UCLA Basic: Analysis Notes

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Based on Notes/Lectures from Sylvester

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Contents

Continuity

With axiom of choice, the following two definitions of continuity are equivalent.

Epsilon Delta Definition of Continuity
Let (X, d) and (Y, d) be metric spaces. Then $f : X \to Y$ is continuous at x if for all $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$d(x, y) < \delta \quad\quad \implies \quad\quad d(f(x), f(y)) < \varepsilon$$

Sequential Definition of Continuity
Let (X, d) and (Y, d) be metric spaces. Then $f : X \to Y$ is continuous at x if for all sequences (x_n) such that $x_n \to x$, $f(x_n) \to f(x)$.

By defining

Oscillation
We define the oscillation of a function f over an interval I as

$$\operatorname{osc}(f, I) = \sup_{x,y\in I} |f(x) - f(y)|$$

Which leads to the following theorem/definition of continuity

Oscillation Definition of Continuity
$f : \mathbb{R} \to \mathbb{R}$ is continuous at x if and only if for all $\varepsilon > 0$ there exists some open interval I such that $x \in I$ and $\operatorname{osc}(f, I) < \varepsilon$.

Theorem. A function f is continuous if and only if $f^{-1}(U)$ is open for all open U.

Theorem. (Intermediate Value Theorem) : For a continuous functioin f with $f(x) = a$ and $f(y) = b$. For all c between a and b there exists some $z \in [x, y]$ such that $f(z) = c$.

Countability

Countability
A set A is countable if there exists an injective map $f : A \to \mathbb{N}$.
(NOTE: this definition includes finite as well)

Theorem. The countable union of countable sets is countable.

Special Classes of Functions

Increasing Functions

Increasing Function
A function $f : I \to \mathbb{R}$ is increasing if for all $x \leq y \in I$, $f(x) \leq f(y)$.
Strictly increasing : $x < y$ implies $f(x) < f(y)$

Theorem. The set of discontinuities of an increasing function is countable.

Proof. Let f be an increasing function. Let D be the set of discontinuities of f. By definition, for all $x \in D$, $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$. Because f is increasing, $\lim_{x\to a^-} f(x) \leq \lim_{x\to a^+} f(x)$. Therefore, $\lim_{x\to a^-} f(x) < \lim_{x\to a^+} f(x)$.
By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists some $q_a \in (\lim_{x\to a^-} f(x), \lim_{x\to a^+} f(x))$. Consider the function $g : D \to \mathbb{Q}$ where $a \mapsto q_a$. Let there exist some $b \in D$ such that $a \neq b$. If $a < b$ then because f is increasing,

$$q_a < \lim_{x\to a^+} \leq \lim_{x\to b^-} < q_b$$

Similarly, if $b < a$ then

$$q_a > \lim_{x\to a^-} \geq \lim_{x\to b^+} > q_b$$

In either case, $q_a \neq q_b$ for $b \neq a$. Therefore $g : D \to \mathbb{Q}$ is injective. Becaue $\mathbb{Q}$ is countable, this implies that D is countable.

Convex Function

Convex Function
A function $f : I \to \mathbb{R}$ is convex if for all $x, y \in I$ and for all $t \in [0, 1]$,

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y)$$

Intuition : The graph of a convex function is below the line between two points on the graph.

Alternate Definition
A function $f : I \to \mathbb{R}$ is convex if for all $x, y \in I$ and for all $t \in [0, 1]$,

$$\frac{f((1-t)x + ty) - f(x)}{t(y-x)} \leq\leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(y) - f((1-t)x + ty)}{(1-t)t(y-x)}$$

Intuition : The slope of a convex function is increasing.

Theorem. Let f be a C^1 function. Then f' is increasing if and only if f is convex.
Corollary. Let f be a C^2 function. Then f is convex if and only if $f'' \geq 0$.

Proof. Follows immediately from the alternative definition and the mean value theorem.

Theorem. If $f : (a, b) \to \mathbb{R}$ is convex, then it is continuous.
Note : f must take on real values only. If f is infinite, then the proof fails.

Proof. Let there exist some $x \in (a, b)$. Let there exist $u \in (a, x)$ and $v \in (x, b)$. Pick some $z \in (a, b)$ such that $z \neq x$. If $x < z$ then by the alternate definition of convexity,

$$\frac{f(z) - f(x)}{z - x} \leq \frac{f(v) - f(x)}{v - x}$$

If $x > z$ then by the alternate definition of convexity,

$$\frac{f(x) - f(z)}{x - z} \leq \frac{f(x) - f(u)}{x - u}$$

Define C as

$$C = \max\left(\left|\frac{f(x) - f(u)}{x - u}\right|, \left|\frac{f(v) - f(x)}{v - x}\right|\right)$$

Then by the previous inequalities,

$$\left|\frac{f(z) - f(x)}{z - x}\right| \leq C$$

$$|f(z) - f(x)| \leq C|z - x|$$

Continuity at x follows immediately from this inequality with $\delta = \varepsilon/(C + 1)$.

