On ill-posedness for the dispersion-managed nonlinear Schrödinger equation

Matthew Kowalski

January 13, 2026

University of California, Los Angeles

Based on arXiv:2510.11887

Overview

The *Gabitov–Turitsyn equation* reads:

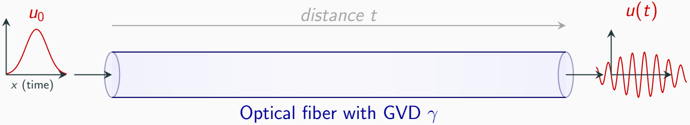
$$i\partial_t u + \langle\gamma\rangle\Delta u + \int_0^1 e^{-i\sigma\Delta}\left[\left|e^{i\sigma\Delta}u\right|^p \cdot e^{i\sigma\Delta}u\right]d\sigma = 0, \qquad \text{(GT)}$$

where $\langle\gamma\rangle \neq 0$ is a constant, $p > 0$ is even, and $e^{i\sigma\Delta}$ is the linear Schrödinger propagator. We will discuss the sharp well-posedness theory for this model.

In this talk, I have four goals:

1. Motivate (GT) as a physically relevant model
2. Motivate the mathematical significance of (GT)
3. Provide a framework for well-posedness and the state of the ill-posedness theory
4. Highlight open problems and future directions

Nonlinear Schrödinger Equation

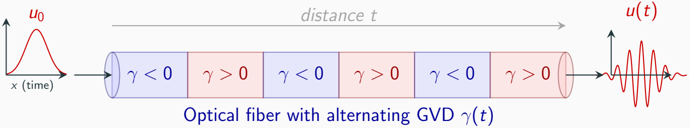


The propagation of pulses in an optical fiber is primarily modeled by the *cubic nonlinear Schrödinger equation*:

$$i\partial_t u + \gamma \partial_{xx} u + |u|^2 u = 0, \quad u(0,x) = u_0(x), \tag{NLS}$$

here γ is the group velocity dispersion (GVD) of the fiber, u is the complex modulation of a quasi-monochromatic carrier wave, and the roles of t and x are flipped from expectation: t represents the *distance* along the fiber and x is a *retarded time*, traveling with the carrier wave.

Dispersion-management



In a typical optical fiber, dispersion dominates the nonlinear effects and causes pulses to broaden. This limits bandwidth as pulses overlap and interact. A common technique to mitigate these effects is *dispersion management*, concatenating fiber segments with opposing GVD.

Mathematically, this gives the *dispersion-managed NLS*:

$$i\partial_t u + \gamma(t)\partial_{xx} u + |u|^2 u = 0, \quad u(0,x) = u_0(x). \qquad \text{(DM-NLS)}$$

Large-scale dynamics — The Gabitov–Turitsyn Equation

Most often, *strong dispersion-management* is employed, alternating quickly between extreme positive and negative GVD:

$$\gamma(t) = \langle\gamma\rangle + \varepsilon^{-1}\gamma_0(t/\varepsilon),$$

where $\gamma_0(t)$ is 2-periodic with mean 0, $\varepsilon \ll 1$, and $\langle\gamma\rangle$ is the net/average GVD.

Taking the limit as $\varepsilon \to 0$, the *Gabitov–Turitsyn equation* emerges:

$$i\partial_t u + \langle\gamma\rangle\partial_x^2 u + \int_0^1 e^{-i\sigma\Delta}\Big[|e^{i\sigma\Delta}u|^2 \cdot e^{i\sigma\Delta}u\Big]d\sigma = 0, \quad u(0,x) = u_0(x).$$

Averaging Process — DM-NLS to GT

We take our dispersion profile to be

$$\gamma(t) = \langle \gamma \rangle + \varepsilon^{-1}\gamma_0(t/\varepsilon),$$

where $\gamma_0(t)$ is periodic with mean 0 and $\varepsilon \ll 1$.
Here we choose $\gamma_0 = \chi_{[0,1)} - \chi_{[1,2)}$ as a model case.

