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Week 1

Solutions to Ordinary Differential Equations

We define an **ordinary differential equation** to be an equation involving an unknown function of a single variable along with its derivatives. Most commonly, the unknown function will be $y(t)$ with t being the independent variable. The **order** of the differential equation is the order of the highest derivative occurring in the equation.

We say that a function $y(t)$ satisfies a differential equation $\phi(t, y, y', y'', \ldots) = 0$ if $\phi(t, y(t), y'(t), y''(t), \ldots) = 0$ for all t where $y, y', \ldots$ are defined.

Exercise 1. Show that the function $y(t)$ is the solution of the corresponding ODE.

(1) $y(t) = Ce^{-t^2/2}$ for $y' = -ty$

(2) $y(t) = \sin(2t) + t^2$ for $y'' + y + y' - 3\sin(2t) - 2\cos(2t) + t^2 - 2t + 2$

Initial Value Problems

A general ordinary differential equation may not have a unique solution. Take for instance $y' = y$. We know from basic calculus that this ODE is satisfied by $y(t) = ce^t$ for any constant c. So for this ODE there actually infinite solutions.

To guarantee a unique solution, we need to specify more information. To do so, we specify an initial value for the problem. For our current purposes, we focus on first order initial value problems, which are defined as

Initial Value Equation
A differential equation together with an initial condition

$$\phi(t, y') = 0, \qquad y(t_0) = y_0$$

is called an **initial value problem**. A solution of the initial value problem is a function $y(t)$ satisfying the ODE for all t in an interval containing t_0.

With this, we define the interval of existence of a solution to a initial value problem to be the largest interval containing the initial point over which the solution is defined.

Integrate Both Sides

The simplest initial value problem is set up as

$$y' = f(t) \qquad y(t_0) = y_0$$

To solve this, we can just integrate both sides and apply the fundamental theorem of calculus. Doing so,

$$\int f(t)dt = \int y'(t)dt = y(t) + c$$

Plugging in t_0 and solving for c then gives the solution.

This can also be done a bit more explicitly with definite integrals. In this manner, the fundamental theorem of calculus gives

$$y(t) - y(t_0) = \int_{t_0}^{t} y'(t')dt' = \int_{t_0}^{t} f(t')dt'$$

Which gives $y(t) = \int_{t_0}^{t} f(t')dt' + y_0$. Here, t' is being integrate against instead of t to distinguish it from the bounds on the integral.

Exercise 2. Solve the following initial value problems

(1) $y'(t) = t$, $y(1) = 2$

(2) $y'(t) = \cos(t)$, $y(0) = 5$

(3) $y'(t) = e^t \cos\left(e^t\right)$, $y(0) = 0$.

Separable Equations

A separable equation is a first order differential equation that can be written as

$$y' = f(t)g(y)$$

To solve this, we can rewrite it as

$$\frac{1}{g(y)} \frac{dy}{dt} = f(t)$$

Abusing notation slightly, we can then separate the dy and dt terms onto either side

$$\frac{1}{g(y)} dy = f(t)dt$$

And then integrate both sides

$$\int \frac{1}{g(y)} dy = \int f(t)dt$$

The left hand side will give some function of y and the left hand side will give some function of t. We can then rearrange things to solve for y as a function of t.

Exercise 3. Solve the following initial value problems

(1) $y' = y/t$, for $y(1) = -2$

(2) $y' = e^{t+y}$, for $y(0) = 0$

Rigorous Derivation

If the idea of separating dy and dt makes you uneasy, then this can be done with change of variables in the same manner. Rewriting the ODE in the same way, we have

$$\frac{1}{g(y(t))}y'(t) = f(t)$$

We can then integrate both sides with respect to t to find

$$\int \frac{1}{g(y(t))}y'(t)dt = \int f(t)dt$$

Just as we did with u-substitutions, the term $y'(y)dt$ can be identified with dy, which changes the integral to be with respect to y instead of t. We then have

$$\int \frac{1}{g(y)}dy = \int f(t)dt$$

And the solution follows from evaluating both sides.

Week 2

Linear Equations

Linear Differential Equation
We call a differential equation **linear** if it can be written in the form

$$b(t)x' = c(t)x + g(t)$$

If $g(t) = 0$ then the linear equation is said to be **homogeneous**. The important part of this definition is that x, x' appear on their own, with coefficients that depend only on t.

If we had a homogeneous linear differential equation, then the solution would be simple from past methods. In this case, we would be considering the equation

$$b(t)x' = c(t)x$$

This equation is separable and so can be solved by separating $x' = dx/dt$ and integrating both sides as we did before.

However, when the $g(t)$ term is nonzero, it makes things much more complicated. For this case, we need to introduce the method of integrating factors. This general method is outlined as follows

Theorem. The equation

$$x' = ax + f$$

can be solved using the following four steps

(1) Rewrite the equation as

$$x' - ax = f$$

(2) Multiply by the integrating factor

$$u(t) = \exp\left(-\int a(t)\,dt\right)$$

Where the $+c$ term is ignored from the integral. With this, the equation becomes

$$(ux)' = u(x' - ax) = uf$$

(3) Integrate this equation to obtain

$$u(t)x(t) = \int u(t)f(t)\,dt + C$$

(4) Solve for $x(t)$.

To illustrate this method, here are some exercises.

Exercise 4. Consider the following first order differential equation

$$y'(t) + \cos(t)\,y(t) = 0$$

(1) Multiply both sides by $e^{\sin(t)}$ and recognize the left hand side as a derivative of a product.

(2) Integrate both sides and arrive to the following form of the solution to the equation

$$y(t) = Ae^{-\sin(t)}$$

where A is a constant.

