On Ill-Posedness for the Gabitov–Turitsyn Equation

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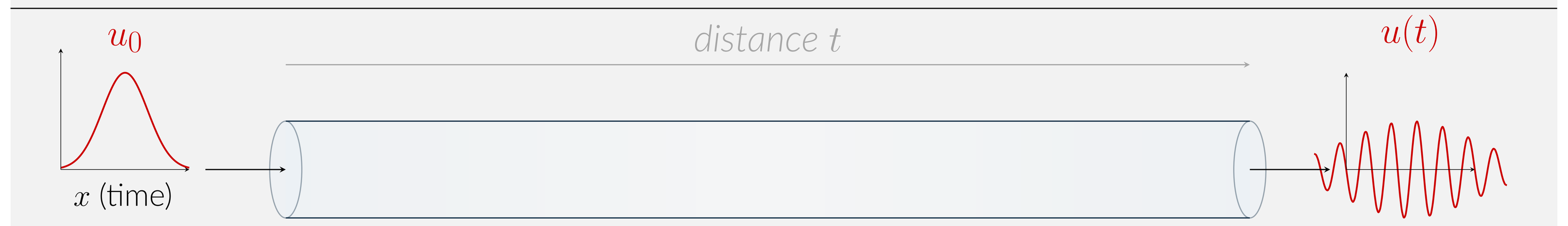
Derivation of the Model

Dispersion-managed optical fibers



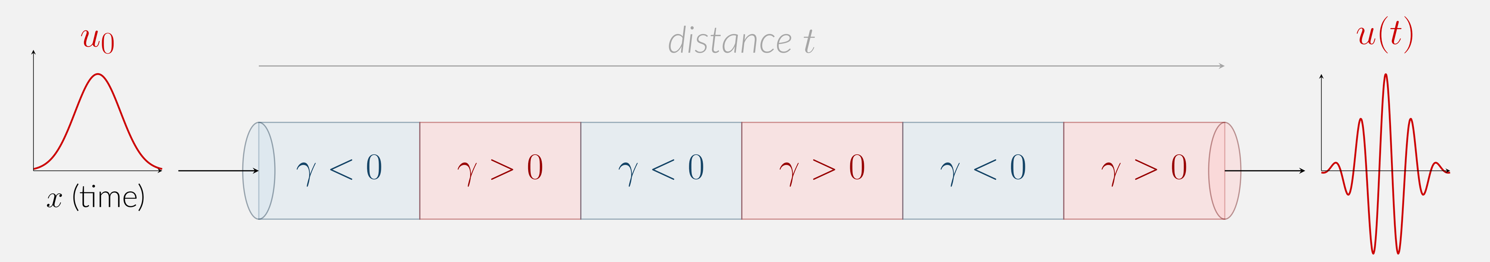
Optical fiber with GVD γ

The propagation of pulses in an optical fiber is primarily modeled by the *cubic nonlinear Schrödinger equation*:

$$i\partial_t u + \gamma \partial_{xx} u + |u|^2 u = 0, \quad u(0,x) = u_0(x), \tag{NLS}$$

here γ is the group velocity dispersion (GVD) and the roles of t and x are flipped from expectation: t represents the *distance* along the fiber and x is a *retarded time*, travelling with the carrier wave.

In a typical optical fiber, dispersion dominates the nonlinear effects and causes pulses to broaden. This limits bandwidth as pulses overlap and interact. A common technique to mitigate these effects is *dispersion-management*, concatenating segments of optical fibers with opposite GVD:



Optical fiber with alternating GVD $\gamma(t)$

Mathematically, this gives the *dispersion-managed NLS*:

$$i\partial_t u + \gamma(t) \partial_{xx} u + |u|^2 u = 0, \quad u(0,x) = u_0(x). \tag{DM-NLS}$$

Large-scale dynamics: The Gabitov–Turitsyn equation (GT)

Most often, *strong dispersion-management* is employed, alternating quickly between extreme positive and negative GVD. Mathematically, this corresponds to

$$\gamma(t) = \langle\gamma\rangle + \varepsilon^{-1}\gamma_0(t/\varepsilon),$$

where $\gamma_0(t)$ is periodic with mean 0 and $\varepsilon \ll 1$.

Taking the limit as $\varepsilon \to 0$, we find that the cubic one-dimensional *Gabitov–Turitsyn equation* emerges as the primary model of the large-scale dynamics. Much like the cubic (NLS), the cubic one-dimensional (GT) introduces additional challenges. For our purposes, we then consider a generalized form:

$$i\partial_t u + \langle\gamma\rangle \Delta u + \int_0^1 e^{-i\sigma\Delta}\Big[|e^{i\sigma\Delta}u|^p \cdot e^{i\sigma\Delta}u\Big]d\sigma = 0, \quad u(0,x) = u_0(x), \tag{GT}$$

where $u : \mathbb{R}_t \times \mathbb{R}^d_x \to \mathbb{C}$.