Let $D(t) = \int_0^t \gamma(s)ds$. Suppose that u_ε solves (DM-NLS) with $u_\varepsilon(0) = u_0$.
Making the change of variables

$$v_\varepsilon = e^{-iD(t/\varepsilon)\Delta} u_\varepsilon,$$

we find that v_ε solves

$$i\partial_t v_\varepsilon + \langle \gamma \rangle \partial_x^2 v_\varepsilon + e^{-iD(t/\varepsilon)\Delta}\left[\left|e^{iD(t/\varepsilon)\Delta} v_\varepsilon\right|^p \cdot e^{iD(t/\varepsilon)\Delta} v_\varepsilon\right] = 0. \qquad (1)$$

Taking the limit as $\varepsilon \to 0$, we find that v_ε approaches a solution v of (GT) in $L_t^\infty H_x^1$ over the maximal lifetime of v; see [CL22].

Breathing pulses under strong dispersion-management

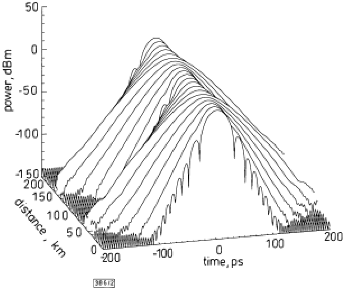


Fig. 2 *Propagation over one cycle for pulse with E = 0.03pJ*
100km sections of $\beta'' = -5.1\,ps^2/km$ and $\beta'' = 4.9\,ps^2/km$

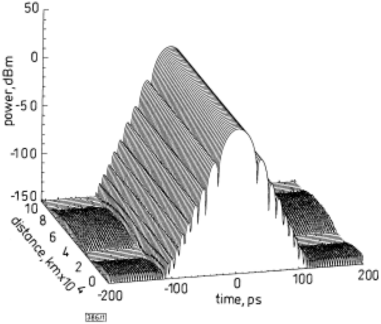


Fig. 1 *Propagation over 100000km for pulse with E = 0.03pJ*
100km sections of $\beta'' = -5.1\,ps^2/km$ and $\beta'' = 4.9\,ps^2/km$
Pulse is shown at the point of anomalous section

Source: numerical study from [NDFK97]

Mathematical Challenges

For the sake of discovering guiding trends, we consider the generalized equation:

$$i\partial_t u + \langle\gamma\rangle\Delta u + \int_0^1 e^{-i\sigma\Delta}\Big[|e^{i\sigma\Delta}u|^p \cdot e^{i\sigma\Delta}u\Big]d\sigma = 0, \quad u(0,x) = u_0(x),$$

where $u : \mathbb{R}_t \times \mathbb{R}_x^d \to \mathbb{C}$ and $p > 0$ is even.

A number of observations:

1. *Highly nonlocal nonlinearity*: It is unclear whether the nonlinearity should be stronger or weaker than (NLS).
2. *Lack of approximate solutions*: Unlike the monomial (NLS), there does not exist explicit solitons nor approximate solutions in the zero dispersion limit $\langle\gamma\rangle = 0$.
3. *Fixed scale*: The integral over $[0, 1]$ fixes a characteristic length of time, destroying any genuine scaling symmetries.

Monomial scaling pseudo-symmetry — Small frequencies

If we restrict attention to the evolution of small frequencies, we expect

$$\int_0^1 e^{-i\sigma\Delta}\Big[|e^{i\sigma\Delta}u|^p \cdot e^{i\sigma\Delta}u\Big]\,d\sigma \approx |u|^p u,$$

and hence (GT) is approximately monomial NLS; this is formalized in [Mur25]. This motivates a critical regularity at

$$s_m = \frac{d}{2} - \frac{2}{p},$$

aligning with the critical regularity of monomial NLS.

Indeed, under the usual scaling for (NLS), we find that a solution u to (GT) satisfies:

$$u_\lambda = \lambda^{-\frac{2}{p}}\, u\Big(\frac{t}{\lambda^2}, \frac{x}{\lambda}\Big) \quad \text{will solve}$$

$$i\partial_t u_\lambda + \langle\gamma\rangle\Delta u_\lambda + \lambda^{-2}\int_0^{\lambda^2} e^{-i\sigma\Delta}\Big[\big|e^{i\sigma\Delta}u_\lambda\big|^p \cdot e^{i\sigma\Delta}u_\lambda\Big]d\sigma = 0.$$

This preserves the *averaging* in the nonlinearity, but changes the interval.

