8/7/25 - (Partial) Derivative Review

Derivatives

given a function $f(x)$, its derivative gives the rate at which $f(x)$ changes as x is varied. This is denoted $f'(x)$ or $\frac{df}{dx}$.

example

- if $x(t)$ gives the position of an object at time t, then $x'(t)$ give the velocity of the object and $x''(t)$ give the acceleration of the object

Formal definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Basic derivatives

constant function

constants: $\frac{d}{dx} 5 = 0$

power rule: $\frac{d}{dx} x^n = n x^{n-1}$

trig functions: $\frac{d}{dx} \sin x = \cos x$

$\frac{d}{dx} \cos x = -\sin x$

exponentials: $\frac{d}{dx} e^x = e^x \longleftarrow$ $f' = f$

logarithm: $\frac{d}{dx} \log x = 1/x$

Derivative Rules

constant multiple: $(a f(x))' = a f'$

sum: $(f(x) + g(x))' = f' + g'$

linearity of d/dx

product: $(f(x) \cdot g(x))' = f'g + fg'$ $(fgh)' = f'gh + fg'h + fgh'$

quotient: $\left(\frac{f(x)}{g(x)}\right)' = \frac{gf' - g'f}{g^2}$

composition

composition (chain rule): $(f(g(x)))' = (f \circ g(x))' = f'(g(x))\, g'(x)$

$(f(g(h)))' = f'(g(h))\, g'(h)\, h'$

Fundamental Theorem of Calculus

$$\frac{d}{dx} \int_0^x f(y)\,dy = f(x)$$

$$\int_0^x \frac{df}{dy}\,dy = f(x) - f(0)$$

examples

- $$(5x^3 + 2x^2 + 3x + 6)' = (\underline{5}x^3)' + (\underline{2}x^2)' + (\underline{3}x)' + (\underline{6})'$$
$$= 5 \cdot 3x^2 + 2 \cdot 2x + 3$$
$$= 15x^2 + 4x + 3$$

- $$\frac{d}{dx}[\sin^2(x)] = (\sin x \cdot \sin x)' = 2\cos x \cdot \sin x$$
$$= ((\sin x)^2)' = 2\sin x \cdot (\sin x)'$$
$$= 2\sin x \cos x$$

- $$\frac{d}{dx}(\sin x \cdot \cos x \cdot x) = x\cos^2 x - x\sin^2 x + \sin x \cdot \cos x$$

- $$\frac{d}{dx}[\tan(x)] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] = \frac{\overbrace{\cos^2 x + \sin^2 x}^{1}}{\cos^2 x} = \sec^2 x$$

$\exp(\sim) = e^{\sim}$

- $$\frac{d}{dx}[\exp(-x^2)] = \exp(-x^2)\,(-x^2)'$$
$$= \underbrace{-2x}_{}\,\underbrace{\exp(-x^2)}_{}$$

Partial Derivatives

given a function of multiple variables $f(x,y)$, a partial derivative $\frac{\partial f}{\partial x}$ is the derivative of f w.r.t. x if y is treated as a constant.

Denoted $\frac{\partial f}{\partial x}$ or $\partial_x f$ or (sometimes) f_x.

examples

- $$\frac{\partial}{\partial x}(\underset{\text{const}}{y}x^2 + \underset{\text{const}}{y^2}x + \underset{\text{const}}{\sin y}) = y\frac{\partial}{\partial x}(x^2) + y^2\frac{\partial}{\partial x}(x) + 0$$
$$= 2yx + y^2$$

- $$\frac{\partial}{\partial y}[\sin(\overset{\text{const}}{x}y)] = \cos(xy)\frac{\partial}{\partial y}(xy) = x\cos(xy$$

- $$\frac{\partial}{\partial z}[\exp(\overset{\text{const}}{x^2}+\overset{\text{const}}{y^2}+z^2)] = \exp(x^2+y^2+z^2)\frac{\partial}{\partial z}(\overset{\text{const}}{x^2}+\overset{\text{const}}{y^2}+z^2)$$
$$= 2z\exp(x^2+y^2+z^2)$$

$$\frac{\partial}{\partial z}(\overbrace{e^{x^2}e^{y^2}}^{\text{const}}e^{z^2}) = 2z\underbrace{e^{x^2}e^{y^2}e^{z^2}}_{\text{product}} = 2z\exp(\underbrace{x^2+y^2+z^2}_{\text{sum}})$$

Application to ODEs

Given a first order ODE in normal form,

$$x'(t) = \underbrace{f(x,t)}_{\text{how "nice"?}}$$

Monday

Module 4

<u>example</u> Suppose $x'(t) = f(x,t) = t^2x + \sin t$.

Find $\frac{\partial f}{\partial x} = t^2$

$\frac{\partial f}{\partial t} = 2tx + \cos t$

Separable ODEs

← really useful for nonlinear ODEs

$$\int \frac{dx/\cancel{dt}}{f}\,\cancel{dt} = \int g(t)\,dt$$

$$\int \frac{1}{f(x)}\,dx =$$

ODEs of the form

$$\boxed{\frac{dx}{dt}} = \underbrace{f(x)}_{\text{dep}}\underbrace{g(t)}_{\text{ind}}$$

can be separated like $\frac{dx/dt}{f(x)} = g(t)$

and then integrated to solve for $x(t)$.

$$\frac{dx/dt}{f(x)} = g(t) \Rightarrow \underbrace{\int \frac{dx}{f(x)}}_{\text{all } x \text{ dep.}} = \underbrace{\int g(t)\,dt}_{\text{all } t \text{ dep}}$$

example: $\frac{dx}{dt} = x^2 \sin t$

$$\int \frac{x'}{x^2}\,dt = \int \sin t\,dt$$

$$\int \frac{dx}{x^2} = \int \sin t\,dt$$

$$+x^{-1} = +\cos t + C$$

$$x(t) = \frac{1}{\cos t + C}$$

general solution

Linear, first order ODEs

given
$$\frac{dy}{dx} = a(x)\,y + f(x)$$

we rearrange
$$\frac{dy}{dx} - a(x)y = f(x)$$

and introduce an integrating factor $\mu = \exp\left(-\int a(x)\,dx\right)$ as

$$\mu\frac{dy}{dx} - \overbrace{\mu a}^{-\mu'} y = \mu f(x)$$

so that
$$\frac{d}{dx}(\mu y) = \mu f(x)$$

then
$$\boxed{y = \frac{1}{\mu}\int \mu(x) f(x)\,dx}$$

$$y' + by = f(x)$$

$$\mu = \exp\left(\int b(x)\,dx\right)$$

example: solve $\cos(x)y' + \sin(x)y = \cos^2(x)$

rearranging, $y' + \tan(x)y = \cos x$

then $\mu = \exp\int \tan x\,dx = \exp(-\log\cos x)$

so $\mu = 1/\cos x$

Then $\frac{d}{dx}(y/\cos x) = 1$

$$y = a\cdot\cos x$$

for some constant a.

Differential forms

a differential form is an expression of the form

$$P(x,y)\,dx + Q(x,y)\,dy \qquad \text{(or more variables)}$$

Similarly, given a function $f(x,y)$, we define its differential as

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

If a differential form can be written as df for some f, then we say it is exact.

Given $P(x,y)\,dx + Q(x,y)\,dy = 0$, a solution is a function $F(x,y)$ such that $dF = \mu(P\,dx + Q\,dy)$ for some integrating factor μ. (More about this on Monday)

__example__ find F s.t. $dF = (2xy - 9x^2)\,dx + (2y + x^2 + 1)\,dy$

one method: integrate both terms

$$\frac{\partial F}{\partial x} = 2xy - 9x^2 \Rightarrow F(x,y) = \int 2xy - 9x^2\,dx$$

important

$$= x^2y - 3x^3 + g(y)$$

and $\frac{\partial F}{\partial y} = 2y + x^2 + 1 \Rightarrow F(x,y) = y^2 + x^2y + y + h(x)$

matching terms,

$$\boxed{F(x,y) = x^2y - 3x^3 + y^2 + y + c}$$

8/12/25

My OH: Tue 1-2pm

Roman: Thr 2-3pm

this zoom.

Linear, first order ODEs

Suppose we have an ODE of the form

$$\boxed{y' = a(x)\,y + b(x)}$$

idea: product rule $(fg)' = f'g + fg'$

$\mu y' - \mu a y = b\mu$ $\qquad \mu' = -a\mu \implies \mu = \exp\left(-\int a\,dx\right)$

$(\mu y)' = b\mu$

<u>example</u> $ty' + 2y = t^2 - t + 1$ w/ $y(1) = 1/2$, $t > 0$

$y' + \underbrace{\frac{2}{t}}_{-a} y = t - 1 + 1/t$

$\mu = \exp \int 2/t\,dt = \exp(2 \log t) = t^2$

$t^2 y' + 2ty = t^3 - t^2 + t$

$\int (t^2 y)' = \int t^3 - t^2 + t\,dt$

$$\boxed{y = \frac{1}{4}t^2 - \frac{1}{3}t^1 + \frac{1}{2} + ct^{-2}}$$

<u>example</u> $y' + (\log t)\,y = 0$

Differential Forms

(warning: googling this may result in more advanced math)

a differential form is an expression of the form

$$\underbrace{P(x,y)\,dx + Q(x,y)\,dy}$$ (or more variables)

Roughly, it is "something that can be integrated"

Given a function $f(x,y)$, we define its differential as

$$df = \underbrace{\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy}$$

If $P\,dx + Q\,dy = df$ for some f, then we say that

$P\,dx + Q\,dy$ is exact.

theorem (condition for exactness)

If $P(x,y)$, $Q(x,y)$ are nice functions (continuosly differentiable)

on a region w/o holes, then $P\,dx + Q\,dy$ is exact if and only if

$$\partial P/\partial y = \partial Q/\partial x$$

examples determine if the following are exact and find f if so

- $(xy^2 + y + e^x)\,dx + (x^2y + x)\,dy = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

  $\downarrow \partial/\partial y$ $\qquad \downarrow \partial/\partial x$

  $2xy + 1 + 0 = 2xy + 1$ ✓ yes, it is exact.

  $\frac{\partial f}{\partial y} = \sim + c'(y)$

  $* \int \frac{\partial f}{\partial x}dx = \int xy^2 + y + e^x\,dx \Rightarrow f = \frac{1}{2}x^2y^2 + xy + e^x + c(y)$

  $\int \frac{\partial f}{\partial y}dy = \int x^2y + x\,dy \Rightarrow f = \frac{1}{2}x^2y^2 + xy + k(x)$

  $f = \frac{1}{2}x^2y^2 + xy + e^x$

- $(2e^x + y^3)\,dx + 3y^2\,dy$

  $\xrightarrow{\partial/\partial y} 3y^2 \qquad \xrightarrow{\partial/\partial x} 0$

  $3y^2 \neq 0 \rightsquigarrow$ not exact

Solving differential forms

Given an equation of the form $P + Q\frac{dy}{dx} = 0$

$$\omega = \underbrace{P(x,y)dx + Q(x,y)dy = 0}$$ $\longrightarrow$ $P\frac{dx}{dy} + Q = 0$

a "solution" is a relationship $\underbrace{F(x,y) = 0}$ between x and y.

