an *initial value problem* is an equation of the form

$$y'(t) = f(t, y), \quad \text{with} \quad y(t_0) = y_0. \tag{1}$$

Here y is an unknown function, $f(t, y)$ is some given function of t and y, and y_0 is the *initial value* at the initial time t_0. Generally speaking, an initial value problem will have just one solution $y(t)$.

We say that a function $y(t)$ satisfies (1)—or is a solution to (1)—if $y'(t) = f(t, y(t))$ is true for t near t_0 and if $y(t_0) = y_0$.

Often, we will consider a differential equation *without* an initial condition:

$$y'(t) = f(t, y).$$

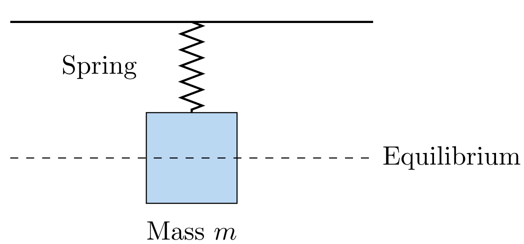
Generally speaking, in this case, we will have *infinitely many* solutions $y(t)$. Often, these different solutions come from the $+c$ from some integral and appear like

$$y(t) = \cdots + c \quad \text{or} \quad y(t) = c \cdot (\dots)$$

so make sure to keep track of those! We call solutions with arbitrary constants like this *general solutions*.

Part 1: Motivation — Springs

Problem 1: A Simple Spring
Imagine attaching a small mass m to a spring and then hanging it from the ceiling. When the weight is at rest, the spring stretches just enough so that the upward spring force balances gravity.



Now suppose you pull the mass down a little and let go. Let $x(t)$ denote the displacement of the spring. At any given moment, the spring pulls upward with a force proportional to how far the weight is stretched beyond its rest length. The force delivered by the spring is then given by

$$F_s = -kx.$$

Using Newton's second law, which states that force is equal to mass times acceleration (acceleration being x'') write a differential equation that models the motion of the mass, $x(t)$.

Solution

$mx''(t) = -kx(t).$

Can you solve this differential equation? If you can, do so! If not, there's no worries, we'll develop the tools later. *Don't spend long on this. Only solve it if you immediately see a solution.*

Solution

$x = a\cos\left(\sqrt{k/m}t\right) + b\sin\left(\sqrt{k/m}t\right).$

Now let's suppose that we add in two additional effects: air resistance and a driving force. Air resistance will push back against the motion of the mass, proportional to its velocity. A driving force can be anything! We'll denote this by $f(t)$. This gives two additional forces acting on the mass,

$$F_a = -cx'(t) \quad \text{and} \quad F_d = f(t).$$

Using Newton's second law, write the new equation of motion for the mass.

Solution

$mx''(t) + cx'(t) + kx(t) = f(t).$

This is definitely too difficult for us now. But by the end of this packet, we'll be able to solve this in full generality. And easily too! But first, we need to develop some machinery.

In Part 2, we'll start with simplified first order equations:

$$cx'(t) + kx(t) = f(t).$$

This will give us a set of tools and solutions to inspire our work in Part 3 on second order equations:

$$mx''(t) + cx'(t) + kx(t) = 0.$$

Then finally in Part 4, we'll be able to solve general springs with a driving force:

$$mx''(t) + cx'(t) + kx(t) = f(t).$$

Finally, we'll talk about the *linear algebra* that makes all of these solutions possible and even extend this to higher order equations!

Part 2: First order equations — Let's just guess

Let's start with a very simple example. First, we'll just consider first order differential equations.

You probably know how to solve these problems.
Instead of solving them like last week, we're just going to guess solutions.

Problem 2:
Consider the following initial value problem,

$$x'(t) = 5x(t), \quad \text{with} \quad x(0) = 3.$$

Guess a solution $x(t) = re^{\lambda t}$ for some unknown constants λ and r. Then solve for r and λ.

