HIGH REGULARITY WELL-POSEDNESS FOR CM-DNLS : STANDARD ENERGY METHODS

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1. Introduction

The *Calogero-Moser derivative NLS equation* is stated

$$iu_t + u_{xx} + 2\Pi^+ D(|u|^2)u = 0 \tag{CM-DNLS}$$

where $D = -i\partial_x$ and Π^+ is the Szegö projector onto non-negative frequencies. We will often abbreviate $\Pi^+ D = D^+$ and $\Pi^+ f = f^+$.

For ease of notation, we will extensively use $f \lesssim g$ to imply that there exists a universal constant $C > 0$ such that $f \leq Cg$. When the constant has additional dependencies, we will indicate those by subscripts. At times, we will use the corresponding notation $\gtrsim$ and $\sim$.

In light of Gerárd and Lenzmann's proof of global well-posedness for small data,[1] we wish to present a self-contained version for personal reference and understanding. Specifically, we aim to prove

Theorem 1.1 (H^2 Global Well-Posedness)**.** *CM-DNLS is globally well-posed for initial data $u_0 \in H^2_+(\mathbb{R})$ with L^2-mass*

$$M(u_0) < 2\pi.$$

Moreover, we have the a-priori bound

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^2} < \infty.$$

2. Local Theory

We study the Cauchy problem for CM-DNLS in $H^2_+(\mathbb{R})$. To do so, we wish to run Kato's classical iterative scheme for quasilinear evolution equations, the details of which will be explained.

Proposition 2.1 (H^2 Local Well-Posedness)**.** *For any $R > 0$ there is some $T(R) > 0$ such that, for every $u^0 \in H^2_+(\mathbb{R})$ with $\|u^0\|_{H^2} \leq R$, there exists a unique solution $u \in C([-T,T]; H^2_+(\mathbb{R}))$ of CM-DNLS with $u(0) = u^0$.*

Moreover, the flow map $u_0 \mapsto u(t)$ is continuous on H^2.

By distributing the derivative and rearranging, we rewrite CM-DNLS as

$$u_t = iu_{xx} + 2u\Pi^+\bar{u}u_x + 2u\Pi^+ u\overline{u_x}. \tag{1}$$

We aim to construct a sequence u^k such that $u^0(t) = u_0$ and

$$u^{k+1} = iu^{k+1}_{xx} + 2u^k\Pi^+\overline{u^k}u^{k+1}_x + 2u^k\Pi^+u^k\overline{u^k_x}$$

To that end, we first find bounds on the inhomogeneous term $2u^k\Pi^+u^k\overline{u^k_x}$ with the following Lemma.

Lemma 2.2. *For all $p \in \{0,1,2\}$, if $u \in H^2_+(\mathbb{R})$ and $v \in H^p$, then $\Pi^+ u\overline{v_x} \in H^p_+$ with*

$$\|\Pi^+ u\overline{v_x}\|_{H^p} \lesssim \|u\|_{H^2}\|v\|_{H^p}.$$

Proof. We first prove a bound on $\|\Pi^+ u\overline{f_x}\|_2$ for $f \in H^1_+$. By direct computation,

$$\begin{aligned}\widehat{u\overline{f_x}}(\xi) &= \int_{\mathbb{R}} \widehat{u}(\xi-\eta)\widehat{\overline{f_x}}(\eta)\frac{d\eta}{\sqrt{2\pi}} \\ &= i\int_{\mathbb{R}} \widehat{u}(\xi+\eta)\eta\overline{\widehat{f}(\eta)}\frac{d\eta}{\sqrt{2\pi}}\end{aligned}$$