If ω is exact, $\omega = dF = 0 \Rightarrow \boxed{F(x,y) = C}$ solution

If ω is not exact, then we need to introduce an integrating factor μ s.t. $\mu\omega = \underbrace{\mu P dx + \mu Q dy}$ is exact.

Then $\mu\omega = dF = 0 \rightsquigarrow \boxed{F(x,y) = C}$ solution

① separable

given $a(x)b(y)dx + c(x)f(y)dy = 0$,

then $\frac{a(x)}{c(x)}dx + \frac{f(y)}{b(y)}dy = 0$, $\mu = \frac{1}{c(x)b(y)}$

$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$

example solve $\frac{\sin x}{y}dx + \frac{y}{\cos x}dy = 0$

extra exercise

② **linear**

$$\underbrace{\frac{dy}{dx} + ay + b = 0}$$ $$\underbrace{\frac{dx}{dy} + fx + g = 0}$$

given $dy + (a(x)y + b(x))dx = 0$ or $dx + (f(y)x + g(y))dy = 0$

then $\mu = \exp\left(\int a\,dx\right)$ or $\mu = \exp\left(\int f\,dy\right)$

example $dy + (e^t y - e^t)dt = 0$

extra exercise

③ **homogeneous**

if $P(\lambda x, \lambda y) = \lambda^n P(x,y)$ and $Q(\lambda x, \lambda y) = \lambda^n Q(x,y)$ for some n

e.g. $x^2 \rightsquigarrow \lambda^2 x^2$ $\quad x^2 + xy + y^2 \rightsquigarrow \lambda^2(x^2 + xy + y^2)$

then make the change of variables $y(x) = xv(x)$ $\quad (dy = v\,dx + x\,dv)$

so $P(x, xv)\,dx + Q(x, xv)(v\,dx + x\,dv)$ is exact

$$= \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial v}dv \quad \text{for some } F(x,v)$$

example solve $\underbrace{x^3}_{\text{ord } 3}dx + \underbrace{xy^2}_{\text{ord } 3}dy = 0$

$y = xv \rightsquigarrow x^3 dx + x(xv)^2(x\,dv + v\,dx) = 0$

$$x^3 dx + x^4 v^2 dv \quad x^3 v^3 dx = 0$$

$$x^3(1 + v^3)dx + x^4 v^2 dv = 0 \qquad \text{separable !!}$$

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = \frac{1}{x}dx + \frac{v^2}{1+v^3}dv = 0$$

$$\boxed{m = \log x + \frac{1}{3}\log\left(1 + \frac{y^3}{x^3}\right)}$$

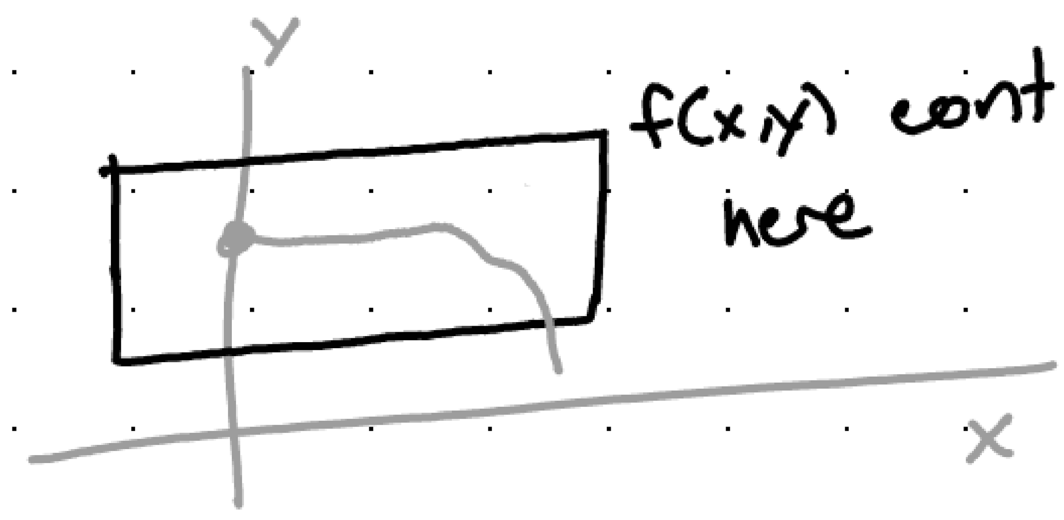
solution

$\frac{\partial F}{\partial x} = 1/x \rightsquigarrow F(x,y) = \log x + c(y)$

$\frac{\partial F}{\partial y} = \frac{v^2}{1+v^3} \rightsquigarrow F(x,y) = \frac{1}{3}\log(1 + v^3) + k(x)$

8/14/25

Existence and Uniqueness (well-posedness)



Consider the ODE $dy/dx = f(x,y)$ w/ an initial condition $y(0) = y_0$.

① if $f(x,y)$ is continuous on some rectangle containing $(0, y(0))$,

the a solution $y(x)$ exists as long as $(x, y(x))$ is in the rectangle

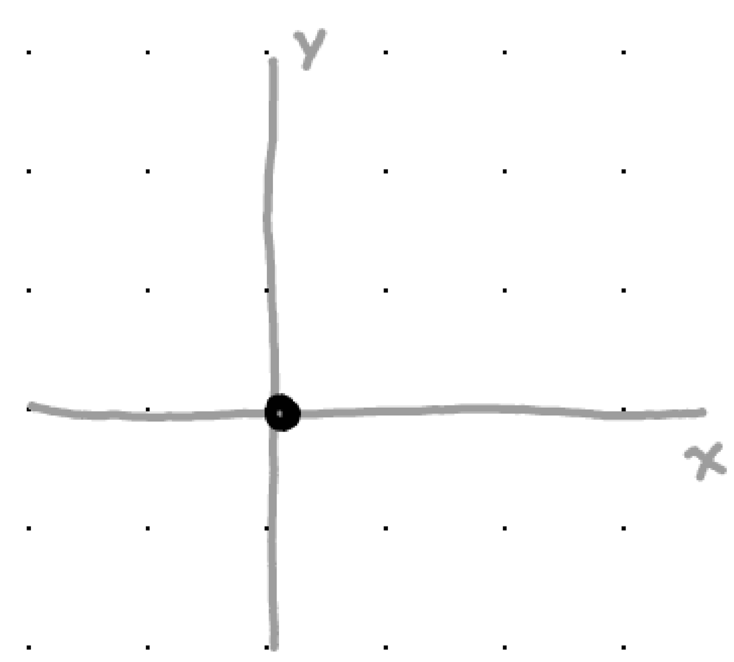
② if $\partial f/\partial y$ is continuous on some rectangle containing $(0, y(0))$,

then a solution $y(x)$ is unique

examples

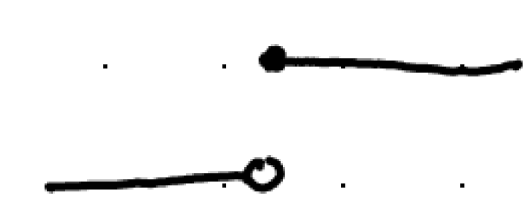
- $y' = y^2$, $y(0) = 0$

  $f(x,y) = y^2$



$f(x,y)$ is cont ✓ exists ✓

look out for y/x



$\frac{\partial f}{\partial y} = 2y$ cont ⇝ unique sol ✓

$y' = y^2$ $\quad \int \frac{y'}{y^2} dx = \int 1 dx$

$-y^{-1} = \int \frac{dy}{y^2} = x + C$

$$\boxed{y(x) = \frac{1}{-x-c}}$$

Non-Uniqueness (modified after discussion)

consider $y' = x\sqrt{|y|}$ w/ $y(0)=0$

existence

the RHS $x\sqrt{|y|}$ is continuous in both x and y so a solution exists.

uniqueness

Using the fact that $|y| = \begin{cases} y, & \text{for } y \geq 0 \\ -y, & \text{for } y < 0 \end{cases}$

We calculate $\frac{\partial f}{\partial y} = \begin{cases} \frac{x}{2\sqrt{y}} & \text{for } y>0 \\ \text{DNE} & \text{for } y=0 \\ \frac{-x}{2\sqrt{-y}} & \text{for } y<0 \end{cases}$

In particular, $\partial f/\partial y$ DNE (and is discont.) at $y=0$.

So uniqueness is not guaranteed.

To show uniqueness fails, we construct 2 solutions.

1. if $y(x)=0$ then $y' = 0 = x\sqrt{|y|}$ and $y(0)=0$.

   So $y=0$ is a solution.

2. Assume $y \geq 0$ so that $|y|=y$. Then the eq is separable

   so we can solve

$$y' = x\sqrt{y} \Leftarrow \frac{y'}{\sqrt{y}} = x$$

$$\Leftrightarrow \int \frac{dy}{\sqrt{y}} = \int x\,dx$$

$$\Leftrightarrow 2\sqrt{y} = \frac{1}{2}x^2 + c$$

different c

$$\Leftrightarrow y = \left(c + \frac{1}{4}x^2\right)^2$$

$y(0)=0$ then implies $c=0$, so $y = \frac{1}{2}x^4$ is another solution.

Autonomous ODEs

$\underline{y' = f(y)}$ ← no dependence on t,

pros: ① separable ⇝ easily find solutions

② qualitatively easy

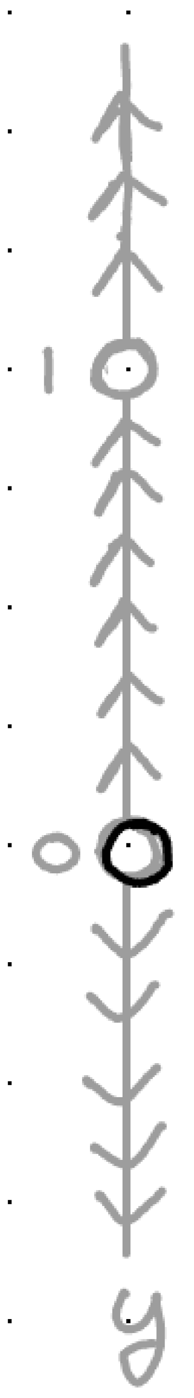
<u>example</u>: $y' = \underbrace{(y-1)^2 y}$

fixed points: (equilibrium sols)