Integrated scaling pseudo-symmetry — High frequencies

If we restrict attention to the evolution of high frequencies, we expect

$$\int_0^1 e^{-i\sigma\Delta}\Big[|e^{i\sigma\Delta}u|^p \cdot e^{i\sigma\Delta}u\Big]d\sigma \approx \int_0^\infty e^{-i\sigma\Delta}\Big[|e^{i\sigma\Delta}u|^p \cdot e^{i\sigma\Delta}u\Big]d\sigma.$$

This is not yet formalized and will be the subject of future work.

This recovers a genuine scaling symmetry and another critical regularity:

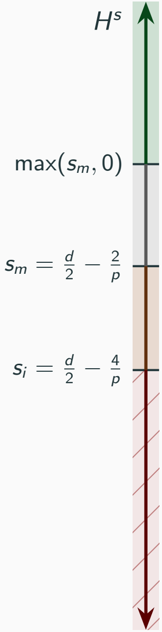
$$u \mapsto u_\lambda = \lambda^{-\frac{4}{p}}\, u\Big(\frac{t}{\lambda^2}, \frac{x}{\lambda}\Big) \quad \text{identifies} \quad s_i = \frac{d}{2} - \frac{4}{p}.$$

Under this modified scaling, we find that a solution u to (GT) satisfies:

$$i\partial_t u_\lambda + \langle\gamma\rangle \Delta u_\lambda + \int_0^{\lambda^2} e^{-i\sigma\Delta}\Big[\big|e^{i\sigma\Delta}u_\lambda\big|^p \cdot e^{i\sigma\Delta}u_\lambda\Big]d\sigma = 0.$$

This preserves the *integral* in the nonlinearity, but breaks the averaging.

Main Theorem [K.]



Analytically Well-posed in H^s $(s \geq \max(s_m, 0))$

For all $u_0 \in \dot{H}^{\max(s_m,0)}$, there exists a unique corresponding solution $u \in C_t \dot{H}_x^{\max(s_m,0)}((-T, T) \times \mathbb{R}^d)$ of (GT) for $T \sim \|u_0\|_{\dot{H}^{\max(s_m,0)}}^{-p}$. Moreover, the data-to-solution map $u_0 \to u$ is analytic.

Open Problem

Failure of uniform continuity is expected; see [KPV01].

Analyticity Fails in H^s $(s < s_m)$

The data-to-solution map fails to be C^{p+1}.

Norm Inflation Expected in H^s $(s < s_i)$

We expect that for all $\varepsilon > 0$, there exists initial data u_0 with $\|u_0\|_{H^s} < \varepsilon$ and a time $|t| < \varepsilon$ such that the corresponding solution $u(t)$ satisfies $\|u(t)\|_{H^s} > \varepsilon^{-1}$.

This is resolved in the *mass-subcritical* $(s < \min(s_i, 0))$ and *energy-supercritical* $(1 \leq s < s_i)$ cases.

Norm Inflation

Norm Inflation

Definition (Norm inflation in H^s)

We say that *norm inflation* occurs in H^s if, for all $\varepsilon > 0$, there exists a solution $u(t)$ to (GT) with smooth initial data u_0 that satisfies

$$\|u_0\|_{\dot{H}^s} < \varepsilon \quad \text{with} \quad \|u(T)\|_{\dot{H}^s} > \varepsilon^{-1}, \quad \text{for some} \quad |T| < \varepsilon.$$

This indicates that the data-to-solution map fails to be continuous at $u_0 = 0$ and hence the equation is ill-posed.

Theorem (Mass-subcritical norm inflation [K.])

For the one-dimensional cubic (GT), *norm inflation occurs in H^s for $s < s_i = -\frac{3}{2}$.*

Theorem (Energy-supercritical norm inflation [K.])