Then

$$\begin{aligned}\|\Pi^+ u\overline{f_x}\|_2^2 &\lesssim \int_0^\infty \left|\int_0^\infty |\eta||\widehat{u}(\xi+\eta)||\widehat{f}(\eta)|d\eta\right|^2 d\xi \\ &\le \|f\|_2^2 \int_0^\infty\int_0^\infty |\eta+\xi|^2|\widehat{u}(\xi+\eta)|^2 d\eta d\xi \qquad \text{(Hölder's)} \\ &= \|f\|_2^2 \int_0^\infty\int_{-\zeta}^{\zeta} |\zeta|^2|\widehat{u}(\zeta)|^2 d\omega d\zeta \qquad (\zeta=\eta+\xi, \omega=\eta-\xi) \\ &\sim \|f\|_2^2\int_0^\infty \zeta^3|\widehat{u}(\zeta)|^2 d\zeta \\ &= \|u\|_{\dot{H}^{3/2}}^2\|f\|_2^2\end{aligned}$$

as desired. By density of H^1_+ in L^2_+, this bound extends to $f \in L^2_+$. In particular, this implies that $\|\Pi^+ u\overline{v_x}\|_2 \lesssim \|u\|_{H^2}\|v\|_2$ for $v \in L^2$.

To prove the original statement, we first note that by Sobolev, for $f \in H^2$, $\|f\|_\infty \le \|f\|_{\dot{H}^{1/2}}$. Using this, we compute for $v \in H^1$,

$$\begin{aligned}\|\Pi^+ u\overline{v_x}\|_{\dot{H}^1} &\lesssim \|\Pi^+ u_x\overline{v_x}\|_2 + \|\Pi^+ u\overline{v_x x}\|_2 \\ &\lesssim \|u_x\|_\infty\|v_x\|_2 + \|u\|_{\dot{H}^{3/2}}\|v_x\|_2 \\ &\lesssim \|u\|_{H^2}\|v\|_{H^1}\end{aligned}$$

Similarly, for $v \in H^2$,

$$\begin{aligned}\|\Pi^+ u\overline{v_x}\|_{\dot{H}^2} &\lesssim \|u_{xx}\overline{v_x}\|_2 + \|u_x\overline{v_{xx}}\|_2 + \|\Pi^+ u\overline{v_{xxx}}\|_2 \\ &\le \|u_{xx}\|_2\|v_x\|_\infty + \|u_x\|_\infty\|v_{xx}\|_2 + \|\Pi^+ u\overline{v_{xxx}}\|_2 \\ &\lesssim \|u\|_{H^2}\|v\|_{H^2}.\end{aligned}$$

Combining these concludes the desired result for all $p \in \{0,1,2\}$. □

We now aim to prove that our iteration scheme is valid, for which we establish the following Lemma.

Lemma 2.3. *Let $u \in C([-T,T]; H^p_+)$ with some $T > 0$, $p \in \{0,1,2\}$ and $w_0 \in H^p_+(\mathbb{R})$, $f \in L^1([-T,T]; H^p_+)$. Then there exists a unique $w \in C([-T,T]; H^p_+)$ such that*

$$w_t = iw_{xx} + 2u\Pi^+\overline{u}w_x + f, \quad w(0) = w_0 \tag{2}$$

Furthermore,

$$\|w\|_{L^\infty_t H^p_x} \lesssim e^{C\int_{-T}^T \|u(t)\|_{H^2}^2 dt}\left(\|w_0\|_{H^p} + \|f\|_{L^1_t H^p}\right) \tag{3}$$

for some constant $C > 0$.

To prove this Lemma, we must employ "standard energy methods" to perturb the problem slightly, solve it in the perturbed case, and then extend this solution to our desired case. To that end, for $\varepsilon > 0$, we introduce the perturbed equation

$$w_t^\varepsilon = iw_{xx}^\varepsilon + 2u(1-\varepsilon\partial_x^2)^{-1}\Pi^+\overline{u}w_x^\varepsilon + f, \quad w^\varepsilon(0) = w_0. \tag{4}$$

Note that here, $(1-\varepsilon\partial_x^2)^{-1}$ is the Fourier multiplier $(1+\varepsilon\xi^2)^{-1}$. We first prove a well-posedness result for equation 4 on the time interval $[-T,T]$.