Scaling pseudo-symmetries

Due to the averaging in the nonlinearity, **(GT) lacks a true scaling symmetry**. Instead, we can identify two *scaling pseudo-symmetries*, which map solutions of (GT) to solutions of a (GT)-variant.

Monomial pseudo-symmetry

Under the usual scaling for (NLS), we find that a solution u to (GT) satisfies:

$$u_\lambda = \lambda^{-\frac{2}{p}} u\Big(\frac{t}{\lambda^2}, \frac{x}{\lambda}\Big) \quad \text{will solve} \quad i\partial_t u_\lambda + \langle\gamma\rangle\Delta u_\lambda + \lambda^{-2}\int_0^{\lambda^2} e^{-i\sigma\Delta}\Big[|e^{i\sigma\Delta}u_\lambda|^p \cdot e^{i\sigma\Delta}u_\lambda\Big]d\sigma = 0.$$

This preserves the *averaging* in the nonlinearity and identifies a critical regularity at

$$s_m = \frac{d}{2} - \frac{2}{p}.$$

Integrated pseudo-symmetry

Under a modified scaling, we find that a solution u to (GT) satisfies:

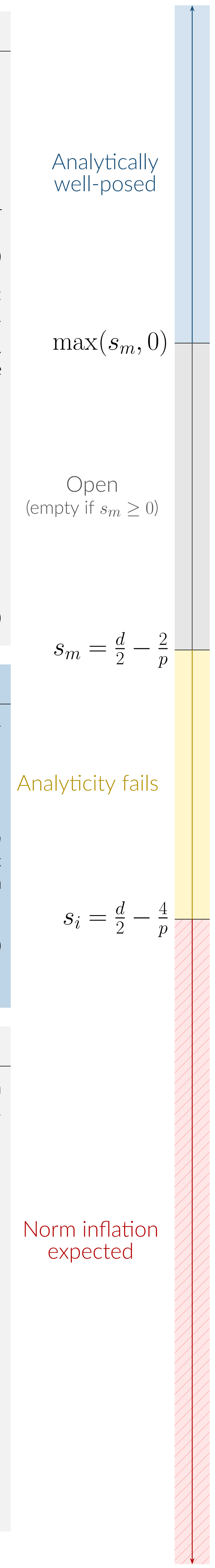
$$u_\lambda = \lambda^{-\frac{4}{p}} u\Big(\frac{t}{\lambda^2}, \frac{x}{\lambda}\Big) \quad \text{will solve} \quad i\partial_t u_\lambda + \langle\gamma\rangle\Delta u_\lambda + \int_0^{\lambda^2} e^{-i\sigma\Delta}\Big[|e^{i\sigma\Delta}u_\lambda|^p \cdot e^{i\sigma\Delta}u_\lambda\Big]d\sigma = 0.$$

This preserves the *integral* in the nonlinearity and identifies a critical regularity at

$$s_i = \frac{d}{2} - \frac{4}{p}.$$

If the interval of (GT) was changed to $[0,\infty)$ or $\mathbb{R}$, then this would be a true scaling symmetry.

Results



Analytically well-posed ($s \geq \max(s_m, 0)$)

Theorem 1. (Local well-posedness) (This appeared earlier by Kawakami–Murphy) *For $u_0 \in \dot{H}^{\max(s_m,0)}$ and $T \sim \|u_0\|^{-p}_{\dot{H}^{\max(s_m,0)}}$, there exists a unique solution to* (GT):

$$u \in C_t\dot{H}^{\max(s_m,0)}((-T,T)\times\mathbb{R}^d) \quad \textit{with} \quad u(0,x) = u_0(x).$$

Moreover, the data-to-solution map is real analytic on a neighborhood of $u_0 = 0$: for $\|u_0\|_{\dot{H}^{\max(s_m,0)}} \leq R$ and $T \sim R^{-p}$ the data-to-solution map

$$u_0 \in B_R\Big(\dot{H}^{\max(s_m,0)}\Big) \quad \longmapsto \quad u \in C_t\dot{H}_x^{\max(s_m,0)}\Big((-T,T)\times\mathbb{R}^d\Big)$$

satisfies a power series expansion; see (3).