Suppose that $s_i > 1$. Then norm inflation occurs in H^s for (GT) *for all $1 \leq s < s_i$.*

Standard Method for Positive Regularity

The usual proof of norm inflation due to Christ–Colliander–Tao [CCT03, CCT] relies on approximate solutions found in the zero dispersion limit. If you consider NLS with variable dispersion,

$$i\partial_t u + \gamma \Delta u \pm |u|^p u = 0, \quad u(0,x) = u_0(x),$$

then in the limit as $\gamma \to 0$, we find an ODE with explicit solution

$$u(t) = e^{\pm it|u_0|^p} u_0.$$

This solution moves to high frequencies in a predictable manner and gives an approximate solution to NLS. Under re-scaling, this frequency cascade can be made to occur arbitrarily quickly for small initial data.

Energy-supercritical case

We fix our attention to the 'defocusing' $\langle\gamma\rangle = -1$ case.

(GT) has a conserved energy, akin to the monomial NLS,

$$E(u) = \underbrace{\tfrac{1}{2}\int |\nabla u|^2}_{\text{kinetic}} + \underbrace{\tfrac{1}{p+2}\iint_0^1 |e^{i\sigma\Delta}u|^{p+2} d\sigma dx}_{\text{potential}} .$$

Rough idea: Construct initial data with high potential energy and low kinetic energy. Potential energy should convert to kinetic, causing a growth in H^s provided $s \geq 1$. To make this formal, we look for a *energy equipartition* phenomenon.

Virial identity

Proposition (Virial identity [CHL25, K.])

Let u be the maximal solution of (GT) for $p \geq \frac{8}{d}$, with initial data $u_0 \in \mathcal{S}(\mathbb{R}^d)$. Define the variance of u as

$$v(t) = \int_{\mathbb{R}^d} |x|^2 |u(t,x)|^2 dx.$$

In the defocusing case, $\langle \gamma \rangle = -1$, we find that for $-\frac{1}{2} \leq t \leq 0$,

$$v(t) \leq v(0) - t\dot{v}_1(0) + C(p,d)E(u)t^2 + \text{error terms} \tag{2}$$

where $C(p,d) > 0$ for $p \geq \frac{8}{d}$ and $\dot{v}_1$ is given by

$$\dot{v}_1(t) = \int \overline{u}(x \cdot \nabla u) dx \geq \dot{v}_1(0) - 16E(u)t.$$

Norm Inflation Proof

Proof. To show energy equipartition and norm inflation in the energy-supercritical case, we work directly with skinny Gaussians. We choose our initial data:

$$u_0(x) = Ae^{-|x|^2/4\sigma^2}.$$

Then

$$\|e^{i\sigma\Delta}u_0\|^{p+2}_{L^{p+2}_{\sigma,x}([0,1])} \sim A^{p+2}\sigma^{d+2} \quad \text{and} \quad \|u_0\|^2_{H^1} \sim A^2\sigma^{d-2}.$$

Provided $s < s_i$, we can then choose $\sigma \ll 1$ and $A \gg 1$ with

$$A^p\sigma^4 \gg 1 \quad \text{and} \quad \|u_0\|_{H^s} \sim A^2\sigma^{d-2s} \ll 1.$$

This implies that potential energy greatly outweighs the kinetic energy.

This seems to imply that $s < s_i$ is necessary, at least for $s \geq 1$.

Norm Inflation Proof (continued)

Proof cont. As u_0 is real-valued, we find that $\dot{v}_1(0) = 0$. Applying the virial identity for $\dot{v}_1$, we then find that

$$-16E(u)t \leq 4\,\mathrm{Im}\int \overline{u}(x \cdot \nabla u(t))dx \lesssim \sqrt{v(t)}\|u(t)\|_{\dot{H}^1}.$$

With the virial identity, we then find that for $-\frac{1}{2} \leq t \leq 0$.

$$\|u(t)\|_{\dot{H}^1}^2 \gtrsim \frac{E(u)^2 t^2}{v(0) + E(u)t^2}. \tag{3}$$

From this identity, we see that at time $T^2 = v(0)/E(u)$ for $T < 0$,

$$\|u(T)\|_{\dot{H}^1}^2 \gtrsim E(u) \sim \|e^{i\sigma\Delta}u_0\|_{L^{p+2}_{\sigma,x}([0,1])}^{p+2}.$$

Finally, we calculate that $T^2 \sim A^{-p} \ll 1$ and hence norm inflation occurs. $\square$