Lemma 2.4. *Fix $\varepsilon > 0$. Let $u \in C([-T,T]; H_+^p)$ with some $T > 0$, $p \in \{0,1,2\}$ and $w_0 \in H_+^p(\mathbb{R})$, $f \in L^1([-T,T]; H_+^p)$. Then there exists a unique $w^\varepsilon \in C([-T,T]; H_+^p)$ such that 4 holds. Furthermore,*

$$\|w^\varepsilon\|_{L_t^\infty H_x^p} \lesssim e^{C\int_{-T}^T \|u(t)\|_{H^2}^2\,dt}\left(\|w_0\|_{H^p} + \|f\|_{L_t^1 H_x^p}\right) \tag{5}$$

for some constant $C > 0$ independent of ε.

Proof. We argue via contraction mapping. We will construct a local solution for small time depending on the size of the initial data and then extend to $[-T,T]$ via uniform bounds.

Let F^ε denote the non-linearity,

$$F^\varepsilon(v) = 2u(1-\varepsilon\partial_x^2)^{-1}\Pi^+\overline{u}v_x$$

As usual, we seek a strong solution of the form

$$w^\varepsilon(t) = e^{it\Delta}w_0 - i\int_0^t e^{i(t-s)\Delta}\left(F^\varepsilon(w^\varepsilon(s)) + f\right)ds.$$

We claim that $F^\varepsilon : H_+^p \to H_+^p$. To see this, we first calculate that for $f, g \in H_+^1$,

$$\begin{aligned}
|\widehat{\overline{f}} * \widehat{g_x}(\xi)| &\leq \int_0^\infty \eta|\widehat{f}(\eta-\xi)||\widehat{g}(\eta)|d\eta \\
&\leq \|\eta\widehat{f}(\eta-\xi)\|_2\|g\|_2 && \text{(Hölder's)} \\
&\leq \left(\|(\eta-\xi)\widehat{f}(\eta-\xi)\|_2 + \xi\|f\|_2\right)\|g\|_2 \\
&\leq (1+\xi)\,\|f\|_{H^1}\|g\|_2.
\end{aligned}$$

Density then extends this bound to $f \in L_+^2$. Applying this to F^ε for $v \in L_+^2$, we find that

$$\begin{aligned}
\|F^\varepsilon(v)\|_2^2 &\leq \|u\|_\infty^2 \int_0^\infty \left|(1+\varepsilon\xi^2)^{-1}\widehat{\overline{u}} * \widehat{v_x}(\xi)\right|^2 d\xi \\
&\leq \|u\|_\infty^2\|u\|_{H^1}^2\|v\|_2^2 \int_0^\infty \left(\frac{1+\xi}{1+\varepsilon\xi^2}\right)^2 d\xi \\
&\lesssim \varepsilon^{-1/2}\|u\|_\infty^2\|u\|_{H^1}^2\|v\|_2^2 \leq \varepsilon^{-1/2}\|u\|_{H^2}^4\|v\|_2^2
\end{aligned}$$

for sufficiently small ε as desired.

Similarly, for $v \in H_+^1$,

$$\begin{aligned}
\|\partial_x F^\varepsilon(v)\|_2 &\lesssim \varepsilon^{-1/4}\|u_x\|_\infty\|u\|_{H^1}\|v\|_2 + \varepsilon^{-1/4}\|u\|_\infty\|u_x\|_{H^1}\|v\|_2 + \varepsilon^{-1/4}\|u\|_\infty\|u\|_{H^1}\|v_x\|_2 \\
&\lesssim \varepsilon^{-1/4}\|u\|_{H^2}^2\|v\|_{H^1}.
\end{aligned}$$