To illustrate this methods use compared to separation of variables, we have the following exercise.

Exercise 5. Solve the following first order linear differential equation

$$\left(1 - t^2\right) y'(t) + 2t\,y(t) = 0, \qquad y(2) = 1$$

(1) By using separation of variables.

(2) By using integration factor.

Variation of Parameters

As an alternative to the integration factor method, you can instead use variation of parameters. This method is summarized as follows

Theorem. The equation

$$y' = ay + f$$

can be solved using the following steps.

(1) Consider the homogeneous portion of the equation, $y_h' = ay_h$ where the subscript h is used to indicate that it is homogeneous. By separation of variables, this has a *particular solution*

$$y_h = \exp\left(\int a(t)\,dt\right)$$

where the $+c$ from the integration is ignored.

(2) Consider the equation $y = uy_h$. Substituting this into the original differential equation, we can solve for u. Doing so, we find that

$$u' = \frac{f}{y_h}$$

and can then solve for u.

(3) With this, the general solution for y is given by $y(t) = u(t)\,y_h(t)$.

The following exercise contrasts the integration factor method with the variation of parameters method.

Exercise 6. Solve the following first order linear differential equation

$$\cos(t)\,y' + y\sin(t) = 1, \qquad y(0) = 5$$

(1) By using integration factor.

(2) By using variation of parameters.

Structure of a Solution

The method of variation of parameters shows an important aspect about linear differential equations. Namely, it shows that any solution has a particular part and a homogeneous part. By moving around constants, we can then show that every solution to the differential equation can be written as the sum

$$y(t) = y_p(t) + A y_h(t)$$

Where A is a constant, y_p is one particular solution to the differential equation, and $y_h(t)$ is one particular solution to the homogeneous portion of the differential equation. This decomposition will be very important in the remainder of this class.

Week 3

Calculus Review

For this week, it will be useful to review some integration techniques and differentiation techniques. For that, we have the following two exercises.

Exercise 7. *(Integration) :* Compute the following indefinite integrals

(a) *u-substitution*

$$\int \frac{2x^2}{1+x^3}\,dx$$

(b) *integration by parts*

$$\int x^2 e^x\,dx$$

(c) *partial fractions*

$$\int \frac{x^2+2x+2}{(x+1)(x+2)(x+3)}\,dx$$

Exercise 8. *(Implicit Differentiation) :* For the given equations, compute y' at the given point (x_0, y_0).

(1)

$$\sin(x-y) = x\cos(y+\pi/4), \quad (x_0, y_0) = (\pi/4, \pi/4)$$

(2)

$$x^{2/3} + y^{2/3} = 2, \quad (x_0, y_0) = (1, 1)$$

Differential Forms

Consider a differential equation of the form

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0$$

As we've found, it is not always possible to find an explicit formula for $y(x)$. In general though, a implicit solution for y can be found. That is, we can find an equation F and a constant C such that $F(x,y) = C$. Here, it is important to note that the roles of x and y are interchangeable. To reflect this, we alter our differential equation by abusing notation and multiplying both sides by dx. Doing so, we arrive at the equation

$$P(x,y)\,dx + Q(x,y)\,dy = 0 \tag{1}$$

Which has no preference for x or y. As earlier, the solution of a differential equation in this form is a relationship between x and y and can be written as $F(x,y) = C$. However, in this form, what it means for F to satisfy equation 1 is no longer clear.

Let's carefully break down what that means. If we have a solution $y = y(x)$ to $P(x,y)dx + Q(x,y)dy = 0$, then $dy = \frac{dy}{dx}dx$ and so $P(x,y(x))dx + Q(x,y(x))y'dx = 0$. Rearranging and eliminating a dx on both sides, this can equivalently be written as

$$\frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)}$$

Now suppose that $y = y(x)$ is defined by $F(x,y) = C$. Then, by implicit differentiation

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \quad \Longrightarrow \quad \frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}$$

Combining these identities for dy/dx, we find that

$$\frac{\partial F/\partial x}{\partial F/\partial y} = \frac{P}{Q} \quad \Longrightarrow \quad \frac{1}{P}\frac{\partial F}{\partial x} = \frac{1}{Q}\frac{\partial F}{\partial y}$$

Let μ denote the final expression. Then $F(x,y) = C$ is a solution to the differential equation $P(x,y)dx + Q(x,y)dy = 0$ if and only if there exists a function μ such that

$$\frac{\partial F}{\partial x} = \mu P \qquad \frac{\partial F}{\partial y} = \mu Q$$

Exact Forms

An expression of the form given in equation 1 will be very important in the remainder of this class. For that reason, we call it a differential form and define it as follows.

Differential Form
A differential form in two variables, x and y is an expression of the type

$$\omega = P(x,y)dx + Q(x,y)dy$$

Where P, Q are differentiable functions of x and y. The simple forms dx and dy are called **differentials**.

Intuitively, differential forms behave like vectors or like little steps in the x or y direction. So the differential form $\omega = P(x,y)dx + Q(x,y)dy$ is saying that ω is a tiny step of magnitude $P(x,y)$ in the x direction and a tiny step of magnitude $Q(x,y)$ in the y direction. It's important to note that these are not actual steps of size $P(x,y)$ and $Q(x,y)$, but rather infinitely small steps times $P(x,y)$ and $Q(x,y)$.