Local Well-posedness

Power Series Expansion

In the standard way, a solution to (GT) satisfies the Duhamel formula

$$u(t) = e^{it\langle\gamma\rangle\Delta}u_0 + i\int_0^t\int_0^1 e^{i\langle\gamma\rangle(t-s)\Delta - i\sigma\Delta}\Big[\big|e^{i\sigma\Delta}u(s)\big|^p \cdot e^{i\sigma\Delta}u(s)\Big]d\sigma ds.$$

We decompose this into a linear component and nonlinear correction as

$$Lf = e^{it\langle\gamma\rangle\Delta}f,$$

$$N_p(f_0, \cdots, f_p) = i\int_0^t\int_0^1 e^{i\langle\gamma\rangle(t-s)\Delta - i\sigma\Delta}\Big[e^{i\sigma\Delta}f_0(s)\cdot \ldots \cdot e^{i\sigma\Delta}f_p(s)\Big]d\sigma ds.$$

With this notation, a solution u of (GT) with initial data u_0 satisfies

$$u = Lu_0 + N_p(u, \cdots, u).$$

Ouroborically substituting this formula into itself, we find the formal expansion

$$u = Lu_0 + N_p(Lu_0, \cdots, Lu_0) + N_p\big[Lu_0, \cdots, Lu_0, N_p(Lu_0, \cdots, Lu_0)\big] + \cdots .$$

Power Series Expansion (continued)

Grouping terms of equal order, we recursively define

$$\Xi_0(u_0) = L(u_0),$$
$$\Xi_j(u_0) = \sum_{\substack{j_0,\dots,j_p \geq 0 \\ j_0+\cdots+j_p=j-1}} N_p(\Xi_{j_0}(u_0), \cdots, \Xi_{j_p}(u_0)).$$

This implies that u has the following (formal) power series expansion

$$u = \sum_{j\geq 0} \Xi_j(u_0). \tag{4}$$

Proposition (Quantitative Well-posedness [K.])

Let $D = \dot{H}^{\max(s_m,0)}$ denote the space of initial data, and $S = C_t\dot{H}^{\max(s_m,0)}((-T,T)\times\mathbb{R}^d)$ the space of solutions. The operators $N_p : S^{p+1} \to S$ and $L : D \to S$ are bounded in the following sense:

$$\|Lg\|_S = \|g\|_D$$
$$\|N_p(f_0,\dots,f_p)\|_S \leq C_p T\|f_0\|_S \dots \|f_p\|_S.$$

Local Well-posedness Statement

Theorem (Local well-posedness, [K., KM26])

For $u_0 \in \dot{H}^{\max(s_m,0)}$ *and* $T \sim \|u_0\|_{\dot{H}^{\max(s_m,0)}}^{-p}$, *there exists a unique solution to* (GT)*:*

$$u \in C_t \dot{H}^{\max(s_m,0)}((-T,T)\times\mathbb{R}^d) \quad \text{with} \quad u(0,x) = u_0(x).$$

Moreover, the data-to-solution map is real analytic on a neighborhood of $u_0 = 0$*: for* $\|u_0\|_{\dot{H}^{\max(s_m,0)}} \leq R$ *and* $T \sim R^{-p}$ *the data-to-solution map*

$$u_0 \in B_R\big(\dot{H}^{\max(s_m,0)}\big) \quad \longmapsto \quad u \in C_t \dot{H}_x^{\max(s_m,0)}\big((-T,T)\times\mathbb{R}^d\big)$$

satisfies the power series expansion (4)*.*

In addition, there exists $\delta = \delta(p,d)$ *such that for all* $\|u_0\|_{\dot{H}^{\max(s_m,0)}} < \delta$*, the associated solution* u *may be extended to a global solution in* $C_t \dot{H}_x^{\max(s_m,0)}(\mathbb{R}\times\mathbb{R}^d)$ *which scatters.*

Failure of Analyticity

Failure of Analyticity

We consider the $(p+1)^{st}$ directional derivative of the data-to-solution map [Bou97, §6]: For $\phi \in \mathcal{S}(\mathbb{R}^d)$, consider the initial value problem (GT) with initial data

$$u_0 = \delta\phi \quad \text{for} \quad \delta > 0.$$

Let $u = u(\delta, \phi)$ be the corresponding local solution. Then,

$$\frac{\partial^{p+1} u}{\partial \delta^{p+1}}\bigg|_{\delta=0} = i \int_0^t \int_0^1 e^{i\langle\gamma\rangle(t-s)\Delta - i\sigma\Delta}\Big[\big|e^{i(s+\sigma)\Delta}\phi\big|^p \cdot e^{i(s+\sigma)\Delta}\phi\Big] d\sigma ds = \Xi_1(\phi).$$