Solution

$x(t) = 3e^{5t}$.

Problem 3: A little more complicated
For constants a and b, consider the following initial value problem

$$x'(t) = 5x(t) + 15, \quad \text{with} \quad x(0) = 0.$$

Earlier, we guessed a solution of the form $x = re^{\lambda t}$ and it worked! But now we have *added a constant* to our differential equation. How might we adapt our guess now that we *added a constant*?

Solution

add a constant to our guess. So
$x(t) = ae^{\lambda t} + b$

Using your guess, figure out the appropriate coefficients to find a solution to the ODE.

Solution

$\implies x(t) = 3e^{5t} - 3.$

Compare this solution to your solution to problem 2. What terms are the same? What terms are different?

Solution

We still have $3e^{5t}$ (this will eventually be the homogeneous solution), but now we also have -3 (this will eventually be the particular solution).

Problem 4: Separating Variables
Ignore the initial condition and solve the differential equation

$$x'(t) = 5x(t) + 15,$$

as a separable equation.
If you didn't get to this part of last week's packet, that's okay! Just ask an instructor for help.

Solution

$x(t) = ce^{5t} - 3.$

Notice the term that comes with an arbitrary, unknown constant. That's the same term that appeared in the previous two problems! This is not a coincidence and we'll make this precise in a second. But first, a more complicated problem.

Problem 5: More complicated
Consider the differential equation

$$x'(t) = 5x(t) + e^{3t}.$$

When we added a constant in problem 5, we added an arbitrary constant to our guess. Now we added an exponential e^{3t}. What should we add to our guess now?
Hint: the extra term you add needs a constant too.

Solution

We should add an exponential. $x(t) = ae^{5t} + be^{3t}$

Use your guess to find a general solution to the differential equation. Because there is no initial condition, you should have an arbitrary constant left in your solution. Based on the previous problem, where do you expect that arbitrary constant to be?

Solution

Based on the previous part, the arbitrary constant should be on e^{5t}. The general solution is $x(t) = ae^{5t} - \frac{1}{2}e^{3t}$.

Theorem 6: Decomposition of solutions
Like we've been doing, consider a differential equation of the form

$$x'(t) = a(t)x(t) + b(t).$$

We can always break down the solution $x(t)$ as

$$x(t) = x_h(t) + x_p(t),$$

where $x_h(t)$ is the general solution to the *homogeneous equation*

$$x_h'(t) = a(t)x_h(t),$$

called the *homogeneous solution*, and $x_p(t)$ is any one solution to the full equation

$$x_p'(t) = a(t)x_p(t) + b(t),$$

called the *particular solution*.

Problem 7: Proof of Theorem 6
Consider the differential equation

$$x'(t) = a(t)x(t) + b(t). \tag{2}$$

(a) Suppose that $x_1(t)$ and $x_2(t)$ both solve (2). Find a differential equation that $f(t) = x_1(t) - x_2(t)$ solves.
Hint: calculate the derivative $f'(t)$ and use (2).

Solution

$f'(t) = a(t)f(t)$.

(b) Based on the Theorem 6, what do we call the differential equation that $f(t) = x_1(t) - x_2(t)$ solves?

Solution

the homogeneous equation

(c) Suppose that x_h is the homogeneous solution to (2) and x_p is any particular solution to (2). Show that $x_h + x_p$ is a solution to (2).

Solution

$y(t) = \frac{1}{2}t^2 + c.$

Together, these parts finish Theorem 6's proof! Part (a) shows that any solution x_1 of (2) can be written as the sum of homogeneous solution $f(t)$ and another solution x_2, i.e. $x_1(t) = f(t) + x_2(t)$. Part (c) shows that any function of this form is a solution to (2).

Remark: The homogeneous equation $x' = ax$ is *linear* in the sense that if x_1, x_2 are solutions, then so is any linear combination $a_1x_1 + a_2x_2$. This means that we can intuitively think of the space of solutions as a one-dimensional linear space, i.e. a line.