Theorem. A convex function $f : (a, b) \to \mathbb{R}$ is differentiable at all but countably many points.

Proof.

Right/Left Hand Derivatives
If they exist, the right and left hand derivatives of f are given by

$$\partial_r f(x) = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$$
$$\partial_\ell f(x) = \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h}$$

Pick some $x \in (a,b)$ and let there exist $u \in (a,x)$ and $v \in (x,b)$. By the alternate definition of convexity, we know that

$$\frac{f(x+h)-f(x)}{h}$$

is monotonically increasing in h and x. Additionally, we know that

$$\frac{f(x)-f(u)}{x-u} \leq \frac{f(x+h)-f(x)}{h} \leq \frac{f(v)-f(x)}{v-x}$$

Therefore, $\frac{f(x+h)-f(x)}{h}$ is monotonic and bounded in h, so $\partial_r f(x)$ and $\partial_\ell f(x)$ exist.
As shown, $\frac{f(x+h)-f(x)}{h}$ is monotonically increasing in x. Therefore, $\partial_r f(x)$ and $\partial_\ell f(x)$ are monotonically increasing. This impliees that $\partial_r f(x)$ is continuous except at countably many points. Let $\partial_r f$ and $\partial_\ell f$ be continuous at x and let there exist $\varepsilon > 0$. By definition of continuity, there exists some δ such that

$$|\partial_r f(x+\delta) - \partial_r f(x)| \leq \varepsilon$$
$$|\partial_r f(x) - \partial_r f(x-\delta)| \leq \varepsilon$$

By definition of convexity, we know that $\partial_\ell f(z) \leq \partial_r f(z)$ for all z. Therefore, by the monotonicity of $\partial_\ell f$, for all $z \in B_\delta(x)$,

$$\partial_r f(x-\delta) \leq \partial_\ell f(z) \leq \partial_r f(x+\delta)$$

Which implies that for all $z \in B_\delta(x)$,

$$|\partial_r f(z) - \partial_r f(x)| \leq \varepsilon$$

As this holds for all ε and $\partial_\ell f$ and $\partial_r f$ are continuos at x, this implies that $\partial_\ell f(x) = \partial_r f(x) = f'(x)$. As this holds for all but countably many x, this implies that f is differentiable at all but countably many points.

Riemann Integration

Partition
A finite set $P \subset [a, b]$ is a partition of $[a, b]$ if $P = \{x_1, \ldots, x_n\}$ with $x_1 = a$, $x_n = b$ and $x_i < x_{i+1}$.

Upper and Lower Sum
Given any function $f : [a, b] \to \mathbb{R}$, we define

$$U(P, f) = \sum_{i=1}^{n-1} \sup_{x \in [x_i, x_{i+1}]} f(x)(x_{i+1} - x_i)$$
$$L(P, f) = \sum_{i=1}^{n-1} \inf_{x \in [x_i, x_{i+1}]} f(x)(x_{i+1} - x_i)$$

Refinement
A partition P is a refinement of a partition P' if $P \subset P'$.

Theorem. Let P be a refinement of P'. Then

$$U(f, P) \geq U(f, P')$$
$$L(f, P) \leq L(f, P')$$

Proof. Follows immediately from supremum and infimum.

Riemann Integrability
A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if $\sup_P L(f, P) \inf_P U(f, P)$.
Note : This immediately implies that f must be bounded.

Theorem. For any partitions P, Q of $[a, b]$ and function $f : [a, b] \to \mathbb{R}$,

$$L(f, P) \leq U(f, Q)$$

Corollary :

$$\underline{\int} f = \sup_P L(f, P) \leq \inf_P U(f, P) = \overline{\int} f$$

Proof. Let P, Q be partitions of $[a, b]$. Then by definition, P and Q are refinements of $P \cup Q$. Therefore

$$\begin{aligned} L(f, P) &\leq L(f, Q \cup P) \\ &\leq U(f, Q \cup P) \\ &\leq U(f, Q) \end{aligned}$$

The corollary follows immediately by applying an infimum to the left and then a supremum to the right.

Alternate Riemann Integrability Definition
A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if for all $\varepsilon > 0$ there exists some partition P such that

$$U(f, P) - L(f, P) < \varepsilon$$

Note : The equivalence of this definition to the original can be found with a ε proof utilizing common refinements.

Continuity and Integrability

Theorem. (Riemann-Lebesgue Condition) : A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if the set of discontinuities has measure zero.

The proof of this is too long for the basic exam. No need to know it.