To show that the data-to-solution map $u_0 \mapsto u$ fails to be C^{p+1}, it then suffices to show that Ξ_1 fails to be bounded $H^s \to L_t^\infty H_x^s((-T, T))$ for all $T > 0$. We fix $\langle\gamma\rangle = 1$ for simplicity.

Failure of Analyticity (statement)

Theorem (Failure of Analyticity [K.])

For any $T > 0$ and $s < s_m$, the $(p+1)^{st}$ derivative of the data-to-solution map, Ξ_1, fails to be bounded $H^s \to L^\infty_t H^s_x((-T,T))$ for Schwartz data.

Proof. Using the dual formulation of H^s and testing $\Xi_1(\phi)$ against $e^{it\Delta}\phi$, we may estimate $\Xi_1(\phi)$ from below as

$$\left\|\Xi_1(\phi)\right\|_{L^\infty_t H^s_x([0,T))} \gtrsim \frac{1}{\|\phi\|_{H^{-s}_x}} \int_0^T \int_0^1 \int_{\mathbb{R}^d} \left|e^{i(s+\sigma)\Delta}\phi\right|^{p+2} dx\, d\sigma\, ds.$$

Failure of Analyticity (proof)

Proof cont. We make the change of variables $\tau = \sigma + s$ and $\rho = s - \sigma$. Assuming that $T < 1$, we then find

$$\left\|\Xi_1(\phi)\right\|_{L_t^\infty H_x^s([0,T))} \gtrsim \frac{1}{\|\phi\|_{H_x^{-s}}} \int_{\frac{T}{2}}^{\frac{3T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{\mathbb{R}^d} \left|e^{i\tau\Delta}\phi\right|^{p+2} dx d\rho d\tau$$
$$= \frac{T}{\|\phi\|_{H_x^{-s}}} \int_{\frac{T}{2}}^{\frac{3T}{2}} \int_{\mathbb{R}^d} \left|e^{i\tau\Delta}\phi(x)\right|^{p+2} dx d\tau.$$

Shifting our initial data $e^{iT\Delta}\phi \mapsto \phi$, we then find

$$\left\|\Xi_1(\phi)\right\|_{L_t^\infty H_x^s([0,T))} \gtrsim \frac{T}{\|\phi\|_{H_x^{-s}}} \int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{\mathbb{R}^d} \left|e^{i\tau\Delta}\phi(x)\right|^{p+2} dx d\tau.$$

Suppose that $\Xi_1 : H^s \to L_t^\infty H_x^s((-T,T))$ is bounded. Then

$$\frac{T}{\|\phi\|_{H_x^{-s}}} \int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{\mathbb{R}^d} \left|e^{i\tau\Delta}\phi(x)\right|^{p+2} dx d\tau \lesssim \|\phi\|_{H^s}^{p+1}.$$

Failure of Analyticity (conclusion)

Proof cont. Given

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{\mathbb{R}^d} \left| e^{i\tau\Delta}\phi(x) \right|^{p+2} dx d\tau \lesssim_T \|\phi\|_{H^s}^{p+1} \|\phi\|_{H^{-s}},$$

we now choose ϕ Schwartz such that

$$\widehat{\phi}(\xi) = \begin{cases} 1 & |\xi| \in [N/2, 2N] \\ 0 & |\xi| \notin [N/4, 4N] \end{cases}$$

Standard scaling arguments then imply that

$$N^{d(p+2)-d-2} \lesssim_T N^{(p+1)(s+\frac{d}{2})+\frac{d}{2}-s}$$
$$N^{p(\frac{d}{2}-\frac{2}{p}-s)} \lesssim_T 1.$$