Combining this with the L^2 bound, we find that $\|F^\varepsilon(v)\|_{H^1} \lesssim \varepsilon^{-1/4}\|u\|_{H^2}^2\|v\|_{H^1}$. Similarly, for $v \in H_+^2$,

$$\|\partial_x^2 F^\varepsilon(v)\|_2 \lesssim \|u_{xx}\|_2\|(1-\varepsilon\partial_x^2)^{-1}\Pi^+\overline{u}v_x\|_\infty + \|u_x\|_\infty\|\partial_x(1-\varepsilon\partial_x^2)^{-1}\Pi^+\overline{u}v_x\|_2$$

$$+ \|u\|_\infty \|\partial_x^2 (1-\varepsilon\partial_x^2)^{-1}\Pi^+ \overline{u} v_x\|_2$$
$$\lesssim \varepsilon^{-1/4}\|u\|_{H^2}^2\|v\|_2 + \|u\|_{H^2}^2\|v\|_{H^1} + \|u\|_{H^2}^2\|v\|_{H^2}$$
$$\lesssim \varepsilon^{-1/4}\|u\|_{H^2}^2\|v\|_{H^2}$$

Therefore $\|F^\varepsilon\|_{H^p\to H^p} \lesssim \varepsilon^{-1/4}\|u\|_{H^p}^2$ as desired.

With this, we construct our contraction. Define

$$B = \left\{ v \in C_t H^p_{+x}([-\tilde{T},\tilde{T}]\times\mathbb{R}) \;:\; \|v\|_{L_t^\infty L_x^2} \le 2\|w_0\|_{H^p} + \|f\|_{L_t^1 H_x^p} \right\}$$

for $\tilde{T}$ to be chosen later and $\Phi : B \to B$ such that

$$\Phi(v) = e^{it\Delta}w_0 + \int_0^t e^{i(t-s)\Delta}\left(F^\varepsilon(v(s)) + f\right) ds.$$

To see that $\Phi : B \to B$ is well-defined, we first show that $\Phi(v) \in C_t H^p_{+x}$. We note that $\Phi(v) \in L^2_+$ by construction and recall that $t \mapsto e^{it\Delta}w_0$ is $C_t(\mathbb{R}, H^p_x)$. Furthermore, for $t > \tau$, Strichartz yields

$$\left\| \left(\int_0^t - \int_0^\tau \right) e^{i(t-s)\Delta}\left(F^\varepsilon(v(s)) + f\right) \right\|_{H^p} \lesssim \|f\|_{L_t^1 H_x^p([\tau,t]\times\mathbb{R})} + \varepsilon^{-1/4}(t-\tau)\|u\|^2_{L_t^\infty H_x^2}\|v\|_{L_t^\infty H_x^p}.$$

Combining these facts, we find that $\Phi(v) \in C_t H^p_x$. To conclude that $\Phi : B \to B$, we compute via Strichartz estimates that for $v \in B$,

$$\|\Phi(v)\|_{L_t^\infty H_x^p} \lesssim \|w_0\|_2 + \|f\|_{L_t^1 H_x^p} + \|F^\varepsilon(v(s))\|_{L_t^1 H_x^p}$$
$$\lesssim \|w_0\|_2 + \|f\|_{L_t^1 H_x^p} + \varepsilon^{-1/4}\tilde{T}\|u\|^2_{L_t^\infty H_x^p}\|v\|_{L_t^\infty H_x^p}.$$

Choosing $\tilde{T}$ sufficiently small then implies $\Phi : B \to B$ is well-defined. To now show that Φ is a contraction, we note that Φ is affine and so Strichartz implies

$$\|\Phi(v) - \Phi(\tilde{v})\|_2 \lesssim \varepsilon^{-1/4}\tilde{T}\|u\|_\infty \|u\|_{L_t^\infty H_x^1}\|v-\tilde{v}\|_{L_t^\infty H_x^p}.$$

Again choosing $\tilde{T}$ sufficiently small concludes that Φ is a contraction on B. This implies that there exists a unique solution w^ε to 4 on $[-\tilde{T},\tilde{T}]$.