Theorem. (Fundamental Theorem of Calculus) : Suppose $f \in C^1[a, b]$. Then

$$f(b) - f(a) = \int_a^b f'(x)\, dx$$

Theorem. (Mean Value Theorem for Integrals) : Suppose $f : [a, b] \to \mathbb{R}$ is continuous and $g : [a, b] \to [0, \infty)$ is Riemann integrable. Then there exists $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)\, dx = f(c) \int_a^b g(x)\, dx$$

Theorem. (Integration by Parts) : Suppose there exists $f, g \in C^1[a, b]$. Then

$$\int_a^b f(x)g'(x)\, dx = f(x)g(x)\Big|_{x=a}^{b} - \int_a^b f'(x)g(x)\, dx$$

Using the fundamental theorem of calculus, we can prove a weak version of the mean value theorem

Theorem. (Weak Mean Value Theorem) : Let there exist $f \in C^1[a,b]$. Then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Utilizing the extreme value theorem, it can be shown that the mean value theorem holds for any differentiable function.

Utilizing integration by parts, we also can arrive at a function approximation, Taylor's theorem.

Theorem. (Taylor's Theorem) : Let there exist $a < b$, $f : [a,b] \to \mathbb{R}$, and $n \in \mathbb{N}$ such that $f^{(n-1)}$ is continuous on $[a,b]$ and differentiable on (a,b). Then for all $x_0 \in [a,b]$ there exists some ξ between x_0 and x such that

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n)}(\xi)}{n!}(x - x_0)^n$$

Metric Space

Metric
Let X be a space. A function $d : X \times X \to [0, \infty)$ is a metric if

(1) *Positive Definiteness :* For all $x, y \in X$, $d(x, y) \geq 0$ with equality if and only if $x = y$.

(2) *Symmetry :* For all $x, y \in X$, $d(x, y) = d(y, x)$.

(3) *Triangle :* For all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

Metric Space
A space X along with a metric $d : X \times X \to [0, \infty)$ is a metric spcae.

Open
A set U is open if for all $x \in U$ there exixts $\delta > 0$ such that $B(x, \delta) \subset U$.

Convergence
A sequence $(x_n) \subset X$ converges to x if $\lim_{n\to\infty} d(x, x_n) = 0$. This is denoted by $x_n \to x$.

Closed
A set $E \subset X$ is closed if for all convergent sequences $x_n \to x$ where $(x_n) \subset E$ then $x \in E$.

Ultrametric Space

Ultrametric Space
A metric space whose metric such that for all $y \in X$,

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}$$

Normed Vector Space

Norm
Let V be a vector space. A norm $\| * \| : V \to [0, \infty)$ is a function satisfying

(1) *Positive Definiteness :* For all $x \in V$, $\|x\| = 0$ if and only if $x = 0$.

(2) *Homogeneity :* For all $x \in v$ and scalar c, $\|cx\| = |c|\|x\|$.

(3) *Triangle Inequality :* For all $x, y \in V$, $\|x + y\| \leq \|x\| + \|y\|$.

Normed Vector Space
A vector space equipped with a norm.

Theorem. Any normed vector space can be made into a metric space with $d(x, y) = \|x - y\|$.

Inner Product Space

Real Inner Product Space
A real vector space V equipped with a function $\langle *, * \rangle : V \times V \to \mathbb{R}$ such that $\langle *, * \rangle$ satisfies

(1) *Symmetry :* $\langle x, y \rangle = \langle y, x \rangle$

(2) *Bi-Linearity :* $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$

(3) *Positive Definiteness :* $\langle x, x \rangle \geq 0$ with equality if and only if $x = 0$

Theorem. Any real inner product space can be made into a normed space with $\|x\| = \sqrt{\langle x, x \rangle}$.

Completeness

Cauchy
A sequence $x_n \in X$ is Cauchy if for all $\varepsilon > 0$ there exists N such that for all $n, m \geq N$, $d(x_n, x_m) \leq \varepsilon$.

Complete
A space X is complete if every Cauchy sequence has a limit.

Banach Space

Banach Space
A Banach space is a complete normed space.

Hilbert Space

Hilbert Space
A Hilbert space is a complete normed inner product space.

Compactness

In a metric space, the following three definitions are equivalent for compactness.

Compactness
A space X is compact if every open covers admits a finite subcover.

Sequentially Compact
For all sequences $x_n \in X$, there exists a convergent subsequence.

Finite Intersection Property
Let there exist $F_i \subset X$ with $i \in I$ such that F_i are closed and such that $\cap_{i \in G \subset I} F_i \neq \emptyset$. Then $\cap_{i \in I} F_i \neq \emptyset$.

Theorem. A set X is compact if and only if X is totally bounded and X is complete.

Totally Bounded
A set X is totally bounded if for all $\varepsilon > 0$ there exists $s_1, \ldots, x_n$ such that $X \subset \cup_{i=1}^{n} B(x_i, \varepsilon)$.