Let's use this theorem now.

Problem 8: Application
Consider the differential equation

$$x'(t) = -3x(t) + 10\sin(t)$$

(a) What is the homogeneous equation? Find the *general* solution to the homogeneous equation. You can either guess the solution or separate variables.

Solution

$x_h' = -3x_h \implies x_h = ae^{-3t}.$

(b) To solve for the particular solution, we're going to need to guess. Since we added a trigonometric function of to our differential equation, we're going to guess that the particular solution is also a trigonometric function, of the form

$$x_p(t) = a\sin(t) + b\cos(t).$$

Find a, b such that x_p is a particular solution. This method is called *undetermined coefficients*.

Solution

$x_p = 3\sin(t) - \cos(t).$

(c) Using parts (a) and (b) and Theorem 6, write out the general solution to the differential equation.

Solution

$x(t) = ae^{-3t} + 3\sin(t) - \cos(t)$.

(d) Consider the *altered* differential equation

$$x'(t) = -3x(t) + 6\sin(3t).$$

Note that the homogeneous equation is the same, but that we changed the inhomogeneous part. What should we guess as a particular solution? Then find the general solution.

Solution

Our guess should be $x_p = a\sin(3t) + b\cos(3t)$ this gives $x(t) = ae^{-3t} + \sin(3t) - \cos(3t)$.

Problem 9: Undetermined coefficients

The general idea of undetermined coefficients is to guess that our particular solution looks like the inhomogeneous portion of the differential equation (everything that isn't the homogeneous part).

Let's try it with a few examples. For each of the following, find the general solution. Each of these equations has the same homogeneous part, so you only need to find the right particular solution.

(a) $x'(t) = 2x(t) + e^{3t}$

Solution

$x_p(t) = be^{3t} \implies x_p(t) = e^{3t} \implies x(t) = ae^{2t} + e^{3t}$.

(b) $x'(t) = 2x(t) + 4t^2$

Hint: the inhomogeneous part is a *degree 2 polynomial*

Solution

$x_p(t) = bt^2 + ct + d \implies x_p(t) = -2t^2 - 2t - 1 \implies x(t) = ae^{2t} - 2t^2 - 2t - 1$.

(c) $x'(t) = 2x(t) + e^{3t} + 4t^2$
Hint: remember how we added solutions earlier

Solution

$x(t) = ae^{2t} + e^{3t} - 2t^2 - 2t - 1.$

Problem 10: More challenging
Consider the differential equation

$$x'(t) = 2x(t) + te^t.$$

What is the general structure of the inhomogeneity? What should you guess for the particular solution? Find the general solution to the ODE.

Solution

linear polynomial times an exponential.
guess $x_p = (at + b)e^t \implies x_p = -(t + 1)e^t.$

Problem 11: Resonant solutions

Consider the differential equation

$$x'(t) = -5x(t) + e^{-5t}.$$

(a) Find the homogeneous solution.

Solution

$x_h(t) = ae^{-5t}$.

(b) What should you guess as the particular solution? Does this work? Why or why not?

Solution

You would normally guess $x_p = be^{-5t}$. This doesn't work because it is already a homogeneous solution.

(c) In the case where our guess is already a homogeneous solution, we need to make our guess more complicated by multiplying by t. That is,

$$\text{normal guess} : x_p = ae^{-5t} \quad \Longrightarrow \quad \text{new guess} : x_p = ate^{-5t}$$

Use this new guess to find a particular solution.

Solution

$x_p(t) = te^{-5t}$

(d) Write the full general solution.

Solution

$x(t) = ae^{-5t} + te^{-5t}$.

Part 3: Second order equations become polynomials

Definition 12: n^{th}-order differential equation
We say that a differential equation is n^{th} order if it can be written as

$$x^{(n)}(t) = f(t, x, x', \dots, x^{(n-1)}).$$

Here $x^{(k)}$ is the k^{th} derivative of x. So the *order* of a differential equation is the highest derivative that appears.