Thee most common use of differential forms up until now has been integration and u-substitutions. When integrating, we can define a function $u = u(x)$ and then we get the differential form $du = u'(x)dx$. With this in mind, we define the differential of a continuously differential function as follows

Differential

The differential of a continuously differentiable function F is the differential form

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

Certain differential forms admit easier differential equations. These differential forms are called exact and are defined as follows

Exact
A differential form ω is said to be exact if it is the differential of a continuously differential function. That is to say that ω is exact if $\omega = dF$.
For the differential form

$$\omega = P(x, y)dx + Q(x, y)dy$$

This is equivalent to the existence of a continuously differentiable function F satisfying

$$\frac{\partial F}{\partial x} = P(x, y) \qquad \frac{\partial F}{\partial y} = Q(x, y)$$

Comparing this to what it means for $F(x, y) = C$ to satisfy $P(x, y)dx + Q(x, y)dy = 0$, we find that a form is exact if it has a solution $F(x, y) = C$ with $\mu = 1$.

It should be noted that exactness is very similar to a vector field being conservative from 32b. Unsurprisingly then, we have a theorem that relates exactness and the cross partials.

Theorem. Let $\omega = P(x, y)dx + Q(x, y)dy$ be a differential form where P, Q are continuously differentiable.

(a) If ω is exact then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{2}$$

(b) If equation 2 is true in a rectangle R, then ω is exact in R.

For practice in recognizing exact forms, we have the following exercise

Exercise 9. *(Exact Differential Equations) :* Check exactness of the following differential equations

(1)

$$(2y\cos(2x) - 2x)dx + \sin(2x)dy = 0$$

(2)

$$(2x + \ln(y))dx + xydy = 0$$

(3)

$$(2xy^2 + 4x^3)dx + 2x^2 y dy = 0$$

(4)

$$y' = \frac{2 - y/x}{\ln(x)}$$

Solving exact differential equations

Theorem. If the equation $P(x, y)dx + Q(x, y)dy = 0$ is exact, then the solution is given by $F(x, y) = C$ where F is found by the steps

(1) Solve $\partial F/\partial x = P$ by integration

$$F(x, y) = \int P(x, y)dx + \phi(y)$$

(2) Solve $\partial F/\partial y = Q$ by choosing ϕ so that

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\int P(x, y)dx + \phi'(y) = Q(x, y)$$

Exercise 10. Check that the following differential equations are exact and find the solutions to the corresponding initial value problems

(1)

$$\frac{y}{1+t}dt + \left(\ln(1+t) + 3y^2\right)dy = 0 \qquad y(0) = 1$$

(2)

$$(2x + y)dx + (x - 6y)dy = 0 \qquad y(0) = 1$$

Integrating Factor

Though exact differential equations are easier to solve, they are not the only differential equations of the form $P(x, y)dx + Q(x, y)dy = 0$ that can be solved.

Earlier, we found that $F(x, y) = C$ is a solution to the differential equation $P(x, y)dx + Q(x, y)dy = 0$ if there exists a function μ such that

$$\frac{\partial F}{\partial x} = \mu P \qquad \frac{\partial F}{\partial y} = \mu Q$$

If $P(x, y)dx + Q(x, y)dy$ is exact, then $\mu = 1$. However, finding such an F with $\mu = 1$ is not always possible. In these cases, we need to choose μ carefully to make our lives easier. The right choice of such a μ is called an integrating factor and is defined as

Integrating Factor
An integrating factor for the differential equation $\omega = Pdx + Qdy = 0$ is a function $\mu(x, y)$ such that $\mu\omega$ is exact.

Once an integrating factor is found, it is easy to solve the differential equation because it is exact. However, finding the integrating factor can be difficult. We summarize some methods for finding them here.

Separable Equations

A differential equation is said to be **separable** if there is an integrating factor μ such that μP is a function of x and μQ is a function of y. In this case, the final solution is given by

$$F(x, y) = \int P(x, y)\mu(x, y)dx + \int Q(x, y)\mu(x, y)dy = C$$

A differential equation will be separable if $P(x, y) = f_1(x)g_1(y)$ and $Q(x, y) = f_2(x)g_2(y)$. Then $\mu(x, y) = 1/g_1(y)f_2(x)$.

Linear Equations

For linear equations, the integrating factor μ aligns with the integration factor u from last week.

Integrating Factor in One Variable

As shown earlier, for a form to be exact it suffices to show that it satisfies equation 2. Therefore, for μ to be an integrating factor, μ must satisfy

$$\frac{\partial}{\partial y}(\mu P) = \frac{\partial}{\partial x}(\mu Q) \tag{3}$$

If we assume that μ depends only on x, then this simplifies to

$$\mu\frac{\partial P}{\partial y} = \frac{\partial}{\partial x}(\mu Q) = \frac{\partial \mu}{\partial x}Q + \mu\frac{\partial Q}{\partial x} \quad \Longrightarrow \quad \frac{d\mu}{dx} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\mu \tag{4}$$

and so

$$\mu = \exp\left(\int \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)dx\right)$$

Is an integrating factor. Likewise, if

$$h = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

is an equation of only x, then μ can be chosen to be a function of only x given by $\mu = \exp\left(\int h(x)dx\right)$.

Summarizing this method and the similar method for μ depending only on y, we find that

Theorem. The form $Pdx + Qdy$ has an integrating factor depending on one variable under the following conditions

(1) If
$$h = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$
is a function of x only, then $\mu(x) = \exp\left(\int h(x)dx\right)$ is an integrating factor.

(2) If
$$g = \frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$
is a function of y only, then $\mu(y) = \exp\left(-\int g(y)dy\right)$ is an integrating factor.

Note that the $-$ in $\mu(y)$ comes from the order of the partials in the definition of g.

Homogeneous Equations

Homogeneous Function
A function $G(x, y)$ is homogeneous of degree n if
$$G(tx, ty) = t^n G(x, y)$$
For all $t > 0$ and $x, y \neq 0$.