Separability

Dense
A set $D \subset X$ is dense if for all $x \in X$ and for all $\varepsilon > 0$, we have $B(x, \varepsilon) \cap D \neq \emptyset$.

Basis
A collection of open sets $\mathcal{B}$ in X is a basis for the topology if for all open sets $U \subset X$, for all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \in \mathcal{B}$.

Separability
A set X is separable if there exists a countable dense subset $D \subset X$ or if there exists a countable basis for the topology.

Epsilon Nets

One very useful tool in determining compactness and separability is the epsilon-net.

ε-Net
An ε-net is a subset $S \subset X$ such that S is a maximal ε-separated set. Maximal here implies that if S cannot have points added to it and remain ε-separated.

Theorem. A complete space X is compact if it has a finite ε net for every $\varepsilon > 0$.

Proof. Follows immediately by restating totally bounded.

Theorem.

(1) X is separable if every ε separated set is countable.

(2) X is non-separable if there exists $\varepsilon > 0$ and an ε-separated set S which is uncountable.

Proof.

1. Let $S_{1/n}$ be a countable $1/n$-net. Define $S = \cup_n S_{1/n}$. Then S is dense.
2. Let there exist an uncountable ε-net S_ε. Assume that X is separable, with countable dense set D. Then for all $s \in S_\varepsilon$, there exists $d_s \in D$ such that $d(d_s, s) < \varepsilon/2$. Because S_ε is ε-separated, this implies that $d_s \neq d_r$ for all $r \neq s \in S_\varepsilon$. However, because S_ε is uncountable, this implies that D is uncountable, which is a contradiction. Therefore, X is not separable.

Embedding of Banach Space

Theorem. (Kuratowski Embedding) : Every metric space X embeds to a Banach Space isometrically.

Proof. For some $x_0 \in X$, the function

$$\phi : X \to C_b(X) \qquad x \mapsto f_x(z) = d(x, z) - d(x_0, z)$$

Embeds X into $C_n(X)$.

Connectivity

Relatively Open and Relatively Closed
Let (X, d) be a metric space with $E \subset Y \subset X$. We say that E is relatively closed or relatively open with respect to Y if it is closed or open in the metric space $(Y, d|_{Y \times Y})$

Separated
Let (X, d) be a metric space. Two subsets $A, B \subset X$ are separated if $A \cap \overline{B} = \overline{A} \cap B = \emptyset$.

Disconnected
There are many equivalent definitions of disconnected. Let (X, d) be a metric space. X is said to be disconnected if

(1) there exists disjoint non-empty open sets $V, W \subset X$ such that $V \cup W = X$.

(2) there exists disjoint non-empty closed sets $V, W \subset X$ such that $V \cup W = X$.

(3) there exists non-empty separated sets $V, W \subset X$ such that $V \cup W = X$.

Note that the subsets are open/closed/separated RELATIVE TO X. Though that doesn't actually matter for separated I don't think.

Connected
Not disconnected. That's the only way to characterize it and prove it.

Theorem. A metric space (X, d) is disconnected if and only if it contains a non-empty proper subset (not X or $\emptyset$) which is both open and closed.

Proof. Follows immediately from definition. Separated definition appears to be the most straightforward.

Connected Set
Let (X, d) be a metric space with a subset Y. We say that Y is connected if and only if the metric space $(Y, d|_{Y\times Y})$ is connected.

Path Connected
Let (X, d) be a metric space with a subset E. We say that E is path-connected if and only if for all $x, y \in E$ there exists a continuous map $\gamma : [0, 1] \to E$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Other Types of Sets

Perfect Set
Let (X, d) be a metric space with subset E. Then E is perfect if E is closed and every point of E is a limit point.

Hausdorff Space
A topological space X is Hausdorff if for all $x \neq y \in X$ there exists disjoint open sets $U, V \subset X$ with $x \in U$ and $y \in V$.

Continuum
A continuum is a compact connected Hausdorff space.

Existence for ODE's

Consider the ODE

$$y'(t) = f(y(t), t)$$
$$y(t_0) = y_0$$

If f is continuos, then we can rewrite this using the fundamental theorem of calculus as

$$y(t) = y_0 + \int_{t_0}^{t} f(y(s), s)\, ds$$

But how do we know when this ODE has a unique solution? Well the following theorem can certainly be helpful.

Theorem. (Fixed Point Theorem) or **(Contraction Mapping Principle) :** Suppose that X is complete and $T : X \to X$ is a

λ-Contraction
A transformation such that for all $x, y \in X$,

$$d(T(x), T(y)) \leq \lambda d(x, y)$$

for some $\lambda \in (0, 1)$. Then there exists a unique fixed point x^* ($T(x^*) = x^*$).