Let's test out the methods that we developed in the previous section on some second order differential equations.

Problem 13: A simple example
Consider the second order differential equation,

$$x''(t) + x'(t) - 6x(t) = 0.$$

Just like before, let's guess a solution. Once again, we're going to guess an exponential of the form $x = ae^{\lambda t}$.

(a) In order for x to be a solution, what polynomial must λ be a root of? This polynomial is called the *characteristic polynomial* of the differential equation.

Solution

$\lambda^2 - \lambda - 6 = 0.$

(b) What are the possible values of λ? These two values give two different *fundamental solutions* to the differential equation. What are these solutions?

Solution

$\lambda = -3, 2 \implies x = ae^{-3t}$ and $x = ae^{2t}$.

(c) Does the constant a have any restrictions on it?
What if we take a linear combination of the fundamental solutions? Show that for any constants a_1, a_2,

$$x(t) = a_1 e^{-3t} + a_2 e^{2t}$$

is a solution to the differential equation.

Solution

No, a can be anything. To show that any linear combination is a solution, we just expand the equation linearly.

This is the general solution to the second order ODE. Note that there are two arbitrary constants.

Theorem 14: Number of solutions
Generally speaking, a general solution to a second order differential equation will have *two arbitrary constants* in it. Intuitively, this is because we would have to integrate the equation twice to find a solution and each integral produces an arbitrary constant.

If we had a n^{th} order ODE, then we would expect n arbitrary constants.

Problem 15: Initial value problem
Since there are two arbitrary constants, we need two initial conditions to fix a single solution. Solve the following initial value problem.

$$x''(t) + x'(t) - 6x(t) = 0, \quad \text{with} \quad x(0) = 5, x'(0) = 0.$$

Use your general solution from the previous problem and then solve for the arbitrary constants.

Solution

$x(t) = 2e^{3t} + 3e^{-2t}$.

Problem 16: Distinct real roots
Generally, we break these second order equations down into the types of roots that the characteristic polynomial has. The example that we just saw has two distinct real roots. Here are some more examples in this case.

Find the general solutions to the following differential equations. In each case, find the characteristic polynomial, then the roots, and then write the general solution from the roots.

(a) $x'' + 11x' + 24x = 0$

Solution

$\lambda^2 + 11\lambda + 24 = 0 \implies x(t) = ae^{-8t} + be^{-3t}.$

(b) $x'' + 3x' - 10x = 0$

Solution

$\lambda^2 + 3\lambda - 10 = 0 \implies x(t) = ae^{-5t} + be^{2t}.$

Problem 17: One repeated real root

What happens if the characteristic polynomial only has one real root? Let's find out.

Consider the differential equation

$$x'' - 4x' + 4x = 0.$$

(a) Write down the characteristic polynomial for this equation. What are the roots? What solution(s) does that give for the differential equation?

Solution

$(\lambda - 2)^2 = 0$ only gives $x = ae^{2t}$.

(b) Like before, let's make our guess just one step more complicated. Guess the solution $x = te^{\lambda t}$. For which λ is this a solution?

Solution

$0 = te^{\lambda t}(\lambda^2 - 4\lambda + 4) + e^{\lambda t}(2\lambda - 4) = 0$ so $\lambda = 2$.

(c) Show that any linear combination $x(t) = ae^{2t} + bte^{2t}$ is a solution to the differential equation.

Solution

Linearity or plugging in will work here.

This again gives us our general solution!

Problem 18: Imaginary roots

What happens if the characteristic polynomial has no real roots? Let's see what happens in that case.

Consider the differential equation

$$x'' + x = 0.$$

(a) Write down the characteristic polynomial for this equation. What are the roots?
Recall that the imaginary number i satisfies $i^2 = -1$.

Solution

$\lambda^2 + 1 = 0$ gives $\lambda = \pm i$.