We can define a similar concept for differential equations by

Homogeneous Differential Equation
A differential equation $Pdx + Qdy$ is homogeneous if both of the function P, Q are homogeneous of the same degree. This is equivalent to saying that there exists some m such that for all $x, y \neq 0$ and $t > 0$,
$$P(tx, ty)dx + Q(tx, ty)dy = t^m \left(P(x, y)dx + Q(x, y)dy\right)$$

To solve a homogeneous equation, we make the substitution $y = xv$ so that $P(x, y)dx + Q(x, y)dy = 0$ implies that
$$P(x, xv)dx + Q(x, xv)(vdx + xdv) = 0 \implies (P(1, v) + vQ(1, v))\,dx + x^1 Q(1, v)dv = 0$$
Where the x^m terms are divided out. In this case, the equation is separable and so we can use the integrating factor $\mu(x, v)$ defined by
$$\mu(x, v) = \frac{1}{x(P(1, v) + vQ(1, v))}$$
Which separates the equation and makes it exact.

Week 4

Existence of Solutions

Theorem. Suppose that the function $f(t,x)$ is defined and continuous on the rectangle R in the tx plane. Then for any $(t_0, x_0) \in R$, the initial value problem

$$x' = f(t,x) \qquad x(t_0) = x_0$$

has a solution $x(t)$ defined on some interval containing t_0. Additionally, the solution will be defined at least until the solution curve $(t, x(t))$ leaves R.

Uniqueness of Solutions

Theorem. Suppose that the function $f(t,x)$ and its partial derivative $\partial f/\partial x$ are both continuous on the rectangle R in the tx plane. Suppose that $(t_0, x_0) \in R$ and that the solutions

$$x' = f(t,x) \qquad y' = f(t,y)$$

satisfy

$$x(t_0) = y(t_0) = x_0$$

Then as long as $(t, x(t))$ and $(t, y(t))$ stay in R, $x(t) = y(t)$. That is, the solution x is unique.

Autonomous Equation

First Order Autonomous Equation
A first-order autonomous equation is an equation of the form

$$y' = f(y)$$

The important characteristic to note being that there is no dependence on an independent variable t.

An important feature of autonomous equations is their equilibrium points and equilibriums solutions. An equilibrium point is a point where y stops changing, aka reaches equilibrium. Similarly, an equilibrium solution is a solution where y is at equilibrium. These are defined as follows

Equilibrium Point
Given an autonomous first order differential equation $y' = f(y)$, an equilibrium point is a point y_0 such that $f(y_0) = 0$.

Equilibrium Solution

If y_0 is an equilibrium point of the autonomous equation $y' = f(y)$, then $y(t) = y_0$ is an equilibrium solution.

Week 5

Second Order Equations

We are now moving onwards to second order differential equations. As before, these are often expressed in normal form, which for second order ODEs is

$$y'' = f(t, y, y')$$

For our purposes, we are mostly interested in linear second order equations. For these, the standard form to write them in is

$$y'' + p(t)y' + q(t)y = g(t)$$

To solve such an equation, we can break it down into three steeps

(1) Consider the corresponding homogeneous equation, namely

$$y'' + p(t)y' + q(t)y = 0$$

Solve for a general homogeneous solution y_h.

(2) Solve the full ODE $y'' + p(t)y' + q(t)y = g(t)$ for a particular solution y_p.

(3) The general solution is then $y = y_p + y_h$.

The proof that this works is relatively straightforward. In essence, linearity implies that the difference between any two solutions is itself a homogeneous solution. Therefore you can get to any solution by adding the general homogeneous solution to a particular solution.

Much like the existence theorem of last week, we have an existence theorem for second order ODEs.

Theorem. Suppose that the functions $p(t), q(t), g(t)$ are continuous on the interval (α, β). Let t_0 be any point in (α, β). Then for any real numbers y_0, y_1 there is one and only one function $y(t)$ defined on (α, β) which is a solution to

$$y'' + p(t)y' + q(t)y = g(t) \qquad \text{for } \alpha < t < \beta$$

and satisfies the initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_1$.

Note that for second order equations we need two initial conditions.

Linear Dependence

Linearly Dependent
We call two functions u, v linearly dependent if they are constant multiples of each other.
If two functions are not linearly dependent then they are called linearly independent.

Linear Combination

A linear combination of functions u, v is a function $au(t) + bv(t)$ for constants a, b.

Basis

A finite set of functions $f_1, \dots, f_n$ form a basis for a class of functions $\mathscr{F}$ (i.e. solutions to a particular ODE) if all functions in $\mathscr{F}$ can be written as the linear combination of $f_1, \dots, f_n$. That is, for all f in $\mathscr{F}$ there exists constant $a_1, \dots, a_n$ such that

$$f(t) = a_1 f_1(t) + \cdots + a_n f_n(t)$$

The power of linearly independent functions is that they can form a basis for the solutions to differential equations. In particular, we have the following theorem

Theorem. Suppose that y_1, y_2 are linearly independent solutions to the equation

$$y'' + p(t)y' + q(t)y = 0 \tag{5}$$

Then y_1, y_2 form a basis of the solutions to equation 5. This implies that the general solution to equation 5 is

$$y = c_1 y_1 + c_2 y_2$$

Where c_1, c_2 are arbitrary constants.

In order to prove this and eventually extend it to higher order equations, we need to come up with a consistent method of showing functions are linearly independent. This is because at times, computing whether functions are linearly independent, especially over various areas, can be difficult. To help with this, we define a function called the Wronskian that relates the functions and their derivatives.

Wronskian

Given two differentiable functions u, v, we define the Wronskian $W(t)$ to be

$$W(t) = \det\begin{pmatrix} u & v \\ u' & v' \end{pmatrix} = u(t)v'(t) - v(t)u'(t)$$

We will extend the definition of the Wronskian to more functions later in the course. For now, we have a few lemmas which prove the theorem above.