Sending $N \to \infty$ implies that $s \geq s_m = \frac{d}{2} - \frac{2}{p}$, as desired. Hence, if $\Xi_1 : H^s \to L_t^\infty H_x^s((-T, T) \times \mathbb{R}^d)$ is bounded as a $(p+1)$-linear map, then $s \geq s_m$. □

Norm Inflation

Norm Inflation

Definition (Norm inflation in H^s)

We say that *norm inflation* occurs in H^s if, for all $\varepsilon > 0$, there exists a solution $u(t)$ to (GT) with smooth initial data u_0 that satisfies

$$\|u_0\|_{\dot{H}^s} < \varepsilon \quad \text{with} \quad \|u(T)\|_{\dot{H}^s} > \varepsilon^{-1}, \quad \text{for some} \quad |T| < \varepsilon.$$

This indicates that the data-to-solution map fails to be continuous at $u_0 = 0$ and hence the equation is ill-posed.

Theorem (Mass-subcritical norm inflation [K.])

For the one-dimensional cubic (GT), *norm inflation occurs in* H^s *for* $s < s_i = -\frac{3}{2}$.

Theorem (Energy-supercritical norm inflation [K.])

Suppose that $s_i > 1$. *Then norm inflation occurs in* H^s *for* (GT) *for all* $1 \leq s < s_i$.

Mass-subcritical Case

When $s < \min(s_i, 0)$, norm inflation is driven by high-to-low frequency cascades. We consider $\langle\gamma\rangle = -1$, $d = 1$, and $p = 2$ for the sake of exposition, though we expect these results to generalize naturally.

Consider initial data of the form:

$$\widehat{\phi} = R\big(\chi_{[N,N+A]} + \chi_{[2N,2N+A]}\big), \tag{5}$$

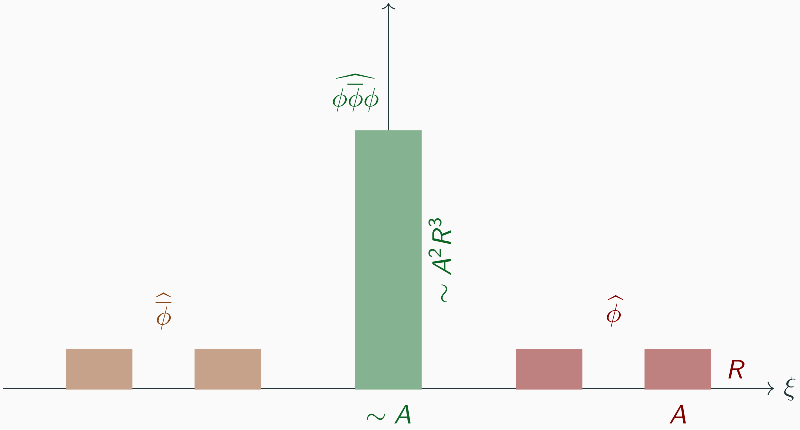
with $A = N^{1-}$ and $R = N^{1+}$. Recall

$$\Xi_1(\phi) = i \int_0^t \int_0^1 e^{i\langle\gamma\rangle(t-s)\Delta - i\sigma\Delta}\Big[\big|e^{i(s+\sigma)\Delta}\phi\big|^2 \cdot e^{i(s+\sigma)\Delta}\phi\Big]\, d\sigma\, ds.$$

The high frequencies of ϕ interact in Ξ_1 to create mass near $\xi = 0$, causing a growth in H^s for $s < 0$. Choosing A, N, T carefully allows for this to occur arbitrarily quickly.

This method is due to [IO15] and [Kis19, Oh17].

Mass-subcritical Case (continued)



Mass-subcritical Case (continued)

When u_0 interacts with itself, a *high-low frequency cascade* occurs, causing growth in the H^s norm for $s < 0$:

Lemma

Fix $s < -\frac{1}{2}$, $R > 0$, $1 \ll A \ll N$, and ϕ as in (5). Then

$$\left\| [\Xi_1(\phi)](t) \right\|_{H^s} \gtrsim t N^{-2} A^2 R^3,$$

uniformly in A, N, R and $0 < t \ll N^{-2}$.