(b) The previous part suggests that our homogeneous solution should be

$$x(t) = ae^{it} + be^{-it}.$$

However, we are only interested in real valued solutions. So, we're going to use *Euler's formula*,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

to turn this into something real.
Using Euler's formula, expand our general solution. Group all $\sin(t)$ and $\cos(t)$ terms together and rename the constants. This gives us our general solution!

Solution

$x(t) = \alpha\cos(t) + \beta\sin(t)$.

Problem 19: Complex roots

The previous problem worked nicely because our roots were purely imaginary. Let's make it more complicated and work with complex roots instead.

Consider the initial value problem

$$x'' - 2x' + 2x = 0.$$

(a) Write down the characteristic polynomial for this equation. What are the roots? What (complex-valued) solutions does this give for the differential equation?

Solution

$\lambda = 1 + i$ and $\lambda = 1 - i$

(b) Once again, we'll use Euler's formula. Breaking down the imaginary exponent, we can write

$$e^{a+ib} = e^a(\cos(b) + i\sin(b)).$$

Grouping sines and cosines like before, use this to write down the general solution for our differential equation.

Solution

$x(t) = \alpha e^t \cos(t) + \beta e^t \sin(t).$

Problem 20: Damped Spring Classification

Consider the equation governing a mass on a damped spring, that we found earlier:

$$mx'' + cx' + kx = 0,$$

where $m, c, k > 0$.

(a) Write down the characteristic polynomial for this differential equation.

Solution

$m\lambda^2 + c\lambda + k = 0.$

(b) Solve for the roots λ in terms of m, c, k.

Solution

$\lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}.$

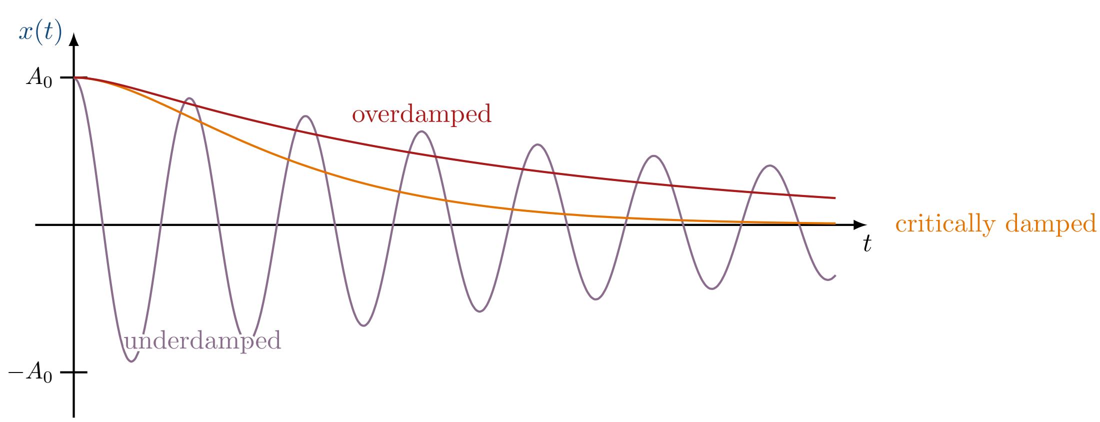
(c) Depending on whether $c^2 - 4mk$ is positive, zero, or negative, our roots will either be distinct real roots, repeated real roots, or complex roots respectively. Each of these cases corresponds to a different physical regime for the equation:

- *Overdamped*: motion returns to equilibrium without oscillating.
- *Critically damped*: fastest return to equilibrium without oscillating.
- *Underdamped*: lots of oscillations, with exponential decay.

Based on what the solutions look like (and context clues) determine which physical case corresponds to $c^2 - 4mk$ being positive, zero, or negative.