Lemma 1. Suppose that u, v are differentiable on (a, b).

(a) If u, v are linearly dependent then $W(t) = 0$ for all $t \in (a, b)$.

(b) If $W(t_0) \neq 0$ for some $t_0 \in (a, b)$ then u, v are linearly independent on (a, b).

Lemma 2. Let y_1, y_2 be solutions to the homogeneous equation $y'' + py' + qy = 0$ on the interval (α, β). Then the Wronskian of y_1, y_2 is either identically equal to zero on (α, β) or it is never equal to zero.

Lemma 3. Let y_1, y_2 be solutions to the homogeneous equation $y'' + py' + qy = 0$. Then linear combinations of y_1, y_2 are also solutions.

Lemma 4. Let y_1, y_2 be solutions to the homogeneous equation $y'' + p(t)y' + q(t)y = 0$ on the interval (α, β). Then y_1, y_2 are linearly dependent if and only if their Wronskian is identically 0 in (α, β).

These four lemmas establish the earlier theorem. Based on this theorem, we establish the following definition

Fundamental Set of Solutions
Given a differential equation $y'' + p(t)y' + q(t)y = g(t)$, we say that two linearly independent solutions form a fundamental set of solutions.

Constant Coefficients

Let p, q be constants and consider the second order linear homogeneous equation

$$y'' + py' + qy = 0$$

This is a particularly easy differential equation that we will become adept at solving. To solve such an equation, we first isolate the characteristic equation, which is defined as follows

Characteristic Equation
Consider a constant coefficients second order linear homogeneous equation

$$y'' + py' + qy = 0$$

With constants p, q. The **characteristic equation** is the corresponding polynomial

$$\lambda^2 + p\lambda + q$$

This is sometimes called the **characteristic polynomial**.
Given a characteristic equation, we call the roots of said equation the **characteristic roots**.

Given a characteristic equation we can find the roots λ_1, λ. We can then break down the solution based on these roots

Distinct Roots

If these roots are distinct, then the equations $e^{\lambda_1 t}, e^{\lambda_2 t}$ form a fundamental set of solutions and so the general solution is $c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$. If these roots are real, then the solution can remain in this form. If the roots are complex, then $\lambda_1 = \overline{\lambda_2} = a + bi$ and the solution can alternatively be written as

$$A_1 e^{at}\cos(bt) + A_2 e^{at}\sin(bt)$$

Repeated Roots

If $\lambda_1 = \lambda_2 = \lambda$, then the equations $e^{\lambda t}, te^{\lambda t}$ form a fundamental set of solutions. This implies that the general solution is

$$c_1 e^{\lambda t} + c_2 t e^{\lambda t}$$

Note that repeated roots cannot occur unless λ is a real number.

Week 6

Inhomogeneous Equations

We now aim to fully consider inhomogeneous second order linear differential equations. Which is quite the mouthful. That is, we consider equations of the form

$$y'' + py' + qy = f$$

Where p, q, f are functions of the independent random variables t.

In general, we know the structure of solutions.

Theorem. *(Structure of a Solution) :* Consider the differential equation

$$y'' + py' + qy = f$$

Where p, q, f are functions of the independent variables t. Let y_h be the general solution to the homogeneous equation $y'' + py' + qy = 0$ and let y_p be a particular solution to the inhomogeneous equation $y'' + py' + qy = f$. Then the general solution is given by

$$y = y_p + y_h$$

The proof of this fact follows immediately from considering the difference of any two solutions and noting that it is a homogeneous solution. This theorem also indicates that our strategy for solving these equations will be to solve the homogeneous and particular portions separately to later combine them.

Undetermined Coefficients

Consider the differential equation

$$y'' + py' + qy = f$$

where p, q are constants and f depends on t. Then we can find the homogeneous solution y_h using the method of constant coefficients and we can use the method of undetermined coefficients to, hopefully, find the particular solution y_p.

The method of undetermined coefficients states, heuristically, that the particular solution for a given $f(t)$ probably the same form as f. So if $f(t)$ is a polynomial of degree n, then we let y_p be a polynomial of degree n and then we attempt to vary the coefficients so that y_p is a particular solution. To summarize this

$$\begin{aligned}
f(t) = e^{rt} \qquad &\Longrightarrow \qquad \text{trial function}: y_p = ae^{rt} \\
f(t) = \cos(\omega t) \text{ or } \sin(\omega t) \qquad &\Longrightarrow \qquad \text{trial function}: y_p = a\cos\omega t + b\sin\omega t \\
f(t) = a_0 + a_1 t + \cdots + a_n t^n \qquad &\Longrightarrow \qquad \text{trial function}: y_p = b_0 + b_1 t + \cdots + b_n t^n
\end{aligned}$$

Combinations of these hold similarly. For instance, if $f(t) = e^{rt}\cos(\omega t) + e^{rt}\sin(\omega t)$ then we use the trial function $y_p = e^{rt}(a\cos(\omega t) + b\sin(\omega t))$.

Variation of Parameters

We now turn our attention to solving equations $y'' + py' + qy = f$ where p, q, f are all functions. We don't currently have a method to solve the homogeneous portion of this equation, but if we have already found the solution of the homogeneous portion then we can use it to find a particular solution. To do so, we use a method called variation of parameters.

Theorem. *(Variation of Parameters) :* Consider the differential equation $y'' + py' + qy = g$ and suppose that you have found a set of fundamental solutions y_1, y_2 to the associated homogeneous equation $y'' + py' + qy = 0$.

(1) Consider the equation $y_p = v_1 y_1 + v_2 y_2$ where v_1, v_2 are functions that are yet to be determined.