want solutions $y = c \iff y' = 0$ $\quad (y-1)^2 y = 0$

zeros: $\boxed{y=1 \quad y=0}$

phase curve



$(y-1)^2 y = y'$

$\quad + \quad + \quad +$

$(y-1)^2 y = y'$

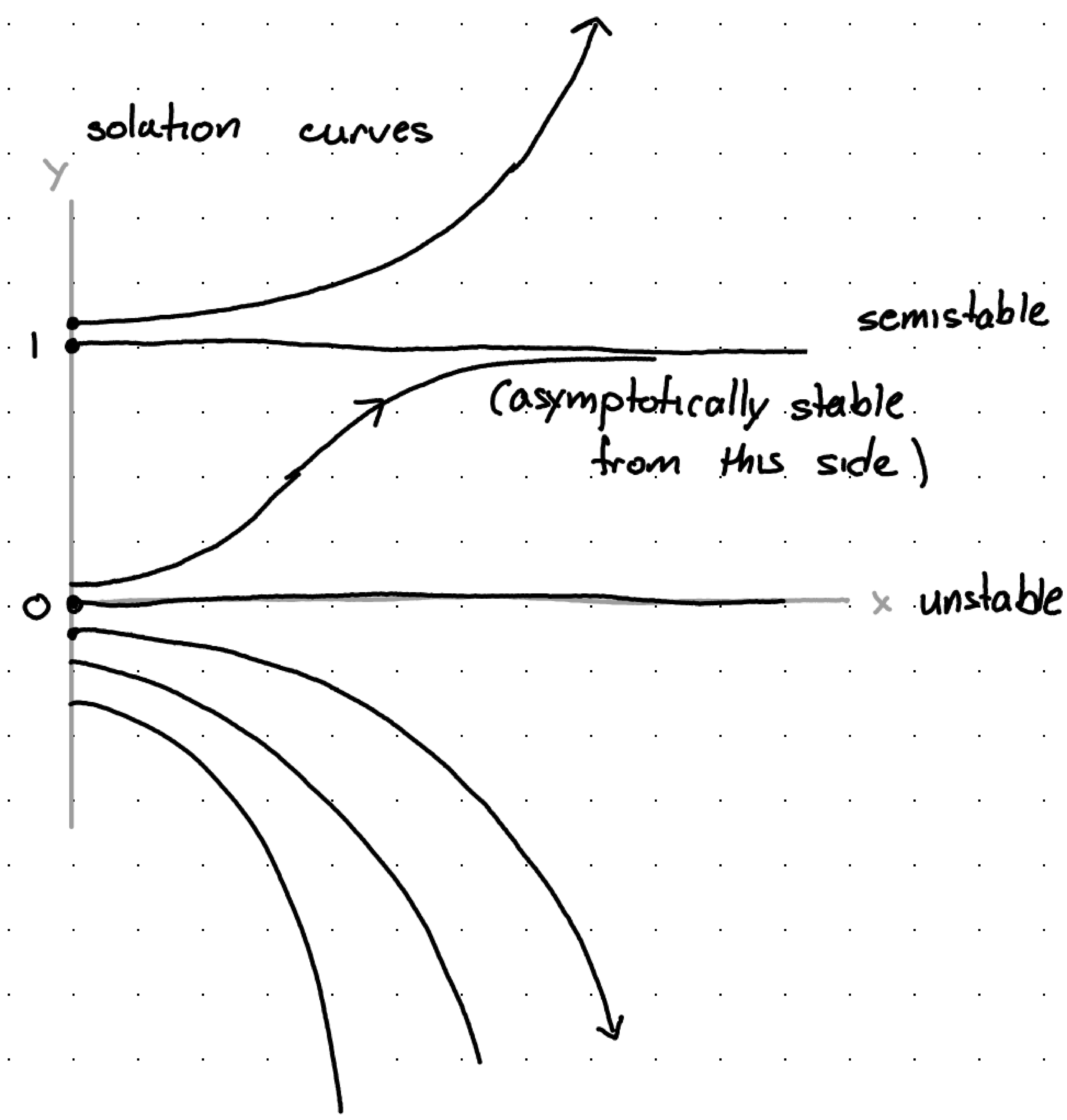
$(-)^2 \quad + \quad +$

$(y-1)^2 y = y'$

$(-)^2 \quad - \quad = \quad -$

stability

solution curves



<u>extra examples</u>

$y' = \sin(y)$, $\quad y' = $ any polynomial

2^{nd} Order Linear ODEs

$$y'' + a(x)y' + b(x)y = c(x) \qquad *$$

2^{nd} order : b/c y''

linear : y_1 y_2 solutions $\Rightarrow$ $y_1 + y_2$ is a solution

linearly independent solutions

given solutions y_1, y_2 to $*$, we know that

$$a_1 y_1 + a_2 y_2$$

is a solution for all constants a_1, a_2

If y_1, y_2 are linearly independent, then all solutions can be written in this form. Then for all solns y, $y = a_1 y_1 + a_2 y_2$

example $y'' - 2y' + y = 0$ has solutions $y_1 = e^t$ and $y_2 = te^t$

show that y_1, y_2 are linearly independent

the vectors $\begin{pmatrix} y_1 \\ y_1' \end{pmatrix}$ and $\begin{pmatrix} y_2 \\ y_2' \end{pmatrix}$ are linearly independent as vectors (not parallel)

① $\begin{pmatrix} e^t \\ e^t \end{pmatrix}$ $\begin{pmatrix} te^t \\ e^t + te^t \end{pmatrix}$ choose specific t

$t = 0$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

not parallel $\Rightarrow$ linearly ind.

② Wronskian $\det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$ $= 0$ not linearly independent; $\neq 0$ linearly independent

$$\det\begin{pmatrix} e^t & te^t \\ e^t & e^t + te^t \end{pmatrix} = \underbrace{e^t(e^t + te^t) - te^{2t}}_{1 \neq 0} = e^{2t} + \cancel{te^{2t}} - \cancel{te^{2t}} = e^{2t} \neq 0$$

$t = 0$

8/19/25

Constant Coefficients

example 1: $y'' - 9y' + 20y = 0$ w/ $y(0) = 0$ $y'(0) = 1$

homogeneous

guess: $y = e^{\lambda t}$

$y' = \lambda e^{\lambda t}$

$y'' = \lambda^2 e^{\lambda t}$

$\lambda^2 e^{\lambda t} - 9\lambda e^{\lambda t} + 20 e^{\lambda t} = 0$

$\underbrace{(\lambda^2 - 9\lambda + 20)}_{0}\underbrace{e^{\lambda t}}_{\neq 0} = 0$

find roots of the char poly $\lambda_1, \lambda_2 \rightsquigarrow e^{\lambda_1 t}$ $e^{\lambda_2 t}$

char poly: $\lambda^2 - 9\lambda + 20 = (\lambda - 4)(\lambda - 5) \rightsquigarrow$ roots: $4, 5$

solutions: e^{4t}, e^{5t} linearly independent ✓

general solution: $y = ae^{4t} + be^{5t}$ constants a, b.

example 2: $y'' - 6y' + 9y = 0$

char poly: $\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$

root: 3 w/ multiplicity 2

solutions: $\boxed{e^{3t}}$

linearly independent

idea: make e^{3t} one step more complicated $\rightsquigarrow$ $\boxed{te^{3t}}$

general solution: $\boxed{y = ae^{3t} + bte^{3t}}$

example 3: $y'' - 2y' + 2 = 0$

char poly: $\lambda^2 - 2\lambda + 2$

roots: $\frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$ complex roots.

solutions: $e^{(1+i)t}$ $e^{(1-i)t}$

real part imaginary part

general solution: $y = ae^{(1+i)t} + be^{(1-i)t}$

Euler's id: $e^{i\theta} = \cos\theta + i\sin\theta$

expand out gen soln and Euler's identify

real part

$y = e^t(a\cos t + b\sin t)$ (new constants a, b)

imaginary part

example: $3 \pm 2i \rightsquigarrow$ gen soln $y = e^{3t}(a\cos 2t + b\sin 2t)$

Facts about (real) polynomials

① if $a+bi$ is a root then $a-bi$ is a root

② a deg n polynomial will always have n roots (counting multiplicity)

Inhomogeneous equations (undetermined coefficients)

example 1: $\underbrace{y'' - 9y' + 20y}_{\text{linear ODE}} = \underbrace{x}_{f(x)}$

idea: solve homogeneous equation: $y'' - 9y' + 20y = 0$

general solution: $y_h = ae^{4x} + be^{5x}$

find 1 particular solution to the full equation y_p

combine these $y = y_h + y_p$ is the general solution to the ODE

undetermined coefficients: informed guessing

RHS: deg 1 polynomial $\rightsquigarrow$ guess $y_p = cx + d$

$$y_p' = c$$

$$y_p'' = 0$$

$$0 - 9c + 20(cx+d) = x \qquad\qquad 20d - 9/20 = 0$$

$$\underline{20cx} + 20d - 9c = \underline{x} \;\_\; \rightsquigarrow \; c = 1/20 \qquad d = 9/400$$

$$y_p = \frac{x}{20} + \frac{9}{400} \implies \boxed{y = ae^{4x} + be^{5x} + \frac{x}{20} + \frac{9}{400}}$$

example 2: $y'' - 9y' + 20y = \cos 3x$ guess $y_p = a\cos 3x + b\sin 3x$

example 3: $y'' - 9y' + 20y = xe^x$ = (deg 1 poly) · exponential

guess $y_p = (ax+b)e^x$

$y_p' = ae^x + (ax+b)e^x = (a+b)e^x + axe^x$

$y_p'' = ae^x + ae^x + (ax+b)e^x = (2a+b)e^x + axe^x$

$axe^x + (2a+b)e^x - 9axe^x - 9(a+b)e^x + 20axe^x + 20be^x = xe^x$

$(a - 9a + 20a)xe^x = xe^x$

$12a = 1 \Rightarrow a = 1/12$

$2a + b - 9a - 9b + 20b =$

$-7a + 12b = 0$

$12b = 7/12 \Rightarrow b = 7/144$

$$\left. \right\} \quad y_p = \left(\frac{x}{12} + \frac{7}{144}\right)e^x$$

Harmonic Oscillator

$$x'' + 2cx' + \omega_0^2 x = 0$$

char poly: $\lambda^2 + 2c\lambda + \omega_0^2$

roots: $\dfrac{-2c \pm \sqrt{4c^2 - 4\omega_0^2}}{2} = -c \pm \sqrt{c^2 - \omega_0^2}$

case 1: $c^2 - \omega_0^2 < 0$

case 2: $c^2 - \omega_0^2 > 0$

case 3: $c^2 - \omega_0^2 = 0$

8/21/25

Midterm Tue 9:20 - 10:50

Secular Solutions

example 1: $\underbrace{y'' - 9y' + 20y}_{\text{homogeneous part}} = e^{4x}$ } inhomogeneous

$y'' - 9y' + 20y = 0 \rightsquigarrow$ char poly: $\lambda^2 - 9\lambda + 20 = (\lambda-4)(\lambda-5)$

roots: 4, 5

$y_h = ae^{4x} + be^{5x}$ constants a, b

undetermined coeffs: guess $y_p = cxe^{4x}$

$y_p' = ce^{4x} + 4cxe^{4x}$ $y_p'' = 8ce^{4x} + 16cxe^{4x}$

$\cancel{16cxe^{4x}} + 8ce^{4x} - 9ce^{4x} - \cancel{36cxe^{4x}} + \cancel{20cxe^{4x}} = e^{4x}$

$-ce^{4x} = e^{4x} \Rightarrow c = -1$

solution: $\boxed{y = ae^{4x} + be^{5x} - xe^{4x}}$

example 2: $y'' - 4y' + 4y = e^{2x}$

homogeneous: $\lambda^2 - 4\lambda + 4 = (\lambda-2)^2$ roots 2 w/ mult 2

$y_h = ae^{2x} + bxe^{2x}$

particular: guess $y_p = cx^2e^{2x}$

$y_p' = 2cxe^{2x} + 2cx^2e^{2x}$

$y_p'' = 2ce^{2x} + 8cxe^{2x} + 4cx^2e^{2x}$

$2ce^{2x} + \cancel{8cxe^{2x}} + \cancel{4cx^2e^{2x}} - \cancel{8cxe^{2x}} - \cancel{8cx^2e^{2x}} + \cancel{4cx^2e^{2x}} = e^{2x}$