Proof. Consider the sequence $x, T(x), T(T(x)), \ldots$. Completeness gives a limit and continuity of the metric implies that the limit is a fixed point. Continuity of the metric follows immediately from $|d(x, y) - d(x, z)| \leq d(y, z)$.

Baire Category Theorem

G_δ **Set**
A set U is a G_δ set if U is the countable intersection of open sets.

F_δ **Set**
A set U is a F_δ set if U is the countable union of closed sets.

Theorem. The set of discontinuities of a function $f : \mathbb{R} \to \mathbb{R}$ is F_δ.

Proof.

$$\begin{aligned}\{x|f \text{ is continuous at } x\} &= \{x|\text{for all } 1/n > 0 \text{ there exists open interval } I \text{ s.t. } \operatorname{osc}(f, I) < 1/n\} \\ &= \bigcap_{n=1}^{\infty} \underbrace{\bigcup_{\substack{\text{open } I \subset \mathbb{R} \\ \text{s.t. } \operatorname{osc}(f,I)<1/n}} I}_{\text{open}} \\ &= F_\delta\end{aligned}$$

Baire Category Theorem

Theorem. (Baire Category Theorem) : If X is a complete metric space, then

(1) If U_i are countably many open sets which are dense in X, then the G_δ set

$$\cap_i U_i$$

is non-empty and dense.

(2) If E_i are countably many closed sets so that $X \subset \cup_i E_i$ then at least one of the E_i has non-empty interior.

Uniform Convergence

Convergence

(1) f_n converges pointwise to f if for all $x \in [a,b]$, $\lim_{n\to\infty} f_n(x) f(x)$

(2) f_n converges uniformly to f if for all $\varepsilon > 0$ there exists N such that for all $n \geq N$ and for all $x \in [a,b]$, $|f_n(x) - f(x)| < \varepsilon$

Supremum Norm
We can treat the space of continuous function as a metric space by defining the supremum norm as

$$\|f\|_\infty = \sup_{t\in[a,b]} |f(t)|$$

Which induces a metric on $C([a,b])$ in the usual way as

$$d(f,g) = \|f - g\|_\infty \tag{1}$$

Often, the ∞ subscript is dropped.

Theorem. (Equivalence of Sup Norm and Uniform Convergence) : f_n converges uniformly to f if and only if

$$\lim_{n\to\infty} \|f_n - f\|_\infty = 0$$

Exchanging Limits

Convergence and Continuity

Theorem. (Uniform Convergence Preserves Continuity) : Let there exist continuous $f_n : [a,b] \to \mathbb{R}$ such that $f_n \rightrightarrows f$. Then f is continuous.

Proof. Bound things by $\varepsilon/3$ and then use the triangle inequality.

Convergence and Integration

Theorem. If there exists continuous $f_n : [a,b] \to \mathbb{R}$ such that $f_n \rightrightarrows f$, then

$$\lim_{n\to\infty} \int_a^b f_n(t)dt = \int_a^b f(t)dt \tag{2}$$

Note : $[a,b]$ can be extended to an unbounded set if for all $\varepsilon > 0$ there exists some M such that for all n, $\left(\int_M^\infty + \int_{-\infty}^M\right) |f_n(x)|dx < \varepsilon$.

Theorem. (Dominated Convergence) : $|f_n(x)| \leq g(x)$ for all x and $\int_0^\infty g(x)dx < \infty$.

Theorem. (Weierstrass M-Test) : Let there exist $f_n : X \to \mathbb{R}$ such that

$$\|f_n\|_\infty \leq M_n$$

and

$$\sum_{n=1}^\infty M_n < \infty$$

Then $\sum_n f_n(x)$ converges uniformly.

Proof. Follows from Cauchy convergence criterion.

Convergence and Derivative

Theorem. Let $K : \mathbb{R} \times [0,1] \to \mathbb{R}$ be continuous such that $K(*, t)$ is differentiable for all t. Such that $|\partial_x K(x,t)| \leq M$ and $\partial_x K$ is continuous for all t.
Let $f(x) = \int_0^1 K(x,t)dt$. Then f is C^1 with $f'(x) = \int_0^1 \partial_x K(x,t)dt$.

Fubini

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Suppose $\int_I \int_J |g(x,y)|dxdy < \infty$. Then

$$\int_I \int_J g(x,y)dxdy = \int_J \int_I g(x,y)dydx$$

Newton's Method

Theorem. Let there exist $f \in C^2(\mathbb{R})$ and suppose $f(x^*) = 0$ with $f'(x^*) \neq 0$. Then there exists δ such that if $x \in (x^* - \delta, x^* + \delta)$ then the Newton iteration

$$x_0 = x$$
$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

converges to x^* quadratically.

If a question asks about Euler's method, then consult the CCLE notes on desktop because idk.