(2) Find v_1, v_2 through one of the following two methods.

(a) We can find v_1, v_2 by direct integration as

$$v_1(t) = \int \frac{-y_2(t)g(t)dt}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}$$
$$v_2(t) = \int \frac{y_1(t)g(t)dt}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}$$

(b) First, we differentiate y_p to find

$$y_p' = (v_1' y_1 + v_2' y_2) + v_1 y_1' + v_2 y_2'$$

Isolating the term $v_1 y_1' + v_2 y_2'$, we consider the equation

$$v_1' y_1 + v_2' y_2 = 0$$

Then, we take the second derivative of y_p and insert y_p, y_p', y_p'' into the differential equation. After simplifying, a second equation of the form

$$v_1' y_1' + v_2' y_2' = g(t)$$

will appear. Considering both of these equations, we can solve for v_1', v_2'. Integrating, we then arrive at solutions for v_1, v_2. This gives a particular solution $y = v_1 y_1 + v_2 y_2$.

Week 7

Higher Order Constant Coefficients

Consider a higher order constant coefficient linear differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = f$$

As usual, the general solution to this equation can be written as

$$y = y_p + y_h$$

Where y_p is a particular solution and y_h is the general solution to the homogeneous equation. To find y_h we follow the same method as in the second order case. Considering the solution $y = e^{\lambda t}$, we arrive at the characteristic equation

$$\lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0$$

Finding the roots of this, we find solutions. If a given root λ is repeated n times, then we find the n solutions

$$e^{\lambda t} \quad te^{\lambda t} \quad \ldots t^{n-1}e^{\lambda t}$$

For complex roots, these can be split into sines and cosines in the same manner as last week. If there are repeated complex roots, then factors of t can be added as above.

To find y_p, we follow the method of undetermined coefficients exactly.

Linear Systems with Constant Coefficients

We now want to consider systems of linear differential equations. For example, we could consider a system consisting of two linear differential equations such as

$$y_1' = t^2(1+t)y_1 + 5y_2 \qquad y_2' = 12e^t y_1 + \sin(t)y_2$$

where we want to solve for equations y_1, y_2 that satisfy both of these equations.

For this week, we restrict ourselves to systems with constant coefficients. That is, we restrict ourselves to looking at systems of the form

$$y_1' = a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \qquad \ldots \qquad y_n' = a_{n1}y_1 + \cdots + a_{nn}y_n$$

Utilizing vectors and matrices, we can let

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \qquad A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_n \end{pmatrix}$$

So that our system of linear differential equations becomes $y' = Ay$.

Just as with constant coefficients the last two weeks, we look for solutions that are in exponential form. Specifically, we look for solutions of the form

$$y(t) = e^{\lambda t} v$$

for some vector v. In this case, we have $y'(t) = \lambda e^{\lambda t} v = \lambda y(t)$ and so our linear system becomes

$$y' = Ay \qquad \Longrightarrow \qquad \lambda y = Ay$$

And so we must find a number λ and a vector v that satisfies the above equation. This concept is a central idea in linear algebra, so we focus on that slightly.

Eigenvalues and Eigenvectors

Eigenvalue and Eigenvector
Suppose that A is an $n \times n$ matrix. A number λ is an eigenvalue of A if there is a nonzero vector v such that

$$Av = \lambda v$$

If λ is an eigenvalue, then any vector v satisfying $Av = \lambda v$ is an eigenvector.

To find eigenvalues and eigenvectors, we can use some linear algebra. Rewriting the condition to be an eigenvalue and eigenvector, we find that λ is an eigenvalue and v is an eigenvector if

$$[A - \lambda I] v = 0$$

Where I is the identity matrix. From linear algebra, which I won't go into here, we know that for a fixed λ, $[A - \lambda I] v = 0$ for some nonzero vector v if and only if $\det(\lambda I - A) = 0$. Expanding this determinant gives the characteristic polynomial,

Characteristic Polynomial
If A is an $n \times n$ matrix, the polynomial

$$p(\lambda) = \det(\lambda I - A)$$

is called the characteristic polynomial of A and the equation

$$p(\lambda) = \det(\lambda I - A) = 0$$

is called the characteristic equation.

With this, we arrive at the theorem

Theorem. The eigenvalues of an $n \times n$ matrix are the roots of its characteristic polynomial.

Once we have the eigenvalues, we still need to find the corresponding eigenvectors. To do so, we recall that v will be an eigenvector if and only if

$$(\lambda I - A)v = 0$$

And so, to find eigenvectors, we look at the matrix $(\lambda I - A)$ and find any nonzero vectors v such that $(\lambda I - A)v = 0$. In doing so, we define the eigenspace of an eigenvalue λ as

Eigenspace
Let λ be an eigenvalue of a matrix A. The set of all eigenvectors corresponding to λ is called the eigenspace of λ. The eigenspace is equivalent to the space of all v such that $(\lambda I - A)v = 0$.
It's important to note that the eigenspace of an eigenvalue is closed under linear combinations. That is to say that the sum or scalar multiple of any eigenvector is still an eigenvector.

Finding Solutions

Now let's once again consider our system of differential equations

$$y' = Ay$$

As shown, $y = e^{\lambda t}v$ is a solution when λ is an eigenvalue with eigenvector v. If there are n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, with eigenvectors $v_1, \ldots, v_n$, then these solutions form a fundamental set of solutions. The general solution is then

$$y = c_1 e^{\lambda_1 t} v_1 + \cdots + c_2 e^{\lambda_n t} v_n$$

We can absorb the constants $c_1, \ldots, c_n$ into the vectors $v_1, \ldots, v_n$ to simplify the form of the solution, but this makes it a little harder to figure out how to solve for initial conditions, so we leave the constants for now.