$2ce^{2x} = e^{2x} \rightsquigarrow c = 1/2$

$\boxed{y = ae^{2x} + bxe^{2x} + \frac{1}{2}x^2e^{2x}}$

Complex Formulation

example: $y'' - 4y' + 4y = e^t \sin t$ → $e^t \sin t = \mathrm{Im}\left(e^{(1+i)t}\right)$

easier: $y'' - 4y' + 4y = e^{(1+i)t}$

$y_h = ae^{2t} + bte^{2t}$

$y_p = (c+di)e^{(1+i)t}$

$y_p' = (c+di)(1+i)e^{(1+i)t}$

$y_p'' = (c+di)(1+i)^2 e^{(1+i)t}$

$= (c+di)(1+2i-1)e^{(1+i)t}$

$= (c+di)(2i)e^{(1+i)t}$

$(c+di)\cancel{e^{(1+i)t}}(2i \cancel{-4} - 4i \cancel{+4}) = \cancel{e^{(1+i)t}}$

$-2i(c+di) = 1 \qquad c+di = \frac{-1}{2i} = \frac{i}{2}$

$y_c = ae^{2t} + bte^{2t} + \frac{i}{2}e^{(1+i)t}$

$y_c = ae^{2t} + bte^{2t} + e^t\left(\frac{i}{2}\cos t - \frac{1}{2}\sin t\right)$ (complex)

real solution: $\boxed{y = ae^{2t} + bte^{2t} + \frac{1}{2}e^t \cos t}$

Variation of Parameters

example 1: find the general solution to

$$\underbrace{t y'' - (t+1)y' + y = t^2}$$

given that $y_1 = e^t$ and $y_2 = t+1$ are homogeneous solutions

Idea: rewrite as $y'' + \underbrace{q(t)y' + r(t)y = g(t)}$

guess a solution of the form $\underbrace{y_p = v_1 y_1 + v_2 y_2}$

then $\underbrace{v_1' = \frac{-y_2 g}{W(t)}}$ and $v_2' = \frac{y_1 g}{W(t)}$

where $W(t) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$ is the Wronskian

$$W(t) = \det\begin{pmatrix} e^t & t+1 \\ e^t & 1 \end{pmatrix}$$

$$= e^t - (t+1)e^t = -te^t$$

$$y'' - \frac{(t+1)}{t} y' + \frac{y}{t} = t$$

$$y_p = v_1 e^t + v_2 (t+1)$$

$$v_1' = \frac{-(t+1)\cancel{t}}{-\cancel{t}e^t} = (t+1)e^{-t}$$

$$v_2' = \frac{e^t t}{-te^t} = -1$$

$$v_1 = \int te^{-t}\,dt + \int e^{-t}\,dt$$

$$v_2 = -t$$

$$v_1 = -te^{-t} - 2e^{-t}$$

$$y_p = v_1 y_1 + v_2 y_2$$

extra notes / problems

Paul's online math notes

example 2: find the general solution to $y'' - 2y' + y = \frac{e^t}{t^2+1}$

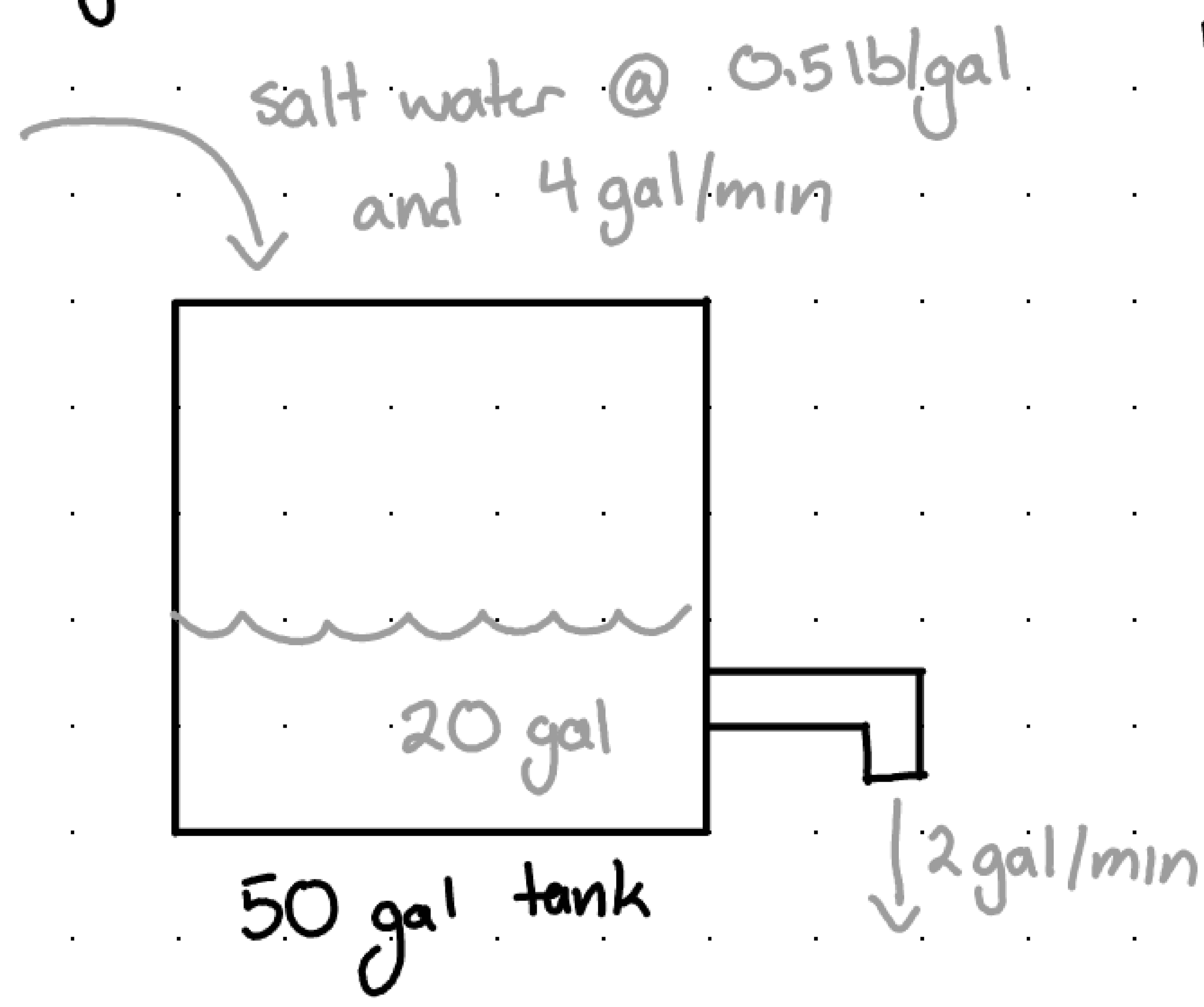
WARNING: don't guess: $\boxed{\frac{ae^t}{bt^2+ct+d}}$

Extra time

Existence/Uniqueness

consider $y' = x\sqrt{|y|}$ w/ $y(0)=0$. Does there exist a unique solution?

Mixing Problems



What is the salt content when the tank is full?

8/28/25

Logistics: Midterms are graded by Professor Roper and the TAs.

Should take ~1 week.

Note: problems 5+6 were intended to be challenging

Eigenvalues + Eigenvectors

Given a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, an eigenvalue is a constant λ s.t. there exists a vector $\vec{v} \neq \vec{0}$ which satisfies

$$A\vec{v} = \lambda \vec{v}.$$

The vector $\vec{v}$ is then an eigenvector of A w/ eigenvalue λ.

<u>example</u> find the eigenvalues/vectors of $A = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$

char poly: $\det \begin{bmatrix} 1-\lambda & 0 \\ 3 & -2-\lambda \end{bmatrix} = (1-\lambda)(-2-\lambda) - \cancel{3 \cdot 0} = -(1-\lambda)(2+\lambda)$

roots: $\underline{1}$, -2 eigenvalues!

find $\vec{v}$ in the kernel of $A - \lambda I$

$\lambda = 1$ $\begin{bmatrix} 0 & 0 \\ 3 & -3 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{matrix} 0a + 0b = 0 \\ 3a - 3b = 0 \end{matrix}$ $\Leftrightarrow$ $a = b$ eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = -2$ $\begin{bmatrix} 3 & 0 \\ 3 & 0 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Leftrightarrow$ $\begin{matrix} 3a = 0 \\ 3a = 0 \end{matrix}$ $a = 0$ eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

<u>example</u> $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ real matrix

if a real matrix has eigenval $a+bi$ then $a-bi$ is an eigenval

char poly: $\det \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$

eigenvals: $\pm i$

a, b are real numbers

$\lambda = i$ $\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{pmatrix} a+bi \\ c+di \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Leftrightarrow$ $\begin{pmatrix} -ia + b + c + di \\ -a - ib - ic + d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$b + c = 0$ $-a + d = 0$ $b = 1$ $c = -1$ $a = 1$ $d = 1$

$-a + d = 0$ $b + c = 0$

$$\boxed{\begin{matrix} \vec{v} = \begin{pmatrix} 1+i \\ -1+i \end{pmatrix} \times i \\ \lambda = -i \quad \vec{v} = \begin{pmatrix} 1-i \\ -1-i \end{pmatrix} \end{matrix}}$$

<u>idea:</u> pick something easy

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-iz + w = 0$ $\quad z = 1 \quad$ $-i + w = 0$

$w = i$

$\lambda = i \quad \begin{pmatrix} 1 \\ i \end{pmatrix}$

example: $A = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix}$

$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

char poly: $\det \begin{pmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{pmatrix} = (7-\lambda)(3-\lambda) + 4$

$= 21 - 10\lambda + \lambda^2 + 4 = 25 - 10\lambda + \lambda^2 = (\lambda-5)^2$

$\lambda = 5$: $\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff$ $\begin{matrix} 2a+b=0 \\ -4a-2b=0 \end{matrix}$ $\iff -2(2a+b)=0$ $\quad b = -2a$

$a = 1 \quad b = -2$ $\quad \boxed{\begin{pmatrix} 1 \\ -2 \end{pmatrix}}$

this one is extra!

example $A = \begin{bmatrix} -4 & -17 \\ 2 & 2 \end{bmatrix}$

again, pick something easy.

Application to ODEs

eigenvals 1 w/ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
-2 w/ $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Eric!