Stone-Weierstrass

Theorem. (Stone-Weierstrass Generalized Theorem) : Suppose $A \subset C(X)$ is an algebra of continuous functions and X is compact. If for all $x \neq y \in X$ there exists f such that $f(x) \neq 0$ and g such that $g(x) \neq g(y)$, then A is dense in $C(X)$.
Then for all $f \in C(X)$ there exists $p_n \in A$ such that $p_n \rightrightarrows f$.

Algebra
A vector space over a field equipped with a bilinear form. Need to show closure under addition, multiplication, scalar multiplication.

Arzelà Ascoli

Equibounded
A family of functions $\mathcal{F}$ is equibounded if there exists some M that bounds all $f \in \mathcal{F}$.

Equicontinuos
A family of functions $\mathcal{F}$ is equicontinuous if for all $\varepsilon > 0$ there exists some $\delta > 0$ such that for all $f \in F$ and for all x, y satisfying $|x - y| < \delta$, $|f(x) - f(y)| < \varepsilon$.
Note that this is perhaps better understood as uniform equicontinuity.

Theorem. (Arzelà Ascoli Theorem) : Let there exist a family of functions $\mathcal{F} \subset C(X)$ where X is compact. $\mathcal{F}$ is compact if and only if

- $\mathcal{F}$ is closed
- $\mathcal{F}$ is equibounded
- $\mathcal{F}$ is equicontinuous

Corollary : A sequence in $C(X)$ is uniformly convergent if and only if it is equicontinuous and it converges pointwise to some limit.
Corollary : A sequence in $C(X)$ has a uniformly convergent subsequence if it is equibounded and equicontinuous.e
Corollary : A family of functions $\mathcal{F} \subset C(X)$ is pre-compact (not closed, but everything else) if and only if $\mathcal{F}$ is equibounded and eqeuicontinuous.

Proof. Pick some arbitrary sequence. We aim to show that there is a uniformly convergent subsequence.
Backwards : The backwards direction follows from the diagonal argument. Pick some countable dense set D and consider the sequence of functions on this set. The diagonal argument shows that there is a subsequence that converges on all points of D.
Using equicontinuity, it is easy to show that this limit function is equicontinuous on D with the rest of the family. Then extend said limit function by continuity to the other points continuously by taking $f(x) = \lim_{n\to\infty}(x_n)$.
Show that the subsequence converges uniformly by considering a finite (from compactness) δ-net inside D of the space. Pointwise convergence and equicontinuity extends the convergence to the remaining points. Maximum over all N_i and done.
f is continuous by uniform convergence. Bounded follows from convergence as well. Then the subsequence converges uniformly within $\mathcal{F}$
Forwards : If $\mathcal{F}$ is compact then it is bounded (equibounded) and closed. It remains to show equicontinuous.
Assume otherwise. Then for all $\varepsilon > 0$ and $1/n$ there exists f_n, x_n, and y_n such that $d(x_n, y_n) < 1/n$ but $|f_n(x_n) - f_n(y_n)| \geq \varepsilon$. By construction, the sequence is not equicontinuous and no subsequence is equicontinuous.
Then compactness shows uniform convergence of subsequence. Because continuous functions converge uniformly to continuous function implies equicontinuity, this is a contradiction. Therefore our assumption is wrong and the space is equicontinuous.

Sequences and Series

Sequences

Convergence of Sequence
A sequence (a_n) converges to a if for all $\varepsilon > 0$ there exists some N such that for all $n \geq N$, $|a_n - a| < \varepsilon$. This is denoted as $\lim_{n\to\infty} a_n = a$.

How to Prove Convergence :

(1) If (a_n) is monotone and bounded, then it converges.

(2) If a_n is Cauchy and we're working in a complete metric space.

Cauchy
For all $\varepsilon > 0$ there exists some N such that for all $n, m \geq N$, $|a_n - a_m| < \varepsilon$

(3) Know the limit and show that $|a_n - a| \leq \varepsilon$ or $\lim_{n\to\infty} |a_n - a| = 0$.

How to Find Limit :

(1) Just know the limit.

(2) Take the log and then find it.

(3) Apply some other continuous function to it and then find the limit.

(4) L'Hopital's rule if you can show that the function can be made differentiable easily.

How to show not convergent :

(1) Unbounded

(2) Not Cauchy

(3) Alternating but in a bad way

There isn't a whole lot of trickery to sequences.

Series

Series Convergence
Consider a series $\sum_{n=1}^{\infty} a_n$. Define $S_m = \sum_{n=1}^{m} a_n$. The series converges if and only if S_m converges with respect to m.

Theorem. (Monotone Test) : If $a_n \geq 0$ for all n and there exists M such that $S_m \leq M$ for all m, then the series converges.

Theorem. (Absolute Convergence) : If $\sum_{n=1}^{\infty} |a_n| < \infty$ then $\sum_{n=1}^{\infty} a_n$ converges.