We now extend w^ε to $[-T,T]$. We note that the choice of $\tilde{T}$ depended only on ε, universal constants, $\|f\|_{L_t^1 H_x^p}$ and $\|w_0\|_{H^p}$. Therefore, to extend w^ε to $[-T,T]$, it suffices to show that the H^p norm is bounded under the flow of 4 and so it suffices to show 5.To do so, we calculate

$$\frac{d}{dt}\|w^\varepsilon(t)\|_{H^p}^2 = \operatorname{Re}\langle w^\varepsilon, iw^\varepsilon_{xx} + F^\varepsilon(w^\varepsilon) + f(t)\rangle_{H^p}$$
$$\le \|f(t)\|_{H_x^p}\|w^\varepsilon\|_{H^p} + |\operatorname{Re}\langle w^\varepsilon, F^\varepsilon(w^\varepsilon)\rangle_{H^p}|.$$

Consider only the final term. Noting that $u(1-\varepsilon\partial_x^2)^{-1}\Pi^+\overline{u}$ is self-adjoint and $(1-\varepsilon\partial_x^2)^{-1}$ is bounded, we find that

$$\operatorname{Re}\langle w^\varepsilon, F^\varepsilon(w^\varepsilon)\rangle_2 = -2\operatorname{Re}\langle u_x(1-\varepsilon\partial_x^2)^{-1}\Pi^+\overline{u}w^\varepsilon, w^\varepsilon\rangle_2 - 2\operatorname{Re}\langle u(1-\varepsilon\partial_x^2)^{-1}\Pi^+\overline{u_x}w^\varepsilon, w^\varepsilon\rangle_2$$
$$- \operatorname{Re}\langle F^\varepsilon(w^\varepsilon), w^\varepsilon\rangle_2$$

$$2\operatorname{Re}\langle w^\varepsilon, F^\varepsilon(w^\varepsilon)\rangle_2 = -2\operatorname{Re}\langle u_x(1-\varepsilon\partial_x^2)^{-1}\Pi^+\overline{u}w^\varepsilon, w^\varepsilon\rangle_2 - 2\operatorname{Re}\langle u(1-\varepsilon\partial_x^2)^{-1}\Pi^+\overline{u_x}w^\varepsilon, w^\varepsilon\rangle_2$$

$$|\operatorname{Re}\langle w^\varepsilon, F^\varepsilon(w^\varepsilon)\rangle_2| \lesssim \|u\|_{H^2}^2\|w^\varepsilon\|_2^2$$

Using this result, we compute

$$|\operatorname{Re}\langle w^\varepsilon, F^\varepsilon(w^\varepsilon)\rangle_{\dot{H}^1}| = |\operatorname{Re}\langle \partial_x w^\varepsilon, \partial_x F^\varepsilon(w^\varepsilon)\rangle_2| = |\operatorname{Re}\langle F^\varepsilon w^\varepsilon_x, w^\varepsilon_x\rangle_2| \lesssim \|u\|_{H^2}^2\|w^\varepsilon\|_{\dot{H}^1}.$$

Similarly,

$$\begin{aligned}|\operatorname{Re}\langle w^\varepsilon, F^\varepsilon(w^\varepsilon)\rangle_{\dot H^2}| &\sim |\operatorname{Re}\langle w^\varepsilon_x, F^\varepsilon w^\varepsilon_x\rangle_{\dot H^1}| + |\operatorname{Re}\langle w^\varepsilon_{xx}, u_{xx}(1-\varepsilon\partial_x^2)^{-1}\Pi^+\overline{u}w^\varepsilon_x\rangle_2| \\ &\quad + |\operatorname{Re}\langle w^\varepsilon_{xx}, u(1-\varepsilon\partial_x^2)^{-1}\Pi^+\overline{u_{xx}}w^\varepsilon_x\rangle_2| \\ &\quad + |\operatorname{Re}\langle w^\varepsilon_{xx}, u_x(1-\varepsilon\partial_x^2)^{-1}\Pi^+\overline{u_x}w^\varepsilon_x\rangle_2| \\ &\lesssim \|u\|_{H^2}^2\|w^\varepsilon\|_{\dot H^2}.\end{aligned}$$