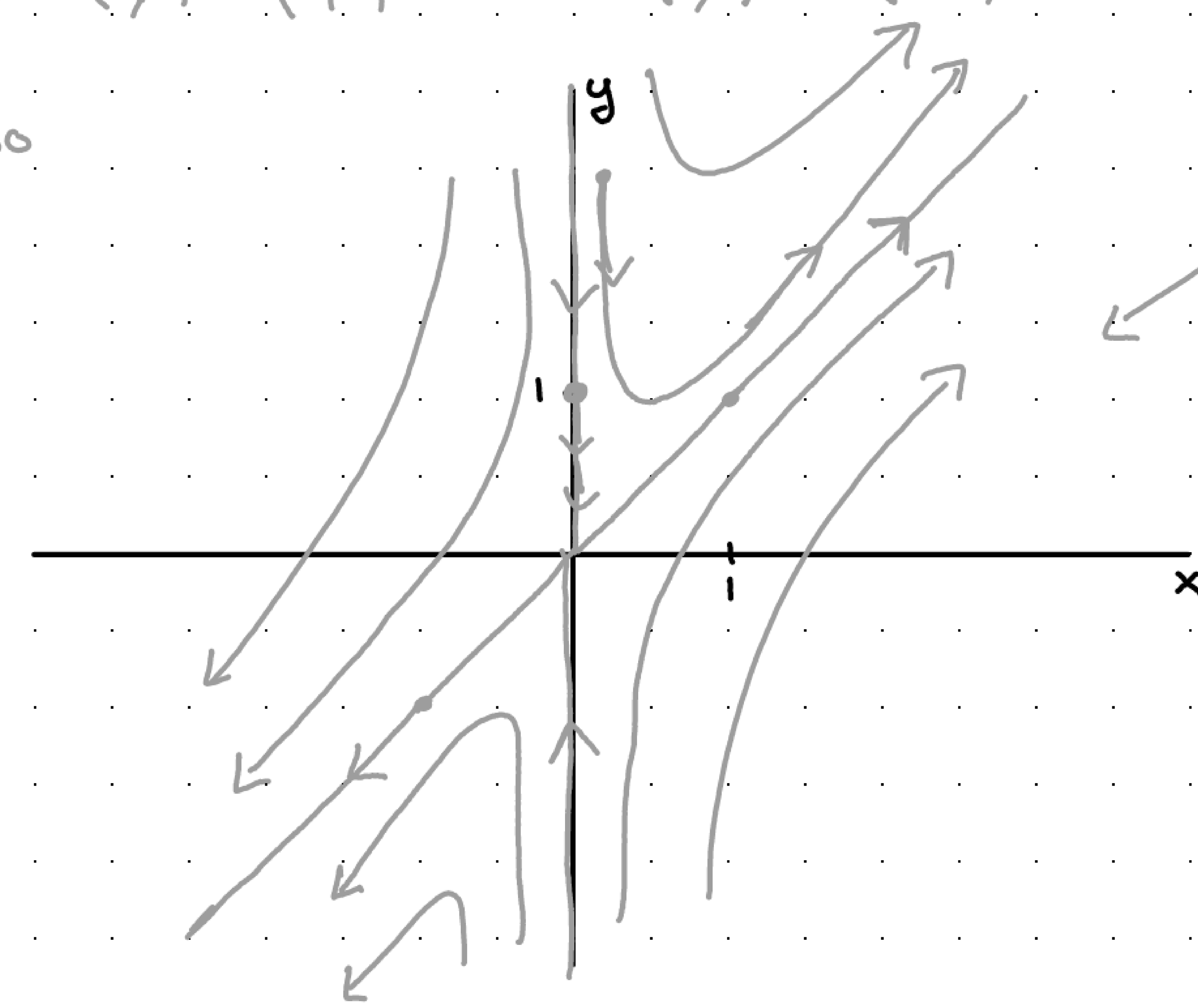
fun note: these kinds of diagrams are the main subject of math 134. It's a fun class!

$$\begin{cases} x' = x \\ y' = 3x - 2y \end{cases} \iff \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

if $\underbrace{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$ then $\begin{pmatrix} x \\ y \end{pmatrix}' = \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

if $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then $\begin{pmatrix} x \\ y \end{pmatrix}' = -2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

so



generalized eigenvecs

$(A - \lambda I)\vec{v} = \vec{0}$

$(A - \lambda I)^2 \vec{v} = \vec{0}$

generalized eigenvector

similar to constant coefficients, we may use these eigenvals/vectors to write the general solution

$e^{\lambda t}\vec{v}$ ... is a sol if $\lambda, \vec{v}$ eigenval/vec pair

general sol: $\vec{x} = a\underbrace{e^{t}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

9/2/25

Logistics: midterms (probably) returned sometime this week

Same-sign real eigenvalues

Consider $\begin{cases} x' = 7x - y \\ y' = -x + 7y \end{cases}$

We will find the general solution and sketch.

$$\begin{bmatrix} 7 & -1 \\ -1 & 7 \end{bmatrix} = A, \qquad \det(A - \lambda I) = \det\begin{pmatrix} 7-\lambda & -1 \\ -1 & 7-\lambda \end{pmatrix}$$

$$= (7-\lambda)^2 - 1$$

$$= 49 - 14\lambda + \lambda^2 - 1$$

$$= \lambda^2 - 14 + 48$$

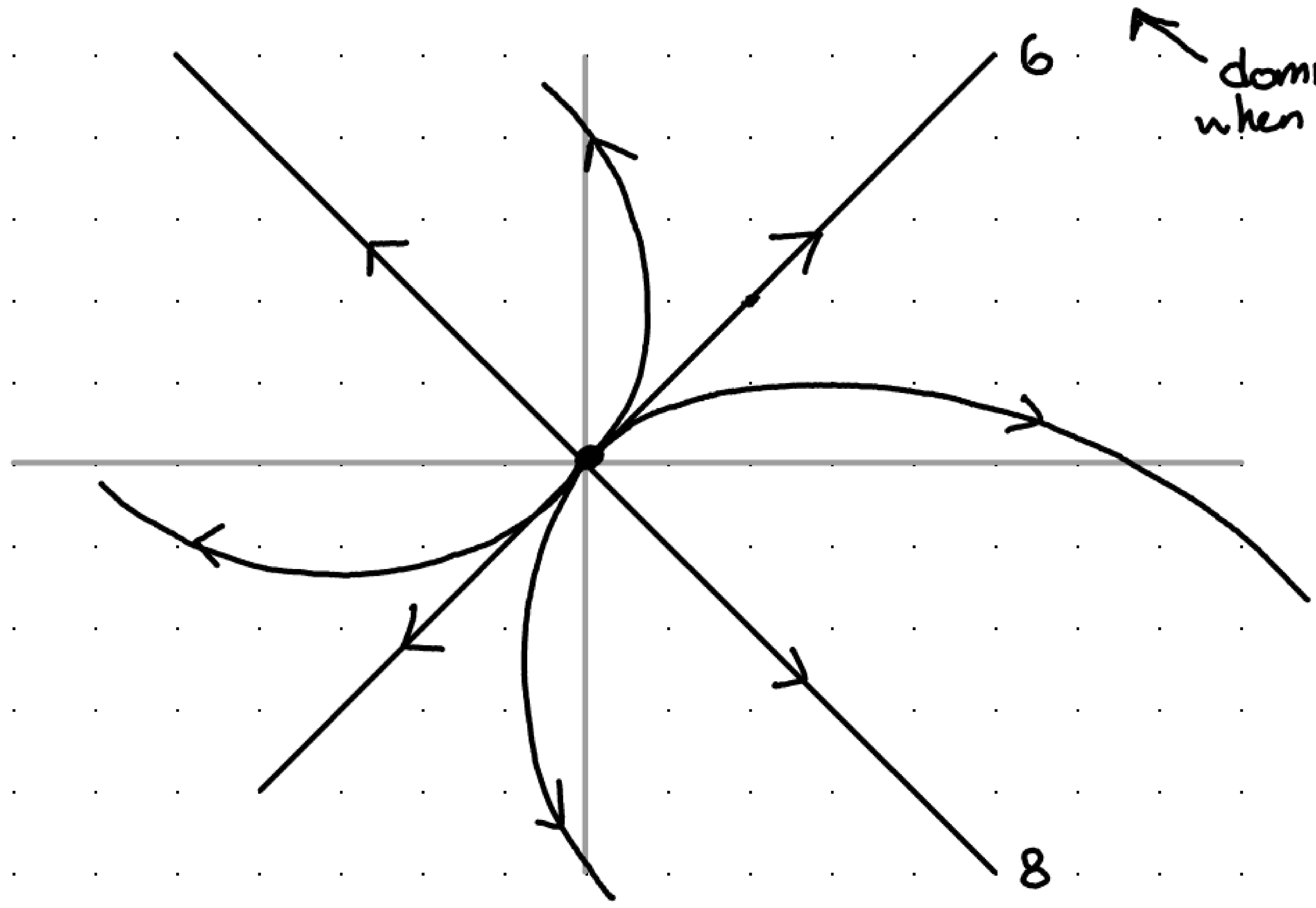
$$= (\lambda - 6)(\lambda - 8) \qquad \text{eigenvals } 6, 8$$

$\underline{\lambda = 6}$ $\quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{matrix} a - b = 0 \\ -a + b = 0 \end{matrix} \rightsquigarrow \underline{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

$\lambda = 8$ $\quad \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{matrix} -a - b = 0 \\ -a - b = 0 \end{matrix} \rightsquigarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

<u>general solution</u>: $\vec{x} = a e^{6t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\, \underline{e^{8t}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

dominant when $t \to -\infty$ | dominant when $t \to \infty$



Imaginary eigenvalues

$$\begin{cases} x' = y \\ y' = -x \end{cases} \qquad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\boxed{i \qquad \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix}}$$

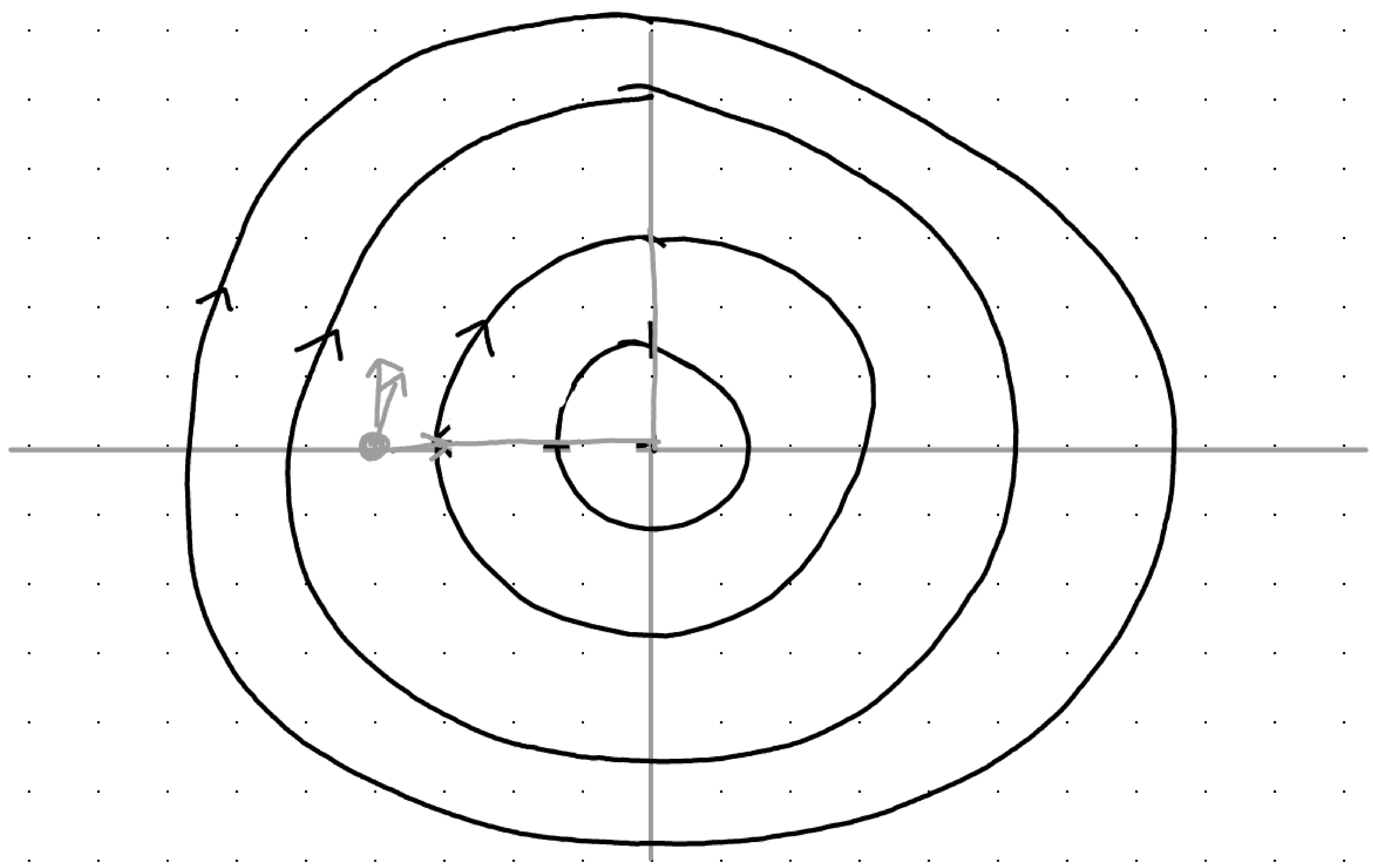
$$-i \qquad \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