Absolutely Convergent
$\sum_n a_n$ is absolutely convergent if $\sum_n |a_n|$ converges. If a series is absolutely convergent then you can basically do whatever the hell you want to it and nothing will change.

Conditionally Convergent
$\sum_n a_n$ is conditionally convergent if $\sum_n |a_n|$ diverges. If a series is conditionally convergent then DO NOT REARRANGE. THE SERIES CAN CONVERGE TO LITERALLY ANYTHING.

Theorem. (Alternating Series Test) : The series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges if $a_n \geq 0$, a_n is decreasing, and $\lim_{n\to\infty} a_n = 0$.

See the wikipedia page for more tests.

Submultiplicative and Subadditive Sequences

Theorem. (Summation by Parts) : Let there exist (a_n) and (b_n) and define $A_n = \sum_{k=1}^{n} a_k$ with $A_0 = 0$. Then

$$\sum_{n=1}^{N} a_n b_n = \sum_{n=1}^{N} (A_n - A_{n-1}) b_n$$
$$= A_N b_N + \sum_{n=1}^{N-1} A_n (b_n - b_{n+1})$$

Note : This is useful if A_n is bounded or if $(b_n - b_{n+1}$ dies quickly.

Theorem. (Dirichlet Test) : Suppose there exists M such that $|\sum_{n=1}^{N} a_n| \le M$ for all N and b_n is a decreasing sequence that converges to 0. Then

$$\sum_{n=1}^{\infty} a_n b_n$$

converges.

Proof. Summation by parts

Infinite Products

Consider the infinite product $\prod_{i=1}^{n} a_n$. The most, and really only, method for evaluating these is

Theorem. (Logarithm Test) : For $a_n > 0$, $\prod_{n=1}^{\infty} a_n$ converges if and only if one of the following holds

(i) $\sum_{n=1}^{\infty} \log(a_n) = -\infty$. Then $\prod_{n=1}^{\infty} a_n = 0$.

(ii) $\sum_{n=1}^{\infty} \log(a_n) = \infty$. Then $\prod_{n=1}^{\infty} a_n = \infty$.

(iii) $\sum_{n=1}^{\infty} \log(a_n) \in (0, \infty)$. Then $\prod_{n=1}^{\infty} a_n = e^{\sum_{n=1}^{\infty} \log(a_n)}$.

Multivariable Stuff

Differentiation

There are three types of derivatives when it comes to multivariable.

Partial Derivative
The directional derivative of f at x in the direction x_i where x_i is a basis vector of the space is given by

$$\partial_i f(x) = \lim_{t\to 0} \frac{f(x+tx_i) - f(x)}{t}$$

A partial derivative is a directional derivative in the direction of a standard basis vector.

Directional Derivative
The directional derivative of f at x in the direction e is given by

$$\partial_e f(x) = \lim_{t\to 0} \frac{f(x+te) - f(x)}{t}$$

Differentiable
$f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at x if there exists a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h\to 0} \frac{|f(x+h) - f(x) - Ah|}{|h|} = 0$$

Then $f'(x) = A$.

Theorem. If $\partial_x f$ exists and is continuous for all i then f is differentiable.

Theorem. (Chain Rule) : If $f : \mathbb{R}^p \to \mathbb{R}^q$ and $G : \mathbb{R}^q \to \mathbb{R}^n$ are continuously differentiable then

$$D(G \circ G)(x_0) = DG(F(x_0)) \cdot DF(x_0)$$

Theorem. (Claytor's Theorem) : If $f : \mathbb{R}^n \to \mathbb{R}$ is C^2 then

$$\partial_i \partial_j f(x) = \partial_j \partial_i f(x)$$

for all $x \in \mathbb{R}^n$.

Inverse Function Theorem

Theorem. (Inverse Function Theorem) : Let there exist some $x_0 \in \Omega \subset \mathbb{R}^n$ where Ω is open. If $F : \Omega \to \mathbb{R}^n$ is C^1 with $J_F(x_0) = \det(DF(x_0)) \neq 0$ then there exists an open U containing x_0 and an open V containing $F(x_0)$ such that $f : U \to V$ is bijective and $F^{-1} : V \to U$ is C^1

Open Function
A function $f : X \to Y$ is open if $f(U)$ is open for open U.

Theorem. (Corollary) : If $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and $J_F(x) \neq 0$ for all $x \in \Omega$, then F is open.

Implicit Function Theorem

Theorem. (Implicit Function Theorem) : Let there exist $F : \mathbb{R}^n \to \mathbb{R}^m$ with $m < n$ such that $\text{rank}(DF) = m$. Let there exist $x_0 \in \mathbb{R}^n$ and let $F(x_0) = y_0$.
Then there exist $i_1, \ldots, i_{n-m}$ and $g(x^{i_1}, \ldots, x^{i_{n-m}})$ such that

$$F(x^{i_1}, \ldots, x^{i_{n-m}}, g(x^{i_1}, \ldots, x^{i_{n-m}}) = 0$$

More clearly, this states that there exists some $g : \mathbb{R}^{n-m} \to \mathbb{R}^m$ such that

$$F(x, g(x)) = y_0$$

For all x in some neighborhood U of x_0.