Combining these results, we find that $|\operatorname{Re}\langle w^\varepsilon, F^\varepsilon(w^\varepsilon)\rangle_{H^p}| \lesssim \|u\|_{H^2}^2\|w^\varepsilon\|_{H^p}$. Therefore

$$\frac{d}{dt}\|w^\varepsilon(t)\|_{H^p}^2 \lesssim \|f(t)\|_{H^p_x}\|w^\varepsilon\|_{H^p} + \|u\|_{H^2}^2\|w^\varepsilon\|_{H^p}^2$$

which concludes the Lemma via Gronwall's inequality. □

We now take the limit as $\varepsilon \to 0$ in L^2 for fixed u, w_0. To do so, we show that w^ε is uniformly Cauchy in ε via a Gronwall argument. Note that $\|w^\varepsilon\|_{L^\infty_t H^2_x} \lesssim 1$ uniformly in ε. Then by definition,

$$\begin{aligned}\frac{d}{dt}\|w^\varepsilon - w^\eta\|_2^2 &= 2\operatorname{Re}\left\langle w^\varepsilon - w^\eta, i(w^\varepsilon - w^\eta)_{xx} + 2u\left((1-\varepsilon\partial_x^2)^{-1}\Pi^+\overline{u}w^\varepsilon_x - (1-\eta\partial_x^2)^{-1}\Pi^+\overline{u}w^\eta_x\right)\right\rangle_2 \\ &= 4\operatorname{Re}\left\langle w^\varepsilon - w^\eta, u\left((1-\varepsilon\partial_x^2)^{-1}\Pi^+\overline{u}w^\varepsilon_x - (1-\eta\partial_x^2)^{-1}\Pi^+\overline{u}w^\eta_x\right)\right\rangle_2 \\ &= 4\operatorname{Re}\left\langle w^\varepsilon - w^\eta, u\frac{(\varepsilon-\eta)\partial_x^2}{(1-\varepsilon\partial_x^2)(1-\eta\partial_x^2)}\Pi^+\overline{u}w^\varepsilon_x - u(1-\eta\partial_x^2)^{-1}\Pi^+\overline{u}(w^\varepsilon - w^\eta)_x\right\rangle_2 \\ &= 4\operatorname{Re}\left\langle w^\varepsilon - w^\eta, u\frac{(\varepsilon-\eta)\partial_x^2}{(1-\varepsilon\partial_x^2)(1-\eta\partial_x^2)}\Pi^+\overline{u}w^\varepsilon_x - F^\eta\left(w^\varepsilon - w^\eta\right)\right\rangle_2 \\ &\lesssim |\varepsilon-\eta|\|w^\varepsilon - w^\eta\|_{H^2}\|w^\varepsilon\|_{H^2}\|u\|_{H^2}^2 + \|u\|_{H^2}^2\|w^\varepsilon - w^\eta\|_2^2 \\ &\lesssim |\varepsilon-\eta| + \|w^\varepsilon - w^\eta\|_2^2.\end{aligned}$$

Since $T > 0$ is finite, Gronwall's inequality then concludes that w^ε is uniformly Cauchy in ε.

Therefore, for fixed w_0, u there exists some $w \in C([-T,T]; L^2)$ such that $w^\varepsilon \to w \in L^\infty_t L^2_x$. Moreover, since w^ε is uniformly bounded in H^p_x, Fatou's lemma implies that $w \in L^\infty_t H^p_x$. To show that w is a strong solution of 2, we then show that w satisfies the Duhamel formula. Let $\tilde w(t)$ denote the Duhamel formula for 2. By definition of w^ε and Strichartz estimates,