To ensure that higher-order terms do not negate this growth, we bound:

Lemma

Fix $s < -\frac{1}{2}$, $R > 0$, $1 \ll A \ll N$, and ϕ as in (5). Then for all j,

$$\| [\Xi_j(\phi)](t) \|_{H^s} \leq C^j t^j R^{2j+1} (\log A)^{2j},$$

for some universal constant C, uniform in R, N, A and $0 < t \ll N^{-2}$.

Constraints on A, R, N, T

A number of relationships are necessary between A, N, R, T:

(i) *Local well-posedness:*

$$T\|\phi\|_{L^2}^2 \ll 1 \quad \Longleftrightarrow \quad TR^2 A \ll 1$$

(ii) *Small initial data:*

$$\|\phi\|_{H^s} \sim N^s R A^{\frac{1}{2}} \ll 1.$$

(iii) *Norm growth / high-to-low frequency cascade in Ξ_1:*

$$TN^{-2}A^2R^3 \gg 1.$$

(iv) *Convergence of higher order terms:*

$$TR^2(\log A)^2 \ll 1 \quad \text{and} \quad TR^2N^2A^{-2}(\log A)^4 \ll 1.$$

(v) *Separation:*

$$1 \ll A \ll N.$$

(vi) *Instantaneity:*

$$T \ll 1.$$

Mass-subcritical Conclusion

To achieve these requirements, we fix some $N \gg 1$ and then choose

$$R = N^{1+3\delta}, \quad A = N^{1-\delta}, \quad \text{and} \quad T = N^{-3-6\delta},$$

where $\delta > 0$ is chosen sufficiently small so that $s + \frac{3}{2} + \frac{5}{2}\delta < 0$.

We note that this differs from the parameters one would choose for NLS. The time T is significant smaller and R is significantly larger; see [Oh17].

Thank you!

Backup Slides

[Bou97] J. Bourgain.
Periodic Korteweg de Vries equation with measures as initial data.
Selecta Mathematica. New Series, 3(2):115–159, 1997.

[CCT] Michael Christ, James Colliander, and Terence Tao.
Ill-posedness for nonlinear Schrodinger and wave equations.
Preprint `arXiv:0311048`.

[CCT03] Michael Christ, James Colliander, and Terrence Tao.
Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations.
American Journal of Mathematics, 125(6):1235–1293, 2003.

[CHL25] Mi-Ran Choi, Younghun Hong, and Young-Ran Lee.
Global existence versus finite time blowup dichotomy for the dispersion managed NLS.
Nonlinear Analysis, 251:113696, 2025.

[CL22] Mi-Ran Choi and Young-Ran Lee.
Averaging of dispersion managed nonlinear Schrödinger equations.
Nonlinearity, 35(4):2121, Mar 2022.

[IO15] Tsukasa Iwabuchi and Takayoshi Ogawa.
Ill-posedness for the nonlinear Schrödinger equation with quadratic non-linearity in low dimensions.
Trans. Amer. Math. Soc., 367(4):2613–2630, 2015.

[K.] Matthew K.
On ill-posedness for the Gabitov–Turitsyn equation.
Preprint `arXiv:2510.11887`.

[Kis19] Nobu Kishimoto.
A remark on norm inflation for nonlinear Schrödinger equations.
Communications on Pure and Applied Analysis, 18(3):1375–1402, 2019.

[KM26] Jumpei Kawakami and Jason Murphy.
Small and large data scattering for the dispersion-managed NLS.
Discrete and Continuous Dynamical Systems, 47(0):256–285, 2026.

[KPV01] Carlos E. Kenig, Gustavo Ponce, and Luis Vega.
On the ill-posedness of some canonical dispersive equations.
Duke Mathematical Journal, 106(3):617–633, 2001.

[Mur25] Jason Murphy.
Large scale limit for a dispersion-managed NLS.
Journal of Differential Equations, 448:113830, 2025.

[NDFK97] J.H.B. Nijhof, N.J. Doran, W. Forysiak, and F.M. Knox.
Stable soliton-like propagation in dispersion managed systems with net anomalous, zero and normal dispersion.
Electronics Letters, 33:1726–1727, 1997.

[Oh17] Tadahiro Oh.
A remark on norm inflation with general initial data for the cubic nonlinear Schrödinger equations in negative Sobolev spaces.
Funkcialaj Ekvacioj. Serio Internacia, 60(2):259–277, 2017.