Solution

- Overdamped: $c^2 > 4mk$, two distinct real negative roots; motion returns to equilibrium without oscillating.
- Critically damped: $c^2 = 4mk$, one repeated real root; fastest return to equilibrium without oscillating.
- Underdamped: $c^2 < 4mk$, complex conjugate roots; oscillatory motion with exponentially decaying amplitude.



Part 4: Springs with a driving force.

Now suppose that we consider an inhomogeneous second order equation. Something like

$$ax'' + bx' + cx = f(t).$$

Thankfully, we can solve this in the exact same way we did previously! Drawing on our earlier example, we can view this as a damped spring with a driving force $f(t)$.

Just like Theorem 6, we can decompose our solution down into a *homogeneous solution* x_h, which solves the homogeneous equation

$$ax_h'' + bx_h' + cx_h = 0,$$

and a *particular solution* x_p which solves the full equation. Then the general solution will be given by

$$x(t) = x_h(t) + x_p(t).$$

We already saw how to find the homogeneous solution and we find the particular solution by guessing via *undetermined coefficients*, just like last time! Let's work through some examples.

Problem 21: An example
Find the general solution to the following second order differential equation,

$$x'' - 4x' - 12x = 3e^{5t}.$$

(a) What is the homogeneous equation? Write down the characteristic polynomial, find the roots, and write down the general homogeneous solution.

Solution

$x_h(t) = ae^{-2t} + be^{6t}$

(b) Like before, use *undetermined coefficients* to guess a solution. Solve for the coefficients to find a particular solution.

Solution

$x_p(t) = -\frac{3}{7}e^{5t}$.

(c) Write the full general solution $x(t)$.

Solution

$x(t) = ae^{-2t} + be^{6t} - \frac{3}{7}e^{5t}$.

(d) Suppose you are given the initial conditions

$$x(0) = \tfrac{18}{7}, \quad x'(0) = -\tfrac{1}{7}.$$

Find the solution $x(t)$ to the initial value problem using your general solution from the previous part.

Solution

$x(t) = 2e^{-2t} + e^{6t} - \frac{3}{7}e^{5t}$.

Problem 22: More examples
For each of the following, find the general solution. You will need to use the tricks that you learned in Part 2. The homogeneous equation is the same for each one, only the particular solution needs to change.

(a) $x'' - 4x' - 12x = \sin(2t)$

Solution

$x_p(t) = \frac{1}{40}\cos(2t) - \frac{1}{20}\sin(2t)$.

(b) $x'' - 4x' - 12x = te^{4t}$

Solution

$x_p(t) = e^{4t}(-\frac{t}{12} - \frac{1}{36})$.

(c) $x'' - 4x' - 12x = e^{6t}$

Solution

$x_p(t) = \frac{t}{8}e^{6t}$.

Part 5: Linearly Independent Solutions

We've been making some assumptions in the previous sections. When solving a second order equation, we found two homogeneous solutions x_1 and x_2 and then assumed that *any* homogeneous solution could be written as

$$a_1x_1 + a_2x_2.$$

But how do we know that this is true? Is this true for *any* homogeneous solutions x_1 and x_2?

Problem 23: Same or Different?

We've found that for many second order equations, there are two distinct exponential solutions. For instance, for

$$x'' - 5x' + 6x = 0,$$

we found

$$x_1(t) = e^{2t}, \qquad x_2(t) = e^{3t}.$$

(a) Check that both x_1 and x_2 satisfy the differential equation.

Solution

Substitute and verify that both give 0.

(b) Can we write $x_2(t)$ as a constant multiple of $x_1(t)$? If not, why not?

Solution

No. $e^{3t} \neq ce^{2t}$ for any constant c.

(c) What if instead we took $x_1(t) = e^{2t}$ and $x_2(t) = 5e^{2t}$? Are those truly different solutions?

Solution

They are just constant multiples, so they describe the same shape.

(d) Can we write e^{3t} as a linear combination of e^{2t} and $5e^{2t}$? Why or why not?

Solution

Same as previous problem.