If there are less than n distinct eigenvectors, then we don't yet have enough solutions to solve the equation. We will have to do more work in that case, which we will do later.

Week 8

Forming Solutions

We now form the solutions of a system of linear equations based on the eigenvectors and eigenvalues. Consider the 2×2 system

$$y_1' = ay_1 + by_2 \qquad y_2' = cy_1 + dy_2$$

Setting this up in the usual way with matrices, this becomes $y' = Ay$. We can now break down our solutions based on the eigenvalues of A.

Two Distinct Eigenvalues

Suppose that A has eigenvalues λ_1, λ_2 that are distinct. Let v_1, v_2 be eigenvectors of λ_1, λ_2 respectively. Then $e^{\lambda_1 t} v_1$ and $e^{\lambda_2 t} v_2$ are linearly independent solutions, which implies that general solution is of the form

$$y = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_1 t} v_1$$

If λ_1, λ_2 are complex numbers, then $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$. Let $w = u + iv$ be the eigenvector of $\lambda_1 = a + bi$. The solution y then becomes

$$y = c_1 e^{at} (\cos(bt)u - \sin(bt)v) + c_2 e^{at} (\sin(bt)u + \cos(bt)v)$$

One Eigenvalue of Multiplicity Two

Suppose that A only has one eigenvalue λ of multiplicity two.

If the eigenspace of λ is two dimensional, that is there exists two linearly independent eigenvectors v_1, v_2 associated with λ, then the solution is given by

$$y = e^{\lambda t}(c_1 v_1 + c_2 v_2)$$

If the eigenspace of λ is one dimensional, then more work must be done. Let v be an eigenvector associated with λ. We can find another vector v^* such that $(A - \lambda I)v^* = v$. Then the general solution is given by

$$y = e^{\lambda t}\left(c_1 v_1 + c_2 \left(v^* + tv\right)\right)$$

Week 9

As this week deals mainly with drawing pictures, the textbook or course lecture notes will be more helpful than these.

Critical Points

Given a 2×2 system of linear equations $y' = Ay$, we want to sketch its solutions to determine its behaviour. As this is an autonomous equation, it will be important to identify equilibrium points and the behaviors around said points. To that end, we define

Critical Points
Given a system of linear equations $y' = Ay$, y is a critical point if $Ay = 0$.

Based on the matrix A, we can determine "how many" critical points there will be. More accurately, we can determine what dimension the space of critical points are.

Theorem. Suppose that $y' = Ay$ is a system of linear equations.

(a) If A is invertible, then $y' = Ay$ has only one critical point at 0.

(b) If $\text{rank}(A) = 1$, then the critical points form a straight line through the origin.

(c) If $\text{rank}(A) = 0$, then the critical points are all of $\mathbb{R}^2$.

It is important to note that critical points correspond to equilibrium solutions, just like for autonomous equations.

Characterizing Critical Points

Stable
Intuitively, a critical point is stable if a solution will fall back into it when pushed slightly. Formally, this happens along a half-line when there is a negative eigenvalue.
A critical point is stable only if all directions are stable.

Unstable
Intuitively, a critical point is unstable if a solution will fall away from it when pushed slightly. Formally, this happens along a half-line when there is a positive eigenvalue.
A critical point is unstable if any direction is unstable.

Saddle Point
Consider the system of linear equations $y' = Ay$ with critical point y_0. If A has real eigenvalues $\lambda_1 \neq \lambda_2$, such that $\lambda_1 < 0 < \lambda_2$, then we say that y_0 is a saddle point.

Saddle Point
Consider the system of linear equations $y' = Ay$ with critical point y_0. If A has real eigenvalues $\lambda_1 \neq \lambda_2$, such that $\lambda_1 < 0 < \lambda_2$, then we say that y_0 is a saddle point.
Intuitively, this corresponds to one direction being stable and another being unstable. Overall, a saddle point is unstable.

Nodal Sink
Consider the system of linear equations $y' = Ay$ with critical point y_0. If A has real eigenvalues $\lambda_1 \neq \lambda_2$, such that $\lambda_1 < \lambda_2 < 0$, then we say that y_0 is a nodal sink.
Intuitively, this corresponds to all directions being stable and so a nodal sink is stable.

Nodal Source
Consider the system of linear equations $y' = Ay$ with critical point y_0. If A has real eigenvalues $\lambda_1 \neq \lambda_2$, such that $0 < \lambda_1 < \lambda_2$, then we say that y_0 is a nodal source.
Intuitively, this corresponds to all directions being unstable and so a nodal source is unstable.

Center
Consider the system of linear equations $y' = Ay$ with critical point y_0. If A has a complex eigenvalue $\lambda = \alpha + i\beta$ with $\beta \neq 0$ that is purely imaginary, that is $\alpha = 0$, then we say that y_0 is a center.
Intuitively, this corresponds to all solutions being closed loops around the critical point. These are neither unstable nor stable.

Spiral Source
Consider the system of linear equations $y' = Ay$ with critical point y_0. If A has a complex eigenvalue $\lambda = \alpha + i\beta$ with $\beta \neq 0$ and $\alpha > 0$, then we say that y_0 is a spiral source.
Intuitively, this corresponds to all solutions spiraling outwards from the critical point. This is an unstable critical point.

Spiral Sink
Consider the system of linear equations $y' = Ay$ with critical point y_0. If A has a complex eigenvalue $\lambda = \alpha + i\beta$ with $\beta \neq 0$ and $\alpha < 0$, then we say that y_0 is a spiral sink.
Intuitively, this corresponds to all solutions spiraling inwards towards the critical point. This is a stable critical point.