this has eigenvalues $\pm i$ w/ eigenvectors $\begin{pmatrix} \mp i \\ 1 \end{pmatrix}$

generally, eigenvalues $a \pm bi$ w/ eigenvectors $\vec{v}_1 \pm i\vec{v}_2$ will have solution

$$\vec{x} = e^{at}\left((c_1 \cos bt + c_2 \sin bt)\vec{v}_1 + (-c_1 \sin bt + c_2 \cos bt)\vec{v}_2 \right)$$

$$\vec{x} = (c_1 \cos t + c_2 \sin t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + (-c_1 \sin t + c_2 \cos t)\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$



to find direction of rotation, check a point.

Complex eigenvalues

$$\begin{cases} x' = -4x - 17y \\ y' = 2x + 2y \end{cases}$$

eigenvalues $-1 \pm 5i$ w/ eigenvectors $\begin{pmatrix} -3 \\ 2 \end{pmatrix} \pm i \begin{pmatrix} 5 \\ 0 \end{pmatrix}$

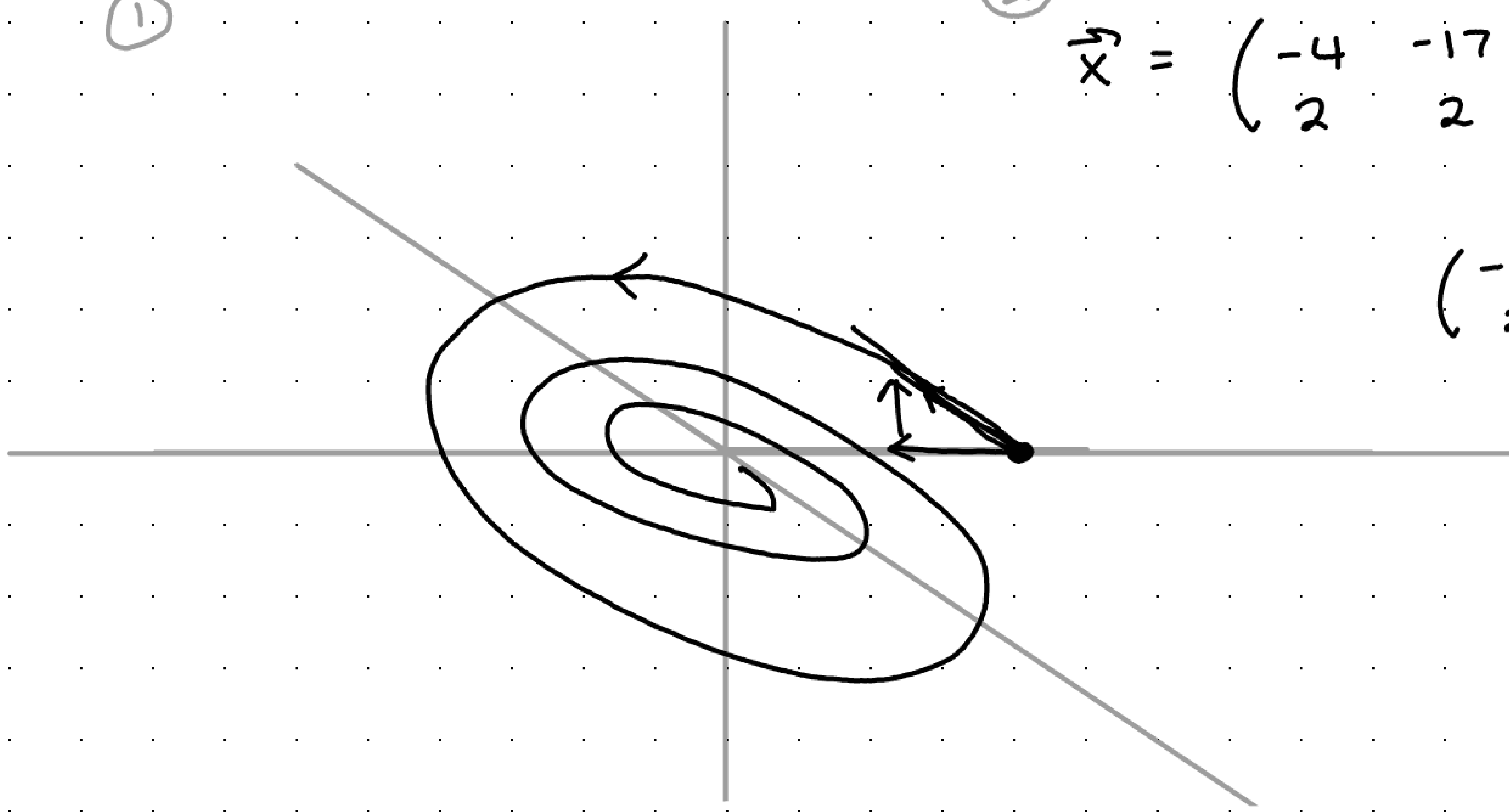
$-1 + \underline{5}i$ $\quad \underline{\begin{pmatrix} -3 \\ 2 \end{pmatrix}} + i \underline{\begin{pmatrix} 5 \\ 0 \end{pmatrix}}$

$$\vec{x} = e^{at}\left(\left(c_1 \cos bt + c_2 \sin bt\right)\vec{v}_1 + \left(-c_1 \sin bt + c_2 \cos bt\right)\vec{v}_2\right)$$

$$\vec{x} = \underbrace{e^{-t}}_{1}\left(\left(c_1 \cos 5t + c_2 \sin 5t\right)\underbrace{\begin{pmatrix} -3 \\ 2 \end{pmatrix}}_{2} + \left(-c_1 \sin 5t + c_2 \cos 5t\right)\begin{pmatrix} 5 \\ 0 \end{pmatrix}\right)$$

$$\vec{x} = \begin{pmatrix} -4 & -17 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad 3$$

$$\begin{pmatrix} -4 & -17 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$$



$$\begin{pmatrix} -4 & -17 \\ 2 & 2 \end{pmatrix}$$

$$\det \begin{pmatrix} -4-\lambda & -17 \\ 2 & 2-\lambda \end{pmatrix} = -8 + 4\lambda - 2\lambda + \lambda^2 + 34 = \lambda^2 + 2\lambda + 26$$

$$\lambda = \frac{-2 \pm \sqrt{4 - 104}}{2}$$

$$\boxed{\lambda = -1 \pm 5i} \longrightarrow \begin{pmatrix} 17 \\ -3-5i \end{pmatrix}$$

$\lambda = -1 + 5i$

$$\begin{pmatrix} -4 - (-1+5i) & -17 \\ 2 & 2-(-1+5i) \end{pmatrix} = \begin{pmatrix} -3-5i & -17 \\ 2 & 3-5i \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3-5i & -17 \\ 2 & 3-5i \end{pmatrix}\begin{pmatrix} 1 \\ w \end{pmatrix} \rightsquigarrow$$

$-3 - 5i - 17w = 0 \rightsquigarrow w = \dfrac{-3-5i}{17}$

$$2 + (3-5i)\frac{-3-5i}{17} = 0$$

$$34 + (3-5i)(-3-5i) = \cancel{34} - \cancel{9} + \cancel{15i} - \cancel{15i} - \cancel{25} = 0$$

9/4/25

Complex eigenvalues

$$\begin{cases} x' = 3x - 13y \\ y' = 5x + y \end{cases} \quad \text{w/} \quad \begin{matrix} x(0) = 3 \\ y(0) = -10 \end{matrix}$$

$$\vec{x}\,' = \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix} \vec{x}$$

$$\det \begin{pmatrix} 3-\lambda & -13 \\ 5 & 1-\lambda \end{pmatrix} = (3-\lambda)(1-\lambda) + 65 = 68 - 4\lambda + \lambda^2$$

eigenvals : $\lambda = \dfrac{4 \pm \sqrt{16 - 272}}{2} = \dfrac{4 \pm \sqrt{-256}}{2} = 2 \pm 8i$

$\lambda = 2 - 8i$

$$\begin{pmatrix} 1+8i & -13 \\ 5 & -1+8i \end{pmatrix} \begin{pmatrix} 1 \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad 1 + 8i - 13w = 0$$