Based on the previous problem, we need to make sure that the solutions that we find are *different enough*. The precise way we define that is through linear independence, just like in linear algebra.

Definition 24: Linear Independence

We say that two solutions x_1, x_2 are *linearly dependent* if there exists a constant c such that

$$x_1 = cx_2 \quad \text{or} \quad cx_1 = x_2.$$

We say that two solutions are *linearly independent* if this is not possible.

Theorem 25:

Suppose that x_1, x_2 are linearly independent solutions of $x''(t) + b(t)x'(t) + c(t)x(t) = 0$. Then any solution $x(t)$ can be written as a linear combination

$$x(t) = a_1 x_1(t) + a_2 x_2(t).$$

Problem 26: Proof of Theorem 25 (Challenge)

Suppose we have a homogeneous second order differential equation, $x'' + b(t)x' + c(t)x = 0$ and we already know two linearly independent solutions

$$x_1(t), \qquad x_2(t).$$

That means x_2 is not just a constant multiple of x_1.

Now imagine we find another solution $x(t)$ to the same differential equation.

(a) Because x_1 and x_2 are both solutions, any linear combination

$$c_1 x_1(t) + c_2 x_2(t)$$

is also a solution. Why is that?

Solution

The equation is linear, so derivatives and sums pass through linearly.

(b) Suppose we know the values of x and x' at $t = 0$. Show that there are constants c_1, c_2 such that

$$x(0) = c_1 x_1(0) + c_2 x_2(0), \qquad x'(0) = c_1 x_1'(0) + c_2 x_2'(0).$$

Hint: this is just a system of two equations for the two unknowns c_1, c_2. What do we know about $x_1(0)$ and $x_2(0)$?

Solution

Because x_1 and x_2 are independent, the system can be solved uniquely for c_1, c_2.

(c) Because we are looking at a second order differential equation, we know that any solution is uniquely determined by two initial conditions. What does this show about *all* solutions to the differential equation?

Solution

Every solution can be written as a linear combination of x_1 and x_2.

Problem 27: The Wronskian Test

When we have two possible solutions $x_1(t)$ and $x_2(t)$ to a homogeneous second order equation, we can check whether they are linearly independent using something called the *Wronskian*:

$$W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix} = x_1(t)x_2'(t) - x_2(t)x_1'(t).$$

(a) Compute the Wronskian for $x_1(t) = e^{2t}$ and $x_2(t) = e^{3t}$.

Solution

$W(t) = e^{2t}(3e^{3t}) - e^{3t}(2e^{2t}) = e^{5t}$.

(b) Since $W(t) \neq 0$ for any t, what can we conclude about x_1 and x_2?

Solution

They are linearly independent.

(c) Try the same for $x_1(t) = e^{2t}$ and $x_2(t) = 5e^{2t}$. What happens now?

Solution

$W(t) = 0$, so they are linearly dependent.

Part 6: Higher order differential equations (Challenge)

The techniques that we developed are equally valid for higher order systems! Try to find the general solutions for the following. Now you will need 3 (or more) arbitrary constants.

Problem 28:
Find the general solution to

$$x''' + x'' - 6x' = 0.$$

Solution

$x(t) = ae^{-3t} + b + ce^{2t}$.

Problem 29:
Find the general solution to

$$x''' + x'' - x' - x = e^{2t}.$$

Solution

$x(t) = ae^{-t} + bte^{-t} + ce^{t} + \frac{1}{9}e^{2t}.$

Problem 30:
Find the general solution to

$$x''' + 3x'' + 3x' + x = 0.$$

Solution

$x_h(t) = ae^{t} + bte^{t} + ct^2e^{t}.$

Problem 31: (Challenge)
Find the general solution to

$$x'''' + x'' = \sin(t).$$

Solution

$x_h(t) = a\cos(t) + b\sin(t) + c + dt.$
$x_p = \frac{t}{2}\cos(t).$