$$\begin{aligned}\|w(t) - \tilde w(t)\|_2 &\le \|w - w^\varepsilon\|_{L^\infty_t L^2_x} + \left\|\int_0^t e^{i(t-s)\Delta}F^\varepsilon(w^\varepsilon - w)ds\right\|_2 + \left\|\int_0^t e^{i(t-s)\Delta}2u\frac{\varepsilon\partial_x^2}{1-\varepsilon\partial_x^2}\Pi^+\overline{u}w_x ds\right\|_2 \\ &\lesssim \|w - w^\varepsilon\|_{L^\infty_t L^2_x} + T\left\|F^\varepsilon(w^\varepsilon - w)\right\|_{L^\infty_t L^2_x} + T\varepsilon\|u\Pi^+\overline{u}w_x\|_{L^\infty_t L^2_x} \\ &\lesssim \|w - w^\varepsilon\|_{L^\infty_t L^2_x} + T\varepsilon^{-1/4}\|u\|_{H^2}^2\|w - w^\varepsilon\|_{L^\infty_t L^2_x} + T\varepsilon\|u\|_{H^2}^2\|w\|_{L^\infty_t H^2_x}\end{aligned}$$

Therefore the Duhamel formula holds for w. Since $w \in L^\infty_t H^p_x$ for $p \in \{0,1,2\}$, the Duhamel formula implies that $w \in C([-T,T]; H^p_x)$ and so w is a solution to 2. Uniqueness then follows from a standard contraction argument, thus proving Lemma 2.3.

With these lemmas, we now prove Proposition 2.1.

Proof of Proposition 2.1. Suppose that $\|u_0\|_{H^2} \le R$. By Lemma 2 with $f = 2u^k\Pi^+u^k\overline{u^k_x}$, we construct by iteration $u^k \in C([-T,T]; H^2_+(\mathbb{R}))$ with $u^0(t,x) =$

$u_0(x)$ for T to be chosen later. We aim to show that (u^k) is bounded in H^2_+ and uniformly convergent in L^2_+. By 3,

$$\|u^{k+1}\|_{L^\infty_t H^2_x} \lesssim e^{2CT\|u^k\|^2_{L^\infty_t H^2_x}} \left(\|u_0\|_{H^2} + 2T\|u^k\|^3_{L^\infty_t H^2_x}\right).$$

A discrete bootstrap argument yields $\|u^k\|_{L^\infty_t H^2_x} \lesssim 1$ uniformly in k for T sufficiently small.

We now show that (u^k) is uniformly Cauchy in $L^\infty_x L^2_x$. To do so, it suffices to show that $\|u^{k+1} - u^k\|_{L^\infty_t L^2_x}$ converges geometrically. We observe that

$$\begin{aligned}(u^{k+1} - u^k)_t &= i(u^{k+1} - u^k)_{xx} + 2u^k\Pi^+\overline{u^k}(u^{k+1} - u^k)_x \\ &\quad + 2\left(u^k\Pi^+\overline{u^k} - 2u^{k-1}\Pi^+\overline{u^{k-1}}\right)u^k_x + 2u^k\Pi^+u^k\overline{u^k_x} - 2u^{k-1}\Pi^+u^{k-1}\overline{u^{k-1}_x}.\end{aligned}$$

Since $\|u^k\|_{L^\infty_t L^2_x} \lesssim 1$ uniformly in k, it follows that the final two terms are in $L^1_t L^2_x([-T,T]\times\mathbb{R})$. Equation 3 with $p = 2$ then implies that

$$\|u^{k+1} - u^k\|_{L^\infty_t L^2_x} \lesssim Te^{2C'T}\left\|2\left(u^k\Pi^+\overline{u^k} - 2u^{k-1}\Pi^+\overline{u^{k-1}}\right)u^k_x + 2u^k\Pi^+u^k\overline{u^k_x} - 2u^{k-1}\right\|_{L^\infty_t L^2_x}$$