In addition, there exists $\delta = \delta(p,d)$ such that for all $\|u_0\|_{\dot{H}^{\max(s_m,0)}} < \delta$, the associated solution u may be extended to a global solution in $C_t\dot{H}_x^{\max(s_m,0)}(\mathbb{R}\times\mathbb{R}^d)$ which scatters.

Open ($s_m \leq s < 0$)

When a Galilean symmetry exists for a dispersive model, it is expected that the data-to-solution map fails to be uniformly continuous in H^s for $s < 0$. For focusing equations, this failure is often proved with solitons. Combined with the scaling symmetry, one can boost sufficiently narrow solitons to high speeds, causing them to be small in H^s for $s < 0$ and to decohere quickly.

For (GT), this story is complicated. The existence of such *dispersion-managed solitons* has been rigorously justified for **non-negative** average GVD by Choi, Hundertmark, and Lee. However, the nonlocal structure of (GT) forces these results to formulate solitons as constrained energy minimizers, leaving their profiles, dynamical properties, and widths opaque. This breaks the usual proof and leaves the status of ill-posedness open.

Analyticity fails ($s < s_m$)

For $s < s_m$, we find that the data-to-solution map fails to be analytic:

Theorem 2. *For any $T > 0$ and $s < s_m$, the $(p+1)^{st}$ derivative of the data-to-solution map,*

$$\Xi_1(\phi) = i\int_0^t\int_0^1 e^{i\langle\gamma\rangle t\Delta - i(s+\sigma)\Delta}\Big[|e^{i\sigma\Delta}u(s)|^p \cdot e^{i\sigma\Delta}u(s)\Big]d\sigma ds,$$

fails to be bounded $H^s \to L_t^\infty H_x^s((-T,T))$.

The operator Ξ_1 can be understood as the $(p+1)^{st}$ *directional derivative* of the data-to-solution map at initial data $u_0 = 0$ in the direction of ϕ.

Mass-subcritical norm inflation ($s < \min(s_i, 0)$)

Definition. (Norm inflation in H^s) *We say that* norm inflation *occurs in H^s if, for all $\varepsilon > 0$, there exists smooth initial data u_0 with corresponding solution $u(t)$ to* (GT) *that satisfies*

$$\|u_0\|_{\dot{H}^s} < \varepsilon \quad \textit{with} \quad \|u(T)\|_{\dot{H}^s} > \varepsilon^{-1}, \quad \textit{for some} \quad |T| < \varepsilon.$$

This indicates that the data-to-solution map fails to be continuous at $u_0 = 0$ and hence the equation is ill-posed in the strongest sense.

Theorem 3. (Mass-subcritical norm inflation) *For the one-dimensional cubic* (GT), *norm inflation occurs in H^s for $s < s_i = -\frac{3}{2}$. As a consequence, for any $T > 0$, the data-to-solution map of* (GT) *fails to be continuous at $u_0 = 0$ from $H^s \to C_tH_x^s((-T,T))$ for $s < s_i$.*

Energy-supercritical norm inflation ($1 \leq s < s_i$)

Theorem 4. (Energy-supercritical norm inflation) *Suppose that $s_i > 1$. Then norm inflation occurs in H^s for* (GT) *for all $1 \leq s < s_i$. Therefore, for any $T > 0$, the data-to-solution map of* (GT) *fails to be continuous at $u_0 = 0$ from $H^s \to C_tH_x^s((-T,T))$ for $1 \leq s < s_i$.*

Our proof of norm inflation in this region relies heavily on the following phenomenon:

Proposition 5. (Energy equipartition) *Suppose that u_0 is real-valued and sufficiently regular to justify the virial identity* (4). *Then for $T^2 \sim E(u_0)/\int|xu_0|^2dx$ with $T < 0$, the corresponding solution u to the defocusing* (GT) *($\langle\gamma\rangle = -1$) satisfies*

$$\|u(T)\|^2_{\dot{H}^1_x} \gtrsim \|e^{i\sigma\Delta}u(T)\|^{p+2}_{L^{p+2}_{\sigma,x}([0,1])}.$$

In other words, should $\|u_0\|^2_{H^1} \ll \|e^{i\sigma\Delta}u_0\|^{p+2}_{L^{p+2}_{\sigma,x}([0,1])}$, then the kinetic and potential energy of u become comparable by time T.