$a+bi$

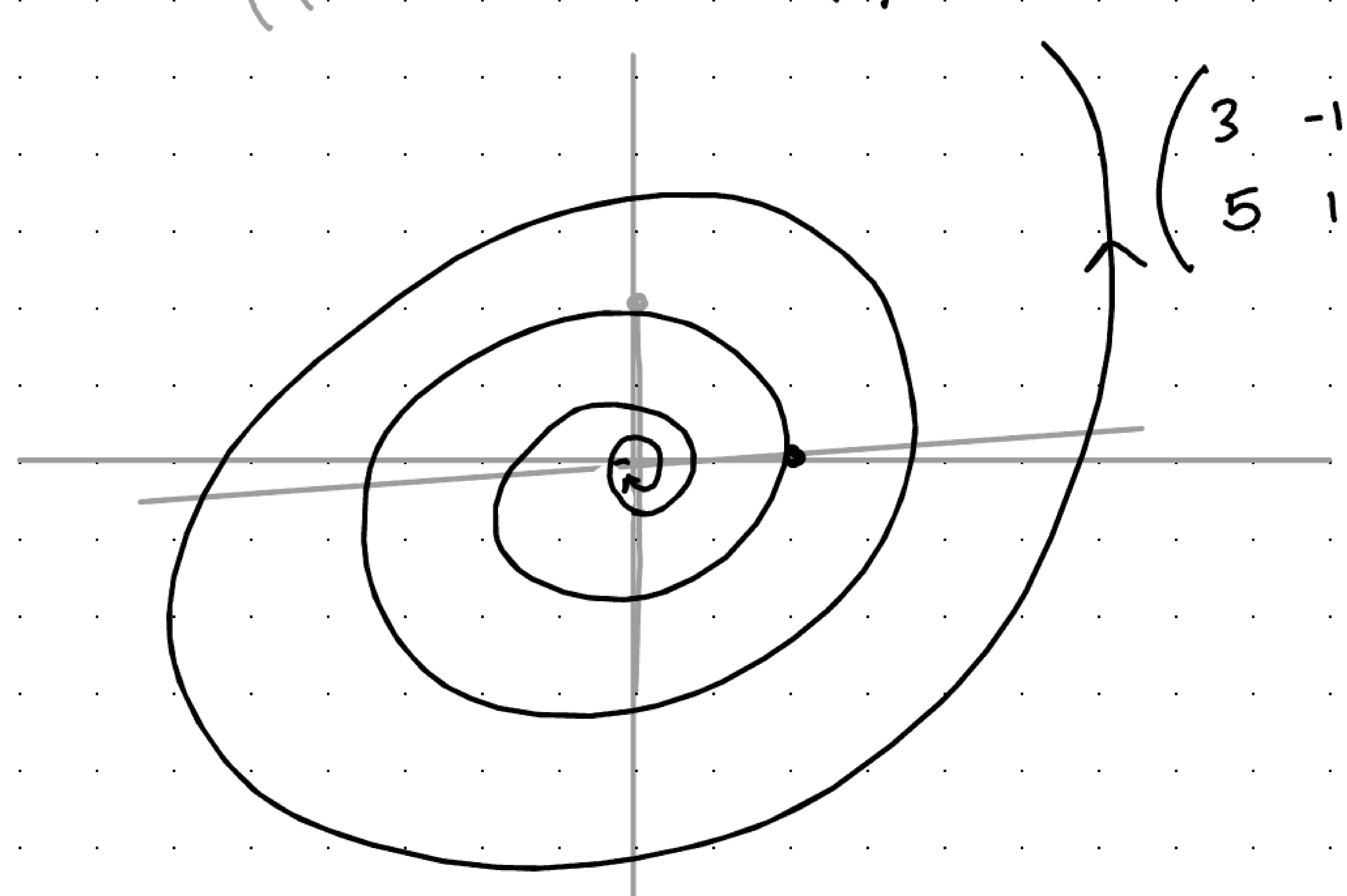
$$w = \frac{1}{13}(1+8i)$$

recommend checking 2^{nd} row

eigenvector: $\begin{pmatrix} 13 \\ 1+8i \end{pmatrix} = \begin{pmatrix} 13 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$2 + 8i \rightsquigarrow \begin{pmatrix} 13 \\ 1 \end{pmatrix} - i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\vec{x} = e^{2t}\left((c_1 \cos 8t + c_2 \sin 8t)\begin{pmatrix} 13 \\ 1 \end{pmatrix} - (-c_1 \sin 8t + c_2 \cos 8t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$



$$\begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Repeated Eigenvalues - 2 eigenvector (am = 2, gm = 2)

$$\begin{cases} x' = 4x \\ y' = 4y \end{cases}$$

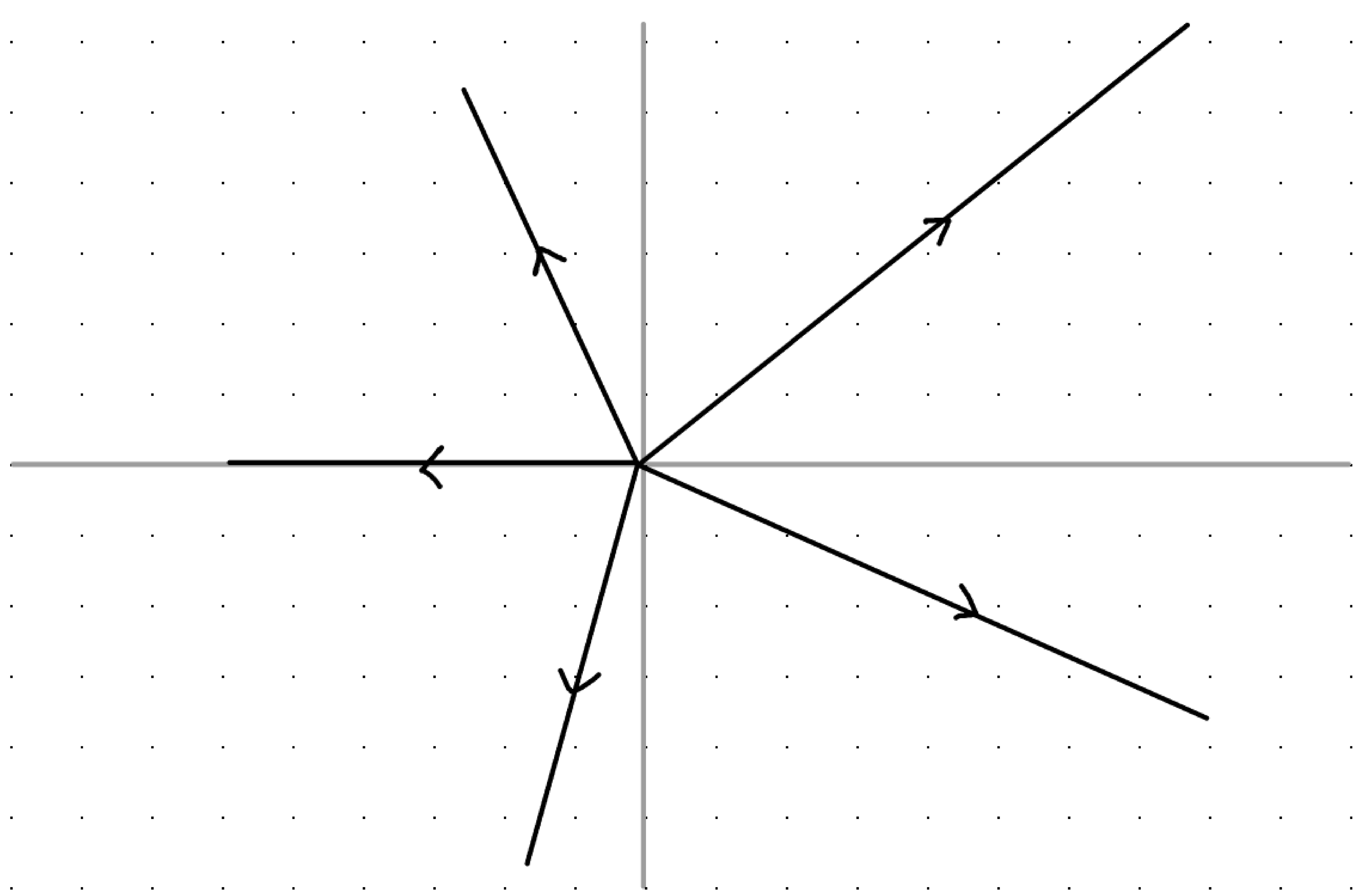
this is the only such example

$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$, $\det\begin{pmatrix} 4-\lambda & 0 \\ 0 & 4-\lambda \end{pmatrix} = (4-\lambda)^2$ eigenvals: 4

$\lambda = 4$ $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

general sol: $\vec{x} = ae^{4t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + be^{4t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\vec{x} = e^{4t}\begin{pmatrix} a \\ b \end{pmatrix}$$



Repeated Eigenvalues - 1 eigenvector $(am = 2,\ gm = 1)$

$$\begin{cases} x' = 7x + y \\ y' = -4x + 3y \end{cases} \qquad \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} \qquad \det\begin{pmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{pmatrix} = 21 - 10\lambda + \lambda^2 + 4 = (\lambda - 5)^2$$

eigenval $\underline{5}$ $\quad \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

to find general solution:

① pick any non-eigenvector $\vec{w}$ $\quad$ generalized eigenvector

$$\underline{\vec{w}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (A - 5I)^2\vec{w} = \vec{0}$$

② calculate $\vec{v}_1 = (A - \lambda I)\vec{w}$ (which is an eigenvector)

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \vec{v}_1$$

③ general solution: $\vec{x} = ae^{\lambda t}\vec{v}_1 + be^{\lambda t}(t\vec{v}_1 + \vec{w})$

$$\vec{x} = ae^{5t}\begin{pmatrix} 2 \\ -4 \end{pmatrix} + be^{5t}\left(t\begin{pmatrix} 2 \\ -4 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

④ solution curves: eigenvector line then test

$$\begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$$

Repeated eigenvalue (again)

$$\begin{cases} x' = -6x + 9y \\ y' = -x - 12y \end{cases}$$

Bonus: non-homogeneous systems (non-examinable unless otherwise stated).

$$\vec{x}' = \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}}\vec{x} + \underbrace{t\begin{pmatrix} 2 \\ -4 \end{pmatrix}}$$

idea: undetermined coefficients, but vectors.

<u>homogeneous</u>: $\det\begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix} = 2-3\lambda+\lambda^2-6 = \lambda^2-3\lambda-4$

$= (\lambda-4)(\lambda+1)$

$\lambda=4 \quad \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \vec{x_h} = ae^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + be^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda=-1 \quad \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

<u>particular</u>: $\vec{x_p} = \vec{v}t + \vec{u}$

$\vec{x}' - A\vec{x} = t\begin{pmatrix} 2 \\ -4 \end{pmatrix}$

$\underline{\vec{v}} - \underbrace{A\vec{v}t} - \underline{A\vec{u}} = \underbrace{t\begin{pmatrix} 2 \\ -4 \end{pmatrix}}$

$-A\vec{v} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

$\vec{v} = A\vec{u}$

$-A^2\vec{u} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 7 & 6 \\ 9 & 10 \end{pmatrix}$$

$A^2\vec{u} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 7 & 6 \\ 9 & 10 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} \rightsquigarrow \vec{u} = \frac{1}{8}\begin{pmatrix} -22 \\ 23 \end{pmatrix}$

so $\vec{x_p} = \frac{t}{8}A\begin{pmatrix} -22 \\ 23 \end{pmatrix} + \frac{1}{8}\begin{pmatrix} -22 \\ 23 \end{pmatrix}$

<u>general</u>: $\vec{x} = ae^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + be^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{t}{8}A\begin{pmatrix} -22 \\ 23 \end{pmatrix} + \frac{1}{8}\begin{pmatrix} -22 \\ 23 \end{pmatrix}$

Bonus: higher order constant coefficients

example: $y'''' - y''' + y'' - y' = x$ (challenging)

extra: $y''' - 3y'' - y' + 3y = e^{2x}$

9/9/25

1. systems of linear ODEs
2. linear first order ODE
3. modelling / related rates
4. constant coefficients + variation of parameters
5. reduction of order complete
6. existence/uniqueness see notes from 8/14
7. homogeneous equations
8. complex eigenvalues complete