Taylor Expansion

Theorem. (Second Order Taylor Expansion) : Let f be C^2 then

$$f(x+h) = f(x) + \nabla f(x) \cdot h + \frac{1}{2}\langle \nabla^2 f(x)h, h\rangle + O(\|x\|^3) \tag{3}$$

or

$$f(x+h) = f(x) + \nabla f(x) \cdot h + \frac{1}{2}\langle \nabla^2 f(x)h, h\rangle + E(x) \tag{4}$$

Where $E(x)$ is an eerror term such that $\lim_{x\to\infty} \|E(x)\|/\|x\|^2 = 0$.

Theorem. (General Taylor Expansion) : Let f be C^k then

$$f(x+h) = \sum_{|\alpha|\leq n} \frac{f^{(\alpha)}(x)}{\alpha!} + \sum_{|\alpha|=k} \frac{f^{(\alpha)}(x+\xi h)}{\alpha!} h^\alpha$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ with

$$|\alpha| = \alpha_1 + \cdots + \alpha_n$$
$$\alpha! = \alpha_1! \ldots \alpha_n!$$
$$x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$$
$$f^{(\alpha)}(x) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$$

Integration

Theorem. (Fubini's Theorem) : If

$$\int_a^b \int_c^d |f| dx dy < \infty$$

Then

$$\int_a^b \int_c^d f dx dy = \int_c^d \int_a^b |f| dy dx$$

Theorem. (Integrals Preserve Inequalities) : If $f \leq g$ then

$$\iint_R f \leq \iint_R g$$

Theorem. (Mean Value Theorem) : If R is connected and f is continuous then there exists $\xi \in R$ such that

$$\iint_R f = |R| f(\xi)$$

Green's Theorem and Others

Theorem. (Green's Theorem) : Let C be a positively oriented, piecewise smooth, simple closed curve in $\mathbb{R}^2$. Let D be the region bounded by C. Then

$$\oint_C f\,dx + g\,dy = \iint_D (\partial_x g - \partial_y f)\,dA$$

Where integration along C is counter-clockwise. (The constraints on C can be a little looser I think.)

Theorem. (Divergence Theorem) : Suppose V is a subset of $\mathbb{R}^3$ which is bounded and has piecewise smooth boundary ∂V. If $\mathbf{F}$ is a C^1 vector field defined on a neighborhood of V, then

$$\iiint_V \nabla \cdot \mathbf{F} dV = \oiint_{\partial V} \mathbf{F} \cdot d\mathbf{S}$$

where ∂V is oriented by outward normals. And $d\mathbf{S} = \mathbf{n} dS$

Theorem. (Stokes Theorem) : For a smooth oriented surface Σ in $\mathbb{R}^3$ with boundary $\partial\Sigma$ and $A : \mathbb{R}^3 \to \mathbb{R}^3$ with continuous partial derivatives,

$$\iint_\Sigma (\nabla \times A) \cdot da = \oint_{\partial\Sigma} A \cdot dl$$

Surfaces

There are three main, equivalent ways to describe (locally) C^1 surfaces in $\mathbb{R}^3$.

Parametric Form
For a surface S and $p \in S$. If there exists

(i) an open neighborhood $U \subset \mathbb{R}^2$ of $(0,0) \in \mathbb{R}^2$

(ii) an open neighborhood $V \subset \mathbb{R}^3$ of $p \in \mathbb{R}^3$.

(iii) A C^1 function $\phi : U \to V$ such that $D\phi$ has rank 2 and $\phi(U) = V \cap S$, $\phi(0,0) = p$.

Then S is locally a C^1 surface about p.

Graph Form
S is locally a graph at $p = (x_0, y_0, z_0)$ if there exists

(i) open neighborhood U of $(x_0, y_0) \in \mathbb{R}^2$

(ii) open neighborhood V of p in $\mathbb{R}^3$ and a C^1 function $z : U \to \mathbb{R}$ such that

$$\{(x, y, z(x, y)) : (x, y) \in U\} = V \cap S \text{ and } z(x_0, y_0) = z_0$$

Level Surface
S is locally a level surface at p if there exists

(i) a neighborhood V of p in $\mathbb{R}^3$

(ii) a function $f : V \to \mathbb{R}$ which is C^1 such that

  (a) $S \cap V = f^{-1}(0) \cap V$

  (b) $\nabla f \neq 0$ on $S \cap V$

C^1 Surface
S is a C^1 surface if and only if any of the previous three definitions are satisfied for all $p \in S$.