Applying the triangle inequality repeatedly pulling out many factors of $\|u^j\|_{L^\infty_{t,x}} \lesssim \|u^j\|_{L^\infty_t H^2_x} \lesssim 1$, we find that

$$\|u^{k+1} - u^k\|_{L^\infty_t L^2_x} \lesssim Te^{2C'T}\|u^k - u^{k-1}\|_{L^\infty_t L^2_x}$$

. Choosing T sufficienty small then implies that $\|u^{k+1} - u^k\|_{L^\infty_t L^2_x}$ converges geometrically to 0. Then u^k converges uniformly in $L^\infty_t L^2_x$ to some $u \in C([-T,T]; L^2_+)$ which solves 1 and hence CM-DNLS. Fatou's lemma then implies that $u(t) \in H^2_+$ for all $t \in [-T,T]$.

To establish uniqueness, we consider two different sequences $u^k, \tilde{u}^k$ each with initial data u_0. Minor adjustments to the Cauchy argument above implies that $\|u^k - \tilde{u}^k\|_{L^\infty_t L^2_x} \to 0$ and so u is unique.

By the Duhamel formula, we can then conclude via a standard argument that $u \in C([-T,T]; H^2_+)$ as desired. □

3. Global Well-Posedness

We do not recreate the proof either of the Lax pair or the hierarchy of conservation laws. Instead, we recall that CM-DNLS has the Lax pair

$$L_u = D - u\Pi^+\overline{u} \quad B_u = -i\partial_x^2 + 2u\partial_x\Pi^+\overline{u}. \tag{6}$$

We note that this is not the operator B_u that [1] uses, but is an equivalent one that is more natural for CM-DNLS since it satisfies $u_t = B_u u$. With these operators, we have the following result from [1],

Proposition 3.1 (Lax Equation)**.** *If $u \in C([0,T]; H^2_+)$ solves CM-DNLS, then it holds*

$$\frac{d}{dt}L_u = [B_u, L_u].$$

Moreover, we have the usual hierarchy of conservation laws

Proposition 3.2 (Hierarchy of Conservation Laws)**.** *Let $u \in C([0,T]; H_+^{n/2})$ be a solution of $CM - DNLS$ for some $n \in \mathbb{N}$. Then the quantities*

$$I_k := \langle L_u^k u, u \rangle \quad \textit{with } k = 0, \dots, n$$

are conserved, where $\langle \cdot, \cdot \rangle$ denotes the usual pairing of $H_+^{-n/2}$ and $H_+^{n/2}$.

We also recall the following lemma from [1],

Lemma 3.3. *For $u \in L_+^2$ and $f \in H_+^{1/2}$, we have $\Pi^+ \overline{u} f \in L_+^2$ with*

$$\|\Pi^+ \overline{u} f\|_2^2 \leq \frac{1}{2\pi} \|u\|_{L^2}^2 \langle Df, f \rangle.$$

With these, we can establish Theorem 1.1 for sub-critical mass.

Proof of Theorem 1.1. We recall that in the proof of local well-posedness, T depended solely on $\|u_0\|_{H^2}$ and universal constants. Therefore, to extend our local solutions globally, it suffices to show a-priori bounds on the H^2 norms. To do so, we utilize the hierarchy of conservation laws.

Suppose that $u_0 \in H_+^2$ with $\|u_0\|_2^2 < 2\pi$. Then by lemma 3.3,

$$|I_1(u)| = |\langle Du, u \rangle - \|\Pi^+ \overline{u} u\|_2^2| \geq \|u\|_{H^{1/2}} - \frac{\|u_0\|_2^2}{2\pi} \|u\|_{H^{1/2}}.$$

Therefore $\|u\|_{L_t^\infty H_x^{1/2}} \lesssim 1$ uniformly in t.

To show a similar bound for H^2, we first show bounds for $H^1, H^{3/2}$ by expanding I_2, I_3 and bounding lower order terms. Doing the same for H^2 yields an a-priori bound and hence Theorem 1.1. □

References

[1] Patrick Gérard and Enno Lenzmann. The calogero–moser derivative nonlinear schrödinger equation, 2023.

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