Trace-Determinant Plane

To better visualize these distinctions, we can break down the characteristic equation of our system. To do so, we define the trace of a matrix as

Trace
The trace of an $n \times n$ matrix is the sum of the diagonal elements. This is written as $\text{Tr}(A)$.

A useful formulation is to express the trace and determinant of a matrix with its eigenvalues

Theorem. Let A be a $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, repeated according to multiplicity. Then

$$\det(A) = \lambda_1 * \lambda_2 * \cdots * \lambda_n \qquad \text{and} \qquad \text{Tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

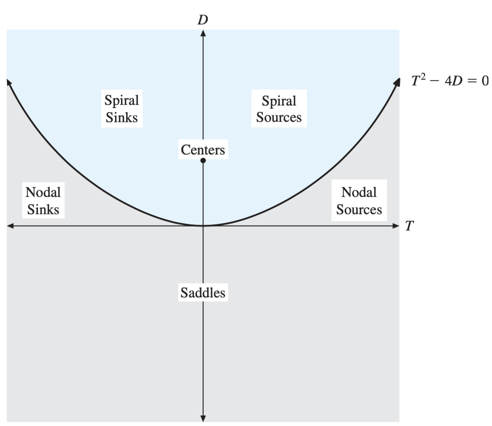
For a matrix A, we can write the characteristic equation of A as $\lambda^2 - \text{Tr}(A)\lambda + \det(A)$. Then quadratic formula tells us that the eigenvalues of A are given by

$$\lambda = \frac{\text{Tr}(A) \pm \sqrt{\text{Tr}(A)^2 - 4\det(A)}}{2}$$

Which means that we can determine exactly what the eigenvalues are based on the trace and determinant of our matrix. In particular, we can determine how many distinct eigenvalues there are through the trace and determinant by

$$\begin{aligned}
\text{Tr}(A)^2 - 4\det(A) > 0, \det(A) \neq 0 \quad &\Longrightarrow \quad \text{two distinct real nonzero eigenvalues} \\
\text{Tr}(A)^2 - 4\det(A) < 0 \quad &\Longrightarrow \quad \text{complex eigenvalues} \\
\text{Tr}(A)^2 - 4\det(A) = 0 \quad &\Longrightarrow \quad \text{real eigenvalue with multiplicity two} \\
\det(A) = 0 \quad &\Longrightarrow \quad \text{one real and one zero eigenvalue}
\end{aligned}$$

Using this breakdown, you can determine which critical point occurs through his helpful illustration. It



should be noted here that there are five open regions that define the **generic cases** - nodal source, spiral source, spiral sink, nodal sink, and saddles. We will discuss the remaining, non-generic cases next week.

Week 10

Nongeneric Cases

Exponential of a Matrix

When we used eigenvalues to form a solution, everything became more difficult when we hit a case where we only had one eigenvalue of higher multiplicity. Things got especially difficult when the multiplicity of that eigenvalue was lower than the dimension of its eigenspace. Here, we hope to simplify that process and generalize it, by introducing the exponential of a matrix.

Exponential of a Matrix
For a square matrix A, the exponential of A is defined as

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k$$

where $A^0 = I$ by convention.

We pretty immediately have some important formulas

Theorem. *(Exponential of Diagonal Matrix)*

$$A = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix} \qquad \Longrightarrow \qquad e^A = \begin{pmatrix} e^{a_1} & 0 & \dots & 0 \\ 0 & e^{a_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{a_n} \end{pmatrix}$$

Theorem. *(Zero Matrix) :* For the zero matrix 0,

$$\frac{d}{dt}e^0 = I$$

Theorem. *(Derivative of Exponential) :* For a square matrix A,

$$\frac{d}{dt}e^{tA} = Ae^{tA}$$

Theorem. *(Law of Exponents) :* For square matrices A, B,

$$e^{A+B} = e^A e^B$$

if and only if $AB = BA$.

Theorem. *(Inverse) :* If A is a square matrix, then e^A is invertible with inverse e^{-A}.

Theorem. *(Solution to IVP) :* Given the IVP $y' = Ay$ with $y(0) = v$, the solution is $y = e^{At}v$.

For computing the solution $y = e^{At}v$, it will be necessary to compute $Av, A^2v, \ldots$. The nice part of this is that if $A^\ell v = 0$, then the expression $e^{At}v$ gets truncated as

$$e^{At}v = v + tAv + \frac{t^2}{2}A^2v + \cdots + \frac{t^{\ell-1}}{(\ell-1)!}A^{\ell-1}v$$

Generalized Eigenvector

To generalize our solution process, we need one final piece, the notion of a generalized eigenvector.

Generalized Eigenvector
If λ is an eigenvalue of A, then we call v a generalized eigenvector if v satisfies

$$[A - \lambda I]^p v = 0$$

for some integer $p \geq 1$.

Solution Procedure

Theorem. *(Forming a Solution) :* For every eigenvalue λ of multiplicity q, we can find q linearly independent solutions through the following process.

(i) Find the smallest p such that the nullspace of $[A - \lambda I]^p$ has dimension q.

(ii) Find a basis $\{v_1, \ldots, v_q\}$ of the nullspace of $[A - \lambda I]^p$.

(iii) For each v_j, we have the solution

$$x_j(t) = e^{tA}v_j = e^{\lambda t}\left(v_j + t[A - \lambda I]v_j + \cdots + \frac{t^{p-1}}{(p-1)!}[A - \lambda I]^{p-1}v_j\right)$$

The expansion of $e^{tA}v_j$ is truncated at the $p-1$ term because $[A - \lambda I]^p v_j = 0$.

With this process, we can find solutions to any system of first order linear equations by finding the eigenvalues and then following this process to find our solutions.