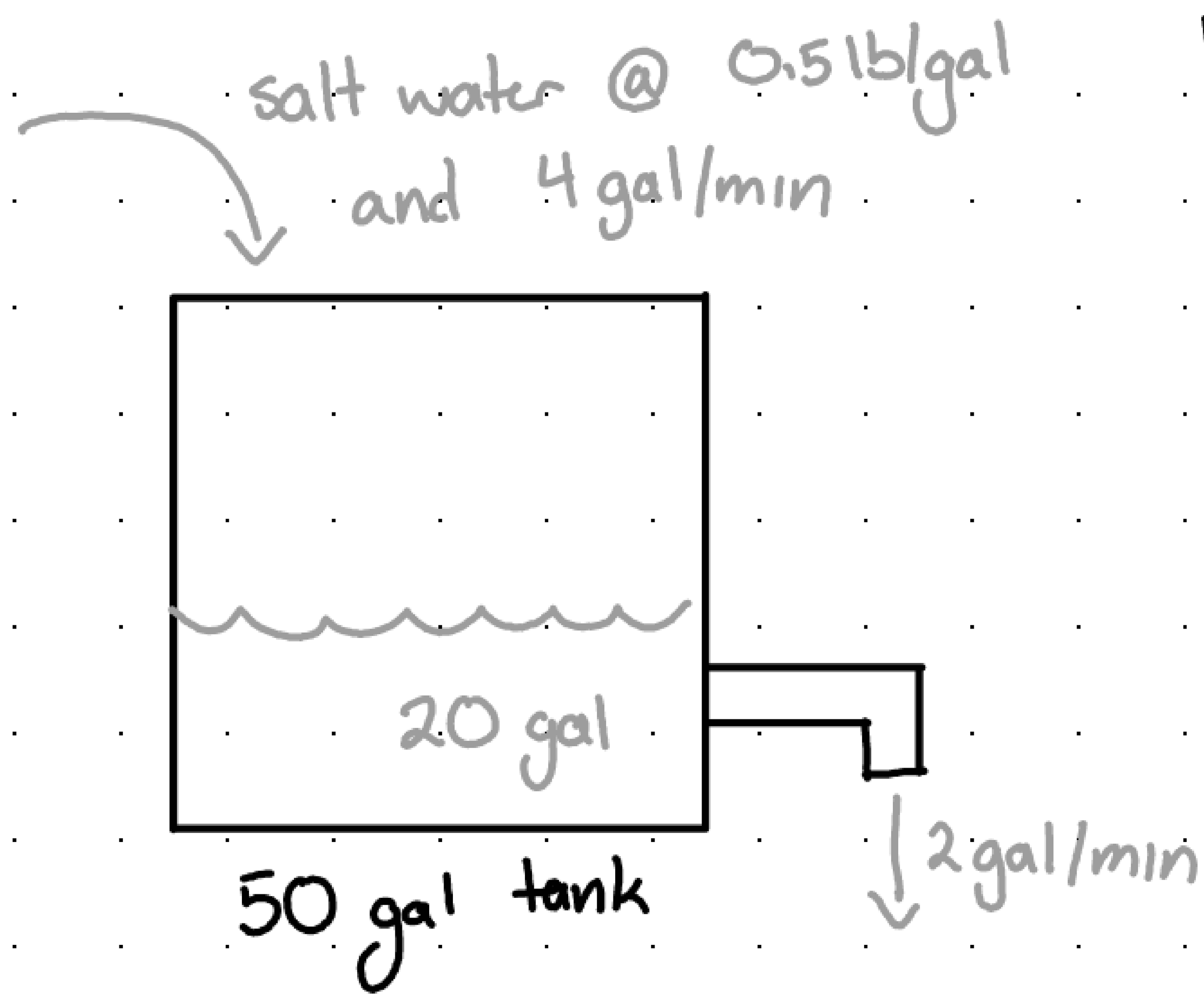
① systems of linear ODEs

$$\begin{cases} x' = -6x + 9y \\ y' = -x - 12y \end{cases} \quad \text{w/} \quad \begin{matrix} x(0) = 1 \\ y(0) = 1 \end{matrix}$$

② linear first order ODE

solve $\cos(x)\, y' + \sin(x)\, y = 2\cos^3(x)\sin(x) - 1 \qquad y(0) = 5$

③ related rates



What is the salt content when the tank is full?

④ Constant coefficients

find the general solution to $y'' - 2y' + y = \frac{e^t}{t^2+1}$

recall: $y_p = v_1 y_1 + v_2 y_2$ w/

$$v_1' = \frac{-y_2 g}{w(t)} \quad \text{and} \quad v_2' = \frac{y_1 g}{w(t)}$$

where $W(t) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$ is the Wronskian

⑤ Reduction of Order

find the general solution to $2t^2y'' + ty' - 3y = 0, \quad t>0$

given that $y_1 = t^{-1}$ is a solution

hint: guess $y_2 = vy_1$

$y_2 = v/t \quad vy_1$

$y_2' = v'/t - v/t^2 \quad vy_1'$

$y_2'' = v''/t - 2v'/t^2 + 2\,v/t^3 \quad vy_1''$

$\cancel{4\,v/t} - 4v' + 2t\,v'' + v' - \cancel{v/t} - \cancel{3\,v/t} = 0$

$2tv'' - 3v' = 0 \qquad w = v'$

$2tw' - 3w = 0$

$\frac{w'}{w} = \frac{3}{2}\,\frac{1}{t}$

$\int \frac{dw}{w} = \frac{3}{2}\int \frac{1}{t}dt$

$\ln w = \frac{3}{2}\ln t + c$

$w = c\,t^{3/2}$

$v' = c\,t^{3/2}$

$\boxed{v = c'\,t^{5/2} + b}$

$y_2 = c\,t^{3/2} + bt^{-1}$

⑥ existence/uniqueness

consider $y' = x\sqrt{|y|}$ w/ $y(0) = 0$

does a sol exist? Is it unique?

⑦ homogeneous equations

solve $xyy' + 4x^2 + y^2 = 0$ w/ $y(2) = -7$

⑧ Complex eigenvalues

$$\begin{cases} x' = 3x - 13y \\ y' = 5x + y \end{cases} \qquad \vec{x}(0) = \begin{pmatrix} 3 \\ -10 \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

real

$$\vec{x}' = \underbrace{\begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix}}_{A} \vec{x}$$

$$\det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & -13 \\ 5 & 1-\lambda \end{pmatrix} = (3-\lambda)(1-\lambda) + 65$$

$$= 3 - 4\lambda + \lambda^2 + 65 = \lambda^2 - 4\lambda + 68$$

136

272

$$\lambda = \frac{4 \pm \sqrt{16 - 272}}{2}$$

$$= \frac{4 \pm \sqrt{-256}}{2}$$

$$= 2 \pm 8i$$

complex

$$w = \frac{1+8i}{13}$$

$$\lambda = 2 - 8i \qquad \begin{pmatrix} 1+8i & -13 \\ 5 & -1+8i \end{pmatrix}\begin{pmatrix} 1 \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$1 + 8i - 13w = 0$$

$$5 + (-1+8i)w = 0 \quad \checkmark$$

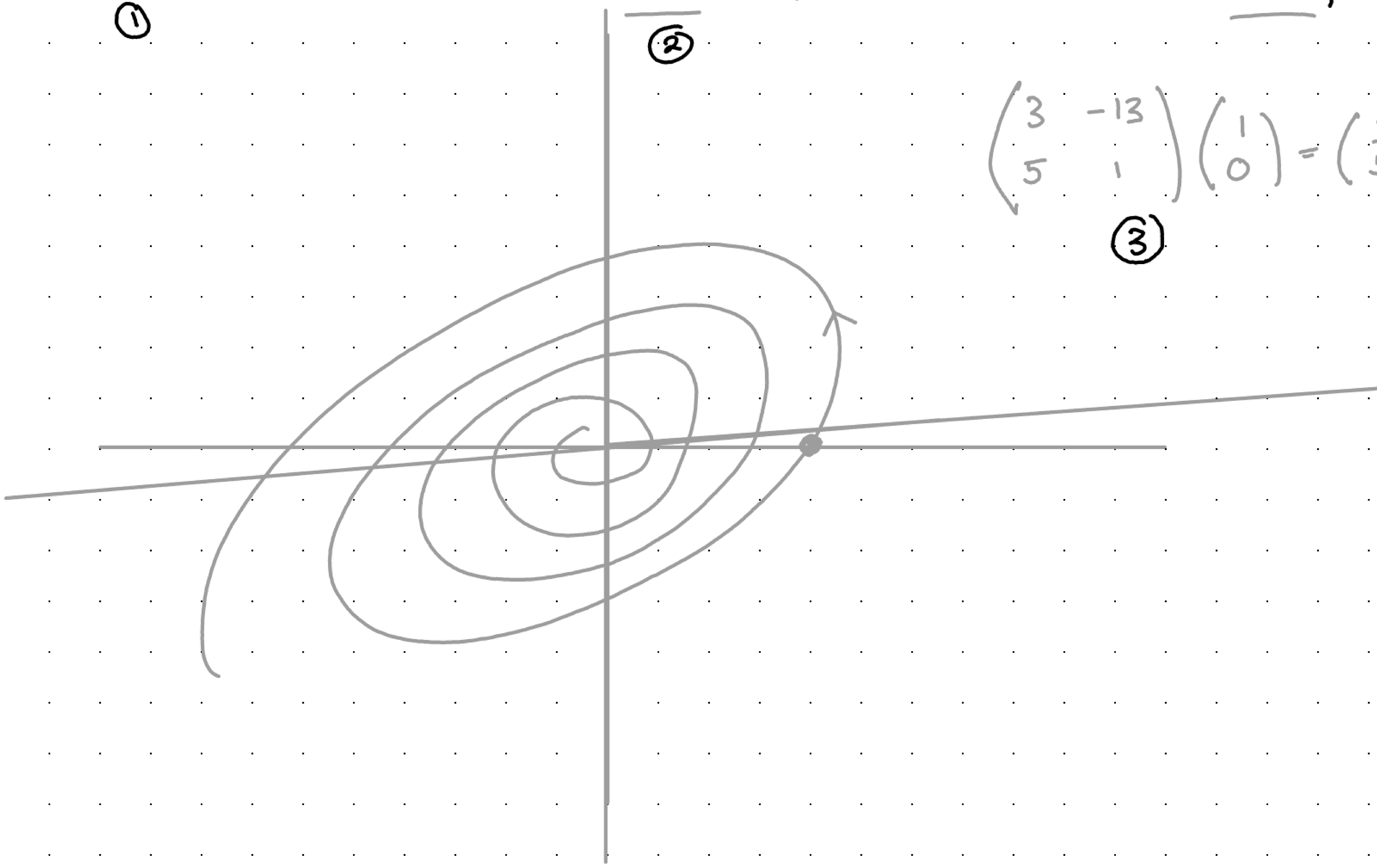
$$\vec{v} = \begin{pmatrix} 13 \\ 1+8i \end{pmatrix} = \begin{pmatrix} 13 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 8 \end{pmatrix}$$

$$\underline{\lambda = 2 + 8i} \rightsquigarrow \vec{u} = \begin{pmatrix} 13 \\ 1 \end{pmatrix} - i\begin{pmatrix} 0 \\ 8 \end{pmatrix}$$

$$\vec{x} = e^{at}\left((c_1 \cos bt + c_2 \sin bt)\vec{v_1} + (-c_1 \sin bt + c_2 \cos bt)\vec{v_2} \right)$$

$$\vec{x} = e^{2t}\left((c_1 \cos 8t + c_2 \sin 8t)\begin{pmatrix} 13 \\ 1 \end{pmatrix} + (-c_1 \sin 8t + c_2 \cos 8t)\begin{pmatrix} 0 \\ -8 \end{pmatrix} \right)$$

$$\vec{x} = \underset{①}{e^{2t}}\left((c_1\cos 8t + c_2\sin 8t)\underset{②}{\begin{pmatrix}13\\1\end{pmatrix}} + (-c_1\sin 8t + c_2\cos 8t)\begin{pmatrix}0\\-8\end{pmatrix}\right)$$



$$\begin{pmatrix}3 & -13\\5 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}3\\5\end{pmatrix}$$

③