Methods and Remarks

Power series expansion

To show local well-posedness and lay the foundation for Theorems 2 and 3, we define

$$Lf = e^{it\langle\gamma\rangle\Delta}f,$$
$$N_p(f_0,\cdots,f_p) = i\int_0^t\int_0^1 e^{i\langle\gamma\rangle(t-s)\Delta - i\sigma\Delta}\Big[e^{i\sigma\Delta}f_0(s)\cdot\overline{e^{i\sigma\Delta}f_1(s)}\cdot\ldots\cdot e^{i\sigma\Delta}f_p(s)\Big]d\sigma ds.$$

With this notation, a solution u of (GT) with initial data u_0 satisfies

$$u = Lu_0 + N_p(u,\cdots,u).$$

Ouroborically substituting this formula into itself, we find the formal expansion

$$u = Lu_0 + N_p(Lu_0,\cdots,Lu_0) + N_p\Big[Lu_0,\cdots,Lu_0,N_p(Lu_0,\cdots,Lu_0)\Big] + \cdots. \tag{1}$$

Grouping terms of equal order, we recursively define

$$\Xi_0(u_0) = L(u_0),$$
$$\Xi_j(u_0) = \sum_{\substack{j_0,\ldots,j_p\geq 0\\ j_0+\cdots+j_p=j-1}} N_p(\Xi_{j_0}(u_0),\cdots,\Xi_{j_p}(u_0)). \tag{2}$$

This implies that u has the following (formal) power series expansion

$$u = \sum_{j\geq 0}\Xi_j(u_0). \tag{3}$$

Proposition 6. (Quantitative well-posedness): *Let $D = \dot{H}^{\max(s_m,0)}$ denote the space of initial data, and $S = L_t\dot{H}^{\max(s_m,0)}((-T,T)\times\mathbb{R}^d)$ the space of solutions. The operators $N_p : S^{p+1}\to S$ and $L : D \to S$ are bounded in the following sense:*

$$\|Lg\|_S = \|g\|_D$$
$$\|N_p(f_0,\ldots,f_p)\|_S \leq C_pT\|f_0\|_S\ldots\|f_p\|_S.$$

This leads to Theorem 1 with standard arguments, see the work of Bejenaru–Tao.

Mass-subcritical norm inflation ($s < \min(s_i, 0)$)

To prove norm inflation in this region, we expand the methods of Nobu Kishimoto and Tadahiro Oh. For a large parameter $N \gg 1$, we construct initial data u_0 of the form:

$$\widehat{u_0} = N^{1+}\big(\mathbb{1}_{[N,N+A]} + \mathbb{1}_{[2N,2N+A]}\big),$$

with $A = N^{1-}$. When u_0 interacts with itself, a *high-low frequency cascade* occurs, causing growth in the H^s norm for $s < 0$. For $s < s_i$, this construction ensures that the growth is driven by Ξ_1 and not negated by higher-order terms.

Energy-supercritical norm inflation ($1 \leq s < s_i$)

The standard method of Christ–Colliander–Tao for (NLS) appears intractable for (GT) due to its nonlocal structure. To surmount this, we instead build on structural identities of (GT), namely the virial identity, first shown for (GT) by Choi–Hong–Lee:

Proposition 7. (Virial identity) *Let u be the maximal solution of* (GT) *for $p \geq \frac{8}{d}$, with initial data $u_0 \in \mathcal{S}(\mathbb{R}^d)$. Define the variance of u as*

$$v(t) = \int_{\mathbb{R}^d}|x|^2|u(t,x)|^2dx.$$

In the defocusing case, $\langle\gamma\rangle = -1$, we find that for $-\frac{1}{2}\leq t\leq 0$,

$$v(t) \leq v(0) - t\int\overline{u_0}(x\cdot\nabla u_0)dx + C(p,d)E(u)t^2 + \textit{error terms} \tag{4}$$

where $C(p,d) > 0$ for $p \geq \frac{8}{d}$.

In the defocusing case, $\langle\gamma\rangle = -1$, we use this identity to prove Proposition 5 and show that suitable solutions undergo *energy equipartition*.

To show norm inflation in the energy-supercritical case, we can then work directly with Gaussians. We choose our initial data:

$$u_0(x) = Ae^{-|x|^2/4\sigma^2}, \quad \text{then} \quad \|e^{i\sigma\Delta}u_0\|^{p+2}_{L^{p+2}_{\sigma,x}([0,1])}/\|u_0\|^2_{H^1} \sim A^p\sigma^4.$$

Provided $s < s_i$, we can then choose $\sigma \ll 1$ and $A \gg 1$ with

$$A^p\sigma^4 \gg 1 \quad \text{and} \quad \|u_0\|_{H^s} \sim A^2\sigma^{d-2s} \ll 1.$$

This implies that potential energy greatly outweighs the kinetic energy. Provided $s \geq 1$, energy equipartition then implies a rapid increase in kinetic energy, and hence H^s norm.